MATRIX GENERALIZATIONS OF MULTIPLE HYPERGEOMETRIC FUNCTIONS

Lalit Mohan Upadhyaya* and H.S. Dhami**
Department of Mathematics
University of Kumaun
Almora(Uttarancahal)
India 263601

AMS Subject Classification no. 33C65, 33D99, 44.

ABSTRACT

Through this introductory paper we announce to the mathematical world our studies of some new hypergeometric functions of several matrix arguments by using Mathai's matrix transform technique. We have attempted the ten Lauricella-Saran functions; the Srivastava functions H_A, H_B ,H_C; the Exton's twenty-one quadruple hypergeometic functions; the Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$, ${}^{(k)}_{(2)}E^{(n)}_{D}$ and ${}^{(k)}_{(1)}E^{(n)}_{C}$ functions; the generalized Horn's functions ${}^{(k)}H^{(n)}_{3}$ and ${}^{(k)}H^{(n)}_{4}$ and the generalized Srivastava $H^{(n)}_{B}$ and $H^{(n)}_{C}$ functions. A number of reducibility cases of these functions and transformation relations have been deduced, wherever possible. Out of these only a limited number of results are being presented here owing to the space constraints. All the matrices appearing in this paper are (p x p) real symmetric positive definite matrices.

INTRODUCTION

Multiple hypergeometric functions constitute a natural generalization of hypergeometric functions of one variable. The development of computers has made it possible to study functions with multiple series representations from a numerical point of view. Lauricella introduced fourteen triple Gaussian series corresponding to the triple series F_1, F_2, F_5 and F_9 and ten further triple series. Saran initiated a systematic study of these ten triple Gaussian series of Lauricella's set and his notations F_E, F_F,…, F_T now prevail in the literature. Since then a few additional triple Gaussian series have been introduced; for example, G_A and G_B by Pandey, H_A, H_B and H_C by Srivastava and so on.

* *Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun(Uttaranchal), India ,248179.*
** *To whom all the correspondence may be addressed.*

A review of these functions can be had from the works of Exton [3] and Srivastava and Karlsson [25]. Mathai [7-20] has considered functions of single matrix arguments including Whittaker functions; Kampé de Fériet's functions, Appell's and Humbert's functions and Lauricella functions of matrix arguments. Among other prominent workers of the field the names of G. Pederzoli [21], R.K. Saxena and P.L. Sethi [22], D.G. Kabe [4-6], Yasuko Chikuse [1-2] etc., may be cited.

The present work is in advancement of the craftsmanship of Mathai where his transform technique has been extended to the ten Lauricella-Saran functions; the Srivastava functions H_A, H_B and H_C; the Exton's twenty-one quadruple hypergeometric functions; the Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$, ${}^{(k)}_{(2)}E^{(n)}_{D}$ and ${}^{(k)}_{(1)}E^{(n)}_{C}$ functions; the generalized Horn's functions ${}^{(k)}H^{(n)}_{3}$ and ${}^{(k)}H^{(n)}_{4}$ and the generalized Srivastava $H^{(n)}_{B}$ and $H^{(n)}_{C}$ functions.

1- The Ten Lauricella-Saran Functions.

Of the ten Lauricella-Saran functions F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S and F_T of matrix arguments, we define the following functions for the sake of illustration:

DEFINITION 1.1: The Lauricella-Saran function $F_K = F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z)$ of matrix arguments is defined as that function which satisfies the following integral equation:

$$M(F_K) = [\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2} \times F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;-X,-Y,-Z)dXdYdZ]$$

$$= [\frac{\Gamma_p(a_1-\rho_1)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(b_1-\rho_1-\rho_3)\Gamma_p(b_2-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)\Gamma_p(b_2)} \times \frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}] \quad \text{............(1.1)}$$

for $\mathrm{Re}(a_1-\rho_1, a_2-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1, c_2-\rho_2, c_3-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2$

DEFINITION 1.2 : $F_G = F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$

$$M(F_G)=[\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)dXdYdZ]$$

$$=[\frac{\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)\Gamma_p(b_1-\rho_1)\Gamma_p(b_2-\rho_2)\Gamma_p(b_3-\rho_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)}\times \frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2-\rho_3)}] \quad \ldots\ldots\ldots\ldots(1.2)$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2-\rho_3,b_1-\rho_1,b_2-\rho_2,b_3-\rho_3,c_1-\rho_1,c_2-\rho_2-\rho_3,\rho_1,\rho_2,\rho_3)>(p-1)/2$

DEFINITION 1.3 : $F_E = F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;-X,-Y,-Z)$

$$M(F_E)=[\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;-X,-Y,-Z)dXdYdZ]$$

$$=[\frac{\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)\Gamma_p(b_1-\rho_1)\Gamma_p(b_2-\rho_2-\rho_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)}\times \frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}] \quad \ldots\ldots\ldots\ldots(1.3)$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2-\rho_3,b_1-\rho_1,b_2-\rho_2-\rho_3,c_1-\rho_1,c_2-\rho_2,c_3-\rho_3,\rho_1,\rho_2,\rho_3)>(p-1)/2$

THEOREM 1.1 : $F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;-X,-Y,-Z)$

$$=\Big[\frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(c_1-a_1)\Gamma_p(c_2-b_2)\Gamma_p(c_3-b_1)}\int_0^I\int_0^I\int_0^I |U|^{a_1-(p+1)/2}\times$$

$$|V|^{b_2-(p+1)/2}|W|^{b_1-(p+1)/2}|I-U|^{c_1-a_1-(p+1)/2}|I-V|^{c_2-b_2-(p+1)/2}\times$$

$$|I-W|^{c_3-b_1-(p+1)/2}\left|I+U^{1/2}XU^{1/2}\right|^{-b_1}\left|I+V^{1/2}YV^{1/2}\right|^{-a_2}\Big|I+(I+V^{1/2}YV^{1/2})^{-1/2}\times$$

$$(I+U^{1/2}XU^{1/2})^{-1/2}W^{1/2}ZW^{1/2}(I+U^{1/2}XU^{1/2})^{-1/2}(I+V^{1/2}YV^{1/2})^{-1/2}\Big|^{-a_2}\times$$

$$dUdVdW\Big] \qquad\cdots\cdots(1.4)$$

for $\mathrm{Re}(c_1-a_1,c_2-b_2,c_3-b_1,a_1,b_1,b_2)>(p-1)/2$.

PROOF: Taking the M-transform of the right side expression of eq. (1.4) with respect to the variables X, Y, Z and the parameters ρ_1,ρ_2,ρ_3, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\left|I+U^{1/2}XU^{1/2}\right|^{-b_1}$$

$$\times\left|I+V^{1/2}YV^{1/2}\right|^{-a_2}\Big|I+(I+V^{1/2}YV^{1/2})^{-1/2}(I+U^{1/2}XU^{1/2})^{-1/2}\times$$

$$W^{1/2}ZW^{1/2}(I+U^{1/2}XU^{1/2})^{-1/2}(I+V^{1/2}YV^{1/2})^{-1/2}\Big|^{-a_2}dXdYdZ\cdots(1.5)$$

Making use of the transformations

$$X_1 = U^{1/2} X U^{1/2}, Y_1 = Y^{1/2} Y V^{1/2} \quad \text{and}$$

$$Z_1 = (I+Y_1)^{-1/2}(I+X_1)^{-1/2} W^{1/2} Z W^{1/2} (I+X_1)^{-1/2}(I+Y_1)^{-1/2}$$

so that $dX_1 = |U|^{(p+1)/2} dX; dY_1 = |V|^{(p+1)/2} dY;$

$$dZ_1 = |I+Y_1|^{-(p+1)/2} |I+X_1|^{-(p+1)/2} |W|^{(p+1)/2} dZ$$

and $|X_1| = |U||X|; |Y_1| = |V||Y|; |Z_1| = |I+Y_1|^{-1} |I+X_1|^{-1} |W||Z|$

the expression (1.5) yields,

$$|U|^{-\rho_1}|V|^{-\rho_2}|W|^{-\rho_3} \int_{X_1>0} \int_{Y_1>0} \int_{Z_1>0} |X_1|^{\rho_1-(p+1)/2} |Y_1|^{\rho_2-(p+1)/2} \times$$

$$|Z_1|^{\rho_3-(p+1)/2} |I+X_1|^{-(b_1-\rho_3)} |I+Y_1|^{-(a_2-\rho_3)} |I+Z_1|^{-a_2} dX_1 dY_1 dZ_1 \cdots (1.6)$$

Integrating out the variables X_1, Y_1, Z_1 by using a type -2 Beta integral the expression (1.6) generates

$$|U|^{-\rho_1}|V|^{-\rho_2}|W|^{-\rho_3} \frac{\Gamma_p(\rho_1)\Gamma_p(b_1-\rho_3-\rho_1)\Gamma_p(\rho_2)\Gamma_p(a_2-\rho_3-\rho_2)\Gamma_p(\rho_3)\Gamma_p(a_2-\rho_3)}{\Gamma_p(b_1-\rho_3)\Gamma_p(a_2-\rho_3)\Gamma_p(a_2)} .. (1.7)$$

Making use of the expression (1.7) on the right side of eq. (1.4), and then integrating out the variables U, V, W by using a type-1 Beta integral, we get

$$\frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(b_1-\rho_3-\rho_1)\Gamma_p(\rho_2)\Gamma_p(a_2-\rho_3-\rho_2)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)} \times$$

$$\Gamma_p(\rho_3)\Gamma_p(a_1-\rho_1)\Gamma_p(b_2-\rho_2) \qquad \cdots\cdots (1.8)$$

which from eq. (1.1) is $M(F_K)$.

In a similar pattern the matrix generalization of an integral of F_G can be obtained as

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$=[\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\iiint |U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}|I-U|^{c_1-b_1-(p+1)/2}\times$$

$$|I-V-W|^{c_2-b_2-b_3-(p+1)/2}\left|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}\right|^{-a_1}\times$$

$$dUdVdW] \qquad \cdots\cdots\cdots(1.9)$$

for $\mathrm{Re}(b_1,b_2,b_3,c_1-b_1,c_2-b_2-b_3)>(p-1)/2$.

The limiting case of which shall have the following form:

$$\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Z}{\alpha})$$

$$=[\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\iiint |U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}|I-U|^{c_1-b_1-(p+1)/2}\times$$

$$|I-V-W|^{c_2-b_2-b_3-(p+1)/2}e^{-\mathrm{tr}(UX+VY+WZ)}dUdVdW] \qquad \cdots(1.10)$$

THEOREM 1.2: A case of reducibility –

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Y)= F_2(a_1,b_1,b_2+b_3;c_1,c_2,;-X,-Y) \qquad \cdots\cdots(1.11)$$

PROOF: Putting $Z=Y$ in eq. (1.9) and observing that

$$\left|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}YW^{1/2}\right|=\left|I+U^{1/2}XU^{1/2}+(V+W)^{1/2}Y(V+W)^{1/2}\right|$$

we get

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Y)$$

$$=[\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\iiint |U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}|I-U|^{c_1-b_1-(p+1)/2}\times$$

$$|I-V-W|^{c_2-b_2-b_3-(p+1)/2}\left|I+U^{1/2}XU^{1/2}+(V+W)^{1/2}Y(V+W)^{1/2}\right|^{-a_1}dUdVdW]\cdots(1.12)$$

which under the transformations $V_1=V; W_1=V+W$ i.e. $W=W_1-V_1$ and $dV_1dW_1=dVdW$ given in the work of Mathai [7] in the range $0<V_1<W_1<I$, produces

$$=[\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\iiint |U|^{b_1-(p+1)/2}|V_1|^{b_2-(p+1)/2}|W_1-V_1|^{b_3-(p+1)/2}|I-U|^{c_1-b_1-(p+1)/2}\times$$

$$|I-W_1|^{c_2-b_2-b_3-(p+1)/2}\left|I+U^{1/2}XU^{1/2}+W_1^{1/2}YW_1^{1/2}\right|^{-a_1}dUdV_1dW_1]\cdots\cdots(1.13)$$

Integrating out V_1 by using a type-1 Beta integral and after simplification the expression (1.13) reduces to

$$=[\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2+b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\int_0^I\int_0^I |U|^{b_1-(p+1)/2}|W_1|^{b_3+b_2-(p+1)/2}|I-U|^{c_1-b_1-(p+1)/2}\times$$

$$|I-W_1|^{c_2-b_2-b_3-(p+1)/2}\left|I+U^{1/2}XU^{1/2}+W_1^{1/2}YW_1^{1/2}\right|^{-a_1}dUdW_1]\cdots(1.14)$$

which is the integral representation of $F_2(a_1,b_1,b_2+b_3;c_1,c_2;-X,-Y)$ according to theorem (3.6) page 58 of Mathai [8].

Another case of reducibility can be seen as under

$$\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Y}{\alpha})={}_1F_1(b_1;c_1;-X).{}_1F_1(b_2+b_3;c_2;-Y)\cdots\cdots(1.15)$$

This result follows by setting Z=Yin eq. (1.10) and by applying the transformations $W_1=V;$ $W_2=V+W$(here$0<W_1<W_2<I$ and $dVdW=dW_1dW_2$) and then integrating out W_1 in the resulting expression by using a type-1 Beta integral which leads us to,

$$[\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2+b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\int_0^I\int_0^I |U|^{b_1-(p+1)/2}|W_2|^{b_3+b_2-(p+1)/2}\times$$

$$|I-U|^{c_1-b_1-(p+1)/2}|I-W_2|^{c_2-b_2-b_3-(p+1)/2}e^{-tr(UX+W_2Y)}dUdW_2]$$

which is the product of the two ${}_1F_1$ functions.

THEOREM 1.3: A transformation theorem-

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$=|I+X|^{-a_1}F_G[a_1,a_1,a_1,c_1-b_1,b_2,b_3;c_1,c_2,c_2;(I+X)^{-1/2}X(I+X)^{-1/2},$$

$$-(I+X)^{-1/2}Y(I+X)^{-1/2},-(I+X)^{-1/2}Z(I+X)^{-1/2}]\qquad\cdots(1.16)$$

PROOF: To prove this theorem we define F_G through an integral representation

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$=[\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\iiint |U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}|I-U|^{c_1-b_1-(p+1)/2}\times$$

$$|I-V-W|^{c_2-b_2-b_3-(p+1)/2}\left|I+X^{1/2}UX^{1/2}+Y^{1/2}VY^{1/2}+Z^{1/2}WZ^{1/2}\right|^{-a_1}dUdVdW]\ldots(1.17)$$

for$\mathrm{Re}(b_1,b_2,b_3,c_1-b_1,c_2-b_2-b_3)>(p-1)/2.$

Now, by observing that

$$\left|I+X^{1/2}UX^{1/2}+Y^{1/2}VY^{1/2}+Z^{1/2}WZ^{1/2}\right|$$

$$=\left|I+X\right|.\left| I-(I+X)^{-1/2}X^{1/2}(I-U)X^{1/2}(I+X)^{-1/2}+(I+X)^{-1/2}Y^{1/2}VY^{1/2}\right.$$
$$\left.\times(I+X)^{-1/2}+(I+X)^{-1/2}Z^{1/2}WZ^{1/2}(I+X)^{-1/2}\right|$$

and applying the transformation $U_1=I-U$, the required theorem can be obtained by a suitable interpretation of eq. (1.17).

THEOREM 1.4: A transformation theorem-

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)=\left|I+Y\right|^{-a_1}F_G[a_1,a_1,a_1,b_1,c_2-b_2-b_3,b_3;c_1,c_2,c_2;$$
$$-(I+Y)^{-1/2}X(I+Y)^{-1/2},(I+Y)^{-1/2}Y(I+Y)^{-1/2},-(I+Y)^{-1/2}(Z-Y)(I+Y)^{-1/2}]\cdots(1.18)$$

where $Z-Y>0$.

PROOF: Applying the transformations $V_1=I-V-W, W_1=W$ to eq. (1.17) and observing that

$$\left|I+X^{1/2}UX^{1/2}+Y^{1/2}(I-V_1-W_1)Y^{1/2}+Z^{1/2}W_1Z^{1/2}\right|$$

$$=\left|I+Y\right|.\left| I+(I+Y)^{-1/2}X^{1/2}UX^{1/2}(I+Y)^{-1/2}-(I+Y)^{-1/2}Y^{1/2}V_1Y^{1/2}\right.$$
$$\left.\times(I+Y)^{-1/2}+(I+Y)^{-1/2}(Z-Y)^{1/2}W_1(Z-Y)^{1/2}(I+Y)^{-1/2}\right|$$

the required theorem follows after a suitable interpretation of eq. (1.17).
Above-mentioned two transformation theorems can be assimilated in order to generate the following result

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$
$$=\left|I+X+Z\right|^{-a_1}F_G[a_1,a_1,a_1,c_1-b_1,b_2,c_2-b_2-b_3;c_1,c_2,c_2;$$
$$(I+X+Z)^{-1/2}X(I+X+Z)^{-1/2},-(I+X+Z)^{-1/2}(Y-Z)(I+X+Z)^{-1/2},$$
$$(I+X+Z)^{-1/2}Z(I+X+Z)^{-1/2}]\qquad\cdots(1.19)$$

where $Y-Z>0$.

This can be obtained by applying the transformations $U_1=I-U;V_1=V;$ $W_1=I-V-W$ to eq. (1.17) and observing that

$$\left|I+X^{1/2}(I-U_1)X^{1/2}+Y^{1/2}V_1Y^{1/2}+Z^{1/2}(I-V_1-W_1)Z^{1/2}\right|$$

$$=|I+X+Z|.\left| I-(I+X+Z)^{-1/2}X^{1/2}U_1X^{1/2}(I+X+Z)^{-1/2}+(I+X+Z)^{-1/2}(Y-Z)^{1/2}\times\right.$$

$$\left.V_1(Y-Z)^{1/2}(I+X+Z)^{-1/2}-(I+X+Z)^{-1/2}Z^{1/2}W_1Z^{1/2}(I+X+Z)^{-1/2}\right|$$

and suitably interpreting eq. (1.17).

The matrix generalization of F_E given in eq. (1.3) can be used as in theorem (1.1) to prove the following result:

$$F_E(a,a,a,b_1,b_2,b_2;c_1,c_2,c_3;-X,-Y,-Z)$$

$$=[\frac{1}{\Gamma_p(a)}\int_{U>0}e^{-tr(U)}|U|^{a-(p+1)/2}{}_1F_1(b_1;c_1;-U^{1/2}XU^{1/2})\times$$

$$\psi_2(b_2;c_2,c_3;-U^{1/2}YU^{1/2},-U^{1/2}ZU^{1/2})dU] \quad \cdots\cdots(1.20)$$

for Re (a) > (p-1)/2.

2. The Srivastava Functions H_A, H_B, H_C.

The Srivastava functions H_A, H_B, H_C of three real symmetric positive definite matrices have been defined by us. One of them is being cited here for illustration.

DEFINITION 2.1: $H_A=H_A(a,b,b';c,c';-X,-Y,-Z)$

$$M(H_A)=\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$

$$H_A(a,b,b';c,c';-X,-Y,-Z)dXdYdZ$$

$$=\frac{\Gamma_p(a-\rho_1-\rho_3)\Gamma_p(b-\rho_1-\rho_2)\Gamma_p(b'-\rho_2-\rho_3)\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2-\rho_3)} \cdots\cdots(2.1)$$

for $Re(a-\rho_1-\rho_3,b-\rho_1-\rho_2,b'-\rho_2-\rho_3,c-\rho_1,c'-\rho_2-\rho_3,\rho_1,\rho_2,\rho_3)>(p-1)/2$.

THEOREM 2.1: $H_A(a,b,b';c,c';-X,-Y,-Z)$

$$=[\frac{1}{\Gamma_p(a)\Gamma_p(b)}\int_{S>0}\int_{T>0} e^{-tr(S+T)}|T|^{a-(p+1)/2}|S|^{b-(p+1)/2}\times$$

$$_0F_1(\,;c;-T^{1/2}S^{1/2}XS^{1/2}T^{1/2})\,_1F_1(b';c';-(S^{1/2}YS^{1/2}+T^{1/2}ZT^{1/2})dSdT]\cdots\cdots(2.2)$$

for Re (a,b)>(p–1)/2.

PROOF: Taking the M-transform of the right side of eq. (2.2) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 we get

$$=[\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$

$$_0F_1(\,;c;-T^{1/2}S^{1/2}XS^{1/2}T^{1/2})\,_1F_1(b';c';-(S^{1/2}YS^{1/2}+T^{1/2}ZT^{1/2})dXdYdZ]\cdots\cdots(2.3)$$

Making use of the transformations

$$X_1=T^{1/2}S^{1/2}XS^{1/2}T^{1/2};Y_1=S^{1/2}YS^{1/2};Z_1=T^{1/2}ZT^{1/2}$$

$$dX_1=|T|^{(p+1)/2}|S|^{(p+1)/2}dX;dY_1=|S|^{(p+1)/2}dY;dZ_1=|T|^{(p+1)/2}dZ$$

$$|X_1|=|T||S||X|;|Y_1|=|S||Y| \text{ and } |Z_1|=|T||Z|$$

The expression (2.3) yields

$$|T|^{-\rho_1-\rho_3}|S|^{-\rho_1-\rho_2}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}|X_1|^{\rho_1-(p+1)/2}|Y_1|^{\rho_2-(p+1)/2}\times$$

$$|Z_1|^{\rho_3-(p+1)/2}\,_0F_1(\,;c;-X_1)\,_1F_1(b';c';-Y_1-Z_1)dX_1dY_1dZ_1\cdots(2.4)$$

Now writing the M-transforms of the $_0F_1$ and the $_1F_1$ functions by using eq. (2.3.5) page 38 and eq. (3.2.7) page 55 of Mathai [8]; the expression (2.4) gives

$$|T|^{-\rho_1-\rho_3}|S|^{-\rho_1-\rho_2}\frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(b'-\rho_2-\rho_3)\Gamma_p(c')\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c-\rho_1)\Gamma_p(b')\Gamma_p(c'-\rho_2-\rho_3)}\cdots\cdots(2.5)$$

Substituting this expression on the right side of eq. (2.2) and integrating out S and T by using a Gamma integral we obtain

$$\frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(b'-\rho_2-\rho_3)\Gamma_p(c')\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(a-\rho_1-\rho_3)\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(c-\rho_1)\Gamma_p(b')\Gamma_p(c'-\rho_2-\rho_3)\Gamma_p(a)\Gamma_p(b)}\cdots(2.6)$$

which is $M(H_A)$. Hence the proof.

We have also established the following results on similar lines:

THEOREM 2.2: $H_B(a,b,b';c_1,c_2,c_3,;-X,-Y,-Z)$

$$=[\frac{1}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')}\int_{S_1>0}\int_{S_2>0}\int_{S_3>0} e^{-tr(S_1+S_2+S_3)}|S_1|^{a-(p+1)/2}|S_2|^{b-(p+1)/2}\times$$

$$|S_3|^{b'-(p+1)/2}\,{}_0F_1(;c_1;-S_2^{1/2}S_1^{1/2}XS_1^{1/2}S_2^{1/2})\,{}_0F_1(;c_2;-S_3^{1/2}S_2^{1/2}YS_2^{1/2}S_3^{1/2})\times$$

$${}_0F_1(;c_3;-S_3^{1/2}S_1^{1/2}ZS_1^{1/2}S_3^{1/2})dS_1dS_2dS_3] \quad \cdots\cdots\cdots(2.7)$$

for Re(a,b,b')>(p–1)/2.

THEOREM 2.3: $H_C(a,b,b';c;-X,-Y,-Z)$

$$=[\frac{1}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')}\int_{S_1>0}\int_{S_2>0}\int_{S_3>0} e^{-tr(S_1+S_2+S_3)}|S_1|^{a-(p+1)/2}|S_2|^{b-(p+1)/2}\times$$

$$|S_3|^{b'-(p+1)/2}\,{}_0F_1(\ ;c;-S_2^{1/2}S_1^{1/2}XS_1^{1/2}S_2^{1/2}-S_3^{1/2}S_2^{1/2}YS_2^{1/2}S_3^{1/2}$$

$$-S_3^{1/2}S_1^{1/2}ZS_1^{1/2}S_3^{1/2})dS_1dS_2dS_3] \quad \cdots\cdots\cdots(2.8)$$

for Re(a,b,b')>(p–1)/2.

We have also generalized many other integrals of Euler and Laplace types available for these functions in chapters four, five and six of Exton [3].

3. The Twenty-One Quadruple Hypergeometric Functions of Exton

The twenty-one quadruple hypergeometric functions $K_1,\ldots\ldots, K_{21}$ of Exton with four real symmetric positive definite matrices as arguments have been defined and all the integrals of Euler and Laplace types available for them in sections 3.3.1, 3.3.2 and in chapters four, five and six of Exton[3] have been generalized and proved. Some cases of reducibility and some transformation relation have also been obtained by us.

4. Exton's generalized hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_D$ and ${}^{(k)}_{(2)}E^{(n)}_D$ and Chandel's ${}^{(k)}_{(1)}E^{(n)}_C$ function of matrix arguments

The above functions have also been defined by us for the matrix arguments case and all the integrals of Euler and Laplace types for these functions given in sections 3.4.2 and 3.4.4 of Exton [3] have been generalized and proved by us.

5. The Generalized Horn's Functions ${}^{(k)}H_3^{(n)}$ and ${}^{(k)}H_4^{(n)}$ of matrix arguments

The approach adopted in sections three and four (given above) has been extended to all the functions under this category and we have also generalized and proved all integrals of Euler and Laplace types for these functions which are available in sections 3.5.2 and 3.5.4 of Exton [3]. One particular thing of interest has been noticed by us in this case is that the three integrals of Euler type available for these functions in section 3.5.2 of Exton [3] can be generalized for the case of 2x2 real symmetric positive definite matrices only, while no such restriction is required for the generalizations of integrals of Laplace types given in sections 3.5.4 of Exton [3].

For instance we define and prove a theorem for the function ${}^{(k)}H_3^{(n)}$ here.

DEFINITION 5.1:

$$ {}^{(k)}H_3^{(n)} = {}^{(k)}H_3^{(n)}(a,b_{k+1},\dots\dots,b_n;c;-X_1,\dots\dots,-X_n) $$

$$ M\left[{}^{(k)}H_3^{(n)}\right] = [\int_{X_1>0}\dots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2}\dots\dots|X_n|^{\rho_n-(p+1)/2}\times $$
$$ {}^{(k)}H_3^{(n)}(a,b_{k+1},\dots\dots,b_n;c;-X_1,\dots\dots,-X_n)dX_1\dots\dots dX_n] $$

$$ =[\frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)\Gamma_p(b_{k+1}-\rho_{k+1})\dots\Gamma_p(b_n-\rho_n)}{\Gamma_p(a)\Gamma_p(b_{k+1})\dots\dots\Gamma_p(b_n)\Gamma_p(c-\rho_1-\cdots-\rho_n)} $$
$$ \Gamma_p(c)\Gamma_p(\rho_1)\dots\dots\Gamma_p(\rho_n)] \qquad \dots\dots\dots\dots\dots\dots\dots(5.1) $$

for $Re(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n;b_{k+1}-\rho_{k+1};\cdots;b_n-\rho_n;c-\rho_1-\cdots-\rho_n;\rho_i)>(p-1)/2$, $i=1,\dots\dots,n$.

THEOREM 5.1:

For p=2 and for $Re[(a+1)/2,b_{k+1},\cdots,b_n,c-b_{k+1}-\cdots-b_n-(a+1)/2]>(p-1)/2$,

$$ {}^{(k)}H_3^{(n)}(a,b_{k+1},\dots\dots,b_n;c;-X_1,\dots\dots,-X_n) $$

$$ =[\frac{\Gamma_p(c)}{\Gamma_p[(a+1)/2]\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)\Gamma_p[c-b_{k+1}-\cdots-b_n-(a+1)/2]}\times $$

$$ \int\dots(n-k+1)\dots\int |U_{k+1}|^{b_{k+1}-(p+1)/2}\dots\dots|U_n|^{b_n-(p+1)/2}\times $$

continued in the next page………………..

$$\left|I-U_{k+1}-\cdots-U_n\right|^{c-b_{k+1}-\cdots-b_n-(p+1)/2}\left|V\right|^{(a+1)/2-(p+1)/2}\times$$

$$\left|I-V\right|^{c-b_{k+1}-\cdots-b_n-(a+1)/2-(p+1)/2}\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a}\times$$

$$\left|I+4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}(I-U_{k+1}-\cdots-U_n)^{1/2}\times\right.$$

$$V^{1/2}(X_1+\cdots+X_k)V^{1/2}(I-U_{k+1}-\cdots-U_n)^{1/2}\times$$

$$\left.(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}\right|^{-(2a+1)/4}dU_{k+1}\ldots\ldots dU_n dV]\ldots\ldots..(5.2)$$

PROOF: Taking the M-transform of the function involving $X_1,\ldots\ldots,X_n$ on the right side of eq.(5.2) with respect to the variables $X_1,\ldots\ldots,X_n$ and the parameters $\rho_1,\ldots\ldots,\rho_n$ respectively, we get

$$\int_{X_1>0}\cdots\int_{X_n>0}\left|X_1\right|^{\rho_1-(p+1)/2}\cdots\left|X_k\right|^{\rho_k-(p+1)/2}\left|X_{k+1}\right|^{\rho_{k+1}-(p+1)/2}\times$$

$$\cdots\left|X_n\right|^{\rho_n-(p+1)/2}\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a}\times$$

$$\left|I+4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}(I-U_{k+1}-\cdots-U_n)^{1/2}\times\right.$$

$$V^{1/2}(X_1+\cdots+X_k)V^{1/2}(I-U_{k+1}-\cdots-U_n)^{1/2}\times$$

$$\left.(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}\right|^{-(2a+1)/4}dX_1\ldots\ldots dX_k dX_{k+1}\ldots\ldots dX_n\cdots(5.3)$$

Making use of the transformations

$$Y_1=X_1;Y_2=X_1+X_2;\ldots\ldots\ldots\ldots,Y_k=X_1+\cdots+X_k;$$

$$Y_j=U_j^{1/2}X_jU_j^{1/2},\qquad j=k+1,\cdots\cdots,n.$$

Then $dY_1\ldots\ldots dY_k=dX_1\ldots\ldots dX_k$ and $dY_j=\left|U_j\right|^{(p+1)/2}dX_j$,

for $j=k+1,\ldots\ldots,n$ (on using eq. (6.7) page 95 of Mathai [7]) and $|X_1| = |Y_1|; |X_2| = |Y_2 - Y_1|; \cdots |X_k| = |Y_k - Y_{k-1}|; |Y_j| = |U_j||X_j|; j = k+1, \cdots, n.$

Here $0 < Y_1 < Y_2 < \cdots < Y_k$ and $Y_j > 0, j = k+1, \cdots, n.$

We can have

$$|U_{k+1}|^{-\rho_{k+1}} \ldots |U_n|^{-\rho_n} \int \ldots (n) \ldots \int |Y_1|^{\rho_1 - (p+1)/2} |Y_2 - Y_1|^{\rho_2 - (p+1)/2} \times$$

$$\ldots |Y_k - Y_{k-1}|^{\rho_k - (p+1)/2} |Y_{k+1}|^{\rho_{k+1} - (p+1)/2} \ldots |Y_n|^{\rho_n - (p+1)/2} |I + Y_{k+1} + \cdots + Y_n|^{-a}$$

$$|I + 4(I + Y_{k+1} + \cdots + Y_n)^{-1} (I - U_{k+1} - \cdots - U_n)^{1/2} V^{1/2} Y_k V^{1/2} (I - U_{k+1} - \cdots - U_n)^{1/2} \times$$

$$(I + Y_{k+1} + \cdots + Y_n)^{-1} |^{-(2a+1)/4} dY_1 \ldots dY_k dY_{k+1} \ldots dY_n \qquad \cdots (5.4)$$

Integrating out the variables $Y_1, \cdots, Y_{k-1}$ by using a type-1 Beta integral and then making use of another transformation

$$Z_k = 4(I + Y_{k+1} + \cdots + Y_n)^{-1} (I - U_{k+1} - \cdots - U_n)^{1/2} V^{1/2} Y_k V^{1/2} (I - U_{k+1} - \cdots - U_n)^{1/2} \times$$
$$(I + Y_{k+1} + \cdots + Y_n)^{-1}$$

so that $dZ_k = 4^{p(p+1)/2} |I + Y_{k+1} + \cdots + Y_n|^{-(p+1)} |I - U_{k+1} - \cdots - U_n|^{(p+1)/2} \times$
$$|V|^{(p+1)/2} dY_k$$

and $|Z_k| = 4^p |I + Y_{k+1} + \cdots + Y_n|^{-2} |I - U_{k+1} - \cdots - U_n| |V| |Y_k|$

the expression (5.4) yields,

$$4^{-p(\rho_1 + \cdots + \rho_k)} \cdot \frac{\Gamma_p(\rho_1) \ldots \Gamma_p(\rho_k)}{\Gamma_p(\rho_1 + \ldots + \rho_k)} |U_{k+1}|^{-\rho_{k+1}} \ldots |U_n|^{-\rho_n} \times$$

$$|I - U_{k+1} - \cdots - U_n|^{-(\rho_1 + \cdots + \rho_k)} |V|^{-(\rho_1 + \cdots + \rho_k)} \times$$

$$\int_{Z_k > 0} \int_{Y_{k+1} > 0} \cdots (n-k) \cdots \int_{Y_n > 0} |Z_k|^{(\rho_1 + \cdots + \rho_k) - (p+1)/2} |Y_{k+1}|^{\rho_{k+1} - (p+1)/2} \times$$

$$\ldots |Y_n|^{\rho_n - (p+1)/2} |I + Z_k|^{-(2a+1)/4} |I + Y_{k+1} + \cdots + Y_n|^{-(a - 2\rho_1 - \cdots - 2\rho_k)} \times$$

$$dZ_k . dY_{k+1} \ldots dY_n \qquad \cdots (5.5)$$

Integrating out the variable Z_k by using a type-2 Beta integral and the variables $Y_{k+1},\dots,Y_n$ by using a type-2 Dirichlet integral, the above expression produces

$$4^{-p(\rho_1+\cdots+\rho_k)}\Gamma_p(\rho_1)\dots\Gamma_p(\rho_n)|U_{k+1}|^{-\rho_{k+1}}\dots|U_n|^{-\rho_n}|V|^{-(\rho_1+\cdots+\rho_k)}\times$$

$$|I-U_{k+1}-\cdots-U_n|^{-(\rho_1+\cdots+\rho_k)}\frac{\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p[(2a+1)/4]\Gamma_p(a-2\rho_1-\cdots-2\rho_k)}\times$$

$$\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n) \qquad \dots\dots\dots\dots(5.6)$$

Substitution of this expression on the right side of eq. (5.2) gives

$$4^{-p(\rho_1+\cdots+\rho_k)}\Gamma_p(\rho_1)\dots\Gamma_p(\rho_n)\frac{\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p[(2a+1)/4]\Gamma_p(a-2\rho_1-\cdots-2\rho_k)\Gamma_p[(a+1)/2]}\times$$

$$\frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)\Gamma_p(c)}{\Gamma_p(b_{k+1})\dots\dots\Gamma_p(b_n)\Gamma_p[c-b_{k+1}-\cdots-b_n-(a+1)/2]}\times$$

$$\int\dots\dots(n-k+1)\dots\int|U_{k+1}|^{b_{k+1}-\rho_{k+1}-(p+1)/2}\dots|U_n|^{b_n-\rho_n-(p+1)/2}\times$$

$$|I-U_{k+1}-\cdots-U_n|^{c-b_{k+1}-\cdots-b_n-\rho_1-\cdots-\rho_k-(p+1)/2}\times$$

$$|V|^{(a+1)/2-\rho_1-\cdots-\rho_k-(p+1)/2}|I-V|^{c-b_{k+1}-\cdots-b_n-(a+1)/2-(p+1)/2}\times$$

$$dU_{k+1}\dots dU_n dV \qquad \dots\dots\dots\dots(5.7)$$

Integrating out the variables $U_{k+1},\dots,U_n$ and V by using a type-1 Dirichlet integral and a type-1 Beta integral respectively and observing that

$$\frac{4^{-p(\rho_1+\cdots+\rho_k)}\Gamma_p[(a+1)/2-\rho_1-\cdots-\rho_k]\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p(a-2\rho_1-\cdots-2\rho_k)\Gamma_p[(a+1)/2]\Gamma_p[(2a+1)/4]}$$

$$=\frac{1}{\Gamma_p(a)} \text{ for } p=2. \qquad \dots\dots\dots\dots(5.8)$$

by using eq. (6.13) page 84 of Mathai [8], simplification of the expression (5.7) finally yields

$$\frac{\Gamma_p(c)\Gamma_p(\rho_1)....\Gamma_p(\rho_n)\Gamma_p(b_{k+1}-\rho_{k+1})...\Gamma_p(b_n-\rho_n)}{\Gamma_p(b_{k+1})....\Gamma_p(b_n)\Gamma_p(a)\Gamma_p(c-\rho_1-\cdots-\rho_n)}\times$$
$$\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n) \qquad ....(5.9)$$

which is $M\left[{}^{(k)}H_3^{(n)}\right]$, hence the proof. It is to be noted that the result of theorem (5.1) is different from the corresponding result in the scalar case.

6. The Generalized Srivastava Functions $H_B^{(n)}$ and $H_C^{(n)}$ of Matrix Arguments.

After defining the generalized Srivastava functions $H_B^{(n)}$ and $H_C^{(n)}$ of real symmetric positive definite matrices as arguments, the Laplace type integrals available for these functions in eqs. (198),(199) and (200) on page 325 of Srivastava and Karlsson [25] have been generalized and proved.

7. On Other Functions of Matrix Arguments.

Besides the above functions a number of integrals of Euler and Laplace types for other types of functions, viz. Kummer's confluent hypergeometric functions, the ordinary and generalized Gauss's hypergeometric functions available in Slater [23,24], the Appell's and the Humbert's functions, the Lauricella functions available in Exton [3] and Srivastava and Karlsson [25] and which have so far not been generalized by Mathai [7-20] and others, have been generalized and proved by us together with some transformation relations and cases of reducibility.

We propose to deal with the definitions, statements and generalizations and proofs of various results of the different types of functions mentioned in this paper in our future communications.

References

1. Chikuse, Yasuko (1992). Properties of Hermite and Laguerre Polynomials in Matrix Arguments and Their Applications; Linear Algebra and its Applications, 176, pp. 237-260.
2. Chikuse, Yasuko (1992). Generalized Hermite and Laguerre Polynominals in Multiple Symmetric Arguments and Their Applications; Linear Algebra and its Applications, 176, pp.261-287.
3. Exton H. (1976). Multiple Hypergeometric Functions and Applications. Ellis Horwood Limited, Publishers, Chichester.
4. Kabe, D.G. (1991). On Some Multiple Integrals Involving Hypergeometric Functions of Matrix Arguments; Indust. Math. 41 (1991), no.1, 29-39 (1992).
5. Kabe, D.G. (1991). Some Results for Multiple Integrals Involving Hypergeometric Functions of Matrix Arguments; Indust. Math. 41 (1991) no.2, 125-136 (1992).

6. Kabe, D.G. (1993). Herz’s, Conjecture for Hypergeometric Series of Matrix Arguments; Indust. Math. 43 (1993), no. 1, 5-10 (1994).
7. Mathai A.M. (1992). Jacobians of Matrix Transformations I, Centre for Mathematical Sciences, Trivandrum, India.
8. Mathai A.M. (1993). Hypergeometric Functions of Several Matrix Arguments, Centre for Mathematical Sciences, Trivandrum, India.
9. Mathai A.M. (1993). Appell's and Humbert's Functions of Matrix Arguments, Linear Algebra and its Applications; 183, pp. 201-221
10. Mathai A.M. (1993). Lauricella Functions of Real Symmetric Positive Definite Matrices, Indian Journal of Pure and Applied Mathematics; 24(9) pp. 513-531.
11. Mathai A.M.,Pederzoli G. (1993). Kampé de Fériet's Functions of Matrix Arguments; Metron, LI (3-4) pp. 3-24.
12. Mathai A.M. (1995). Special Functions of Matrix Arguments-I; Proceedings of the National Academy of Sciences, India; LXV (II) pp. 121-142.
13. Mathai A.M.(1995). Special Functions of Matrix Arguments-II; Proceedings of the National Academy of Sciences, India; LXV (III) pp. 227-251.
14. Mathai A.M. (1995). Special Functions of Matrix Arguments-III; Proceedings of the National Academy of Sciences, India; LXV (IV), pp.367-393.
15. Mathai A.M. (1995). Special Functions of Matrix Arguments-IV; Proceedings of the National Academy of Sciences, India; LXVI (A-I) pp. 1-22.
16. Mathai A.M.,Pederzoli G. (1995). Hypergeometric Functions of Many Matrix Variables and Distributions of Generalized Quadratic Forms; American Journal of Mathematical and Management Sciences; 15(3&4) pp. 343-354.
17. Mathai A.M. (1996). Whittaker and G-Functions of Matrix Arguments in the Complex Case, International Journal of Mathematical and Statistical Sciences; 5,pp. 7-32.
18. Mathai A.M.,Pederzoli G. (1996). Some Transformations for Functions of Matrix Arguments; Indian Journal of Pure and Applied Mathematics; 27(3), pp. 277-284.
19. Mathai A.M.,Pederzoli G. (1997). Some Properties of Matrix Variate Laplace Transform and Matrix Variate Whittaker Functions; Linear Algebra and its Applications; 253, pp. 209-226.
20. Mathai A.M.,Pederzoli G. (1998). On Whittaker Functions of Matrix Argument; Linear Algebra and its Applications; 269, pp. 91-103.
21. Pederzoli G. (1996). Some Properties of Kampé de Fériet's Functions of Matrix Arguments and Applications; Metron (53), no. 3-4; 5-23.
22. Saxena R.K., Sethi P.L. & Gupta O.P. (1997). Appell's Functions of Matrix Arguments. Indian Journal of Pure and Applied Mathematics; 28, no. 3, pp. 371-380.
23. Slater L.J. (1960). Confluent Hypergeometric Functions. Cambridge University Press.
24. Slater L.J. (1966). Generalized Hypergeometric Functions. Cambridge University Press.
25. Srivastava H.M. & Karlsson P.W. (1985). Multiple Gaussian Hypergeometric Series. Ellis Horwood Limited, Publishers, Chichester.