FORMATION OF FINITE MULTIPLICATIVE GROUP FOR
\textit{n-LATERAL MULTIPLE HYPERGEOMETRIC FUNCTIONS}

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FORMATION OF
FINITE MULTIPLICATIVE GROUP FOR n-LATERAL MULTIPLE
HYPERGEOMETRIC FUNCTIONS

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Some properties of special functions from the point of view of group
theory, or more specifically, from the theory of group representations have
been discussed in research monographs. Here an attempt has been made
to generate finite multiplication group for multiple hypergeometric func-
tions involving two and three variables.

Key Words :- Lauricella function / multiplicative group / multiple
hypergeometric function / identity element.

1 :- INTRODUCTION

The fruitful nature of the theory of single hypergeometric functions
has led to generalizations, giving rise to multiple hypergeometric func-
tions compiled by Exton. Equivalent systems of partial differential equa-
tions associated with triple hypergeometric functions have been again con-
structed by Exton. A new function has been generated by authors in
their earlier studies to meet the requirements of statisticians, whose prod-
uct in the Lauricella function pattern has led to the generation of n-lateral
multiple hypergeometric function. This general function, whose particu-
lar cases include almost all known multiple hypergeometric functions, till
date, is being represented in a numeral form for the formation of a finite
multiplicative group.

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Let us express the B- function, defined as

\[
B = \begin{bmatrix}
    a_p & \gamma_r \\
    \beta_q & \delta_r
\end{bmatrix}
\begin{bmatrix}
    x_1, x_2, \ldots, x_n
\end{bmatrix}
\]

\[
= \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} \frac{\left(\frac{(a_p)^n (\gamma_r)^n}{(\beta_q)^n (\delta_r)^n}\right) x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}}{m_1! m_2! \ldots m_n!}
\]

\text{Where} \quad n = m_1 + m_2 + \ldots + m_n

in the following form

\[
\sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} \left(\frac{(a_1^1 a_1 m_1 + a_2 m_2 + \ldots + (a_n^1 a_n m_n)}{a_1 m_1} \right) \frac{(\gamma_r)^n (\delta_r)^n}{m_1! m_2! \ldots m_n!}
\]

\text{When} \quad \sum_{i=0}^{\infty} a_i = 1 \quad \text{and} \quad a_i \text{ may be equal to 1 or 0.}

Supposing \((a_1, a_2, \ldots, a_n) = (1, 0, 0, \ldots, 0)\) the function given by (2.2) gets converted as

\[
B^0 = \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} \left(\frac{(a_1^1 a_1 m_1 + a_2 m_2 + \ldots + (a_n^1 a_n m_n)}{a_1 m_1} \right) \frac{(\gamma_r)^n (\delta_r)^n}{m_1! m_2! \ldots m_n!}
\]

\text{...........(2.3)}
Where \( a^1_p \) shall be obviously \( a_p \) and therefore shall be single hypergeometric function in \( n \) - variables.

In case if we take

\[
(a_1, a_2, \ldots, a_n) = (0,1,0,\ldots,0),
\]

We observe that the \( B \) - function is transmogrified into a Lauricella type function, in which the second parameter is repeated \( (n-1) \) times, having the value

\[
\sum_{m_1,m_2,\ldots,m_n=0}^{\alpha} \left( \frac{\binom{a^1_p}{m_1} \binom{a^2_p}{m_2} \ldots \binom{a_n}{m_n}}{\binom{\beta_q}{m_1+m_2+\ldots+m_n} \binom{\delta_s}{m_1+m_2+\ldots+m_n}} \right) \frac{x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}}{m_1! m_2! \ldots m_n!} \] ..........................(2.4)

Which represents trilateral hypergeometric function in \( n \) - variables.

This process of generalization, when prolonged further, can generate trilateral, quadrilateral, pentalateral and so on up to \( n \) - lateral hypergeometric functions in \( n \)-variables for corresponding values of \( (a_1, a_2, \ldots, a_n) \) equal to \( (1,0,0,\ldots,0), (0,1,0,\ldots,0), (0,0,1,0,\ldots,0), (0,0,0,1,\ldots,0) \) and \( (0,0,0,\ldots,1) \).

3:- FORMATION OF MULTIPLICATIVE GROUP

For the \( B \) -function expressed by (2.2) we shall have

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a & c \\
b & d \\
\end{bmatrix} = \sum_{m,n=0}^{\alpha} \left( \frac{\binom{a}{m+n} \binom{C}{m+n}}{\binom{b}{m+n} \binom{d}{m+n}} \right) \frac{x^m y^n}{m! n!} \] ..........................(3.1)

from which, according to the following explanation

{In
\[
\begin{bmatrix}
a \\
b \\
\end{bmatrix}
\] if upper parameter is changed, the notation shall be 1;
if both parameter are changed, the notation shall be 2;
if only lower parameter is changed, the notation shall be 3
and 0 shall correspond for the normal function.}

P.T.O. [3]
we can produce

\[
B^{10} = (a_1, a_2) = \left[ \begin{array}{c}
(a_2, a_1) \\
(b_1, b_2)
\end{array} \right] = \left[ \begin{array}{c}
(c_1, c_2) \\
(d_1, d_2)
\end{array} \right] = \left[ \begin{array}{c}
(1,0) \\
(1,0)
\end{array} \right]
\]

\[
= \sum_{m, n=0}^{\infty} \left( \frac{\alpha_m (\alpha^1)_n (\gamma)_{(m+n)}}{(\beta)_m (\beta^1)_n (\delta)_{(m+n)}} \right) \frac{x^m y^n}{m! n!} \quad \ldots\ldots\ldots(3.2)
\]

\[
B^{20} = (a_1, a_2) (b_1, b_2) = \left[ \begin{array}{c}
(a_2, a_1) \\
(b_2, b_1)
\end{array} \right] = \left[ \begin{array}{c}
(c_1, c_2) \\
(d_1, d_2)
\end{array} \right]
\]

\[
= \sum_{m, n=0}^{\infty} \left( \frac{\alpha_m (\alpha^1)_n (\gamma)_{(m+n)}}{(\beta)_m (\beta^1)_n (\delta)_{(m+n)}} \right) \frac{x^m y^n}{m! n!} \quad \ldots\ldots\ldots(3.3)
\]

\[
B^{30} = (b_1, b_2) = \left[ \begin{array}{c}
(a_1, a_2) \\
(b_2, b_1)
\end{array} \right] = \sum_{m, n=0}^{\infty} \left( \frac{\alpha_m (\alpha^1)_n (\gamma)_{(m+n)}}{(\beta)_m (\beta^1)_n (\delta)_{(m+n)}} \right) \frac{x^m y^n}{m! n!} \quad \ldots\ldots(3.4)
\]

\[
B^{01} = (c_1, c_2) = \left[ \begin{array}{c}
(a_1, a_2) \\
(b_1, b_2)
\end{array} \right]
\]

\[
= \sum_{m, n=0}^{\infty} \left( \frac{\alpha_m (\alpha^1)_n (\gamma)_{(m+n)}}{(\beta)_m (\beta^1)_n (\delta)_{(m+n)}} \right) \frac{x^m y^n}{m! n!} \quad \ldots\ldots(3.5)
\]

\[
B^{23} = (a_1, a_2) (b_1, b_2) (d_1, d_2) = \sum_{m, n=0}^{\infty} \left( \frac{\alpha_m (\alpha^1)_n (\gamma)_{(m+n)}}{(\beta)_m (\beta^1)_n (\delta)_{(m+n)}} \right) \frac{x^m y^n}{m! n!} \quad \ldots\ldots(3.6)
\]

P.T.O. [4]
Which in total shall be fifteen.
They shall form the elements of the finite multiplicative group \((S, \cdot)\), for which

\[
S = \{ B^{10} = (a_1, a_2), \quad B^{20} = (a_1, a_2) (b_1, b_2), \quad B^{30} = (b_1, b_2), \quad B^{01} = (c_1, c_2), \quad B^{02} = (c_1, c_2) (d_1, d_2), \quad B^{03} = (d_1, d_2), \quad B^{11} = (a_1, a_2) (c_1, c_2), \quad B^{12} = (a_1, a_2) (c_1, c_2) (d_1, d_2), \quad B^{13} = (a_1, a_2) (d_1, d_2), \quad B^{21} = (a_1, a_2) (b_1, b_2) (c_1, c_2), \quad B^{22} = (a_1, a_2) (b_1, b_2) (d_1, d_2), \quad B^{23} = (a_1, a_2) (d_1, d_2), \quad B^{31} = (b_1, b_2) (c_1, c_2), \quad B^{32} = (b_1, b_2) (c_1, c_2) (d_1, d_2), \quad B^{33} = (b_1, b_2) (d_1, d_2) \}\quad (3.7)
\]

The order of the group shall be

\[2^{p+q+r+s} - 1\]

and \(B^{22}\) shall be identity multiplication element.

Similarly for three variables, we can generate, in order to generate a group

\[
B = \begin{bmatrix} 1 & 1 & a & c \\ 1 & 1 & b & d \\ x, y, z \end{bmatrix} = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (c)_{(m+n+p)}}{(b)_{m+n+p} (d)_{(m+n+p)}} \frac{x^m y^n z^p}{m! \ n! \ p!}
\]

\[...............................(3.8)\]

So that the total element of the multiplicative group shall be 80.

Groups for functions involving more then three variables can be proliferated by the generalization of results obtained in this section.
REFERENCES

1- Anand Singh and H.S.Dhami, Generating function of hypergeometric functions from the view point of change in the nature of hypergeometric series, communi cated for publication.

2- Anand Singh and H.S.Dhami, n-lateral multiple hypergeometric functions, communi cated for publication.

3- Harold Exton (1976), Multiple hypergeometric functions and applications, Ellis Horwood Ltd., A division of John Wiley & Sons, Inc.


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