OSCILLATIONS AND CONCENTRATIONS IN SEQUENCES OF GRADIENTS

By

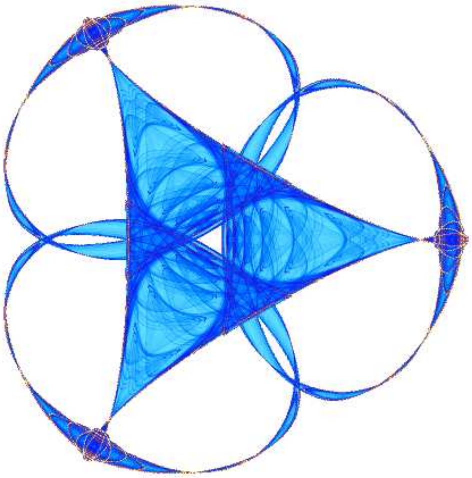
Agnieszka Kałamajska

and

Martin Kružík

IMA Preprint Series # 2069

(October 2005)



INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455–0436
Phone: 612/624-6066 Fax: 612/626-7370
URL: http://www.ima.umn.edu

OSCILLATIONS AND CONCENTRATIONS IN SEQUENCES OF GRADIENTS*

AGNIESZKA KAŁAMAJSKA† AND MARTIN KRUŽÍK‡

Abstract. We use DiPerna's and Majda's generalization of Young measures to describe oscillations and concentrations in sequences of gradients, $\{\nabla u_k\}$, bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ if $p>1$ and $\Omega\subset\mathbb{R}^n$ is a bounded domain with a Lipschitz boundary. Our main results are necessary and also sufficient conditions on a DiPerna-Majda measure to be generated by gradients and, particularly, the explicit characterization of those gradient measures which are generated by gradients of Sobolev maps with the same prescribed trace.

Key words. Bounded sequences of gradients, concentrations, oscillations, quasiconvexity.

AMS subject classifications. 49J45, 35B05

1. Introduction. Oscillations and/or concentrations appear in many problems in the calculus of variations, partial differential equations, or optimal control theory, which admit only L^p but not L^∞ apriori estimates. While Young measures [32] successfully capture oscillatory behavior of sequences they completely miss concentrations. There are several tools how to deal with concentrations. They can be considered as generalization of Young measures, see for example Alibert's and Bouchitté's approach [1], DiPerna's and Majda's treatment of concentrations [6], or Fonseca's method described in [10]. An overview can be found in [25, 29]. Moreover, in many cases, we are interested in oscillation/concentration effects generated by sequences of gradients. A characterization of Young measures generated by gradients was completely given by Kinderlehrer and Pedregal [14, 15], cf. also [22, 23]. To our knowledge, the first attempt to characterize both oscillations and concentrations in sequences of gradients is due to Fonseca, Müller, and Pedregal [11]. They describe concentrations by means of a varifold while oscillations by gradient Young measures. The authors give necessary and sufficient conditions on the varifold, so that they can fully describe effects of concentrations and oscillations on sequences of integrands $\{gv(\nabla u_k)\}_{k\in\mathbb{N}}$ where $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$, $v/(1+|\cdot|^p)$, $p>1$, is real-valued and has a continuous extension on the compactification of $\mathbb{R}^{m\times n}$ by the sphere, and $g:\Omega\to\mathbb{R}$ is continuous and vanishes on the boundary of a bounded domain $\Omega\subset\mathbb{R}^n$.

The aim of our paper is to derive necessary and sufficient conditions for a general DiPerna-Majda measure to be generated by a sequence of gradients. In particular, our main results stated in Theorem 1.5 and Theorem 1.6 generalize necessary and sufficient conditions given in [11], because we describe behavior of sequences $\{gv(\nabla u_k)\}_{k\in\mathbb{N}}$, where $v/(1+|\cdot|^p)$, $p>1$ has continuous extension on a general metrizable compactification of $\mathbb{R}^{m\times n}$, possibly finer than the one by the sphere, and

---

[*] The work of A.K. was supported by the IM PAN and the KBN grant No. 1-PO3A-008-29 while M.K. was supported by the grants IAA 1075402 (GA AV ČR) and VZ6840770021 (MŠMT ČR).

[†] Institute of Mathematics, Warsaw University, ul. Banacha 2, PL-02-097 Warsaw, Poland. This research was done while A.K. was visiting Institute of Mathematics of the Polish Academy of Sciences at Warsaw in the academic year 2004/2005 (`Agnieszka.Kalamajska@mimuw.edu.pl`)

[‡] Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic (corresponding address) & Faculty of Civil Engineering, Czech Technical University, Thákurova 7, CZ-166 29 Praha 6, Czech Republic (`kruzik@utia.cas.cz`) (corresponding author)

$g \in C(\bar\Omega)$. Besides this generalization we state a few results which are of an independent interest. Particularly, it is Lemma 2.4 and Lemma 3.1 showing local and averaging properties of DiPerna-Majda measures, respectively. Our method benefits from the explicit characterization of DiPerna-Majda measures generated by unconstrained sequences given in [18], see also [19] where numerical issues are discussed in detail.

1.1. Basic notation..

Let us start with a few definitions and with the explanation of our notation. Having a bounded domain $\Omega \subset \mathbb{R}^n$ we denote by $C(\Omega)$ the space of continuous functions: $\Omega \to \mathbb{R}$. In what follows "rca(S)" denotes the set of regular countably additive set functions on the Borel σ-algebra on a metrizable set S (cf. [7]), its subset, $\mathrm{rca}_1^+(S)$, denotes regular probability measures on a set S. We write "γ-almost all" or "γ-a.e." if we mean "up to a set with the γ-measure zero". If γ is the n-dimensional Lebesgue measure and $M \subset \mathbb{R}^n$ we omit writing γ in the notation. By $L^p(\Omega,\mu)$ we denote the usual Lebesgue space equipped with the measure μ. We omit μ if it is the Lebesgue measure. Further, $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \le p < +\infty$ denotes the usual space of measurable mappings which are together with their first (distributional) derivatives integrable with the p-th power. The support of a measure $\sigma \in \mathrm{rca}(\Omega)$ is a smallest closed set S such that $\sigma(A) = 0$ if $S \cap A = \emptyset$. Finally, if $\sigma \in \mathrm{rca}(\bar\Omega)$ we write σ_s and d_σ for the singular part and density of σ defined by the Lebesgue decomposition, respectively. Finally, we denote by 'w-lim' the weak limit.

If Ω is a Borel subset of $\mathbb{R}^n$, $\mu \in rca^+(\Omega)$ and $u \in L^1(\Omega,\mu)$ by $\mathcal{L}_u^\mu$ we denote the set of all Lebesgue points of u with respect to μ. If μ is the Lebesgue measure we simply write $\mathcal{L}_u$.

If not said otherwise, we will suppose in the sequel that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary. Some generalizations to less regular domains are possible, however they seem to be technically much more involved.

1.2. Quasiconvex functions.

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. We say that a function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if for any $s_0 \in \mathbb{R}^{m\times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$

$$v(s_0)|\Omega| \le \int_\Omega v(s_0 + \nabla\varphi(x))\,\mathrm{d}x \ .$$

If $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is not quasiconvex we define its quasiconvex envelope $Qv : \mathbb{R}^{m\times n} \to \mathbb{R}$ as

$$Qv = \sup\left\{h \le v;\ h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex } \right\} \ .$$

If v is locally bounded and Borel measurable then for any $s_0 \in \mathbb{R}^{m\times n}$ (see [5])

$$Qv(s_0) = \inf_{\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|}\int_\Omega v(s_0 + \nabla\varphi(x))\,\mathrm{d}x \ . \tag{1.1}$$

If $|v(s)| \le C(1 + |s|^p)$ for some $C > 0$ and all $s \in \mathbb{R}^{m\times n}$ then equivalently

$$Qv(s_0) = \inf_{\varphi\in W_0^{1,p}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|}\int_\Omega v(s_0 + \nabla\varphi(x))\,\mathrm{d}x \ ,$$

as pointed out in [11]. We refer to [3] for the notion of $W^{1,p}$-quasiconvexity.

Let us point out that

$$Qv(s_0) = \inf_{\varphi \in W^{1,p}_{s_0}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|} \int_\Omega v(\nabla\varphi(x))\,\mathrm{d}x \ ,$$

where $W^{1,p}_{s_0}(\Omega;\mathbb{R}^m) = \{\varphi \in W^{1,p}(\Omega;\mathbb{R}^m);\ \varphi(x) = s_0 x \text{ on } \partial\Omega\ \}$.

We will also need the following elementary result. It can be found in a more general form e.g. in [5, Ch. 4, Lemma 2.2] or in [21].

Lemma 1.1. *Let* $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ *be quasiconvex with* $|v(s)| \le C(1 + |s|^p)$, $C > 0$, *for all* $s \in \mathbb{R}^{m\times n}$. *Then there is a constant* $\alpha \ge 0$ *such that for every* $s_1, s_2 \in \mathbb{R}^{m\times n}$ *it holds*

$$|v(s_1) - v(s_2)| \le \alpha(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2| \ . \tag{1.2}$$

1.3. Young measures. For $p \ge 0$ we define the following subspace of the space $C(\mathbb{R}^{m\times n})$ of all continuous functions on $\mathbb{R}^{m\times n}$:

$$C_p(\mathbb{R}^{m\times n}) = \{v \in C(\mathbb{R}^{m\times n}); v(s) = o(|s|^p) \text{ for } |s| \to \infty\} \ .$$

The Young measures on a bounded domain $\Omega \subset \mathbb{R}^n$ are weakly* measurable mappings $x \mapsto \nu_x : \Omega \to \mathrm{rca}(\mathbb{R}^{m\times n})$ with values in probability measures; and the adjective "weakly* measurable" means that, for any $v \in C_0(\mathbb{R}^{m\times n})$, the mapping $\Omega \to \mathbb{R} : x \mapsto \langle \nu_x, v\rangle = \int_{\mathbb{R}^{m\times n}} v(\lambda)\nu_x(\mathrm{d}\lambda)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem, $\mathrm{rca}(\mathbb{R}^{m\times n})$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^{m\times n})^*$, where $C_0(\mathbb{R}^{m\times n})$ stands for the space of all continuous functions $\mathbb{R}^{m\times n} \to \mathbb{R}$ vanishing at infinity. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$. It is known that $\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{m\times n})) \cong L^1(\Omega;C_0(\mathbb{R}^{m\times n}))^*$, where the subscript "w" indicates the property "weakly* measurable". A classical result [28, 31] is that, for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega;\mathbb{R}^{m\times n})$, there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall v \in C_0(\mathbb{R}^{m\times n}) : \qquad \lim_{k\to\infty} v \circ y_k = v_\nu \qquad \text{weakly* in } L^\infty(\Omega) \ , \tag{1.3}$$

where $[v \circ y_k](x) = v(y_k(x))$ and

$$v_\nu(x) = \int_{\mathbb{R}^{m\times n}} v(\lambda)\nu_x(\mathrm{d}\lambda) \ . \tag{1.4}$$

Let us denote by $\mathcal{Y}^\infty(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in $L^\infty(\Omega;\mathbb{R}^{m\times n})$. Note that (1.3) actually holds for any $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ continuous.

A generalization of this result was formulated by Schonbek [26] (cf. also [2]): if $1 \le p < +\infty$: for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ there exists its subsequence (denoted by the same indices) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall v \in C_p(\mathbb{R}^{m\times n}): \quad \lim_{k\to\infty} v\circ y_k = v_\nu \qquad \text{weakly in } L^1(\Omega)\ . \tag{1.5}$$

We say that $\{y_k\}$ generates ν if (1.5) holds.

Let us denote by $\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in $L^p(\Omega;\mathbb{R}^{m\times n})$.

We will use the following lemma from [11] concerning Young measures from $\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ which are generated by sequences of gradients. A similar result was also proved by Kristensen [16].

LEMMA 1.2. *Let* $1 < p < +\infty$ *and* $\Omega \subset \mathbb{R}^n$ *be an open bounded set and let* $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ *be bounded. Then there is a subsequence* $\{u_j\}_{j\in\mathbb{N}}$ *and a sequence* $\{z_j\}_{j\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ *such that*

$$\lim_{j\to\infty} |\{x\in\Omega;\ z_j(x)\neq u_j(x) \text{ or } \nabla z_j(x)\neq\nabla u_j(x)\}| = 0 \tag{1.6}$$

and $\{|\nabla z_j|^p\}_{j\in\mathbb{N}}$ *is relatively weakly compact in* $L^1(\Omega)$*. In particular,* $\{\nabla u_j\}$ *and* $\{\nabla z_j\}$ *generate the same Young measure.*

1.4. DiPerna-Majda measures.

Let us take a complete (i.e. containing constants, separating points from closed subsets and closed with respect to the Chebyshev norm) separable ring $\mathcal{R}$ of continuous bounded functions $\mathbb{R}^{m\times n}\to\mathbb{R}$. It is known [8, Sect. 3.12.21] that there is a one-to-one correspondence $\mathcal{R}\mapsto\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$ between such rings and metrizable compactifications of $\mathbb{R}^{m\times n}$; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$, into which $\mathbb{R}^{m\times n}$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^{m\times n}$ and its image in $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$. Similarly, we will not distinguish between elements of $\mathcal{R}$ and their unique continuous extensions on $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$.

Let $\sigma\in\mathrm{rca}(\bar\Omega)$ be a positive Radon measure on a bounded domain $\Omega\subset\mathbb{R}^n$. A mapping $\hat\nu : x\mapsto\hat\nu_x$ belongs to the space $L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ if it is weakly* σ-measurable (i.e., for any $v_0\in C_0(\mathbb{R}^{m\times n})$, the mapping $\bar\Omega\to\mathbb{R}$: $x\mapsto\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)$ is σ-measurable in the usual sense). If additionally $\hat\nu_x\in\mathrm{rca}_1^+(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ for σ-a.a. $x\in\bar\Omega$ the collection $\{\hat\nu_x\}_{x\in\bar\Omega}$ is the so-called Young measure on $(\bar\Omega,\sigma)$ [32], see also [2, 25, 28, 30, 31].

DiPerna and Majda [6] shown that having a bounded sequence in $L^p(\Omega;\mathbb{R}^{m\times n})$ with $1\le p<+\infty$ and Ω an open domain in $\mathbb{R}^n$, there exists its subsequence (denoted by the same indices) a positive Radon measure $\sigma\in\mathrm{rca}(\bar\Omega)$ and a Young measure $\hat\nu$: $x\mapsto\hat\nu_x$ on $(\bar\Omega,\sigma)$ such that $(\sigma,\hat\nu)$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{m\times n})$ in the sense that $\forall g\in C(\bar\Omega)\ \forall v_0\in\mathcal{R}$:

$$\lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\mathrm{d}x = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} g(x)v_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)\ , \tag{1.7}$$

where

$$v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n}) := \{v_0(1+|\cdot|^p);\ v_0\in\mathcal{R}\}.$$

In particular, putting $v_0 = 1\in\mathcal{R}$ in (1.7) we can see that

$$\lim_{k\to\infty}(1+|y_k|^p)\ =\ \sigma \qquad \text{weakly* in } \mathrm{rca}(\bar\Omega)\ . \tag{1.8}$$

If (1.7) holds, we say that $\{y_k\}_{\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$. Let us denote by $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ the set of all pairs $(\sigma,\hat\nu)\in\mathrm{rca}(\bar\Omega)\times L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ attainable by sequences from $L^p(\Omega;\mathbb{R}^{m\times n})$; note that, taking $v_0=1$ in (1.7), one can see that these sequences must be inevitably bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$. The explicit description of the elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$, called DiPerna-Majda measures, for unconstrained sequences was done in [18, Theorem 2]. The central question which we are about to answer in this contribution is which $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ are generated by gradients, i.e., by $y_k:=\nabla u_k$, for $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ bounded. We denote the set of DiPerna-Majda measures from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ which are generated by gradients $\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Here we solve the case $1<p<+\infty$. Meanwhile $p=+\infty$ excludes concentrations and is completely described by gradient Young measures ([14]), the case $p=1$ is much more involved because of the loss of reflexivity and one must work with fine extensions of $W^{1,1}$.

Alternatively, DiPerna and Majda [6] worked with measures from $\mathrm{rca}(\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$; let us put here

$$\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})=\Big\{\eta\in\mathrm{rca}(\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n});\ \exists\{y_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{m\times n})$$

$$\forall h_0\in C(\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n}):\ \langle\eta,h_0\rangle=\lim_{k\to\infty}\int_\Omega h_0(x,y_k(x))(1+|y_k(x)|^p)\mathrm{d}x\Big\}\,.$$

Without causing any misunderstanding, the elements of $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ will be addressed as DiPerna-Majda measures too. We write $\eta\cong(\sigma,\hat\nu)$ for $\eta\in\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ if $\langle\eta,h_0\rangle\equiv\int_{\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}h_0(x,s)\eta(\mathrm{d}x\mathrm{d}s)=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}h_0(x,s)\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)$ for any $h_0\in C(\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$. It is known [25] that $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ is a convex, closed, non-compact but locally compact and locally sequentially compact subset of the locally convex space $\mathrm{rca}(\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ considered in its weak* topology. We say that $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ is homogeneous if $x\mapsto\hat\nu_x$ is constant. This implies that σ is absolutely continuous with respect to the Lebesgue measure with a constant density d_σ. See formula (1.11) below.

Let us recall that for any $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ there is precisely one $(\sigma^\circ,\hat\nu^\circ)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\int_\Omega\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x)\qquad(1.9)$$

for any $v_0\in C_0(\mathbb{R}^{m\times n})$ and any $g\in C(\bar\Omega)$ and $(\sigma^\circ,\hat\nu^\circ)$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}}$ such that the set $\{|y_k|^p;\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$; see [18, 25] for details. We call $(\sigma^\circ,\hat\nu^\circ)$ the nonconcentrating modification of $(\sigma,\hat\nu)$. We call $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ nonconcentrating if

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)=0\,.$$

There is a one-to-one correspondence between nonconcentrating DiPerna-Majda measures and Young measures; cf. [25].

We wish to emphasize the following fact: if $\{y_k\}\in L^p(\Omega;\mathbb{R}^{m\times n})$ generates $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and σ is absolutely continuous with respect to the Lebesgue measure it generally `does not` mean that $\{|y_k|^p\}$ is weakly relatively compact in $L^1(\Omega)$. A simple examples can be found e.g. in [19, 25].

The following lemma recalls some facts about of the p-nonconcentrating modification. Proofs can be found in [18, Lemma 1., Th. 1,2] and [25, Prop. 3.2.17].

LEMMA 1.3. *Let* $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *and let* σ_s *be the singular part of* σ *(in the sense of Lebesgue's decomposition). Then*

$$\mathrm{supp}\ \sigma_s\subset A_{\hat\nu}:=\left\{x\in\bar\Omega;\ \int_{\mathbb{R}^{m\times n}}\hat\nu_x(\mathrm{d}s)=0\right\}\ .$$

The support of σ_s*, denoted by* supp σ_s*, equals to*

$$A^\sigma=A_{\hat\nu}\setminus\bigcup_{B\subset A_{\hat\nu},\sigma(B)=0}B$$

in the sense that $\sigma_s(A)=\sigma_s(\bar\Omega)$ *for any Borel set* A *such that* $A^\sigma\subset A\subset\bar\Omega$.

Moreover, if $(\sigma^\circ,\hat\nu^\circ)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *be the* p*-nonconcentrating modification of* $(\sigma,\hat\nu)$ *then for almost all* $x\in\Omega$

$$d_{\sigma^\circ}(x)=\left(\int_{\mathbb{R}^{m\times n}}\hat\nu_x(\mathrm{d}s)\right)d_\sigma(x)$$

and

$$\hat\nu^\circ_x(\mathrm{d}s)=\frac{[\hat\nu_x|_{\mathbb{R}^{m\times n}}](\mathrm{d}s)}{\int_{\mathbb{R}^{m\times n}}\hat\nu_x(\mathrm{d}s)}\ ,$$

where d_{σ° *and* d_σ *are densities (with respect to the Lebesgue measure) of* σ° *and* σ*, respectively.*

Having a sequence bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ generating a DiPerna-Majda measure $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ it also generates an L^p-Young measure $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$. It easily follows from [25, Th. 3.2.13] that

$$\nu_x(\mathrm{d}s)=d_{\sigma^\circ}(x)\frac{\hat\nu^\circ_x(\mathrm{d}s)}{1+|s|^p}\quad\text{for a.a. } x\in\Omega\ . \tag{1.10}$$

Note that (1.10) is well-defined as $\hat\nu^\circ_x$ is supported on $\mathbb{R}^{m\times n}$. As pointed out in [18, Remark 2] for almost all $x\in\Omega$

$$d_\sigma(x)=\left(\int_{\mathbb{R}^{m\times n}}\frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\right)^{-1}\ . \tag{1.11}$$

In view of Lemma 1.3 we see that (1.9) can be even improved to

$$\int_\Omega\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x) \tag{1.12}$$

for any $v_0\in\mathcal{R}$ and any $g\in C(\bar\Omega)$. The one-to-one correspondence between Young and DiPerna-Majda measures, in particular (see (1.10) and (1.12))

$$\int_{\mathbb{R}^{m\times n}}v(s)\nu_x(ds)=d_\sigma(x)\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(ds)$$

whenever $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$, finally yields that $\forall g\in C(\bar\Omega)\ \forall v_0\in\mathcal{R}$:

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\mathrm{d}x=&\int_\Omega\int_{\mathbb{R}^{m\times n}}v(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x\\&+\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)\ ,\end{aligned} \tag{1.13}$$

where $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ and $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ are Young and DiPerna-Majda measures generated by $\{y_k\}_{k\in\mathbb{N}}$, respectively. The following proposition from [18] explicitly characterizes elements of $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$.

PROPOSITION 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, $\mathcal{R}$ be a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^{m\times n}$ and $(\sigma,\hat\nu) \in \mathrm{rca}(\bar\Omega) \times L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ and $1 \le p < +\infty$. Then the following two statements are equivalent with each other:*

(i) *the pair $(\sigma,\hat\nu)$ is the DiPerna-Majda measure, i.e. $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$,*

(ii) *The following properties are satisfied simultaneously:*
   1. *σ is positive,*
   2. *$\sigma_{\hat\nu} \in \mathrm{rca}(\bar\Omega)$ defined by $\sigma_{\hat\nu}(\mathrm{d}x) = (\int_{\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s))\sigma(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure ($d_{\sigma_{\hat\nu}}$ will denote its density),*
   3. *for a.a. $x \in \Omega$ it holds*

$$\int_{\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s) > 0, \qquad d_{\sigma_{\hat\nu}}(x) = \left(\int_{\mathbb{R}^{m\times n}} \frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\right)^{-1} \int_{\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s)\ ,$$

   4. *for σ-a.a. $x \in \bar\Omega$ it holds*

$$\hat\nu_x \ge 0, \qquad \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s) = 1\ .$$

REMARK 1.1. *As pointed out to us by M. Hušek and T. Roubíček having a metrizable compactification of $\mathbb{R}^{m\times n}$ we can construct a finer one as follows*

Consider a metrizable compactification $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$ of $\mathbb{R}^{m\times n}$ and the corresponding separable complete closed ring $\mathcal{R}$ with its dense subset $\{v_k\}_{k\in\mathbb{N}}$. We take a bounded continuous function $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$, $\psi \notin \mathcal{R}$ and take a closure (in the Chebyshev norm) of $\{\psi^j\}_{j\in\mathbb{N}\cup\{0\}} \cup \{\psi^j v_k\}_{k\in\mathbb{N}}^{j\in\mathbb{N}\cup\{0\}}$. As $\{\psi^j\} \cup \{\psi^j v_k\}$ is again countable the corresponding compactification is metrizable but strictly finer than $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$.

Our main result can be summarized to the following two theorems. The first one explicitly characterizes DiPerna-Majda measures generated by gradients of maps with the same trace.

THEOREM 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $1 < p < +\infty$ and $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Then then there is $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and a bounded sequence $\{u_k - u\}_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega;\mathbb{R}^m)$ such that $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$ if and only if following three conditions hold*

$$\text{for a.a. } x \in \Omega\text{: } \nabla u(x) = d_\sigma(x) \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\ , \tag{1.14}$$

for almost all $x \in \Omega$ and for all $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ the following inequality is fulfilled

$$Qv(\nabla u(x)) \le d_\sigma(x) \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\ , \tag{1.15}$$

for σ*-almost all* $x \in \bar{\Omega}$ *and all* $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ *with* $Qv > -\infty$ *it holds that*

$$0 \le \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\ . \tag{1.16}$$

Our next theorem addresses an arbitrary domain and DiPerna-Majda measures generated by gradients of maps with possibly different traces.

Theorem 1.6. *Let* Ω *be an arbitrary bounded domain,* $1 < p < +\infty$ *and* $(\sigma,\hat{\nu}) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *be generated by* $\{\nabla u_k\}_{k\in\mathbb{N}}$ *such that* $w\text{-}\lim_{k\to\infty} u_k = u$ *in* $W^{1,p}(\Omega;\mathbb{R}^m)$*. Then the conditions (1.14), (1.15) hold, and (1.16) is satisfied for* σ*-a.a.* $x \in \Omega$*.*

Example 1.2. *Consider* $n = m = 1$*. In this case, quasiconvexity reduces to usual convexity and the quasiconvex envelope of a function to its convex envelope. Further, take* $\mathcal{R}$*, the ring of continuous bounded functions* $\mathbb{R} \to \mathbb{R}$ *possessing limits at infinity, i.e.,* $v_0 \in \mathcal{R}$ *if* $\lim_{|s|\to\infty} v_0(s) =: v_0(\infty) \in \mathbb{R}$*. This ring corresponds to the Alexandroff (one-point) compactification of* $\mathbb{R}$*,* $\beta_{\mathcal{R}}\mathbb{R} \cong \mathbb{R}\cup\{\infty\}$*. Let* $\Omega = (0,1)$*,* $p = 2$*, and take* $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *such that*

$$\sigma(\mathrm{d}x) = \mathrm{d}x + \delta_1, \quad \hat{\nu}_x = \begin{cases} \delta_0 & \text{if} \quad x \in [0,1), \\ \delta_\infty & \text{if} \quad x = 1. \end{cases} \tag{1.17}$$

Recalling that $v(s) = v_0(s)(1+|s|^p)$ *and examining conditions in Theorem 1.5 we see that (1.14) is satisfied for* $u = 0$*, (1.15) holds if* $Qv(0) \le v_0(0)$ *and (1.16) requires* $v_0(\infty) \ge 0$ *if* $Qv > -\infty$*. We have* $v_0(0) = v(0)$ *and by the definition* $Qv(0) \le v(0)$*. Moreover,* $v_0(\infty) < 0$ *means that the convex envelope of* v *equals* $-\infty$*. Therefore, by Theorem 1.5* $(\sigma,\hat{\nu}) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$*. Its generating sequence* $\{u'_k\}$ *is*

$$u'_k(x) = \begin{cases} \sqrt{k} & \text{if} \quad x \in (1-\frac{1}{k},1] \\ 0 & \text{if} \quad x \in (\frac{1}{\sqrt[4]{k}}, 1-\frac{1}{k}] \\ -\frac{1}{k^{\frac14}} & \text{if} \quad x \in [0,\frac{1}{\sqrt[4]{k}}]\ , \end{cases} \tag{1.18}$$

so that

$$u_k(x) = \begin{cases} x\sqrt{k}-\sqrt{k} & \text{if} \quad x \in (1-\frac{1}{k},1] \\ -\frac{1}{\sqrt{k}} & \text{if} \quad x \in (\frac{1}{\sqrt[4]{k}}, 1-\frac{1}{k}] \\ -x\frac{1}{k^{\frac14}} & \text{if} \quad x \in [0,\frac{1}{\sqrt[4]{k}}]\ . \end{cases} \tag{1.19}$$

Finally, note that $u_k(0) = u_k(1) = 0$ *for all* k*.*

2. Necessary conditions. This section is devoted to the analysis of necessary conditions on $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ to be generated by gradients. We start with an easy Lemma whose proof is left to the reader.

Lemma 2.1. *Let* $M \subset \mathbb{R}^n$ *be a bounded Borel measurable set and* $\sigma\ ,\gamma \in \mathrm{rca}(M)$ *be nonnegative and such that for any* $g \in C(M)$*,* $g \ge 0$*,* $\int_M g(x)\,\sigma(\mathrm{d}x) \ge \int_M g(x)\,\gamma(\mathrm{d}x)$*. Then for any measurable* $A \subset M$ $\sigma(A) \ge \gamma(A)$*.*

LEMMA 2.2. *Let* $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *and an open domain* $\omega\subseteq\Omega$ *be such that* $\sigma(\partial\omega)=0$. *Let* $\{y_k\}_{k\in\mathbb{N}}$ *generate* $(\sigma,\hat\nu)$ *in the sense (1.7). Then for all* $v_0\in\mathcal{R}$ *and all* $g\in C(\bar\Omega)$

$$\lim_{k\to\infty}\int_\Omega v(y_k)g(x)\chi_\omega(x)\,\mathrm{d}x=\int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\chi_\omega(x)\,\sigma(\mathrm{d}x)\ , \tag{2.1}$$

where χ_ω *is the characteristic function of* ω *in* Ω.

Proof. Let $C^\infty_0(\omega)$ be the space of infinitely smooth functions on ω whose supports are in ω. First, we extend $\eta\in C^\infty_0(\omega)$, on $\bar\Omega$ by zero. We have possibly for a subsequence of $\{y_k\}$, any $g\in C(\bar\Omega)$ and any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega v(y_k)g(x)\eta(x)\,\mathrm{d}x&=\int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\eta(x)\,\sigma(\mathrm{d}x)\\&=\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\eta(x)\,\sigma(\mathrm{d}x)\end{aligned} \tag{2.2}$$

and

$$\lim_{k\to\infty}\int_\omega v(y_k)g(x)\eta(x)\,\mathrm{d}x=\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\eta(x)\,\tau(\mathrm{d}x)\ , \tag{2.3}$$

where $(\tau,\hat\mu)\in\mathcal{DM}^p(\omega;\mathbb{R}^{m\times n})$ is generated by the restriction of $\{y_k\}_k$ on ω, i.e., by $\{y_k|_\omega\}_{k\in\mathbb{N}}$.

As $C^\infty_0(\omega)$ is dense in $L^1(\omega,\tau+\sigma)$ (here we take the restriction of σ to ω) we have

$$\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\eta_j(x)\,\sigma(\mathrm{d}x)=\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\eta_j(x)\,\tau(\mathrm{d}x) \tag{2.4}$$

for a sequence $\{\eta_j\}\subset C^\infty_0(\omega)$, $\lim_{j\to\infty}\|\eta_j-1\|_{L^1(\omega;\tau+\sigma)}=0$. Finally, noticing that $x\mapsto\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)$ is in $L^\infty(\omega,\sigma)$ for any fixed $v_0\in\mathcal{R}$ and any $g\in C(\bar\Omega)$ we have by the Hölder inequality

$$\begin{aligned}\left|\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)(\eta_j(x)-1)\,\sigma(\mathrm{d}x)\right|&\le C\|\eta_j-1\|_{L^1(\omega,\sigma)}\\&\le C\|\eta_j-1\|_{L^1(\omega,\tau+\sigma)}\to0\ .\end{aligned}$$

We proceed similarly with the right-hand side term in (2.4) to obtain that

$$\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)=\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\,\tau(\mathrm{d}x)\ . \tag{2.5}$$

As (2.5) holds for an arbitrary subsequence of $\{y_k\}$ we see that it holds for the whole sequence $\{y_k\}$ generating $(\sigma,\hat\nu)$. Using Tietze's extension theorem we see that any $g\in C(\bar\omega)$ can be extended to a function belonging to $C(\bar\Omega)$ and therefore (2.5) reads that $\forall\ g\in C(\bar\omega)\ ,v_0\in\mathcal{R}$:

$$\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)=\int_\omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\,\tau(\mathrm{d}x)\ . \tag{2.6}$$

Consider a subsequence of $\{y_k\chi_\omega\}$ (not relabeled) and suppose that it generates $(\gamma,\hat\alpha)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Then we get

$$1+|y_k|^p\to\sigma\ ,\ 1+|y_k\chi_\omega|^p\to\gamma \text{ weakly* } .$$

Therefore by Lemma 2.1 $\gamma\le\sigma$ and thus, γ is absolutely continuous with respect to σ and it means $\gamma(\partial\omega)=0$. We have for any $g\in C(\bar\Omega)$ and any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$\lim_{k\to\infty}\int_\Omega v(y_k\chi_\omega)g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\alpha_x(\mathrm{d}s)g(x)\,\gamma(\mathrm{d}x)\ . \tag{2.7}$$

On the other hand, for any $g\in C(\bar\Omega)$ and any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$\begin{aligned}\int_{\Omega\setminus\omega}v(0)g(x)\,\mathrm{d}x+\lim_{k\to\infty}\int_\omega v(y_k)g(x)\,\mathrm{d}x&=\int_{\Omega\setminus\omega}v(0)g(x)\,\mathrm{d}x\\&\quad+\int_{\bar\omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\,\tau(\mathrm{d}x)\ .\end{aligned} \tag{2.8}$$

As the left hand sides of (2.7) and (2.8) are equal it implies that for any $g\in C(\bar\Omega)$ and any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\alpha_x(\mathrm{d}s)g(x)\,\gamma(\mathrm{d}x)=\int_{\Omega\setminus\omega}v(0)g(x)\,\mathrm{d}x+\int_{\bar\omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\,\tau(\mathrm{d}x)$$

we see that the whole sequence $\{y_k\chi_\omega\}$ generates $(\gamma,\hat\alpha)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and we get for $v_0(s)=1$, i.e., $v(s)=1+|s|^p$ and any $g\in C(\bar\Omega)$ that

$$\int_{\bar\Omega\setminus\bar\omega}g(x)\,\mathrm{d}x+\int_{\bar\omega}g(x)\,\tau(\mathrm{d}x)=\int_{\bar\Omega}g(x)\gamma(\mathrm{d}x)\ .$$

Therefore

$$\int_{\overline{\omega}}g(x)\tau(\mathrm{d}x)\le\int_{\overline{\Omega}}g(x)\gamma(\mathrm{d}x),$$

for any $0\le g\in C(\overline{\Omega})$. By an easy approximation argument (we approximate $g\chi_{\bar\omega}$ by continuous uniformly bounded functions) we derive

$$\int_{\overline{\omega}}g(x)\tau(dx)\le\int_{\overline{\omega}}g(x)\gamma(dx),$$

and Lemma 2.1 yields $\tau(\partial\overline{\omega})\le\gamma(\partial\overline{\omega})\le\sigma(\partial\overline{\omega})=0$. Hence we write instead of (2.6) $\forall\ g\in C(\bar\omega)\ ,v_0\in\mathcal{R}$:

$$\int_{\bar\omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)=\int_{\bar\omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\,\tau(\mathrm{d}x)\ . \tag{2.9}$$

Combining (2.3) and (2.9) keeping in mind that $\sigma(\partial\omega)=\tau(\partial\omega)=0$ we get

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega v(y_k)g(x)\chi_\omega(x)\,\mathrm{d}x&=\int_{\bar\omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\mu_x(\mathrm{d}s)g(x)\,\tau(\mathrm{d}x)\\&=\int_{\bar\omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\\&=\int_{\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\chi_\omega(x)\,\sigma(\mathrm{d}x)\ .\end{aligned}$$

□

Lemma 2.3. *Let* $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *be such that* σ *is absolutely continuous with respect to the Lebesgue measure and let* $(\sigma,\hat\nu)$ *be generated by a sequence* $\{\nabla u_k\}_{k\in\mathbb{N}}$ *where* $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$, $+\infty>p>1$ *and* $w\text{-}\lim_{k\to\infty}u_k=u$ *in* $W^{1,p}(\Omega;\mathbb{R}^m)$. *Then there is* $\{h_k-u\}_{k\in\mathbb{N}}\subset W^{1,p}_0(\Omega;\mathbb{R}^{m\times n})$ *such that* $\{\nabla h_k\}_{k\in\mathbb{N}}$ *generates* $(\sigma,\hat\nu)$.

Proof. Let $\{\eta_j\}_{j\in\mathbb{N}}$ be a sequence of continuous bounded functions such that for any $j\in\mathbb{N}$ $\eta_j(x)=0$ if $x\in\partial\Omega$, $\eta_j(x)=1$ if $x\in\Omega_j=\{x\in\Omega;\ \mathrm{dist}(x,\partial\Omega)>1/j\}$, $|\nabla\eta_j|\le cj$, $c>0$, and $|\eta_j|\le 1$ for $j\in\mathbb{N}$. Thus $\eta_j(x)\to\chi_\Omega(x)$ for all $x\in\Omega$. Let $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ be generated by $\{\nabla u_k\}$ such that $\{u_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$, $p>1$, is bounded and $\text{w-}\lim_{k\to\infty}u_k=u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$.

Consider $f_{jk}=\eta_ju_k+(1-\eta_j)u$. Then $\nabla f_{jk}=\eta_j\nabla(u_k-u)+\nabla u+(u_k-u)\otimes\nabla\eta_j$. In particular $\{f_{jk}-u\}\subset W^{1,p}_0(\Omega;\mathbb{R}^m)$. We denote DiPerna-Majda measures generated by (a subsequence in k) $\{\nabla f_{jk}\}_{k\in\mathbb{N}}$ $(\sigma^j,\hat\nu^j)$. We will use the inequality $|\sum_{i=1}^{\ell}a_i|^p\le C\sum_{i=1}^{\ell}|a_i|^p$, $p\ge 1$, $C=\ell^{p-1}$. Let us further fix $j\in\mathbb{N}$ and $0\le g\in C(\bar\Omega)$ and calculate a measure σ^j generated by $\{1+|\nabla f_{jk}|^p\}_{k\in\mathbb{N}}$

$$
\begin{aligned}
&\int_{\bar\Omega}g(x)\sigma^j(\mathrm{d}x)=\lim_{k\to\infty}\int_\Omega(1+|\nabla f_{jk}(x)|^p)g(x)\,\mathrm{d}x\\
&=\lim_{k\to\infty}\int_\Omega(1+|\eta_j(x)(\nabla u_k(x)-\nabla u(x))+\nabla u(x)+(u_k(x)-u(x))\otimes\nabla\eta_j(x)|^p)g(x)\,\mathrm{d}x\\
&\le C\lim_{k\to\infty}\int_\Omega(1+|\eta_j(x)\nabla u_k(x)|^p+|\nabla u(x)|^p)g(x)\,\mathrm{d}x\\
&+C\lim_{k\to\infty}\int_\Omega|(u_k(x)-u(x))\otimes\nabla\eta_j(x)|^pg(x)\,\mathrm{d}x\\
&\le C\lim_{k\to\infty}\int_\Omega(1+|\nabla u_k(x)|^p+|\nabla u(x)|^p)g(x)\,\mathrm{d}x\\
&+C\lim_{k\to\infty}\int_\Omega|(u_k(x)-u(x))\otimes\nabla\eta_j(x)|^pg(x)\,\mathrm{d}x\\
&=C\int_{\bar\Omega}g(x)\sigma(\mathrm{d}x)+C\int_\Omega|\nabla u(x)|^pg(x)\,\mathrm{d}x\\
&+C\lim_{k\to\infty}\int_\Omega(|(u_k(x)-u(x))\otimes\nabla\eta_j(x)|^p)g(x)\,\mathrm{d}x\\
&=C\int_\Omega g(x)\pi(\mathrm{d}x)\\
&+C\lim_{k\to\infty}\int_\Omega(|(u_k(x)-u(x))\otimes\nabla\eta_j(x)|^p)g(x)\,\mathrm{d}x=\int_{\bar\Omega}g(x)\pi(\mathrm{d}x)\ .
\end{aligned}
$$

where π is a measure on $\bar\Omega$ which is absolutely continuous to the Lebesgue measure with the density d_π, where for a.a. $x\in\Omega$, $d_\pi(x)=C(d_\sigma(x)+|\nabla u(x)|^p)$. We have noticed that $\lim_{k\to\infty}\int_\Omega|(u_k(x)-u(x))\otimes\nabla\eta_j(x)|^pg(x)\,\mathrm{d}x=0$ because $u_k\to u$ strongly in $L^p(\Omega;\mathbb{R}^m)$. Thus, we see that $\sigma^j\le\pi$. Since π is absolutely continuous with respect to the Lebesgue measure, so is σ^j. Let us denote its density by d_{σ^j}.

Lemma 2.2 applied to Ω_j and to $\Omega\backslash\bar{\Omega}_j$ says that for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$, $g \in C(\bar{\Omega})$

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega v(\nabla f_{jk}(x))g(x)\,\mathrm{d}x\\
&= \lim_{k\to\infty}\int_\Omega v(\eta_j(x)(\nabla u_k(x)-\nabla u(x))+\nabla u(x)+(u_k(x)-u(x))\otimes\nabla\eta_j(x))g(x)\,\mathrm{d}x\\
&= \lim_{k\to\infty}\int_\Omega v(\nabla u_k(x))\chi_{\Omega_j}(x)g(x)\,\mathrm{d}x\\
&+ \lim_{k\to\infty}\int_\Omega v(\nabla f_{jk})\chi_{\Omega\backslash\Omega_j}(x)g(x)\,\mathrm{d}x\\
&= \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)\chi_{\Omega_j}(x)d_\sigma(x)\,\mathrm{d}x\\
&+ \lim_{k\to\infty}\int_\Omega v(\nabla f_{jk}(x))\chi_{\Omega\backslash\Omega_j}(x)g(x)\,\mathrm{d}x\ .
\end{aligned}$$

It follows from Lemma 2.2 (if $\Omega\setminus\Omega_j$ is not connected we apply Lemma 2.2 to every its connected component) that

$$\begin{aligned}
&\lim_{k\to\infty}\left|\int_\Omega v(\nabla f_{jk}(x))\chi_{\Omega\backslash\Omega_j}(x)g(x)\,\mathrm{d}x\right|\\
&= \left|\int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}^j_x(\mathrm{d}s)g(x)\chi_{\Omega\backslash\Omega_j}(x)d_{\sigma^j}(x)\,\mathrm{d}x\right|\\
&\le \|g\|_{C(\bar{\Omega})}\|v_0\|_{C(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})}\int_\Omega\chi_{\Omega\backslash\Omega_j}(x)d_\pi(x)\,\mathrm{d}x\ .
\end{aligned}$$

The Lebesgue dominated convergence theorem yields

$$\lim_{j\to\infty}\lim_{k\to\infty}\left|\int_\Omega v(\nabla f_{jk}(x))\chi_{\Omega\backslash\Omega_j}(x)g(x)\,\mathrm{d}x\right| = 0$$

and

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_\Omega v(\nabla f_{jk}(x))g(x)\,\mathrm{d}x = \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)d_\sigma(x)\,\mathrm{d}x\ .$$

As $\{\nabla f_{jk}\}_j^{k>k(j)}$ is uniformly bounded and $\mathcal{R}$ and $C(\bar{\Omega})$ are separable we can use a suitable diagonalization to extract a sequence $\{h_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $h_k(x)=u(x)$ for $x\in\partial\Omega$ and

$$\lim_{k\to\infty}\int_\Omega v(\nabla h_k(x))g(x)\,\mathrm{d}x = \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)d_\sigma(x)\,\mathrm{d}x\ .$$

□

Lemma 2.4. *Let* $(\sigma,\hat{\nu})\in\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$, $1<p<+\infty$. *Let* σ_s *be the singular part of* σ *in the Lebesgue decomposition. Then for almost all* $a\in\Omega\setminus\operatorname{supp}\sigma_s$, *the couple* $(\pi,\hat{\mu})$, *where* $\hat{\mu}_x=\hat{\nu}_a$ *for a.a.* $x\in\Omega$ *and* $\pi(\mathrm{d}x)=d_\sigma(a)\mathrm{d}x$, *is a gradient DiPerna-Majda measure, i.e.* $(\pi,\hat{\mu})\in\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. *Moreover,*

$$\pi(\mathrm{d}x) = \left(\int_{\mathbb{R}^{m\times n}}\frac{\hat{\nu}_a(\mathrm{d}s)}{1+|s|^p}\right)^{-1}\mathrm{d}x\ . \tag{2.10}$$

Proof. Notice that $(\pi,\hat\mu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ by Proposition 1.4. Formula (2.10) comes from (1.11). Let $\{\nabla u_k\}$ be a generating sequence for $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ with $\{u_k\}$ bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$. We look for a sequence $\{u^a_{k,j}\}_{k\in\mathbb{N},\ j>0}$ uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\nabla u^a_{k,j}(x)=\nabla u_k(a+j^{-1}x)\ ,\ j>0\ ,x\in\Omega\ . \tag{2.11}$$

We proceed similarly as in [23, Th. 7.2] and apply Lemma 2.2 for any $\omega:=a+j^{-1}\Omega$. with j large enough. First we choose $a\in\Omega$. Define $\bar V_\ell(y)=d_\sigma(y)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}v_0^\ell(s)\hat\nu_y(\mathrm{d}s)$ where $\{v_0^\ell\}_{\ell\in\mathbb{N}}$ is a dense subset of $\mathcal{R}$. Then we take $a\in\Omega$, $a\in\mathcal{L}_u\cap\mathcal{L}_{d_\sigma}\cap_{\ell=1}^\infty\mathcal{L}_{V_\ell}$ for any $\ell\in\mathbb{N}$. The set of such points has the full Lebesgue measure.

We know that $\{\nabla u_k\}$ is bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$. Moreover, $\mathrm{w}^*-\lim_{k\to\infty}1+|\nabla u_k|^p=\sigma$. In other words, for any $\xi\in C(\bar\Omega)$

$$\lim_{k\to\infty}\int_\Omega\xi(x)(1+|\nabla u_k(x)|^p)\,\mathrm{d}x=\int_\Omega\xi(x)\,\sigma(\mathrm{d}x)\ .$$

We take $\xi_{a,j}\in C_0(\Omega)$ such that

$$0\le\chi_{a+j^{-1}\Omega}(x)\le\xi_{a,j}(x)\le\chi_{a+2j^{-1}\Omega}(x)\ ,\ x\in\Omega\ .$$

Then for some constant $C>0$ one gets

$$\begin{aligned}&\limsup_{j\to\infty}\limsup_{k\to\infty}j^n\int_\Omega(1+|\nabla u_k(x)|^p)\chi_{a+j^{-1}\Omega}(x)\,\mathrm{d}x\\ &\le\limsup_{j\to\infty}\limsup_{k\to\infty}j^n\int_\Omega(1+|\nabla u_k(x)|^p)\xi_{a,j}(x)\,\mathrm{d}x\\ &=\limsup_{j\to\infty}j^n\int_\Omega\xi_{a,j}(x)\,\sigma(\mathrm{d}x)\\ &\le\limsup_{j\to\infty}j^n\int_\Omega\chi_{a+2j^{-1}\Omega}(x)\,\sigma(\mathrm{d}x)\le Cd_\sigma(a)\ .\end{aligned}$$

This and the Lebesgue differentiation theorem in the form

$$\lim_{j\to\infty}j^n\int_{a+\Omega/j}|V(x)-V(a)|\mathrm{d}x=0, \tag{2.12}$$

whenever $V\in L^1(\Omega)$ and for almost all a (see e.g. [9, p. 9], [12, p. 9], or [23, 120]), give

$$\begin{aligned}&\limsup_{j\to\infty}\limsup_{k\to\infty}j^n\int_\Omega|\nabla u_k(x)|^p\chi_{a+j^{-1}\Omega}(x)\,\mathrm{d}x\\ &=\limsup_{j\to\infty}\limsup_{k\to\infty}\int_\Omega|\nabla u_k(a+j^{-1}x)|^p\,\mathrm{d}x<+\infty\ .\end{aligned} \tag{2.13}$$

Suppose that $\mathrm{w}\text{-}\lim_{k\to\infty}u_k=u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, $u_a:\Omega\to\mathbb{R}^m$ is given by $u_a(x)=\nabla u(a)x$ and denote $C_a=|\Omega|^{-1}\int_\Omega u_a(x)\ dx$. Take

$$u^a_{k,j}(x)=j(u_k(a+j^{-1}x)-M_{a,k,j})\ , \tag{2.14}$$

where $M_{a,k,j}$ is a constant chosen so that $\int_\Omega u^a_{k,j}(x)\,\mathrm{d}x=C_a$. By the Poincaré inequality $\{u^a_{k,j}\}_{k\in\mathbb{N},j>0}$ is uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$.

Taking $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ and $g \in C(\bar\Omega)$ we have

$$\begin{aligned}\int_\Omega v(\nabla u^a_{k,j}(x))g(x)\,\mathrm{d}x &= \int_\Omega v(\nabla u_k(a+j^{-1}x)g(x)\,\mathrm{d}x\\ &= j^n\int_\Omega v(\nabla u_k(y))\chi_{a+j^{-1}\Omega}(y)g\left(\frac{y-a}{j^{-1}}\right)\,\mathrm{d}y\ .\end{aligned}$$

Using Lemma 2.2 we get for all $v^\ell = v^\ell_0(1+|\cdot|^p)$ and all $g \in C(\bar\Omega)$ that

$$\lim_{k\to\infty}\int_\Omega v^\ell(\nabla u^a_{k,j}(x))g(x)\,\mathrm{d}x = j^n\int_\Omega \bar V_\ell(y)\chi_{a+j^{-1}\Omega}(y)g\left(\frac{y-a}{j^{-1}}\right)\,\mathrm{d}y\ .$$

Passing to the limit for $j \to \infty$ we get by the Lebesgue differentiation theorem (2.12)

$$\begin{aligned}\lim_{j\to\infty}\lim_{k\to\infty}\int_\Omega v^\ell(\nabla u^a_{k,j}(x))g(x)\,\mathrm{d}x &= \lim_{j\to\infty}\int_\Omega \bar V_\ell(a+j^{-1}x)g(x)\,\mathrm{d}x\\ &= \bar V_\ell(a)\int_\Omega g(x)\,\mathrm{d}x\\ &= \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v^\ell_0(s)\hat\nu_a(\mathrm{d}s)g(x)d_\sigma(a)\,\mathrm{d}x\\ &= \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v^\ell_0(s)\hat\mu_x(\mathrm{d}s)g(x)\,\pi(\mathrm{d}x)\ .\end{aligned}$$

The proof is finished by a diagonalization giving us a sequence $\{\nabla u^a_k\}_{k\in\mathbb{N}}$ and generating $(\pi,\hat\mu)$. Note that $C(\bar\Omega)$ and $\mathcal{R}$ are separable. □

Lemma 2.5. *Let $\sigma \in rca(\overline{\Omega})$ and $\omega \subseteq \Omega$ be an arbitrary connected domain with a Lipschitz boundary. Let us further denote for every $r \in \mathbb{R}$ the set $\omega_r := \{x \in \omega : \mathrm{dist}(x,\partial\omega) > \mathrm{diam}\,\omega - r\}$. Then $\sigma(\partial\omega_r) \neq 0$ for at most a countable number of r.*

Proof. If $|\sigma|(\overline{\omega}) = 0$ where $|\sigma|$ is the total variation of σ there is nothing to prove. In the other case we define the probability measure μ on $\mathbb{R}$ by defying its distribution function:

$$F_\mu(r) := \mu((-\infty,r]) := \frac{|\sigma|(\overline{\omega}_r)}{|\sigma|(\overline{\omega})}.$$

As F_μ is nondecreasing, $\lim_{r\to\infty}F_\mu(r) = 1$, $\lim_{r\to-\infty}F_\mu(r) = 0$ and F_μ is right-continuous, it follows that μ is uniquely defined on a certain σ–body containing $\{(-\infty,r] : r \in \mathbb{R}\}$, see e.g. [4, Th. 14.1, p. 188]. An easy computation shows that $\mu(\{r\}) = \frac{|\sigma|(\partial\omega_r)}{|\sigma|(\overline{\omega})}$. Now lemma follows from the fact that a probability measure can have at most a countable number of atoms. □

Lemma 2.6. *Let $(\sigma,\hat\nu) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$, $1 < p < +\infty$. Then for σ-almost all $x \in \Omega$*

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s) \geq 0\ . \tag{2.15}$$

for all $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ with $Qv > -\infty$.

Proof. Let $\{\nabla u_k\}$ generate $(\sigma,\hat\nu)$ and let $\{z_k\}$ be the sequence constructed in Lemma 1.2. Denoting $w_k=u_k-z_k$ for any $k\in\mathbb{N}$ we set $R_k=\{x\in\Omega;\ \nabla w_k(x)\neq 0\}$. Lemma 1.2 asserts that $|R_k|\to 0$ as $k\to\infty$. We get from Lemma 1.1 that for any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ quasiconvex with $v(0)=0$ and any $g\in C(\bar\Omega)$

$$\begin{aligned}
&\left|\int_\Omega g(x)v(\nabla w_k(x))\,\mathrm{d}x-\int_\Omega g(x)(v(\nabla u_k(x))-v(\nabla z_k(x)))\,\mathrm{d}x\right|\\
&=\left|\int_{R_k} g(x)(v(\nabla u_k(x)-\nabla z_k(x))-v(\nabla u_k(x))+v(\nabla z_k(x)))\,\mathrm{d}x\right|\\
&\le \|g\|_{C(\bar\Omega)}\left(\int_{R_k}|v(\nabla u_k(x)-\nabla z_k(x))-v(\nabla u_k(x))|\,\mathrm{d}x+\int_{R_k}|v(\nabla z_k(x))|\,\mathrm{d}x\right)\\
&\le C\|g\|_{C(\bar\Omega)}\int_{R_k}\left[(1+|\nabla u_k(x)-\nabla z_k(x)|^{p-1}+|\nabla u_k|^{p-1})|\nabla z_k(x)|+(1+|\nabla z_k|^p)\right]\,\mathrm{d}x\\
&\le C'\left(\left(\int_{R_k}|\nabla z_k(x)|^p\,\mathrm{d}x\right)^{1/p}+\int_{R_k}1+|\nabla z_k(x)|^p\,\mathrm{d}x+\int_{R_k}|\nabla z_k(x)|\,\mathrm{d}x\right)
\end{aligned}$$

for constants $C,C'>0$. The last term goes to zero as $k\to\infty$ because $\{|\nabla z_k|^p\}$ is relatively weakly compact in $L^1(\Omega)$ and $|R_k|\to 0$ as $k\to\infty$. This calculation shows that for $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ quasiconvex we can separate oscillation and concentration effects of $\{\nabla u_k\}$ independently of the used compactification of $\mathbb{R}^{m\times n}$. Indeed, due to (1.13) we have for any $g\in C(\bar\Omega)$ and any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ quasiconvex that

$$\begin{aligned}
\lim_{k\to\infty}\int_\Omega v(\nabla w_k(x))g(x)\,\mathrm{d}x&=v(0)\int_\Omega g(x)\,\mathrm{d}x\\
&\quad+\int_{\bar\Omega}\int_{\beta_{\mathbb{R}^{m\times n}}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\ . \qquad (2.16)
\end{aligned}$$

Let $x_0\in\Omega$ and let $\zeta\in C_0^\infty(B(x_0,r))$, $0\le\zeta\le 1$. We have for any fixed $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ with $Qv>-\infty$ that $|Qv(s)|\le c(1+|s|^p)$, for all $s\in\mathbb{R}^{m\times n}$ with a constant $c>0$, cf. [17, Lemma 2.5]. Therefore if $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ with $Qv>-\infty$ we get by Lemma 1.1

$$
\begin{aligned}
&|B(x_0,r)|Qv(0) \le \int_{B(x_0,r)} Qv(\nabla(\zeta(x)w_k(x)))\,\mathrm{d}x \\
&= \int_{B(x_0,r)} Qv(\zeta(x)\nabla w_k(x) + w_k(x)\otimes\nabla\zeta(x))\,\mathrm{d}x \le \int_{B(x_0,r)} Qv(\zeta(x)\nabla w_k(x))\,\mathrm{d}x \\
&+ \alpha\int_{B(x_0,r)} (1 + |\zeta(x)\nabla w_k(x) + w_k(x)\otimes\nabla\zeta(x)|^{p-1})|w_k(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x \\
&+ \alpha\int_{B(x_0,r)} (|\zeta(x)\nabla w_k(x)|^{p-1})|w_k(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x \\
&\le \int_{B(x_0,r)} Qv(\zeta(x)\nabla w_k(x))\,\mathrm{d}x \qquad (2.17)\\
&+ \alpha\int_{B(x_0,r)} (1 + 2^{p-1})|\zeta(x)\nabla w_k(x)|^{p-1})|w_k(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x \\
&+ \alpha\int_{B(x_0,r)} (2^{p-1}|w_k(x)\otimes\nabla\zeta(x)|^{p-1})|w_k(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x \\
&\le \int_{B(x_0,r)} Qv(\zeta(x)\nabla w_k(x))\,\mathrm{d}x \\
&+ \alpha(1+2^{p-1})\|\zeta\nabla w_k\|^{p-1}_{L^p(\Omega;\mathbb{R}^{m\times n})}\|w_k\otimes\nabla\zeta\|_{L^p(\Omega;\mathbb{R}^m)} \\
&+ 2^{p-1}\alpha\|w_k\otimes\nabla\zeta\|^p_{L^p(\Omega;\mathbb{R}^n)}\ .
\end{aligned}
$$

Since $w_k \to 0$ strongly in $L^p(\Omega;\mathbb{R}^n)$ and $\{\nabla w_k\}_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ the last two terms tend to zero if $k\to\infty$. Therefore we have

$$
|B(x_0,r)|Qv(0) \le \liminf_{k\to\infty}\int_{B(x_0,r)} Qv(\zeta(x)\nabla w_k(x))\,\mathrm{d}x\ . \qquad (2.18)
$$

Let us choose such $r>0$ that $\sigma(\partial B(x_0,r))=0$. This is possible due to Lemma 2.5. We continue with the following estimate for a suitable subsequence of $\{\nabla w_k\}$ (not relabeled). Note that we use Lemma 2.2 with $\omega := B(x_0,r)$.

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{B(x_0,r)} Qv(\zeta(x)\nabla w_k(x))\,\mathrm{d}x \le \lim_{k\to\infty}\int_{B(x_0,r)} Qv(\nabla w_k(x))\,\mathrm{d}x \\
&+ \alpha\lim_{k\to\infty}\int_{B(x_0,r)} (1-\zeta(x))(1+\zeta^{p-1}(x))|\nabla w_k(x)|^p\,\mathrm{d}x \\
&+ \alpha\lim_{k\to\infty}\int_{B(x_0,r)} (1-\zeta(x))|\nabla w_k(x)|\,\mathrm{d}x \\
&= \lim_{k\to\infty}\int_{B(x_0,r)} Qv(\nabla w_k(x))\,\mathrm{d}x \\
&+ \alpha\int_{B(x_0,r)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{|s|^p}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)(1-\zeta(x))(1+\zeta^{p-1}(x))\,\sigma(\mathrm{d}x) \qquad (2.19)\\
&+ \alpha\int_{B(x_0,r)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{|s|}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)(1-\zeta(x))\,\sigma(\mathrm{d}x)\ .
\end{aligned}
$$

Taking into account (2.18) and (2.19) and a sequence $\{\zeta_j\}_{j\in\mathbb{N}} \subset C_0^\infty(B(x_0,r))$, $0\le\zeta_j\le 1$ pointwise tending to $\chi_{B(x_0,r)}$ σ-a.e. we have by Lebesgue's dominated convergence theorem

$$|B(x_0,r)|Qv(0) \leq \lim_{k\to\infty}\int_{B(x_0,r)} Qv(\nabla w_k(x))\,\mathrm{d}x \ .$$

The right-hand side is not greater than

$$|B(x_0,r)|Qv(0) + \int_{B(x_0,r)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \ . \tag{2.20}$$

Indeed, we can consider a complete separable ring $\mathcal{S}$ of bounded continuous functions such that $\frac{v}{1+|\cdot|^p} \in \mathcal{S}$ as well as $\frac{Qv}{1+|\cdot|^p} \in \mathcal{S}$. The metrizable compactification $\beta_{\mathcal{S}}\mathbb{R}^{m\times n}$ may be possibly finer than $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$; cf. Remark 1.1 for the construction. Then we have (perhaps up to a subsequence; cf.(2.16))

$$\begin{aligned}\lim_{k\to\infty}\int_{B(x_0,r)} Qv(\nabla w_k(x))\,\mathrm{d}x &= |B(x_0,r)|Qv(0)\\ &\quad+\int_{B(x_0,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{Qv(s)}{1+|s|^p}\tilde{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\end{aligned}$$

for $(\sigma,\tilde{\nu}) \in \mathcal{GDM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$. Notice that by (1.8) σ is independent of the used ring $\mathcal{S}$. Since $Qv \leq v$ we have

$$\begin{aligned}\lim_{k\to\infty}\int_{B(x_0,r)} Qv(\nabla w_k(x))\,\mathrm{d}x &\leq |B(x_0,r)|Qv(0)\\ &\quad+\int_{B(x_0,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\tilde{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \ .\end{aligned} \tag{2.21}$$

As $v_0 = v/(1+|\cdot|^p) \in \mathcal{S}$, too, we have using (1.13)

$$\begin{aligned}\lim_{k\to\infty}\int_{B(x_0,r)} v(\nabla u_k(x))\,\mathrm{d}x &= \int_{B(x_0,r)}\int_{\mathbb{R}^{m\times n}} v(s)\nu_x(\mathrm{d}s)\\ &\quad+\int_{B(x_0,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\tilde{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\\ &= \int_{B(x_0,r)}\int_{\mathbb{R}^{m\times n}} v(s)\nu_x(\mathrm{d}s)\\ &\quad+\int_{B(x_0,r)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \ ,\end{aligned}$$

where $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ is the Young measure generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$. Therefore,

$$\begin{aligned}&\int_{B(x_0,r)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\\ &= \int_{B(x_0,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\tilde{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \ .\end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22) we arrive at (2.20).

Thus it yields

$$0 \leq \int_{B(x_0,r)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x) \ .$$

Therefore, by Lebesgue-Besicovitch differentiation theorem [9, p. 43] for any σ-Lebesgue point x_0 of $x \mapsto \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)$ and any sequence $\{r_j\}_{j\in\mathbb{N}}$ such that $B(x_0,r_j)\subset\Omega$, $\sigma(\partial B(x_0,r_j))=0$, and $\lim_{j\to\infty} r_j = 0$ we get

$$\begin{aligned}0 &\le \lim_{j\to\infty}\frac{1}{\sigma(B(x_0,r_j))}\int_{B(x_0,r_j)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\\ &= \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}_{x_0}(\mathrm{d}s)\ .\end{aligned}$$

We continue similarly as in [11]. The previous calculation yields the existence of a σ-null set $E_v\subset\Omega$ such that

$$0\le \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)$$

if $x\notin E_v$. Let $\{v_0^k\}_{k\in\mathbb{N}}$ be a dense subset of $\mathcal{R}$, so that $\{v^k\}_{k\in\mathbb{N}} = \{v_0^k(1+|\cdot|^p)\}_{k\in\mathbb{N}} \subset \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$. We define

$$E = \bigcup_k \bigcup_{\{j\in\mathbb{N};\ Q(v^k+(1/j)(1+|\cdot|^p))>-\infty\}} E_{v^k+(1/j)(1+|\cdot|^p)}\ .$$

Clearly $\sigma(E)=0$. Fix $x\in(\Omega\setminus E)$, $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ such that $Qv>-\infty$ and choose a subsequence (not relabeled) $\{v_0^k\}_{k\in\mathbb{N}}$ such that

$$v_0^k\to v_0 \text{ in } C(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}) \text{ and } \|v_0^k - v_0\|_{C(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})} < \frac{1}{j(k)}\ ,$$

where $j(k)\to\infty$ if $k\to\infty$. We have

$$\begin{aligned}v^k(s)+\frac{1}{j(k)}(1+|s|^p) &\ge v^k(s)+(1+|s|^p)\|v_0^k-v_0\|_{C(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})}\\ &\ge v^k(s)+|v_0^k(s)-v_0(s)|(1+|s|^p)\ge v(s)\ .\end{aligned}$$

Thus, $Q(v^k+\frac{1}{j(k)}(1+|s|^p))+>-\infty$, as well, and because $x\notin E$ then $x\notin E_{v^k+(1/j(k))(1+|\cdot|^p)}$ and

$$\begin{aligned}0&\le\lim_{k\to\infty}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\left(v_0^k(s)+\frac{1}{j(k)}\right)\hat{\nu}_x(\mathrm{d}s) = \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)\\ &= \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s).\end{aligned}$$

□

We are now to formulate necessary conditions for a gradient DiPerna-Majda measure.

PROPOSITION 2.7. *Let $\Omega\subset\mathbb{R}^n$ be an arbitrary bounded domain. Let $\{u_k\}\subset W^{1,p}(\Omega;\mathbb{R}^{m\times n})$, $1<p<+\infty$ be bounded. Let further $\{\nabla u_k\}$ generate $(\sigma,\hat{\nu})\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ Let d_σ be the density of σ with respect to the Lebesgue measure.*

Then the following three conditions hold:
for a.a. $x \in \Omega$

$$\exists\ u \in W^{1,p}(\Omega;\mathbb{R}^m):\ \nabla u(x) = d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\ , \tag{2.23}$$

for a.a. $x \in \Omega$ *and all* $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ *the following Jensen inequality is valid*

$$Qv(\nabla u(x)) \le d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s) \tag{2.24}$$

and for σ*-almost all* $x \in \Omega$

$$0 \le \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s) \tag{2.25}$$

for all $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ *with* $Qv > -\infty$.

Moreover, if Ω *has extension property in* $W^{1,p}$ *and additionally* $\{u_k - u\}_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega;\mathbb{R}^m)$ *then (2.25) holds for* σ*-almost all* $x \in \bar{\Omega}$.

Proof. We start with the proof of the first part of the proposition deriving conditions (2.23), (2.24), (2.25).

(i) Suppose first that Ω is Lipschitz. As $p > 1$ we assume that $\{u_k\}_k$ converges weakly to $u \in W^{1,p}(\Omega;\mathbb{R}^{m\times n})$. Thus for any $g \in C(\bar{\Omega})$

$$\lim_{k\to\infty}\int_\Omega \nabla u_k(x)g(x)\,\mathrm{d}x = \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)\mathrm{d}_\sigma(x)\,\mathrm{d}x\ ,$$

which gives (2.23) by the density argument.

Let us take a fixed $a \in \Omega \setminus \operatorname{supp}\sigma_s$, a Lebesgue point of ∇u, and denote $Y := \nabla u(a)$. By Lemma 2.4 $(\pi, \hat{\mu}_x) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$, $\hat{\mu}_x = \hat{\nu}_a$ and $\pi(\mathrm{d}x) = d_\sigma(a)\mathrm{d}x$ is a homogeneous DiPerna-Majda measure with a generating sequence $\{\nabla\tilde{w}_k\}$, where $\{\tilde{w}_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$. Using Lemma 2.3 we can suppose that $\tilde{w}_k(x) = Yx$ if $x \in \partial\Omega$ and $k \in \mathbb{N}$. We have for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$\int_\Omega v(\nabla\tilde{w}_k(x))\,\mathrm{d}x \ge |\Omega|Qv(Y)\ . \tag{2.26}$$

Hence, we calculate for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ with the finite quasiconvex envelope.

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega v(\tilde{w}_k(x))\,\mathrm{d}x &= d_\sigma(a)|\Omega|\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_a(\mathrm{d}s)\\ &\ge |\Omega|Qv(Y)\ ,\end{aligned}$$

which proves the first part of the statement for Lipschitz Ω because (2.25) follows from Lemma 2.6.

(ii) Assume now that Ω is an arbitrary bounded domain. Let us overlap Ω by the sequence of its subdomains $\Omega_j \subseteq \Omega$ with a Lipschitz boundary such that $\operatorname{dist}(\Omega_j, \partial\Omega) < \frac{1}{j}$. Using Lemma 2.5 we may additionally assume that $\sigma(\partial\Omega_j) = 0$. We use Lemma 2.2 and deduce that if $\{\nabla u_k\}$ generates $(\sigma, \hat{\nu})$ then the same sequence restricted to each Ω_j generates $(\sigma, \hat{\nu})$ restricted to Ω_j. Therefore (2.23), (2.24), and

(2.25) are satisfied on each Ω_j with the same $(\sigma,\hat\nu)$ and u and it remains to let $j\to+\infty$.

Now we prove the last statement in the proposition.

Let $\tilde u$ be an extension of u to $\mathbb{R}^n$. Let us extend each function u_k to $\mathbb{R}^n$ by plugging $\tilde u_k(x):=\tilde u(x)$ outside Ω. Nikodým ACL Characterization Theorem (see e.g. [20, Sec. 1.1.3, Th. 2]) ensures us that each $\tilde u_k$ belongs to $W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)$. Let $\tilde\Omega$ be an arbitrary bounded domain with Lipschitz boundary such that $\overline\Omega\subseteq\tilde\Omega$ and let $(\tilde\sigma,\tilde\nu_x)$ be generated by $\{\nabla\tilde u_k\}_{k\in\mathbb{N}}$ restricted to $\tilde\Omega$. Decomposing for any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ and $g\in C(\overline\Omega)$:

$$\int_{\tilde\Omega} v(\nabla\tilde u_k(x))g(x)dx=\int_{\tilde\Omega\setminus\Omega} v(\nabla\tilde u(x))g(x)dx+\int_{\Omega} v(\nabla u_k(x))g(x)dx$$

and letting k converge to $+\infty$ we observe that $\{\nabla\tilde u_k\}_{k\in\mathbb{N}}$ generates DiPerna Majda measure $(\tilde\sigma,\tilde\nu)$ on $\tilde\Omega$ such that

$$\tilde\sigma=\begin{cases}(1+|\nabla\tilde u(x)|^p)dx & \text{on}\quad \tilde\Omega\setminus\overline\Omega\\ \sigma & \text{on}\quad \overline\Omega\end{cases},\ \tilde\nu_x=\begin{cases}\delta_{\nabla u(x)} & \text{if } x\in\tilde\Omega\setminus\overline\Omega\\ \nu_x & \text{if } x\in\overline\Omega.\end{cases}$$

As $\tilde\Omega$ is a bounded domain with a Lipschitz boundary, we observe by Lemma 2.6 that (2.15) holds true for $\tilde\sigma$–almost all $x\in\tilde\Omega$. In particular it holds true for σ–almost all $x\in\overline\Omega$.

□

A remark is in order.

Remark 2.1. *(i) In fact, (2.25) together with the characterization of gradient Young measures by Kinderlehrer and Pedregal [15] always imply (2.24). Namely, the characterization of gradient Young measures gives for v continuous, $v(s)\le C(1+|s|^p)$, that*

$$Qv(\nabla u(x))\le d_\sigma(x)\int_{\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s),$$

for almost all $x\in\Omega$. This together with (2.25) implies (2.24).

On the other hand, if σ is absolutely continuous with respect to the Lebesgue measure we see that (2.24) implies (2.25). To see this, decompose $\{u_k\}$ by means of Lemma 1.2 and observe that $\{\nabla w_k\}\to 0$ weakly in $L^p(\Omega;\mathbb{R}^{m\times n})$. Moreover, taking $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ with $Qv>-\infty$, $Qv(0)=0$, we have applying (2.24) from Proposition 2.7 to $\{\nabla w_k\}_{k\in\mathbb{N}}$ and in view of (2.16) and Lemma 2.2 that

$$0\le\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{Qv(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\le\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\ .$$

which gives (2.25). Note that the requirement $Qv(0)=0$ does not restrict generality because we can always put $\tilde v=v-Qv(0)$ for $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$, $Qv>-\infty$ and clearly

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)=\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{\tilde v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\ .$$

Saying otherwise, (2.25) gives an extra condition only if σ has a singular part.

(ii) An arbitrary bounded domain with Lipschitz boundary has the extension property in $W^{1,p}$. *It is shown e.g. in [27, Sect. VI.3].*

(iii) Condition (2.25) is analogous to the formula (5.1) in [11]. Particularly, if $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$ *is the compactification by the sphere (2.25) coincides with [11, Formula (5.1)]. As* $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *must be such that* σ *is nonnegative our conditions (2.24) and (2.25) imply conditions (i) and (ii) in Step 1 [11, p. 748]. Note that as they use functions* $g:\Omega\to\mathbb{R}$ *vanishing on* $\partial\Omega$ *they do not need to take care about the behavior of the varifold for* $x\in\partial\Omega$.

3. Sufficient conditions. This section is devoted to deriving sufficient conditions on a DiPerna-Majda measure to be generated by gradients.

Lemma 3.1. *Let* $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$, $1<p<+\infty$, *be bounded and such that* $u_k(x)=Yx$ *for any* $k\in\mathbb{N}$, *any* $x\in\partial\Omega$ *and* $Y\in\mathbb{R}^{m\times n}$ *fixed. Let* $(\sigma,\hat\nu)\in\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *be generated by* $\{\nabla u_k\}$. *Then there is* $\{w_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ *bounded,* $\{u_k-w_k\}_{k\in\mathbb{N}}\subset W^{1,p}_0(\Omega;\mathbb{R}^{m\times n})$ *such that* $\{\nabla w_k\}_{k\in\mathbb{N}}$ *generates* $\left(\bar\sigma,\bar{\hat\nu}\right)\in\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ *where* $\bar\sigma$ *is absolutely continuous with respect to the Lebesgue measure, its density* $d_{\bar\sigma}(x)=\sigma(\bar\Omega)/|\Omega|$ *for any* $x\in\Omega$ *and for any* $v_0\in\mathcal{R}$ *and almost all* $x\in\Omega$

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\bar{\hat\nu}_x(\mathrm{d}s)=\frac{1}{\sigma(\bar\Omega)}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\ , \tag{3.1}$$

i.e, $\left(\bar\sigma,\bar{\hat\nu}\right)$ *is homogeneous.*

Proof. We follow the proof of [23, Th. 7.1]. The family

$$\mathcal{A}=\left\{x\in a+\epsilon\bar\Omega\subset\Omega\ ;\ a\in\Omega\ ,\ \epsilon\le j^{-1}\right\}$$

is a covering of Ω. There exists a countable collection $\{x\in a_{ij}+\epsilon_{ij}\bar\Omega\}$, $\epsilon_{ij}\le 1/j$ of pairwise disjoint sets and

$$\Omega=\bigcup_i\{x\in a_{ij}+\epsilon_{ij}\bar\Omega\}\bigcup N_j\ ,\ |N_j|=0\ .$$

We see that $\sum_i\epsilon_{ij}^n=|\Omega|/|\Omega|=1$. We now take for $u_Y(x)=Yx$, $x\in\Omega$, the following sequence of mappings

$$w_k(x)=\begin{cases}\epsilon_{ik}u_k\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)+u_Y(a_{ik}) & \text{if } x\in a_{ik}+\epsilon_{ik}\Omega\\ u_Y(x) & \text{otherwise .}\end{cases}$$

Therefore, $w_k=u_Y$ on $\partial\Omega$ and for a.a. $x\in\Omega$

$$\nabla w_k(x)=\nabla u_k\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\ .$$

We have

$$\int_\Omega|\nabla w_k(x)|^p\,\mathrm{d}x=\sum_i\int_{a_{ik}+\epsilon_{ik}\Omega}\left|\nabla u_k\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right|^p\mathrm{d}x=\sum_i\epsilon_{ik}^n\int_\Omega|\nabla u_k(x)|^p\,\mathrm{d}x<C\ .$$

Hence, the Poincaré inequality yield the bound on $\{w_k\}$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Further, for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ and $g \in C(\bar\Omega)$ we get

$$\begin{aligned}\int_\Omega v(\nabla w_k(x))g(x)\,\mathrm{d}x &= \sum_i \epsilon^n_{ik}\int_\Omega v(\nabla u_k(y))g(a_{ik}+\epsilon_{ik}y)\,\mathrm{d}y\\ &= \frac{1}{|\Omega|}\sum_i |\Omega|\epsilon^n_{ik} g(a_{ik}+\epsilon_{ik}\bar y_{ik})\int_\Omega v(\nabla u_k(y))\,\mathrm{d}y\ .\end{aligned}$$

Above, we used the mean value theorem. The first term is the Riemann sum for $\int_\Omega g(y)\,\mathrm{d}y$, hence,

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega v(\nabla w_k(x))g(x)\,\mathrm{d}x &= \int_\Omega g(x)\,\mathrm{d}x\frac{1}{|\Omega|}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)\\ &= \int_\Omega g(x)\,\mathrm{d}x\frac{\sigma(\bar\Omega)}{|\Omega|}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\bar{\hat\nu}_x(\mathrm{d}s)\\ &= \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\bar{\hat\nu}_x(\mathrm{d}s)g(x)\bar\sigma(\mathrm{d}x)\ .\end{aligned}$$

□

It is well known, see e.g. [23], that the set of homogeneous $W^{1,p}$-gradient Young measures ν given for any $v \in C_p(\mathbb{R}^{m\times n})$ by

$$\int_{\mathbb{R}^{m\times n}} v(s)\nu(\mathrm{d}s) = \frac{1}{|\Omega|}\int_\Omega v(\nabla u(x))\,\mathrm{d}x\ ,\ \ u\in W^{1,p}(\Omega;\mathbb{R}^m), u(x)=Yx\ ,\ \ x\in\partial\Omega \tag{3.2}$$

is convex. Let us denote it by M_Y. As Young measures generated by sequences bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ can be embedded into $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ (see [25, Remark 3.2.16]) we get that M_Y is mapped into a subset $\hat m_Y$ of $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ where $(\pi,\hat\mu)\in\hat m_Y$ if for some $u\in W^{1,p}(\Omega;\mathbb{R}^m)$, $u(x)=Yx$ on $x\in\partial\Omega$ we have

$$d_\pi = \frac{1}{|\Omega|}\int_\Omega(1+|\nabla u(x)|^p)\,\mathrm{d}x \tag{3.3}$$

and for any $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\mu(\mathrm{d}s) = \frac{1}{d_\pi|\Omega|}\int_\Omega v(\nabla u(x))\,\mathrm{d}x\ . \tag{3.4}$$

Thus we can define $\eta_u\in\mathrm{rca}(\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ by

$$\langle\eta_u, g\otimes v_0\rangle = \frac{1}{|\Omega|}\int_\Omega v(\nabla u(x))\,\mathrm{d}x\int_\Omega g(y)\,\mathrm{d}y\ , \tag{3.5}$$

where $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ and $g\in C(\bar\Omega)$. Here we used the fact that $\mathbb{R}^{m\times n}$ is a Borel subset of $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$ and that the linear hull of $\{g\otimes v_0;\ g\in C(\bar\Omega)\ ,\ v_0\in\mathcal{R}\}$ is dense in $C(\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$. We see by the inspection of M_Y that η_u is a gradient DiPerna-Majda measure from $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Namely, if $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates ν from (3.2) then the same sequence generates η_u. Let us also introduce $\hat\eta_u\in\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ defined for any $v_0\in C(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ by

$$\langle\hat\eta_u, v_0\rangle = \langle\eta_u, 1\otimes v_0\rangle = \int_\Omega v(\nabla u(x))\,\mathrm{d}x\ .$$

Clearly as M_Y is convex, so is

$$\hat{M}_Y := \{\hat{\eta}_u;\ u \in W^{1,p}(\Omega;\mathbb{R}^m)\ ,\ u(x) = Yx \text{ on } \partial\Omega\} \subset \mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})\ .$$

LEMMA 3.2. *Let* $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$, $1 < p < +\infty$, *be homogeneous, i.e.,* $\hat{\nu}_x = \hat{\nu}_y$ *for all* $x, y \in \Omega$ *and* σ *be absolutely continuous with respect to Lebesgue's measure with the constant density*

$$d_\sigma = \left(\int_{\mathbb{R}^{m\times n}} \frac{\hat{\nu}(\mathrm{d}s)}{1+|s|^p}\right)^{-1} \tag{3.6}$$

such that for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$d_\sigma \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}(\mathrm{d}s) \geq Qv(Y)\ , \tag{3.7}$$

where

$$Y = d_\sigma \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat{\nu}(\mathrm{d}s)\ .$$

Then $(\sigma,\hat{\nu})$ *is a homogeneous gradient DiPerna-Majda measure.*

Proof. Multiplying (3.7) by $|\Omega|$ and defining $\xi \in \mathrm{rca}(\bar{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ by

$$\langle \xi, g\otimes v_0\rangle = \int_\Omega d_\sigma \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}(\mathrm{d}s) g(x)\,\mathrm{d}x\ , \tag{3.8}$$

for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ and $g \in C(\bar{\Omega})$ we get that (3.7) is equivalent to

$$\langle T_\xi, v_0\rangle = \langle \xi, 1\otimes v_0\rangle \geq |\Omega| Qv(Y)\ , \tag{3.9}$$

where $T_\xi \in \mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ is defined by the relation $\langle T_\xi, v_0\rangle = \langle \xi, 1\otimes v_0\rangle$. We will use the Hahn–Banach theorem to show that two subsets of $\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$: $\hat{M}_Y$ and $\hat{T}$ where $\hat{T}$ is given by

$$\hat{T} := \{T_\xi;\ \xi \text{ given by (3.8) \& } (d_\sigma \mathrm{d}x, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n}) \text{ is homogeneous }\},$$

considered as sets of functionals on the space $C(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ (with the weak* topology), cannot be separated by an element of $C(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$.

Suppose that there is $a \in \mathbb{R}$ such that for a fixed $v_0 \in \mathcal{R}$ $\langle \hat{\eta}_u, v_0\rangle \geq a$ for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $u(x) = Yx$ if $x \in \partial\Omega$. This means that $\int_\Omega v(\nabla u(x))\,\mathrm{d}x \geq a$ for any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $u(x) = Yx$ if $x \in \partial\Omega$ and therefore $Qv(Y)|\Omega| \geq a$; cf.(1.1). Hence, by (3.9)

$$\langle T_\xi, v_0\rangle = \langle \xi, 1\otimes v_0\rangle \geq |\Omega| Qv(Y) \geq a\ .$$

Hahn-Banach theorem implies that $T_\xi \in \overline{\hat{M}_Y}$, where the closure is in the weak* topology. Therefore there is a sequence (recall that $\mathcal{R}$ is separable) $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, $u_k(x) = Yx$ on the boundary such that $\lim_{k\to\infty}\langle \eta_{u_k}, 1\otimes v_0\rangle = \langle \xi, 1\otimes v_0\rangle$. In other words, for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$

$$\lim_{k\to\infty}\int_\Omega v(\nabla u_k(x))\,\mathrm{d}x = d_\sigma|\Omega| \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}(\mathrm{d}s)\ . \tag{3.10}$$

Let $(\tau,\hat\alpha)\in\mathcal{DM}^p_{\mathcal R}(\Omega;\mathbb{R}^{m\times n})$ be generated by $\{\nabla u_k\}$ or its subsequence. Then for any $v\in\Upsilon^p_{\mathcal R}(\mathbb{R}^{m\times n})$ and $g\in C(\bar\Omega)$

$$\lim_{k\to\infty}\int_\Omega v(\nabla u_k(x))g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal R}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\alpha_x(\mathrm{d}s)g(x)\,\tau(\mathrm{d}x)\ . \tag{3.11}$$

Now we are going to apply Lemma 3.1 to $(\tau,\hat\alpha)$. It gives us the existence of $\{w_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ with the same boundary conditions as $\{u_k-w_k\}\subset W^{1,p}_0(\Omega;\mathbb{R}^{m\times n})$ such that

$$\lim_{k\to\infty}\int_\Omega v(\nabla w_k(x))g(x)\,\mathrm{d}x=\int_\Omega g(x)\,\mathrm{d}x\frac{1}{|\Omega|}\int_{\bar\Omega}\int_{\beta_{\mathcal R}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\alpha_x(\mathrm{d}s)\tau(\mathrm{d}x)\ . \tag{3.12}$$

Expressing the equality (3.11) for $g=1$ by means of (3.10) and plugging it into (3.12) yields

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega v(\nabla w_k(x))g(x)\,\mathrm{d}x&=d_\sigma\int_\Omega g(x)\,\mathrm{d}x\int_{\beta_{\mathcal R}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu(\mathrm{d}s)\\&=\int_\Omega\int_{\beta_{\mathcal R}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)\ ,\end{aligned}$$

which implies the thesis. □

LEMMA 3.3. *(see [23, Lemma 7.9] for a more general case) Let $\Omega\subset\mathbb{R}^n$ be an open domain with $|\partial\Omega|=0$ and let $N\subset\Omega$ be of the zero Lebesgue measure. For $r_k:\Omega\setminus N\to(0,+\infty)$ and $\{f_k\}_{k\in\mathbb{N}}\subset L^1(\Omega)$ there exists a set of points $\{a_{ik}\}\subset\Omega\setminus N$ and positive numbers $\{\epsilon_{ik}\}$, $\epsilon_{ik}\le r_k(a_{ik})$ such that $\{a_{ik}+\epsilon_{ik}\bar\Omega\}$ are pairwise disjoint for each $k\in\mathbb{N}$, $\bar\Omega=\cup_i\{a_{ik}+\epsilon_{ik}\bar\Omega\}\cup N_k$ with $|N_k|=0$ and for any $j\in\mathbb{N}$ and any $g\in L^\infty(\Omega)$*

$$\lim_{k\to\infty}\sum_i f_j(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x=\int_\Omega f_j(x)g(x)\,\mathrm{d}x\ .$$

PROPOSITION 3.4. *Let $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal R}(\Omega;\mathbb{R}^{m\times n})$, $1<p<+\infty$, be such that σ is absolutely continuous with respect to Lebesgue's measure and let d_σ be its density. Let further the following two conditions hold:*

$$\exists\ u\in W^{1,p}(\Omega;\mathbb{R}^m)\ :\ \nabla u(x)=d_\sigma(x)\int_{\mathbb{R}^{m\times n}}\frac{s}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\ , \tag{3.13}$$

for a.a. $x\in\Omega$ and all $v\in\Upsilon^p_{\mathcal R}(\mathbb{R}^{m\times n})$ the following inequality is valid

$$Qv(\nabla u(x))\le d_\sigma(x)\int_{\beta_{\mathcal R}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\ . \tag{3.14}$$

Then $(\sigma,\hat\nu)$ is generated by gradients, i.e., belongs to $\mathcal{GDM}^p_{\mathcal R}(\Omega;\mathbb{R}^{m\times n})$.

Moreover, its generating sequence, $\{\nabla u_k\}_{k\in\mathbb{N}}$, can be chosen in the way that $\{u_k-u\}_{k\in\mathbb{N}}\subset W^{1,p}_0(\Omega,\mathbb{R}^m)$.

Proof. We will divide the proof into two steps. Although step (ii) is a generalization of (i), we believe that it is instructive to look first at a simpler case.

(i) Suppose first that u in (3.13) and (3.14) is zero. We are looking for a sequence $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ satisfying

$$\lim_{k\to\infty}\int_\Omega v(\nabla u_k(x))g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)$$

for all $g\in\Gamma$ and any $v=v_0(1+|\cdot|^p)$, $v_0\in S$, where Γ and S are countable dense subsets of $C(\bar\Omega)$ and $\mathcal{R}$.

Take $r_k=1/k$ and using Lemma 3.3 find $a_{ik}\in\Omega\setminus N$, $\epsilon_{ik}\le 1/k$ such that for $v_0\in S$ and $g\in C(\bar\Omega)$

$$\lim_{k\to\infty}\sum_i\bar V(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x=\int_\Omega\bar V(x)g(x)\,\mathrm{d}x\ , \tag{3.15}$$

where

$$\bar V(x)=d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)\ .$$

We may assume that $a_{ik}\notin N$, $|N|=0$, by (3.14) and by Lemma 3.2 we can assume that $(d_\sigma(a_{ik})\mathrm{d}x,\hat\nu_{a_{ik}})$ is a homogeneous gradient DiPerna-Majda measure living in $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and we call $\{u^{ik}_j\}_{j\in\mathbb{N}}$ its generating sequence. Recall that $u=0$, so $\text{w-lim}_{j\to\infty}u^{ik}_j=0$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and by Lemma 2.3 we can even suppose that $\{u^{ik}_j\}_{j\in\mathbb{N}}\subset W^{1,p}_0(\Omega;\mathbb{R}^{m\times n})$ and

$$\lim_{j\to\infty}\int_\Omega v(\nabla u^{ik}_j(x))g(x)\,\mathrm{d}x=\bar V(a_{ik})\int_\Omega g(x)\,\mathrm{d}x\ . \tag{3.16}$$

Define the sequence

$$u_k(x)=\begin{cases}\epsilon_{ik}u^{ik}_j\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) & \text{if } x\in a_{ik}+\epsilon_{ik}\Omega\\ 0 & \text{otherwise}\ .\end{cases}$$

Let $\Gamma\times S=\cup_k E_k$ with $E_k\subset E_{k+1}$. For k,i fixed we take $j=j(k,i)$ so large that for all $(g,v_0)\in E_k$

$$\left|\epsilon^n_{ik}\int_\Omega g(a_{ik}+\epsilon_{ik}y)v(\nabla u^{ik}_j(y))\,\mathrm{d}y-\bar V(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x\right|\le\frac{1}{2^ik}\ .$$

Here we exploited (3.16) written for $\tilde g(y)=g(a_{ik}+\epsilon_{ik}y)$ instead of g. Using this estimate and (3.15) we get for any $(g,v_0)\in\Gamma\times S$

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega g(x)v(\nabla u_k(x))\,\mathrm{d}x&=\lim_{k\to\infty}\sum_i\epsilon^n_{ik}\int_\Omega g(a_{ik}+\epsilon_{ik}y)v(\nabla u^{ik}_j(y))\,\mathrm{d}y\\&=\lim_{k\to\infty}\sum_i\bar V(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x=\int_\Omega\bar V(x)g(x)\,\mathrm{d}x\\&=\int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\end{aligned}$$

as we wish. It is clear that $u_k-u\in W^{1,p}_0(\Omega,\mathbb{R}^m)$ for every k.

(ii) If $u \neq 0$ the proof is more technical. We follow [15]. As $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we take $a \in \Omega$ and for $\epsilon > 0$ small enough define

$$w_{a,\epsilon}(y) = \epsilon^{-1}[u(a+\epsilon y) - u(a) - \epsilon\nabla u(a)y] \ .$$

We have that $w_{a,\epsilon} \in W^{1,p}(\Omega;\mathbb{R}^m)$ and

$$\nabla w_{a,\epsilon}(y) = \nabla u(a+\epsilon y) - \nabla u(a) \ .$$

Based on Reshetnyak's result (see Theorem 1 in [24] for Ω being a ball, an arbitrary case follows easily from this particular one), we have that for $\varepsilon \to 0$ and a.a. $a \in \Omega$

$$\|\frac{1}{\epsilon}[u(a+\epsilon y) - u(a) - \epsilon\nabla u(a)y]\|_{W^{1,p}(\Omega)} \to 0.$$

Thus, for almost all $a \in \Omega$,

$$\lim_{\epsilon\to 0} \|\nabla w_{a,\epsilon}\|_{L^p(\Omega;\mathbb{R}^{m\times n})} = 0 \ ,$$

and by the embedding theorem

$$\lim_{\epsilon\to 0} \|w_{a,\epsilon}\|_{L^{p*}(\Omega;\mathbb{R}^m)} = 0 \ .$$

Let's say that this is true for all $a \in \Omega \setminus N$, where $|N| = 0$. Then for $a \in \Omega \setminus N$ and any $k \in \mathbb{N}$ there is $r_k(a) > 0$ such that if $\epsilon < r_k(a)$ then

$$\left(\int_\Omega \left(\epsilon^{-1}[u(a+\epsilon y) - u(a) - \epsilon\nabla u(a)y]\right)^{p^*} \mathrm{d}y\right)^{1/p^*} \leq \frac{1}{k} \ . \tag{3.17}$$

We are looking for a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ satisfying

$$\lim_{k\to\infty} \int_\Omega v(\nabla u_k(x))g(x)\,\mathrm{d}x = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v_s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)$$

for all $g \in \Gamma$ and any $v = v_0(1+|\cdot|^p)$, $v_0 \in S$, where Γ and S are countable dense subsets of $C(\bar\Omega)$ and $\mathcal{R}$.

Take $r_k : \Omega \setminus N \to \mathbb{R}$ and using Lemma 3.3 find $a_{ik} \in \Omega \setminus N$, $\epsilon_{ik} \leq r_k(a_{ik})$ such that for all $v_0 \in S$ and all $g \in C(\bar\Omega)$

$$\lim_{k\to\infty} \sum_i \bar V(a_{ik}) \int_{a_{ik}+\epsilon_{ik}\Omega} g(x)\,\mathrm{d}x = \int_\Omega \bar V(x)g(x)\,\mathrm{d}x \ , \tag{3.18}$$

and

$$\lim_{k\to\infty} \sum_i |\bar V(a_{ik})| \int_{a_{ik}+\epsilon_{ik}\Omega} g(x)\,\mathrm{d}x = \int_\Omega |\bar V(x)|g(x)\,\mathrm{d}x \ , \tag{3.19}$$

where

$$\bar V(x) = d_\sigma(x) \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s) \ .$$

We can assume by Lemma 3.2 that $(d_\sigma(a_{ik})\mathrm{d}x, \hat\nu_{a_{ik}})$ is a homogeneous gradient DiPerna-Majda measure living in $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and we call $\{\nabla u_j^{ik}\}_{j\in\mathbb{N}}$ its generating sequence. It means that

$$\lim_{j\to\infty}\int_\Omega v(\nabla u_j^{ik}(x))g(x)\,\mathrm{d}x = \bar{V}(a_{ik})\int_\Omega g(x)\,\mathrm{d}x\ . \tag{3.20}$$

We have that

$$\mathrm{w}-\lim_{j\to\infty} u_j^{ik} = L^{ik} \text{ in } W^{1,p}(\Omega;\mathbb{R}^m)\ , \tag{3.21}$$

where for almost all x $L^{ik}(x) = \nabla u(a_{ik})x$. Next we define a sequence of smooth cut-off functions $\{\eta_\ell\}_{\ell\in\mathbb{N}}$ such that

$$\eta_\ell(x) = \begin{cases} 0 & \text{in } \Omega_\ell = \{x\in\Omega;\ \mathrm{dist}(x,\partial\Omega)\geq \ell^{-1}\}\ , \\ 1 & \text{on } \partial\Omega \end{cases}$$

and $|\nabla\eta_\ell|\leq C\ell$ for some $C>0$.

In view of Lemma 2.2 and (3.20) we have

$$\lim_{j\to\infty}\int_{\Omega\setminus\Omega_\ell} v(\nabla u_j^{ik}(x))g(x)\,\mathrm{d}x = \bar{V}(a_{ik})\int_{\Omega\setminus\Omega_\ell} g(x)\,\mathrm{d}x\ . \tag{3.22}$$

Particularly,

$$\lim_{\ell\to\infty}\lim_{j\to\infty}\int_{\Omega\setminus\Omega_\ell} v(\nabla u_j^{ik}(x))g(x)\,\mathrm{d}x = \bar{V}(a_{ik})\lim_{\ell\to\infty}\int_{\Omega\setminus\Omega_\ell} g(x)\,\mathrm{d}x = 0\ . \tag{3.23}$$

By Lemma 3.3

$$\bar\Omega = \bigcup_i\{x\in a_{ik}+\epsilon_{ik}\bar\Omega\}\bigcup N_k\ ,\ |N_k|=0\ .$$

Further, take a sequence $\{u_k^\ell\}_{k,\ell\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$u_k^\ell(x) = \begin{cases} \left[u(a_{ik})+\epsilon_{ik}u_j^{ik}\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right]\left(1-\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right) & \\ +u(x)\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) & \text{if } x\in a_{ik}+\epsilon_{ik}\Omega, \\ u(x) & \text{otherwise}\ , \end{cases}$$

where $j=j(i,k,\ell)$ will be chosen later. Note that for every k and l we have $u_k^l-u\in W_0^{1,p}(\Omega,\mathbb{R}^m)$.

We calculate for $x\in a_{ik}+\epsilon_{ik}\Omega$

$$\begin{aligned}\nabla u_k^\ell(x) &= \nabla u_j^{ik}\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\left(1-\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right)\\ &+\nabla u(x)\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\\ &+\frac{1}{\epsilon_{ik}}\left[u(x)-u(a_{ik})-\epsilon_{ik}\nabla u(a_{ik})\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right]\otimes\nabla\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\\ &+\left[\nabla u(a_{ik})\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)-u_j^{ik}\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\right]\otimes\nabla\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\\ &= A_k^\ell(x)+B_k^\ell(x)+C_k^\ell(x)+D_k^\ell(x)\ .\end{aligned} \tag{3.24}$$

Obviously, $\{|B_k^\ell|^p\}_{k\in\mathbb{N}}$ is weakly compact in $L^1(\Omega;\mathbb{R}^{m\times n})$. Further, for $\lim_{\ell\to\infty}\lim_{k\to\infty}\|C_k^\ell\|^p_{L^p(\Omega;\mathbb{R}^{m\times n})} = 0$ if $k\to\infty$ fast enough as it is bounded in $L^{p^*}(\Omega;\mathbb{R}^{m\times n})$ for $p^*>p$; cf. (3.17). Finally, $\lim_{\ell\to\infty}\lim_{k\to\infty}\|D_k^\ell\|^p_{L^p(\Omega;\mathbb{R}^{m\times n})} = 0$ due to (3.21) if j is taken large enough with respect to k.

Let us fix k,i,ℓ and take $j=j(i,k,\ell)$ so large that for any $(g,v_0)\in E_k$

$$\left|\epsilon_{ik}^n\int_\Omega g(a_{ik}+\epsilon_{ik}y)v(\nabla u_j^{ik}(y))\,\mathrm{d}y-\bar V(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x\right|\le\frac{1}{2^ik} \tag{3.25}$$

and

$$\left|\epsilon_{ik}^n\int_{\Omega\setminus\Omega_\ell} g(a_{ik}+\epsilon_{ik}y)v(\nabla u_j^{ik}(y))\,\mathrm{d}y-\epsilon_{ik}^n\bar V(a_{ik})\int_{\Omega\setminus\Omega_\ell}g(a_{ik}+\epsilon_{ik}y)\,\mathrm{d}y\right|\le\frac{1}{2^ik}\ .$$

We have

$$\begin{aligned}\int_\Omega g(x)v(\nabla u_k^\ell(x))\,\mathrm{d}x &= \sum_i\epsilon_{ik}^n\int_\Omega g(a_{ik}+\epsilon_{ik}y)v(\nabla u_j^{ik}(y))\,\mathrm{d}y\\ &-\sum_i\epsilon_{ik}^n\int_{\Omega\setminus\Omega_\ell}g(a_{ik}+\epsilon_{ik}y)v(\nabla u_j^{ik}(y))\,\mathrm{d}y\\ &+\sum_i\epsilon_{ik}^n\int_{\Omega\setminus\Omega_\ell}g(a_{ik}+\epsilon_{ik}y)v(\nabla u_k^\ell(a_{ik}+\epsilon_{ik}y))\,\mathrm{d}y\\ &=T^1_{k\ell}-T^2_{k\ell}+T^3_{k\ell}\ .\end{aligned}$$

We see that

$$\begin{aligned}\lim_{\ell\to\infty}\lim_{k\to\infty}T^1_{k\ell} &= \lim_{k\to\infty}\sum_i\bar V(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x=\int_\Omega\bar V(x)g(x)\,\mathrm{d}x\\ &=\int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\ .\end{aligned}$$

Applying (3.19) with $g=1$ yields

$$\lim_{k\to\infty}\sum_i|\bar V(a_{ik})|\epsilon_{ik}^n|\Omega|=\int_\Omega|\bar V(x)|\,\mathrm{d}x\ .$$

Therefore, we have

$$\begin{aligned}\lim_{\ell\to\infty}\lim_{k\to\infty}|T^2_{k\ell}| &= \lim_{\ell\to\infty}\lim_{k\to\infty}\left|\sum_i\epsilon_{ik}^n\bar V(a_{ik})\int_{\Omega\setminus\Omega_\ell}g(a_{ik}+\epsilon_{ik}y)\,\mathrm{d}y\right|\\ &\le\lim_{\ell\to\infty}\lim_{k\to\infty}\|g\|_{C(\bar\Omega)}\frac{|\Omega\setminus\Omega_\ell|}{|\Omega|}\sum_i\epsilon_{ik}^n|\Omega||\bar V(a_{ik})|\\ &=\lim_{\ell\to\infty}\frac{|\Omega\setminus\Omega_\ell|}{|\Omega|}\|g\|_{C(\bar\Omega)}\int_\Omega|\bar V(x)|\,\mathrm{d}x=0\end{aligned}$$

because $|\Omega\setminus\Omega_\ell|\to 0$. Finally, $\lim_{\ell\to\infty}\lim_{k\to\infty}T^3_{k\ell}=0$, as well, because for a constant

$\bar{C} > 0$ we have

$$\begin{aligned}
&\left|\sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell} g(a_{ik}+\epsilon_{ik}y)v(\nabla u_k^\ell(a_{ik}+\epsilon_{ik}y))\,\mathrm{d}y\right| \\
&\le \tilde{C}\sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell} (1+|\nabla u_j^{ik}(y)|^p)\,\mathrm{d}y \\
&+ \tilde{C}\sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell} |B_k^\ell(a_{ik}+\epsilon_{ik}y)|^p\,\mathrm{d}y \\
&+ \tilde{C}\sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell} |C_k^\ell(a_{ik}+\epsilon_{ik}y)|^p\,\mathrm{d}y \\
&+ \tilde{C}\sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell} |D_k^\ell(a_{ik}+\epsilon_{ik}y)|^p\,\mathrm{d}y \\
&= J_{kl}^1 + J_{kl}^2 + J_{kl}^3 + J_{kl}^4
\end{aligned}$$

and $\lim_{l\to\infty}\lim_{k\to\infty} J_{kl}^t = 0$ for every $t \in \{1,2,3,4\}$. Indeed,

$$\begin{aligned}
\lim_{\ell\to\infty}\lim_{k\to\infty} J_{kl}^3/\bar{C} &= \lim_{\ell\to\infty}\lim_{k\to\infty}\sum_i \int_{a_{ik}+\epsilon_{ik}(\Omega\setminus\Omega_\ell)} |C_k^\ell(y)|^p\,\mathrm{d}y \\
&\le \lim_{\ell\to\infty}\lim_{k\to\infty}\sum_i \int_{a_{ik}+\epsilon_{ik}\Omega} |C_k^\ell(y)|^p\,\mathrm{d}y \\
&= \lim_{\ell\to\infty}\lim_{k\to\infty} \int_\Omega |C_k^\ell(y)|^p\,\mathrm{d}y = 0
\end{aligned}$$

by our previous discussion. The same holds for the term J_{kl}^4 because $\lim_{l\to\infty}\int_{\cup_i\{a_{ik}+\epsilon_{ik}(\Omega\setminus\Omega_l)\}} |\nabla u(x)|^p\mathrm{d}x = 0$. The first term, $J_{kl}^1 = \bar{C}\sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell}(1+|\nabla u_j^{ik}(y)|^p\,\mathrm{d}y$, tends to zero due to same reasons like $T_{k\ell}^2$ because it is (up to $\bar{C}$) $T_{k\ell}^2$ written for $v_0 = 1$ and $g = 1$ and consequently for all $(g, v_0) \in \Gamma \times S$

$$\lim_{\ell\to\infty}\lim_{k\to\infty}\int_\Omega g(x)v(\nabla u_k^\ell(x))\,\mathrm{d}x = \int_\Omega \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\ .$$

The proof is finished by a diagonalization giving us a sequence $\{\nabla u_k\}$ generating $(\sigma,\hat{\nu})$. Note that $\{\nabla u_k^\ell\}_{k>k_0(\ell)}^{\ell\in\mathbb{N}}$ is uniformly bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ due to (3.24). The fact that $\{u_k\}$ can be chosen to have the same boundary conditions as u follows from construction of u_k^l.

□

Proposition 3.5. *Let $1 < p < +\infty$ and $(\sigma,\hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega;\mathbb{R}^{m\times n})$ be such that the following three conditions hold:*

$$\exists\ u \in W^{1,p}(\Omega;\mathbb{R}^m)\ :\ \textit{for a.a.} x \in \Omega\ \nabla u(x) = d_\sigma(x)\int_{\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\ , \tag{3.26}$$

for almost all $x \in \Omega$ and for any $v \in \Upsilon_{\mathcal{R}}^p(\mathbb{R}^{m\times n})$ the following inequality is fulfilled

$$Qv(\nabla u(x)) \le d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\ , \tag{3.27}$$

for σ-almost all $x \in \bar\Omega$ and all $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ with $Qv > -\infty$ it holds that

$$0 \le \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\ . \tag{3.28}$$

Then $(\sigma,\hat\nu) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Moreover, its generating sequence, $\{\nabla u_k\}_{k\in\mathbb{N}}$, can be chosen in the way that $\{u_k - u\}_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega,\mathbb{R}^m)$.

Proof. Notice that if the singular part of σ vanishes then the assertion follows from Proposition 3.4. Hence, we suppose that $\sigma_s \neq 0$. The proof is divided into two steps.

(i) We first suppose that the singular part of σ, σ_s, consists of a finite sum of atoms, i.e., $\sigma_s = \sum_{i=1}^N a_i\delta_{x_i}$, where $a_i > 0$ and $x_i \in \Omega$, $1 \le i \le N$.

First, note that by Lemma 1.3 inevitably $\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \hat\nu_{x_i}(\mathrm{d}s) = 1$ for $1 \le i \le N$. We define $B(x_i,r) \subset \Omega$ such that $B(x_i,r) = \{x \in \Omega;\ |x_i - x| < r\}$ for $r > 0$ sufficiently small , $i = 1,\dots,N$, and $B(x_i,r)\cap B(x_j,r) = \emptyset$ if $i \neq j$. We define for $i = 1,\dots,N$

$$\lambda_i(r) = \frac{1}{a_i}\int_{B(x_i,r)} (1 + |\nabla u(x)|^p)\,\mathrm{d}x\ .$$

As $\lim_{r\to 0}\lambda_i(r) = 0$ we will only consider $r < r_0$ for $r_0 > 0$ so small that $0 < \lambda_i(r) < 1$.

Further, put for a.a. $x \in \Omega$

$$\hat\nu^r_x = \begin{cases} \hat\nu_x & \text{if } x \in \bar\Omega \setminus \cup_{i=1}^N B(x_i,r) \\ \lambda_i(r)\delta_{\nabla u(x)} + (1-\lambda_i(r))\hat\nu_{x_i} & \text{if } x \in B(x_i,r) \end{cases} \tag{3.29}$$

and the measure $\sigma_r = d_{\sigma_r}\mathrm{d}x$ defined through its density d_{σ_r} as

$$d_{\sigma_r}(x) = \begin{cases} d_\sigma(x) & \text{if } x \in \bar\Omega \setminus \cup_{i=1}^N B(x_i,r) \\ \frac{1+|\nabla u(x)|^p}{\lambda_i(r)} & \text{if } x \in B(x_i,r)\ . \end{cases} \tag{3.30}$$

It is easy to verify by means of Proposition 3.4 that $(\sigma_r,\hat\nu^r) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. We see that for almost all $x \in \Omega$

$$d_{\sigma_r}(x)\int_{\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat\nu^r_x(\mathrm{d}s) = \nabla u(x)$$

and that due to (3.28) for almost all $x \in B(x_i,r)$

$$\frac{\lambda_i(r)(Qv(\nabla u(x)) - v(\nabla u(x)))}{(1-\lambda_i(r))(1+|\nabla u(x)|^p)} \le 0 \le \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_{x_i}(\mathrm{d}s)\ .$$

Altogether we have for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$, $Qv > -\infty$

$$Qv(\nabla u(x)) \le d_{\sigma_r}(x)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu^r_x(\mathrm{d}s)$$

and by Proposition 3.4 there is $\{u^r_k\} \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\{\nabla u^r_k\}_{k\in\mathbb{N}}$ generates $(\sigma_r,\hat\nu^r) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and u_k can even coincide with u on the boundary of Ω for any k natural.

We calculate for any $v_0 \in \mathcal{R}$ and $g \in C(\bar\Omega)$

$$\begin{aligned}
&\lim_{r\to 0}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu^r_x(\mathrm{d}s)g(x)\,\sigma_r(\mathrm{d}x)\\
&= \lim_{r\to 0}\int_{\bar\Omega\setminus\cup_{i=1}^N B(x_i,r)}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)d_\sigma(x)\,\mathrm{d}x\\
&+ \lim_{r\to 0}\sum_{i=1}^N\int_{B(x_i,r)} v_0(\nabla u(x))g(x)\,\mathrm{d}x\\
&+ \lim_{r\to 0}\sum_{i=1}^N\frac{1-\lambda_i(r)}{\lambda_i(r)}\int_{B(x_i,r)} g(x)(1+|\nabla u(x)|^p)\,\mathrm{d}x\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_{x_i}(\mathrm{d}s)\\
&= \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)d_\sigma(x)\,\mathrm{d}x\\
&+ \lim_{r\to 0}\sum_{i=1}^N\frac{1-\lambda_i(r)}{\lambda_i(r)}\int_{B(x_i,r)} g(x)(1+|\nabla u(x)|^p)\,\mathrm{d}x\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_{x_i}(\mathrm{d}s)\\
&= \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)d_\sigma(x)\,\mathrm{d}x + \sum_{i=1}^N a_i g(x_i)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_{x_i}(\mathrm{d}s)\\
&= \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\ ,
\end{aligned}$$

where we used the definition of $\lambda_i(r)$, continuity of g on $\cup_{i=1}^N\overline{B(x_i,r)}$ and the mean-value theorem. Finally, it yields

$$\lim_{r\to 0}\lim_{k\to\infty}\int_\Omega v(\nabla u^r_k(x))g(x)\,\mathrm{d}x = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\ . \tag{3.31}$$

We see that for any $r < r_0$

$$\begin{aligned}
\lim_{k\to\infty}\int_\Omega(1+|\nabla u^r_k(x)|^p)\,\mathrm{d}x = \sigma_r(\Omega) &= \int_{\Omega\setminus\cup_{i=1}^N B(x_i,r)} d_\sigma(x)\,\mathrm{d}x\\
&+\sum_{i=1}^N\int_{B(x_i,r)}\frac{1+|\nabla u(x)|^p}{\lambda_i(r)}\,\mathrm{d}x \le \sigma(\Omega)\ .
\end{aligned}$$

Hence, for a fixed $r < r_0$ there is $k_0(r) \in \mathbb{N}$ such that if $k > k_0(r)$ then $\|\nabla u^r_k\|_{L^p(\Omega;\mathbb{R}^{m\times n})} \le \sigma(\Omega) + 1$. Therefore $\{\nabla u^r_k\}^{r<r_0}_{k>k_0(r)}$ is uniformly bounded and $\{u^r_k - u\}_k \subset W^{1,p}_0(\Omega;\mathbb{R}^m)$. Poincare inequality implies that a sequence $\{u^r_k\}$ uniformly bounded in $W^{1,p}(\Omega,\mathbb{R}^m)$. Taking into account (3.31), boundedness of $\{\nabla u^r_k\}^{r<r_0}_{k>k_0(r)}$, and separability of $\mathcal{R}$ and $C(\bar\Omega)$ a suitable diagonalization implies the existence of a bounded sequence $\{\nabla u_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\int_\Omega v(\nabla u_k(x))g(x)\,\mathrm{d}x = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\ , \tag{3.32}$$

whenever $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ and $g \in C(\bar\Omega)$.
(ii) Now we are going to prove a general case. Take $l \in \mathbb{N}$. There exists a finite partition $\mathcal{P}_l = \{\Omega^l_j\}_{j=1}^{J(l)}$ of $\bar\Omega$ such that $\Omega^l_{J_1}\bigcap\Omega_{j_2} = \emptyset$, $1 \le j_1 < j_2 \le J(l)$ and

moreover all Ω_j^l are measurable with $\mathrm{diam}(\Omega_j^l) < 1/l$. Besides, we may suppose that, for any $l \in \mathbb{N}$, the partition $\mathcal{P}_{l+1}$ is a refinement of $\mathcal{P}_l$ and that $\mathrm{int}(\Omega_j^l) \neq \emptyset$ for all j. Let σ_s be the singular part of σ. We set $a_i^l = \sigma_s(\Omega_i^l)$, where σ_s is the singular part of σ. Let us put

$$N(l) = \{1 \leq j \leq J(l);\ a_j^l \neq 0\}\ .$$

take if $i \in N(l)$ take $x_i \in\ \mathrm{int}(\Omega_i^l)$ and define a measure $(\sigma^l, \hat{\nu}^l)$ by the formula $\sigma^l(\mathrm{d}x) = d_\sigma(x) + \sum_{i\in N(l)} a_i^l \delta_{x_i}$. and

$$\hat{\nu}_x^l = \begin{cases} \hat{\nu}_x & \text{if } x \neq x_i \\ \hat{\nu}_{x_i}^l & \text{if } x = x_i\ , \end{cases} \tag{3.33}$$

where $\mathrm{supp}\ \hat{\nu}_{x_i}^l \subset \beta_{\mathcal{R}}\mathbb{R}^{m\times n} \setminus \mathbb{R}^{m\times n}$ and for any $v_0 \in \mathcal{R}$

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_{x_i}^l(\mathrm{d}s) = \frac{1}{\sigma_s(\Omega_i^l)} \int_{\Omega_i^l} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)\,\sigma_s(\mathrm{d}x)\ . \tag{3.34}$$

As σ_s is supported on $A_{\hat{\nu}}$ from Lemma 1.3 we can equivalently rewrite (3.34) as

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_{x_i}^l(\mathrm{d}s) = \frac{1}{\sigma_s(\Omega_i^l)} \int_{\Omega_i^l} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)\,\sigma_s(\mathrm{d}x)\ .$$

Using part (i) we show $(\sigma^l, \hat{\nu}^l) \in \mathcal{GDM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m\times n})$. Indeed, the fact that $(\sigma^l, \hat{\nu}^l) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m\times n})$ is checked by using Proposition 1.4. Moreover, an easy verification shows that (3.26),(3.27), and (3.28) are also satisfied for $(\sigma^l, \hat{\nu}^l)$ and (3.26) holds with the same function u.

Let $\{u_k^l\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ be such that $\{\nabla u_k^l\}_{k\in\mathbb{N}}$ generates $(\sigma^l, \hat{\nu}^l)$. We can additionally assume (as proved in part (i)) that $\{u_k^l - u\}_k \subseteq W_0^{1,p}(\Omega, \mathbb{R}^m)$. We have for any $l \in \mathbb{N}$

$$\lim_{k\to\infty} \int_\Omega 1 + |\nabla u_k^l(x)|^p\,\mathrm{d}x = \sigma^l(\bar{\Omega}) = \sigma(\bar{\Omega}) \tag{3.35}$$

and for any $v_0 \in \mathcal{R}$ and any $g \in C(\bar{\Omega})$

$$\begin{aligned}
&\lim_{l\to\infty} \left| \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x^l(\mathrm{d}s)g(x)\,\sigma^l(\mathrm{d}x) - \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x) \right| \\
&= \lim_{l\to\infty} \Bigg| \sum_{i\in N(l)} g(x_i)\sigma_s(\Omega_i^l) \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_{x_i}^l(\mathrm{d}s) \\
&- \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\,\sigma_s(\mathrm{d}x) \Bigg| \\
&= \lim_{l\to\infty} \Bigg| \sum_{i\in N(l)} \Bigg(\int_{\Omega_i^l} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x_i)\,\sigma_s(\mathrm{d}x) \\
&- \int_{\Omega_i^l} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\,\sigma_s(\mathrm{d}x) \Bigg) \Bigg| \\
&\leq \lim_{l\to\infty} \sum_{i\in N(l)} \int_{\Omega_i^l} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} |v_0(s)|\hat{\nu}_x(\mathrm{d}s)|g(x) - g(x_i)|\,\sigma_s(\mathrm{d}x) \\
&\leq C\sigma_s(\bar{\Omega}) \lim_{l\to\infty} M_g(\frac{1}{l}) = 0\ ,
\end{aligned}$$

where $|v_0| \leq C$ and M_g is the modulus of continuity of the uniformly continuous $g \in C(\bar{\Omega})$. Hence, we get for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ and any $g \in C(\bar{\Omega})$

$$\lim_{l\to\infty}\lim_{k\to\infty}\int_{\Omega} v(\nabla u^l_k(x))g(x)\,\mathrm{d}x = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\ .$$

Taking into account (3.35) we finish the proof similarly as in (i). □

Proof of Theorem 1.5. It easily follows directly from Proposition 2.7, Proposition 3.5 and Remark 2.1, part (ii). □

Remark 3.1. Theorem 1.6 is the part of Proposition 2.7.

Acknowledgment: This work was initiated during M.K's visit of the Institute of Mathematics of the Warsaw University and the visit of both authors in the Institute of Mathematics of the Polish Academy of Sciences and continued during the visit of A.K. in the Inst. of Inf. Theory and Automation of the Academy of Science of the Czech Republic and of Charles University, Prague. Moreover, M.K. thanks the IMA at the University of Minnesota and Caesar, Bonn for the support. The hospitality of all involved institutions is gratefully acknowledged. We thank M. Hušek, J. Kristensen, and T. Roubíček for fruitful discussions.

REFERENCES

[1] Alibert, J.J., Bouchitté, G.: Non-uniform integrability and generalized Young measures. . *J. Convex Anal.* **4** (1997), 125–145.

[2] Ball, J.M.: A version of the fundamental theorem for Young measures. In: *PDEs and Continuum Models of Phase Transition.* (Eds. M.Rascle, D.Serre, M.Slemrod.) Lecture Notes in Physics **344**, Springer, Berlin, 1989, pp.207–215.

[3] Ball, J.M., Murat, F.: $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.* **58** (1984), 225–253.

[4] Billingsley, P.: *Probability and Measure.* 3rd ed., John Wiley & Sons Ltd., Chichester, 1995.

[5] Dacorogna, B. *Direct Methods in the Calculus of Variations.* Springer, Berlin, 1989.

[6] DiPerna, R.J., Majda, A.J.: Oscillations and concentrations in weak solutions of the incompressible fluid equations. **108** (1987), 667–689.

[7] Dunford, N., Schwartz, J.T.: *Linear Operators.*, Part I, Interscience, New York, 1967.

[8] Engelking, R.: *General topology* 2^{nd} ed., PWN, Warszawa, 1985.

[9] Evans, L.C., Gariepy, R.F.: *Measure Theory and Fine Properties of Functions.* CRC Press, Inc. Boca Raton, 1992.

[10] Fonseca, I.: Lower semicontinuity of surface energies. *Proc. Roy. Soc. Edinburgh* **120A** (1992), 95–115.

[11] Fonseca, I., Müller, S., Pedregal, P.: Analysis of concentration and oscillation effects generated by gradients. *SIAM J. Math. Anal.* **29** (1998), 736–756.

[12] de Guzmán, M.: *Differentiation of integrals in* $\mathbb{R}^n$, Lecture Notes in Math. **481**, Springer, Berlin, 1975.

[13] Halmos, P.R.: *Measure Theory*, D. van Nostrand, 1950.

[14] Kinderlehrer, D., Pedregal, P.: Characterization of Young measures generated by gradients. *Arch. Rat. Mech.Anal.* **115** (1991), 329–365.

[15] Kinderlehrer, D., Pedregal, P.: Gradient Young measures generated by sequences in Sobolev spaces. *J. Geom. Anal.* **4** (1994), 59–90.

[16] Kristensen, J.: *Finite functionals and Young measures generated by gradients of Sobolev functions.* Mat-report **1994-34**, Math. Institute, Technical University of Denmark, 1994.

[17] Kristensen, J.: Lower semicontinuity in spaces of weakly differentiable functions. *Math. Ann.* **313** (1999), 653-710.

[18] Kružík, M., Roubíček, T.: On the measures of DiPerna and Majda. *Mathematica Bohemica*, **122** (1997), 383–399.

[19] Kružík, M., Roubíček, T.: Optimization problems with concentration and oscillation effects: relaxation theory and numerical approximation. *Numer. Funct. Anal. Optim.* **20** (1999), 511-530.

[20] Mazja, V.G.: *Sobolev Spaces.* Springer, Berlin, 1985.

[21] Morrey, C.B.: *Multiple Integrals in the Calculus of Variations.* Springer, Berlin, 1966.

[22] Müller, S.: *Variational models for microstructure and phase transisions.* Lecture Notes in Mathematics **1713** (1999) pp. 85–210.

[23] Pedregal, P.: *Parametrized Measures and Variational Principles.* Birkäuser, Basel, 1997.

[24] Reshetnyak, Yu.G.: The generalized derivatives and the a.e. differentiability, *Mat. Sb.* **75** (1968), 323–334. (in Russian).

[25] Roubíček, T.: *Relaxation in Optimization Theory and Variational Calculus.* W. de Gruyter, Berlin, 1997.

[26] Schonbek, M.E.: Convergence of solutions to nonlinear dispersive equations. *Comm. in Partial Diff. Equations* **7** (1982), 959–1000.

[27] Stein, E.M.: *Singular Integrals and Differentiability Properties of Functions.* Princeton university Press, Princeton, 1970.

[28] Tartar, L.: Compensated compactness and applications to partial differential equations. In: *Nonlinear Analysis and Mechanics* (R.J.Knops, ed.) Heriott-Watt Symposium IV, Pitman Res. Notes in Math. **39**, San Francisco, 1979.

[29] Tartar, L.: Mathematical tools for studying oscillations and concentrations: From Young measures to H-measures and their variants. In: *Multiscale problems in science and technology. Challenges to mathematical analysis and perspectives.* (N.Antonič et al. eds.) Proceedings of the conference on multiscale problems in science and technology, held in Dubrovnik, Croatia, September 3-9, 2000. Springer, Berlin, 2002.

[30] Valadier, M.: Young measures. In: Methods of Nonconvex Analysis (A.Cellina, ed.) Lecture Notes in Math. **1446**, Springer, Berlin, 1990, pp. 152–188.

[31] Warga, J.: *Optimal Control of Differential and Functional Equations.* Academic Press, New York, 1972.

[32] Young, L.C.: Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie*, Classe III **30** (1937), 212–234.