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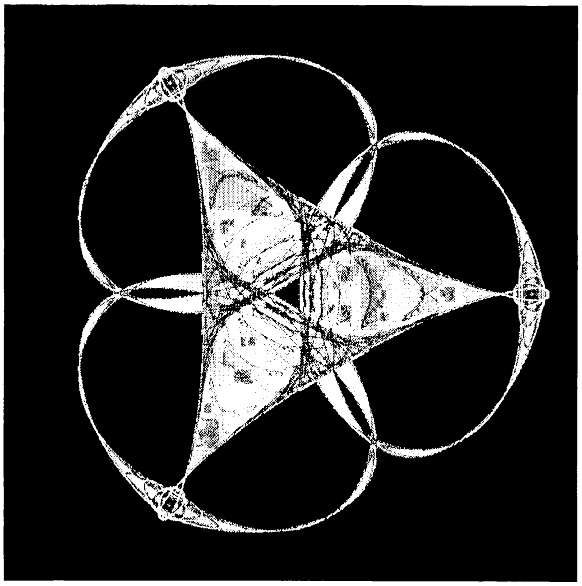

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EXPRESSIONS FOR THE GENERATING FUNCTION OF THE HYPERGEOMETRIC DISTRIBUTIONS WHEN THE PARAMETERS ARE OF DIFFERENT NATURE.

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Our aim in the present paper is to express the distribution function explicitly in terms of parameters of the two categories. The generalised integral and differential coefficient of the finite hypergeometric distribution function have been evaluated and differential equation has been generated.

Key Words : Hypergeometric distributions/ Summation formula replacement and non replacement categories/ Probability of a random variable.

1. INTRODUCTION

It is a well established fact that if there be a given finite population of N elements, exactly M of Which belong to the Category U, then the probability of a random sample of n elements, which includes exactly x elements of category U, is defined as a hypergeometric distribution. The user of the distribution may come across such situations where some of the elements be infinite and others are finite, we may cite the example of the case of an urn containing boxes of balls of different colours, out of which replacement is permissible to a particular group.

The hypergeometric function which shall arise in such type of distributions shall be of following nature.

$$F_{q\,s}^{p\,r}\left[\begin{matrix}(a_1, n_1), (a_2, n_2), \ldots\ldots\ldots\ldots, (a_p, n_p); a_1', a_2', \ldots\ldots\ldots, a_r' \\ \\ (b_1, m_1), (b_2, m_2), \ldots\ldots\ldots\ldots, (b_q, m_q); b_1', b_2', \ldots\ldots\ldots\ldots b_s'\end{matrix}\; x\right] \ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

Where the number of chances $n_1, n_2, \ldots\ldots\ldots\ldots, n_p; m_1, m_2, \ldots\ldots\ldots\ldots, m_q$ (of both categories) are in ascending order. The parameters $(a_1, a_2, \ldots\ldots\ldots\ldots, a_p)$; $(b_1, b_2, \ldots\ldots\ldots\ldots, b_q)$ belong

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to the non replacement group and $(a_1', a_2', a_3' \ldots\ldots\ldots, a_r'); (b_1', b_2, b_3', \ldots\ldots\ldots, b_s')$ fall in its reverse group.

Moments and generating functions and approximations for the ordinary class of hypergeometric distribution can be seen in the work of Johnson & Kotz [7], kendall stuar [6] & Mathai [1] and others etc., but for the conditions discussed by us in the present paper the methods discussed so far are inappropriate. In order to cope up with such situations different special functions expressed in matrix forms also play meaningful roles as seen in the work of Mathai [2-5].

We feel that the distribution function shall have to be estimated explicitly in terms of parameters of the two categories discussed in the beginning of this section. The moment generating functions, Cummulants and other properties for the distribution can be derived only after investigation of differential and integral properties of the function in their classical arenas and generation of the differential equation and it has been the point of concern of the present paper.

2. GENERAL FORMULATION

The function defined by relation (1.1) can be expressed as

$$\sum_{(ni,\, mj,\, m)=(0,0,0)}^{(n_i,\, m_j,\, \infty)} \frac{\prod_{i=1}^{p} (a_i, n_i) \prod_{i=1}^{r} (a'_i, m)}{\prod_{j=1}^{q} (b_j, m_j) \prod_{j=1}^{s} (b'_j, m)} \left(\frac{x^m}{m!}\right) \quad \ldots\ldots\ldots\ldots (2.1)$$

Where $(a_i, n_i) = a_i (a_i + 1) \ldots\ldots\ldots\ldots (a_i + n_i - 1)$ and range for i, j are 1 to p & 1 to q respectively.

This summation formula can be used in the explicit represantation of the distribution functions in the two categories of terminating and non terminating range of parameters. As an example we may cite the simplest case where the balls of the two boxes belong to non replacement category and the one to the replacement category. The distribution function in this case shall be

$$_2F_1 \quad [(a, n_1), (b, n_2); c; x] \qquad \ldots\ldots\ldots\ldots (2.2)$$

One interesting situation may arise when all the three elements belong to the non replacement categories for which the distribution function shall be

$$_2F_1 \; [(a_1, n_1), (a_2, n_3); (b, n_2); x] \qquad \ldots\ldots\ldots\ldots (2.3)$$

Whose value is expressible as

$$_2F_1^{n_1} \; [a_1, a_2 ; b ; x]$$

$$+ \{[(a_1, n_1)(a_2, n_1+1)\, x^{n_1+1}] / [(b_1, n_1+1)(n_1+1)!]\}\; {}_2F_2^{\,n_2-n_1-1} [a_2+n_1+1, 1; b+n_1+1, n_1+2 ; x]$$

$$+ \{[(a_1, n_1)(a_2, n_2+1)\, x^{n_2+1}] / [(b, n_2)(n_2+1)!]\}\; {}_2F_1^{\,n_3-n_2-1} \; [a_2+n_2+1, 1; n_2+2; x] \qquad \ldots\ldots\ldots\ldots (2.4)$$

The expression given in (1.1) for the generating function of the hypergeomedric distribution (when all the parameters are of different nature) shall have the following value in summation form.

$$_pF_q^{\,n_1} + Ax^{n_1+1}\; {}_pF_{q+1}^{\,m_1-n_1-1} + Bx^{m_1+1}\; {}_pF_q^{\,n_2-m_1-1} + C\,x^{n_2+1}\; {}_{p-1}F_q^{\,m_2-n_2-1} + \ldots\ldots\ldots\ldots (2.5)$$

When $n_1 < m_1 < n_2 < m_2 < \ldots\ldots\ldots\ldots$

3. GENERALIZED INTEGRAL AND DIFFERENTIAL COEFFICIENT OF THE FINITE HYPERGEOMETRIC DISTRIBUTION FUNCTION

The m[th] integral for the distribution function given by (2.3) shall be

$$\int_0^x \int_0^x \ldots\ldots\ldots \int_0^x {}_2F_1^{\,n_1}[a_1, a_2; b; x]\,(dx)^m$$

$$= (a_1)_{-m}\,(a_2)_{-m} / (b)_{-m} \; \{{}_2F_1^{\,n_1+m}\,[a_1-m, a_2-m; b-m; x] \; -1\}$$

$$- \sum_{r=1}^{m} \{[(a_1)_{-r}\,(a_2)_{-r}] / (b)_{-r}\,(m-r)!\}\; x^{m-r} \qquad \ldots\ldots (3.1)$$

Which exhibits the specific nature of the parameters in the form of hypergeometric distribution function where the parameters are decreased by m while the chances are increased by the same amount.

This peculiar property is reversed in the case of m^{th} differential coefficient, as is evident from the following result.

$$d^m/dx^m \{{}_2F_1^{\,n_1}[a_1, a_2; b; x]\} = \{(a_1)_m (a_2)_m / (b)_m\}\ {}_2F_1^{\,n_1-m}[a_1+m, a_2+m; b+m; x] \ldots\ldots (3.2)$$

4. GENERATION OF DIFFERENTIAL EQUATION

Considering the result (2.4) as the sum of the three different functions

$$W = w_1 + w_2 + w_3 \ldots\ldots\ldots\ldots\ldots\ldots (4.1)$$

We shall have following differential equation for w_1

$$[\theta(\theta+b-1) - x(\theta+a_1)(\theta+a_2)]\, w_1 - \{(a_1)_{n_1} (a_2)_{n_1} (n_1+a_1)(n_1+a_2)/(b)_{n_1}\} \left(x^{n_1+1}/(n_1)!\right) = 0 \ldots\ldots (4.2)$$

Similar results can be seen in case of w_2 and w_3.

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