ON SOME GENERALIZED MULTIPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS

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ABSTRACT

In the present study three results concerning the Exton's generalized quadruple hypergeometric function ${}^{(k)}_{(2)}E^{(n)}_{D}$ of matrix arguments have been given and a transformation relation has been given for the Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ function as was stated in our previous paper [7]. A transformation relation and two cases of reducibility have also been discussed for the $\Phi^{(n)}_{D}$ - function of matrix arguments and a result has been established for the generalized Srivastava function $H^{(2m)}_{C}$ of matrix arguments along with some special cases of the Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ and the Chandel's ${}^{(k)}_{(1)}E^{(n)}_{C}$ functions of matrix arguments.

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INTRODUCTION

The Exton's generalized quadruple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ of matrix arguments were introduced by us in our previous studies [7,12]. In this paper we have established more results concerning these functions besides some results for the $\Phi^{(n)}_{D}$ - function of matrix arguments and the generalized Srivastava function $H^{(2m)}_{C}$ of matrix arguments. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [2,3].

1. The Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ Functions

THEOREM 1.1:

$$
\begin{aligned}
&{}^{(k)}_{(2)}E^{(n)}_{D}(a, a', b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) \\
&= \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(c-a-a')} \int\int |U|^{a-(p+1)/2} |V|^{a'-(p+1)/2} \times \\
&|I-U-V|^{c-a-a'-(p+1)/2} \left|I+U^{1/2}X_1U^{1/2}\right|^{-b_1} \cdots \left|I+U^{1/2}X_kU^{1/2}\right|^{-b_k} \times \\
&\left|I+V^{1/2}X_{k+1}V^{1/2}\right|^{-b_{k+1}} \cdots \left|I+V^{1/2}X_nV^{1/2}\right|^{-b_n} dUdV \qquad \cdots\cdots (1.1)
\end{aligned}
$$

where $U = U' > 0$, $V = V' > 0$, and $0 < U + V < I$ and

for $\mathrm{Re}(a, a', c-a-a') > (p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(1.1) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\cdots\times$$

$$|X_n|^{\rho_n-(p+1)/2}\left|I+U^{1/2}X_1U^{1/2}\right|^{-b_1}\cdots\left|I+U^{1/2}X_kU^{1/2}\right|^{-b_k}\times$$

$$\left|I+V^{1/2}X_{k+1}V^{1/2}\right|^{-b_{k+1}}\cdots\left|I+V^{1/2}X_nV^{1/2}\right|^{-b_n}dX_1\cdots dX_n \qquad \cdots\cdots(1.2)$$

Applying the transformations,

$Y_i=U^{1/2}X_iU^{1/2}$, $Y_j=V^{1/2}X_jV^{1/2}$; with $dY_i=|U|^{(p+1)/2}dX_i$,

$dY_j=|V|^{(p+1)/2}dX_j$, and $|Y_i|=|U||X_i|$, $|Y_j|=|V||X_j|$

for $i=1,\cdots,k$ and $j=k+1,\cdots,n$;

to the expression (1.2) and then integrating out the variables $Y_1,\cdots,Y_n$ by using a type-2 Beta integral produces,

$$|U|^{-\rho_1-\cdots-\rho_k}|V|^{-\rho_{k+1}-\cdots-\rho_n}\frac{\Gamma_p(\rho_1)\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\cdots\times$$

$$\frac{\Gamma_p(\rho_n)\Gamma_p(b_n-\rho_n)}{\Gamma_p(b_n)} \qquad \cdots(1.3)$$

Using this expression on the right side of eq.(1.1) and then integrating out U and V in the resulting expression by using a type-1 Dirichlet integral produces $M[{}^{(k)}_{(2)}E^{(n)}_D]$ as given by eq.(3.1) of the authors' paper [12].

THEOREM 1.2:

$$\lim_{\alpha\to\infty}{}^{(k)}_{(2)}E^{(n)}_D(a,a',\alpha,\cdots,\alpha;c;\frac{-X_1}{\alpha},\cdots,\frac{-X_n}{\alpha})$$

$$=\Phi_2(a,a';c;-X_1-\cdots-X_k,-X_{k+1}-\cdots-X_n) \qquad \cdots\cdots(1.4)$$

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$$= \lim_{\beta\to\infty} F_1[\beta, a, a'; c; \frac{-(X_1+\cdots+X_k)}{\beta}, \frac{-(X_{k+1}+\cdots+X_n)}{\beta}] \qquad \cdots\cdots(1.5)$$

PROOF: This theorem is a limiting case of the theorem (1.1) and the result in eq.(1.4) then follows by using the theorem (4.3) page 63 of Mathai [3] while, that in eq.(1.5) follows by the use of the theorem (4.8) page 65 of Mathai [3].

THEOREM 1.3:

$${}^{(k)}_{(2)}E_D^{(n)}(a, a', b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$$

$$= \frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)} \int_{U_1>0}\cdots\int_{U_n>0} e^{-tr(U_1+\cdots+U_n)} \left|U_1\right|^{b_1-(p+1)/2} \cdots \times$$

$$\left|U_n\right|^{b_n-(p+1)/2} \Phi_2(a, a'; c; -U_1^{1/2}X_1U_1^{1/2} - \cdots - U_k^{1/2}X_kU_k^{1/2},$$

$$-U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2} - \cdots - U_n^{1/2}X_nU_n^{1/2})dU_1\cdots dU_n \qquad \cdots\cdots(1.6)$$

for $Re(b_1, \cdots, b_n) > (p-1)/2$.

PROOF: Taking the M-transform of the Φ_2 - function on the right side of eq.(1.6) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we obtain,

$$\int_{X_1>0}\cdots\int_{X_n>0} \left|X_1\right|^{\rho_1-(p+1)/2} \cdots \left|X_k\right|^{\rho_k-(p+1)/2} \left|X_{k+1}\right|^{\rho_{k+1}-(p+1)/2} \cdots \times$$

$$\left|X_n\right|^{\rho_n-(p+1)/2} \Phi_2(a, a'; c; -U_1^{1/2}X_1U_1^{1/2} - \cdots - U_k^{1/2}X_kU_k^{1/2},$$

$$-U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2} - \cdots - U_n^{1/2}X_nU_n^{1/2})dX_1\cdots dX_k dX_{k+1}\cdots dX_n \qquad \cdots\cdots(1.7)$$

Making use of the transformations

$$Y_j = U_j^{1/2}X_jU_j^{1/2} \text{ with } dY_j = \left|U_j\right|^{(p+1)/2} dX_j \text{ and } \left|Y_j\right| = \left|U_j\right|\left|X_j\right| \text{ for } j = 1, \cdots, n$$

in the last expression gives,

$$\left|U_1\right|^{-\rho_1}\cdots\left|U_n\right|^{-\rho_n}\int_{Y_1>0}\cdots\int_{Y_n>0}\left|Y_1\right|^{\rho_1-(p+1)/2}\cdots\left|Y_k\right|^{\rho_k-(p+1)/2}\times$$

$$\left|Y_{k+1}\right|^{\rho_{k+1}-(p+1)/2}\cdots\left|Y_n\right|^{\rho_n-(p+1)/2}\Phi_2(a,a^{'};c;-Y_1-\cdots-Y_k,$$

$$-Y_{k+1}-\cdots-Y_n)dY_1\cdots dY_k dY_{k+1}\cdots dY_n \qquad \cdots\cdots(1.8)$$

Now applying another sets of transformations,

$Z_1=Y_1, Z_2=Y_1+Y_2,\cdots,Z_k=Y_1+\cdots+Y_k; Z_{k+1}=Y_{k+1}, Z_{k+2}=Y_{k+1}+Y_{k+2},$ $\cdots, Z_n=Y_{k+1}+\cdots+Y_n$; with $dY_1\cdots dY_k=dZ_1\cdots dZ_k$ and $dY_{k+1}\cdots dY_n$ $=dZ_{k+1}\cdots dZ_n$; where $0<Z_1<\cdots<Z_k$ and $0<Z_{k+1}<\cdots<Z_n$,

to the expression (1.8) and then integrating out the variables $Z_1,\cdots,Z_{k-1}$ and $Z_{k+1},\cdots,Z_{n-1}$ one- by- one and in order by using a type-1 Beta integral and then using the M-transform of a Φ_2- function yields,

$$\left|U_1\right|^{-\rho_1}\cdots\left|U_n\right|^{-\rho_n}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(a^{'})}\frac{\Gamma_p(a-\rho_1-\cdots-\rho_k)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)}\times$$

$$\Gamma_p(a^{'}-\rho_{k+1}-\cdots-\rho_n) \qquad \cdots\cdots(1.9)$$

Substituting this expression on the right side of eq.(1.6) and then integrating out $U_1,\cdots,U_n$ in the resulting expression by using a Gamma integral gives $M[{}^{(k)}_{(2)}E^{(n)}_{D}]$ as given by eq.(3.1) of the authors' paper [12].

THEOREM 1.4: A transformation theorem:

$${}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$

$$=\left|I+X_1\right|^{-a}\,{}^{(k)}_{(1)}E^{(n)}_{D}[a,c-b_1-\cdots-b_k,b_2,\cdots,b_n;c,c';(I+X_1)^{-1/2}X_1\times$$

$$(I+X_1)^{-1/2},-(I+X_1)^{-1/2}(X_2-X_1)(I+X_1)^{-1/2},\cdots,$$

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$$-(I+X_1)^{-1/2}(X_k-X_1)(I+X_1)^{-1/2},-(I+X_1)^{-1/2}X_{k+1}(I+X_1)^{-1/2},\cdots,$$

$$-(I+X_1)^{-1/2}X_n(I+X_1)^{-1/2}] \qquad \cdots\cdots(1.10.1)$$

where $X_i-X_1>0$ for $i=2,\cdots,k$.

$$\vdots \qquad \vdots \qquad \vdots$$

$$=\left|I+X_k\right|^{-a}\,{}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\cdots,b_{k-1},c-b_1-\cdots-b_k,b_{k+1},\cdots,b_n;c,c';$$

$$-(I+X_k)^{-1/2}(X_1-X_k)(I+X_k)^{-1/2},\cdots,-(I+X_k)^{-1/2}(X_{k-1}-X_k)(I+X_k)^{-1/2},$$

$$(I+X_k)^{-1/2}X_k(I+X_k)^{-1/2},-(I+X_k)^{-1/2}X_{k+1}(I+X_k)^{-1/2},\cdots,$$

$$-(I+X_k)^{-1/2}X_n(I+X_k)^{-1/2}] \qquad \cdots\cdots(1.10.k)$$

where $X_i-X_k>0$ for $i=1,\cdots,k-1$.

$$=\left|I+X_{k+1}\right|^{-a}\,{}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\cdots,b_k,c'-b_{k+1}-\cdots-b_n,b_{k+2},\cdots,b_n;c,c';$$

$$-(I+X_{k+1})^{-1/2}X_1(I+X_{k+1})^{-1/2},\cdots,-(I+X_{k+1})^{-1/2}X_k(I+X_{k+1})^{-1/2},$$

$$(I+X_{k+1})^{-1/2}X_{k+1}(I+X_{k+1})^{-1/2},-(I+X_{k+1})^{-1/2}(X_{k+2}-X_{k+1})\times$$

$$(I+X_{k+1})^{-1/2},\cdots,-(I+X_{k+1})^{-1/2}(X_n-X_{k+1})(I+X_{k+1})^{-1/2}]\cdots(1.10.(k+1))$$

where $X_j-X_{k+1}>0$ for $j=k+2,\cdots,n$.

$$\vdots \qquad \vdots \qquad \vdots$$

$$=\left|I+X_n\right|^{-a}\,{}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\cdots,b_{n-1},c'-b_{k+1}-\cdots-b_n;c,c';$$

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$$-(I+X_n)^{-1/2}X_1(I+X_n)^{-1/2},\cdots,-(I+X_n)^{-1/2}X_k(I+X_n)^{-1/2},$$

$$-(I+X_n)^{-1/2}(X_{k+1}-X_n)(I+X_n)^{-1/2},\cdots,-(I+X_n)^{-1/2}(X_{n-1}-X_n)\times$$

$$(I+X_n)^{-1/2},(I+X_n)^{-1/2}X_n(I+X_n)^{-1/2}] \quad \cdots\cdots(1.10.n)$$

where $X_j-X_n>0$ for $j=k+1,\cdots,n-1.$

$$=\left|I+X_i+X_{k+j}\right|^{-a}\,{}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\cdots,b_{i-1},c-b_1-\cdots-b_k,b_{i+1},\cdots,b_k,b_{k+1},$$

$$\cdots,b_{k+j-1},c'-b_{k+1}-\cdots-b_n,b_{k+j+1},\cdots,b_n;c,c';$$

$$-(I+X_i+X_{k+j})^{-1/2}(X_1-X_i)(I+X_i+X_{k+j})^{-1/2},\cdots,-(I+X_i+X_{k+j})^{-1/2}\times$$

$$(X_{i-1}-X_i)(I+X_i+X_{k+j})^{-1/2},(I+X_i+X_{k+j})^{-1/2}X_i(I+X_i+X_{k+j})^{-1/2},$$

$$-(I+X_i+X_{k+j})^{-1/2}(X_{i+1}-X_i)(I+X_i+X_{k+j})^{-1/2},\cdots,-(I+X_i+X_{k+j})^{-1/2}\times$$

$$(X_k-X_i)(I+X_i+X_{k+j})^{-1/2},-(I+X_i+X_{k+j})^{-1/2}(X_{k+1}-X_{k+j})\times$$

$$(I+X_i+X_{k+j})^{-1/2},\cdots,-(I+X_i+X_{k+j})^{-1/2}(X_{k+j-1}-X_{k+j})\times$$

$$(I+X_i+X_{k+j})^{-1/2},(I+X_i+X_{k+j})^{-1/2}X_{k+j}(I+X_i+X_{k+j})^{-1/2},$$

$$-(I+X_i+X_{k+j})^{-1/2}(X_{k+j+1}-X_{k+j})(I+X_i+X_{k+j})^{-1/2},\cdots,$$

$$-(I+X_i+X_{k+j})^{-1/2}(X_n-X_{k+j})(I+X_i+X_{k+j})^{-1/2}] \quad \cdots\cdots(1.11)$$

where $X_l - X_i > 0$, for $l = 1, \cdots, i-1$; $X_m - X_i > 0$, for $m = i+1, \cdots, k$;
$X_r - X_{k+j} > 0$, for $r = k+1, \cdots, k+j-1$; $X_s - X_{k+j} > 0$, for $s = k+j+1, \cdots, n$;
and for $1 \le i \le k$ and $1 \le j \le n-k$.

PROOF: To prove this theorem we first define the ${}^{(k)}_{(1)}E^{(n)}_{D}$ - function through an integral representation:

$$
\begin{aligned}
&{}^{(k)}_{(1)}E^{(n)}_{D}(a, b_1, \cdots, b_n; c, c'; -X_1, \cdots, -X_n) \\
&= \frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c - b_1 - \cdots - b_k)\Gamma_p(c' - b_{k+1} - \cdots - b_n)} \int \cdots (n) \cdots \int \times \\
&\left|U_1\right|^{b_1 - (p+1)/2} \cdots \left|U_n\right|^{b_n - (p+1)/2} \left|I - U_1 - \cdots - U_k\right|^{c - b_1 - \cdots - b_k - (p+1)/2} \times \\
&\left|I - U_{k+1} - \cdots - U_n\right|^{c' - b_{k+1} - \cdots - b_n - (p+1)/2} \left| I + X_1^{1/2} U_1 X_1^{1/2} + \cdots \right. \\
&\left. + X_n^{1/2} U_n X_n^{1/2} \right|^{-a} dU_1 \cdots dU_k dU_{k+1} \cdots dU_n \qquad \cdots\cdots(1.12)
\end{aligned}
$$

for $\operatorname{Re}(b_1, \cdots, b_n, c - b_1 - \cdots - b_k, c' - b_{k+1} - \cdots - b_n) > (p-1)/2$; where
$U_i = U_i' > 0$, for $i = 1, \cdots, n$; $0 < U_1 + \cdots + U_k < I$; and $0 < U_{k+1} + \cdots + U_n < I$.

To obtain the result in eq.(1.10.1) we apply the transformations,
$U_1 = I - V_1 - \cdots - V_k, U_2 = V_2, \cdots, U_n = V_n$; with $dU_1 \cdots dU_n = dV_1 \cdots dV_n$;
to eq.(1.12) and by observing that

$$
\begin{aligned}
&\left| I + X_1^{1/2}(I - V_1 - \cdots - V_k)X_1^{1/2} + X_2^{1/2} V_2 X_2^{1/2} + \cdots + X_n^{1/2} V_n X_n^{1/2} \right| \\
&= \left|I + X_1\right| \left| I - (I + X_1)^{-1/2} X_1^{1/2} V_1 X_1^{1/2} (I + X_1)^{-1/2} + (I + X_1)^{-1/2} (X_2 - X_1)^{1/2} \times \right.
\end{aligned}
$$

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$$V_2(X_2-X_1)^{1/2}(I+X_1)^{-1/2}+\cdots+(I+X_1)^{-1/2}(X_k-X_1)^{1/2}V_k(X_k-X_1)^{1/2}\times$$
$$(I+X_1)^{-1/2}+(I+X_1)^{-1/2}X_{k+1}^{1/2}V_{k+1}X_{k+1}^{1/2}(I+X_1)^{-1/2}+\cdots+(I+X_1)^{-1/2}X_n^{1/2}\times$$
$$\left. V_nX_n^{1/2}(I+X_1)^{-1/2}\right| \quad \text{where } X_i-X_1>0 \text{ for } i=2,\cdots,k$$

and then suitably interpreting the resulting expression in the light of eq.(1.12).

Similarly the result in eq.(1.10.k) follows from eq.(1.12) by the use of the transformations

$U_1=V_1, U_2=V_2,\cdots,U_k=I-V_1-\cdots-V_k, U_{k+1}=V_{k+1},\cdots,U_n=V_n.$

To obtain the result in eq.(1.10.(k+1)) the transformations are

$U_1=V_1,\cdots,U_k=V_k, U_{k+1}=I-V_{k+1}-\cdots-V_n, U_{k+2}=V_{k+2},\cdots,U_n=V_n;$

while those for the result in eq.(1.10.n) are

$U_1=V_1,\cdots,U_{n-1}=V_{n-1}, U_n=I-V_{k+1}-\cdots-V_n.$

The result in eq.(1.11) is a combination of the above two categories of results. It is obtained from eq.(1.12) by the application of the transformations,

$U_1=V_1,\cdots,U_{i-1}=V_{i-1}, U_i=I-V_1-\cdots-V_k, U_{i+1}=V_{i+1},\cdots,U_k=V_k,$

$U_{k+1}=V_{k+1},\cdots,U_{k+j-1}=V_{k+j-1}, U_{k+j}=I-V_{k+1}-\cdots-V_n, U_{k+j+1}=$

$V_{k+j+1},\cdots,U_n=V_n$ where $1\le i\le k$ and $1\le j\le n-k$;

and by observing that,

$$\left| I+X_1^{1/2}V_1X_1^{1/2}+\cdots+X_{i-1}^{1/2}V_{i-1}X_{i-1}^{1/2}+X_i^{1/2}(I-V_1-\cdots-V_k)X_i^{1/2}+\right.$$
$$X_{i+1}^{1/2}V_{i+1}X_{i+1}^{1/2}+\cdots+X_k^{1/2}V_kX_k^{1/2}+X_{k+1}^{1/2}V_{k+1}X_{k+1}^{1/2}+\cdots+X_{k+j-1}^{1/2}V_{k+j-1}\times$$
$$X_{k+j-1}^{1/2}+X_{k+j}^{1/2}(I-V_{k+1}-\cdots-V_n)X_{k+j}^{1/2}+X_{k+j+1}^{1/2}V_{k+j+1}X_{k+j+1}^{1/2}+\cdots+$$
$$\left. X_n^{1/2}V_nX_n^{1/2}\right|$$

$$
\begin{aligned}
&= \left|I + X_i + X_{k+j}\right| \Big| I + (I + X_i + X_{k+j})^{-1/2} (X_1 - X_i)^{1/2} V_1 (X_1 - X_i)^{1/2} \times \\
&(I + X_i + X_{k+j})^{-1/2} + \cdots + (I + X_i + X_{k+j})^{-1/2} (X_{i-1} - X_i)^{1/2} V_{i-1} (X_{i-1} - X_i)^{1/2} \\
&\times (I + X_i + X_{k+j})^{-1/2} - (I + X_i + X_{k+j})^{-1/2} X_i^{1/2} V_i X_i^{1/2} (I + X_i + X_{k+j})^{-1/2} + \\
&(I + X_i + X_{k+j})^{-1/2} (X_{i+1} - X_i)^{1/2} V_{i+1} (X_{i+1} - X_i)^{1/2} (I + X_i + X_{k+j})^{-1/2} + \\
&\cdots + (I + X_i + X_{k+j})^{-1/2} (X_k - X_i)^{1/2} V_k (X_k - X_i)^{1/2} (I + X_i + X_{k+j})^{-1/2} + \\
&(I + X_i + X_{k+j})^{-1/2} (X_{k+1} - X_{k+j})^{1/2} V_{k+1} (X_{k+1} - X_{k+j})^{1/2} \times \\
&(I + X_i + X_{k+j})^{-1/2} + \cdots + (I + X_i + X_{k+j})^{-1/2} (X_{k+j-1} - X_{k+j})^{1/2} V_{k+j-1} \times \\
&(X_{k+j-1} - X_{k+j})^{1/2} (I + X_i + X_{k+j})^{-1/2} - (I + X_i + X_{k+j})^{-1/2} X_{k+j}^{1/2} V_{k+j} X_{k+j}^{1/2} \\
&\times (I + X_i + X_{k+j})^{-1/2} + (I + X_i + X_{k+j})^{-1/2} (X_{k+j+1} - X_{k+j})^{1/2} V_{k+j+1} \times \\
&(X_{k+j+1} - X_{k+j})^{1/2} (I + X_i + X_{k+j})^{-1/2} + \cdots + (I + X_i + X_{k+j})^{-1/2} \times \\
&(X_n - X_{k+j})^{1/2} V_n (X_n - X_{k+j})^{1/2} (I + X_i + X_{k+j})^{-1/2} \Big|
\end{aligned}
$$

where $X_l - X_i > 0$, for $l = 1, \cdots, i-1$; $X_m - X_i > 0$, for $m = i+1, \cdots, k$; $X_r - X_{k+j} > 0$, for $r = k+1, \cdots, k+j-1$; $X_s - X_{k+j} > 0$, for $s = k+j+1, \cdots, n$; and for $1 \le i \le k$ and $1 \le j \le n-k$.

THEOREM 1.5: Special Cases:

(i) $ {}^{(0)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n) = {}^{(0)}_{(2)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$

$= F^{(n)}_{D}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n) \qquad \cdots\cdots(1.13)$

(ii) $ {}^{(1)}_{(1)}E^{(2)}_{D}(a,b_1,b_2;c,c';-X_1,-X_2) = F_2(a,b_1,b_2;c,c';-X_1,-X_2) \cdots\cdots(1.14)$

(iii) $ {}^{(1)}_{(1)}E^{(3)}_{D}(a,b_1,b_2,b_3;c,c';-X_1,-X_2,-X_3)$

$= F_G(a,a,a,b_1,b_2,b_3;c,c',c';-X_1,-X_2,-X_3) \qquad \cdots\cdots(1.15)$

(iv) $ {}^{(3)}_{(1)}E^{(4)}_{D}(a,b_1,b_2,b_3,b_4;c,c';-X,-Y,-Z,-T)$

$= K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,c';-X,-Y,-Z,-T) \qquad \cdots\cdots(1.16)$

(v) $ {}^{(0)}_{(1)}E^{(n)}_{C}(a,b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)$

$= F^{(n)}_{C}(a,b;c_1,\cdots,c_n;-X_1,\cdots,-X_n) \qquad \cdots\cdots(1.17)$

(vi) $ {}^{(1)}_{(1)}E^{(2)}_{C}(a,a',b;c_1,c_2;-X_1,-X_2) = F_2(b,a,a';c_1,c_2;-X_1,-X_2) \cdots\cdots(1.18)$

(vii) $ {}^{(1)}_{(1)}E^{(3)}_{C}(a,a',b;c_1,c_2,c_3;-X_1,-X_2,-X_3)$

$= F_E(b,b,b,a,a',a';c_1,c_2,c_3;-X_1,-X_2,-X_3) \qquad \cdots\cdots(1.19)$

PROOF: (i)This result follows by putting k=0 in eq.(3.1) of the authors' papers [7,12] and then comparing the result with eq.(1.4) of the authors' paper [8].
(ii) This result is obtained by putting k=1 and n=2 in eq.(3.1) of the authors' paper [7] and then comparing the result with eq.(1.2) of the authors' paper [10].
(iii) To obtain this result we put k=1 and n=3 in eq.(3.1) of the authors' paper [7] and then comparing the result with eq.(1.2) of the authors' paper [6].
(iv) The result in eq.(1.16) can be had by putting k=3, n=4 in eq.(3.1) of the authors' paper [7] and then comparing the result with eq.(2.3) of the same paper.
(v) Putting k=0 in eq.(3.2) of the authors' paper [7] and then comparing the result with eq.(1.2) of the same paper produces this result.
(vi) This result can be obtained by putting k=1 and n=2 in eq.(3.2) of the authors' paper [7] and then comparing the outcome with eq.(1.2) of the authors' paper[10].
(vii) Putting k=1 and n=3 in eq.(3.2) of the authors' paper [7] and then comparing

the result with eq.(1.3) of the authors' paper [6] produces this result.

2. The $\Phi_D^{(n)}$-Function of Matrix Arguments

This function was introduced in eq.(1.7) of the authors' paper [8]. Here we give a transformation relation and two cases of reducibility of this function.

THEOREM 2.1:

(i) $\Phi_D^{(n)}(a, b_1, \cdots, b_{n-1}; c; -X_1, \cdots, -X_n)$

$$= e^{-tr(X_n)} \left|I + X_1\right|^{-b_1} \cdots \left|I + X_{n-1}\right|^{-b_{n-1}} \Phi_D^{(n)}[c-a, b_1, \cdots, b_{n-1}; c;$$

$$(I+X_1)^{-1/2} X_1 (I+X_1)^{-1/2}, \cdots, (I+X_{n-1})^{-1/2} X_{n-1} (I+X_{n-1})^{-1/2}, X_n] \quad \cdots\cdots (2.1)$$

(ii) $\Phi_D^{(n)}(a, b_1, \cdots, b_{n-1}; c; -X, \cdots (n-1) \cdots, -X, -X_n)$

$$= \Phi_1(a, b_1 + \cdots + b_{n-1}; c; -X, -X_n) \quad \cdots\cdots (2.2)$$

(iii) $\Phi_D^{(n)}(a, b_1, \cdots, b_{n-1}; c; I, \cdots (n-1) \cdots, I, -X_n)$

$$= \frac{\Gamma_p(c)\Gamma_p(c - b_1 - \cdots - b_{n-1} - a)}{\Gamma_p(c-a)\Gamma_p(c - b_1 - \cdots - b_{n-1})} \times {}_1F_1(a; c - b_1 - \cdots - b_{n-1}; -X_n) \quad \cdots\cdots (2.3)$$

PROOF: To prove the result in eq.(2.1), we define the $\Phi_D^{(n)}$ - function through an integral representation:

$\Phi_D^{(n)}(a, b_1, \cdots, b_{n-1}; c; -X_1, \cdots, -X_n)$

$$= \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I \left|U\right|^{a-(p+1)/2} \left|I - U\right|^{c-a-(p+1)/2} \left|I + X_1^{1/2} U X_1^{1/2}\right|^{-b_1} \cdots \times$$

$$\left|I + X_{n-1}^{1/2} U X_{n-1}^{1/2}\right|^{-b_{n-1}} e^{-tr(UX_n)} dU \quad \cdots\cdots (2.4)$$

for $Re(a, c-a) > (p-1)/2$ and $0 < U < I$.

Now the result in eq.(2.1) follows from eq.(2.4) by applying the transformation $I-U=V$ and by observing that

$$\left|I+X_i^{1/2}(I-V)X_i^{1/2}\right|=\left|I+X_i\right|\left|I-(I+X_i)^{-1/2}X_i^{1/2}VX_i^{1/2}(I+X_i)^{-1/2}\right|$$

for $i=1,\cdots,n-1$; and then suitably interpreting the resulting expression in the light of eq.(2.4).

(ii) The result in eq.(2.2) follows by putting $X_1=\cdots=X_{n-1}=X$ in eq.(2.1) of the authors' paper [9] and then using the theorem (4.1) page 62 of Mathai [3].

(iii) The result in eq.(2.3) follows by letting $X_1\to -I,\cdots,X_{n-1}\to -I$ in eq.(2.1) of the authors' paper [9] and then using the theorem (2.3.4) page 42 of Mathai [3], or, alternatively, by letting $X\to -I$ in eq.(2.2) above and then using eq.(2.18) of the authors' paper [11].

3. The Srivastava $H_C^{(2m)}$ - Function of Matrix Arguments

The generalized Srivastava $H_C^{(n)}$ -function of matrix arguments was introduced in eq.(5.2) of the authors' paper [7]. Here a result is being established for the $H_C^{(2m)}$ -function.

THEOREM 3.1:

$$H_C^{(2m)}(\alpha_1,\cdots,\alpha_{2m};\gamma;-X_1,\cdots,-X_{2m})$$

$$=\frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_3)\cdots\Gamma_p(\alpha_{2m-1})}\int_{T_1>0}\cdots(m)\cdots\int_{T_m>0}\Phi_2^{(m)}(\alpha_2,\alpha_4,\cdots,\alpha_{2m};$$

$$\gamma;-T_1^{1/2}X_1T_1^{1/2}-T_2^{1/2}X_2T_2^{1/2},-T_2^{1/2}X_3T_2^{1/2}-T_3^{1/2}X_4T_3^{1/2},\cdots,-T_{m-1}^{1/2}X_{2m-3}\times$$

$$T_{m-1}^{1/2}-T_m^{1/2}X_{2m-2}T_m^{1/2},-T_m^{1/2}X_{2m-1}T_m^{1/2}-T_1^{1/2}X_{2m}T_1^{1/2})\times e^{-tr(T_1+\cdots+T_m)}$$

$$\times\left|T_1\right|^{\alpha_1-(p+1)/2}\left|T_2\right|^{\alpha_3-(p+1)/2}\cdots\left|T_{m-1}\right|^{\alpha_{2m-3}-(p+1)/2}\left|T_m\right|^{\alpha_{2m-1}-(p+1)/2}$$

$$\times dT_1\cdots dT_m \qquad \cdots\cdots(3.1)$$

for $\mathrm{Re}(\alpha_1,\alpha_3,\cdots,\alpha_{2m-1})>(p-1)/2$.

PROOF: Taking the M-transform of the $\Phi_2^{(m)}$ - function on the right side of eq. (3.1) with respect to the variables $X_1,\cdots,X_{2m}$ and the parameters $\rho_1,\cdots,\rho_{2m}$ respectively, we have,

$$\int_{X_1>0}\cdots(2m)\cdots\int_{X_{2m}>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_{2m}|^{\rho_{2m}-(p+1)/2}\times$$

$$\Phi_2^{(m)}(\alpha_2,\alpha_4,\cdots,\alpha_{2m};\gamma;-T_1^{1/2}X_1T_1^{1/2}-T_2^{1/2}X_2T_2^{1/2},-T_2^{1/2}X_3T_2^{1/2}-T_3^{1/2}X_4\times$$

$$T_3^{1/2},\cdots,-T_{m-1}^{1/2}X_{2m-3}T_{m-1}^{1/2}-T_m^{1/2}X_{2m-2}T_m^{1/2},-T_m^{1/2}X_{2m-1}T_m^{1/2}-T_1^{1/2}X_{2m}\times$$

$$T_1^{1/2})dX_1\cdots dX_{2m} \qquad \cdots\cdots(3.2)$$

Applying the transformations,

$Z_1=T_1^{1/2}X_1T_1^{1/2};Z_2=T_2^{1/2}X_2T_2^{1/2};Z_3=T_2^{1/2}X_3T_2^{1/2};Z_4=T_3^{1/2}X_4T_3^{1/2};\cdots;$
$Z_{2m-3}=T_{m-1}^{1/2}X_{2m-3}T_{m-1}^{1/2};Z_{2m-2}=T_m^{1/2}X_{2m-2}T_m^{1/2};Z_{2m-1}=T_m^{1/2}\times$
$X_{2m-1}T_m^{1/2};Z_{2m}=T_1^{1/2}X_{2m}T_1^{1/2};$ with, $dZ_1=|T_1|^{(p+1)/2}dX_1;$
$dZ_2=|T_2|^{(p+1)/2}dX_2;dZ_3=|T_2|^{(p+1)/2}dX_3;dZ_4=|T_3|^{(p+1)/2}dX_4;\cdots;$
$dZ_{2m-3}=|T_{m-1}|^{(p+1)/2}dX_{2m-3};dZ_{2m-2}=|T_m|^{(p+1)/2}dX_{2m-2};$
$dZ_{2m-1}=|T_m|^{(p+1)/2}dX_{2m-1};dZ_{2m}=|T_1|^{(p+1)/2}dX_{2m};$ and, $|Z_1|=|T_1||X_1|;$
$|Z_2|=|T_2||X_2|;|Z_3|=|T_2||X_3|;|Z_4|=|T_3||X_4|;\cdots;|Z_{2m-3}|=|T_{m-1}||X_{2m-3}|;$
$|Z_{2m-2}|=|T_m||X_{2m-2}|;|Z_{2m-1}|=|T_m||X_{2m-1}|;|Z_{2m}|=|T_1||X_{2m}|;$
to eq.(3.2) yields,

$$|T_1|^{-\rho_1-\rho_{2m}}|T_2|^{-\rho_2-\rho_3}\cdots|T_m|^{-\rho_{2m-2}-\rho_{2m-1}}\int_{Z_1>0}\cdots(2m)\cdots\int_{Z_{2m}>0}\times$$

Continued to the next page ………………………….

$$\left|Z_1\right|^{\rho_1-(p+1)/2}\cdots\left|Z_{2m}\right|^{\rho_{2m}-(p+1)/2}\Phi_2^{(m)}(\alpha_2,\alpha_4,\cdots,\alpha_{2m};\gamma;-Z_1-Z_2,$$
$$-Z_3-Z_4,\cdots,-Z_{2m-3}-Z_{2m-2},-Z_{2m-1}-Z_{2m})dZ_1\cdots dZ_{2m} \qquad \cdots\cdots(3.3)$$

Now, making use of another set of transformations,
$U_1 = Z_1, U_2 = Z_1 + Z_2; U_3 = Z_3, U_4 = Z_3 + Z_4; \cdots; U_{2m-3} = Z_{2m-3}, U_{2m-2} = Z_{2m-3} + Z_{2m-2}; U_{2m-1} = Z_{2m-1}, U_{2m} = Z_{2m-1} + Z_{2m}$; with $dU_1 dU_2 = dZ_1 dZ_2; dU_3 dU_4 = dZ_3 dZ_4; \cdots; dU_{2m-3} dU_{2m-2} = dZ_{2m-3} dZ_{2m-2}$; $dU_{2m-1} dU_{2m} = dZ_{2m-1} dZ_{2m}$; where, $0 < U_1 < U_2; 0 < U_3 < U_4; \cdots;$
$0 < U_{2m-3} < U_{2m-2}; 0 < U_{2m-1} < U_{2m}$
in the expression (3.3) yields,

$$\left|T_1\right|^{-\rho_1-\rho_{2m}}\left|T_2\right|^{-\rho_2-\rho_3}\cdots\left|T_m\right|^{-\rho_{2m-2}-\rho_{2m-1}}\int_{U_1>0}\cdots(2m)\cdots\int_{U_{2m}>0}\times$$
$$\left|U_1\right|^{\rho_1-(p+1)/2}\left|U_2-U_1\right|^{\rho_2-(p+1)/2}\left|U_3\right|^{\rho_3-(p+1)/2}\left|U_4-U_3\right|^{\rho_4-(p+1)/2}\cdots\times$$
$$\left|U_{2m-3}\right|^{\rho_{2m-3}-(p+1)/2}\left|U_{2m-2}-U_{2m-3}\right|^{\rho_{2m-2}-(p+1)/2}\times$$
$$\left|U_{2m-1}\right|^{\rho_{2m-1}-(p+1)/2}\left|U_{2m}-U_{2m-1}\right|^{\rho_{2m}-(p+1)/2}\Phi_2^{(m)}(\alpha_2,\alpha_4,\cdots,\alpha_{2m};\gamma;$$
$$-U_2,-U_4,\cdots,-U_{2m})dU_1\cdots dU_{2m} \qquad \cdots\cdots(3.4)$$

Integrating out the m variables $U_1, U_3, \cdots, U_{2m-3}, U_{2m-1}$ in the above expression by using a type-1 Beta integral and then writing the M-transform of the $\Phi_2^{(m)}$ - function as per eq.(1.4) of the authors' paper [7], we have,

$$\left|T_1\right|^{-\rho_1-\rho_{2m}}\left|T_2\right|^{-\rho_2-\rho_3}\cdots\left|T_m\right|^{-\rho_{2m-2}-\rho_{2m-1}}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_{2m})\times$$
$$\frac{\Gamma_p(\alpha_2-\rho_1-\rho_2)}{\Gamma_p(\alpha_2)}\frac{\Gamma_p(\alpha_4-\rho_3-\rho_4)}{\Gamma_p(\alpha_4)}\cdots\frac{\Gamma_p(\alpha_{2m}-\rho_{2m-1}-\rho_{2m})}{\Gamma_p(\alpha_{2m})}\times$$
$$\frac{\Gamma_p(\gamma)}{\Gamma_p(\gamma-\rho_1-\cdots-\rho_{2m})} \qquad \cdots\cdots(3.5)$$

Substituting this expression on the right side of eq.(3.1) and then integrating out the variables $T_1, \cdots, T_m$ in the resulting expression by using a Gamma integral produces,

$$\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_{2m})}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_3)\cdots\Gamma_p(\alpha_{2m-1})}\frac{\Gamma_p(\alpha_2-\rho_1-\rho_2)}{\Gamma_p(\alpha_2)}\frac{\Gamma_p(\alpha_4-\rho_3-\rho_4)}{\Gamma_p(\alpha_4)}\cdots\times$$

$$\frac{\Gamma_p(\alpha_{2m}-\rho_{2m-1}-\rho_{2m})}{\Gamma_p(\alpha_{2m})}\frac{\Gamma_p(\gamma)\Gamma_p(\alpha_1-\rho_1-\rho_{2m})}{\Gamma_p(\gamma-\rho_1-\cdots-\rho_{2m})}\Gamma_p(\alpha_3-\rho_2-\rho_3)\cdots\times$$

$$\Gamma_p(\alpha_{2m-1}-\rho_{2m-2}-\rho_{2m-1}) \qquad \cdots\cdots(3.6)$$

which is $M[H_C^{(2m)}]$ as can be seen from eq.(5.2) of the authors' paper [7] when interpreted for the case n=2m.

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