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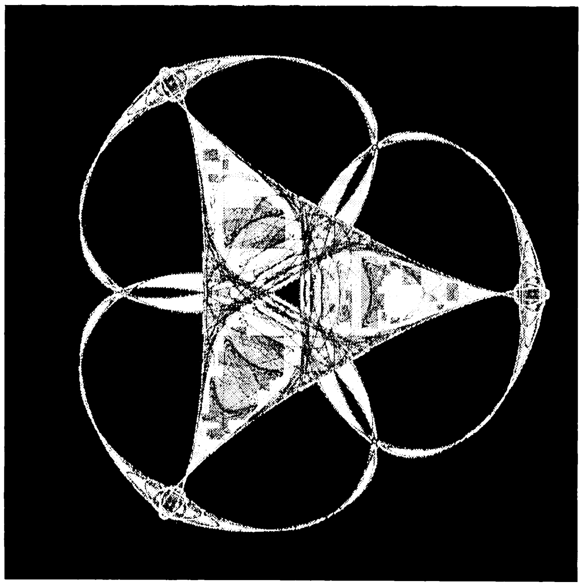
IMA Preprint Series # 1619

June 1999



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CONSTRUCTION OF GENERALIZED MULTIPLE HYPERGEOMETRIC FUNCTION AND CONTRIVANCE OF APPELL FUNCTIONS AS ITS SPECIAL CASES

ANAND SINGH AND H.S. DHAMI *

Abstract : In the present paper an attempt has been made to formulate generalized function, whose particular cases are Appell functions and other multiple hypergeometric functions. Differential equation for the generalized function has been investigated and its validity has been established with the help of known results.

1. Introduction : Gauss series has been generalized by increasing the number of parameters and by making the series infinite in the both directions. The detailed account of work in this context can be seen in the work of Slater (1966). The another way of generalization is of increasing the number of variables leading to the formation of Appell and other similarly generalized functions given in the book of Exton (1976). Some finite summation formulas involving generalized hypergeometric functions can be seen in the work of Srivastava (1985). A general class of mixed trilinear generating relations for Gauss hypergeometric function has been investigated by Chakrabarty and Hazra (1993). The two papers of Exton (1994 &1995) and books of Agarwal (1996) also gives sufficient material on generalized hypergeometric functions.

We in our earlier studies (1998) have formed finite multiplicative group for n - lateral multiple hypergeometric functions. In the present work an attempt has been made to acquire product of two generating functions in order to obtain generalized multiple hypergeometric functions and its differential equation.

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2. General Formulation : **We have expressed the generating function for the hypergeometric series, in our earlier communication , as**

$$B\begin{matrix}p & r\\ & \\ q & s\end{matrix}\left[\begin{matrix}a_p & & b_r & & \\ & ; & & ; & x\\ a'_q & & b'_s & & \end{matrix}\right]$$

$$= \sum_{n=0}^{\propto}\left\{\frac{(a_p)_n \quad (b_r)_{-n}}{(a'_q)_n \quad (b'_s)_{-n}}\right\}\frac{x^n}{n!} \qquad \text{.................(1)}$$

Now considering two functions defined in the following manner

$$B\begin{matrix}p & r\\ & \\ q & s\end{matrix}\left[\begin{matrix}a_p & & a'_r & & \\ & ; & & ; & x\\ b_q & & b'_s & & \end{matrix}\right]\cdot B\begin{matrix}p & r\\ & \\ q & s\end{matrix}\left[\begin{matrix}c_p & & c'_r & & \\ & ; & & ; & y\\ d_q & & d'_s & & \end{matrix}\right]$$

$$= \sum_{m=0}^{\propto}\sum_{n=0}^{\propto}\left(\frac{(a_p)_m \quad (a'_r)_{-m} \quad (c_p)_n \quad (c'_r)_{-n}}{(b_q)_m \quad (b'_s)_{-m} \quad (d_q)_n \quad (d'_s)_{-n}}\right)\frac{x^m \; y^n}{m! \; n!} \quad \text{......(2)}$$

We can deduce the following result for the product of two B - functions

$$\sum_{m=0}^{\propto}\sum_{n=0}^{\propto}\left(\frac{(A_p)_{m+n} \quad (C_r)_{-(m+n)}}{(B_q)_{m+n} \quad (D_s)_{-(m+n)}}\right)\frac{x^m \; y^n}{m! \; n!} \qquad \text{..................................(3)}$$

Where

$$(a_p)_m \; (c_p)_n = (A_p)_{m+n} \quad , \qquad (a'_r)_{-m}(c'_r)_{-n} = (C_r)_{-(m+n)}$$

$$(b_q)_m \; (d_q)_n = (B_q)_{m+n} \quad \& \quad (b'_s)_{-m} \; (d'_s)_{-n} = (D_s)_{-m-n} \qquad \text{.............(4)}$$

Result (2) can be further written as

$$\sum_{m=0}^{\propto}\sum_{n=0}^{\propto}\left(\frac{(A_p)_m \quad (C_r)_{-m}}{(B_q)_m \quad (D_s)_{-m}}\right)\frac{x^{m-n} \qquad y^n}{(m-n)\,! \qquad n!} \qquad \text{............(5)}$$

$$\sum_{m=0}^{\propto} \left(\frac{(A_p)_m \quad (C_r)_{-m}}{(B_q)_m \quad (D_s)_{-m}} \right) \frac{(x+y)^m}{m!} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

which finally shall be

$$B^{p \quad r}_{q \quad s}\left[\begin{matrix} A_p & ; & C_r & ; & x+y \\ B_q & & D_s & & \end{matrix}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This process of generalization when enhanced further leads to the formation of a general type B - function of the form

$$B^{(p,\, p_1) \quad (r,\, r_1)}_{(q,\, q_1) \quad (s,s_1)}$$

which stand for a function of type (7) in which p_1,q_1,r_1 and s_1 variables assume two values simultaneously and p-p_1, q-q_1, r-r_1, s-s_1 are single valued with the pre-requisite that at least one of p_1, q_1, r_1 and s_1 is non- zero.
Thus we can conclude that

$$B^{(p,\, p_1) \quad (r,\, r_1)}_{(q,\, q_1) \quad (s,\, s_1)} = \sum_{m,n=0}^{\propto} \left[\frac{\prod_{i=1}^{p_1} (a_i)_m \prod_{i=1}^{p_1} (a'_i)_n \prod_{j=1}^{p-p_1} (a_{p_1+j})_{m+n}}{\prod_{i=1}^{q_1} (b_i)_m \prod_{i=1}^{q_1} (b'_i)_n \prod_{j=1}^{q-q_1} (b_{q_1+j})_{m+n}} \right.$$

$$\left. \cdot \frac{\prod_{i=1}^{r_1} (c_i)_{-m} \prod_{i=1}^{r_1} (c'_i)_{-n} \prod_{j=1}^{(r-r_1)} (c_{r_1+j})_{-m-n}}{\prod_{i=1}^{s_1} (d_i)_{-m} \prod_{i=1}^{s_1} (d'_i)_{-n} \prod_{j=1}^{(s-s_1)} (d_{s_1+j})_{-m-n}} \right] \frac{x^m \quad y^n}{m! \quad n!} \qquad \ldots(8)$$

3. *Partial differential equation for the general B-function :*

Let us set $\theta = x\,\partial/\partial x$ *and* $\phi = y\,\partial\,/\,\partial y$
so that

$$\theta\ (x^m\ y^n) = m\ x^m y^n \quad \text{and} \quad \phi\ (x^m y^n) = n\ x^m\ y^n$$

So that for the function given by relation (8), we shall have

$$\left\{\prod_{i=1}^{p_1}(\theta+a_i)\prod_{j=1}^{p-p_1}(\theta+\phi+a_{p_1+j})\prod_{i=1}^{s_1}(d_i-\theta-1)\prod_{j=1}^{s-s_1}(d_{s_1+j}-\theta-\phi-1)\right\}B=\sum_{m,n=0}^{\propto}\frac{R}{S} \qquad \ldots\ldots\ldots\ldots(9)$$

where

$$R=\left\{\prod_{i=1}^{p_1}(m+a_i)\prod_{j=1}^{p-p_1}(m+n+a_{p_1+j})\prod_{i=1}^{s_1}(d_i-m-1)\prod_{j=1}^{s-s_1}(d_{s_1+j}-m-n-1)\right.$$

$$\prod_{i=1}^{p_1}(a_i)_m\prod_{i=1}^{p_1}(a'_i)_n\prod_{j=1}^{p-p_1}(a_{p_1+j})_{m+n}$$

$$\left.\prod_{i=1}^{r_1}(c_i)_{-m}\prod_{i=1}^{r_1}(c'_i)_{-n}\prod_{j=1}^{r-r_1}(c_{r_1+j})_{-m-n}\,x^m\,y^n\right\}$$

&

$$S=\left\{\prod_{i=1}^{q_1}(b_i)_m\prod_{i=1}^{q_1}(b'_i)_n\prod_{j=1}^{q-q_1}(b_{q_1+j})_{m+n}\prod_{i=1}^{s_1}(d_i)_{-m}\cdot\prod_{i=1}^{s_1}(d'_i)_{-n}\prod_{j=1}^{s-s_1}(d_{s_1+j})_{-m-n}\;m!\;n!\right\}$$

which can be delineated as $\sum_{m,n=0}^{\propto}\frac{R_1}{S_1}$ ....(10)

where

$$R_1=\left\{\prod_{i=1}^{p_1}\left[(m+a_i)(a_i)_m\right]\prod_{j=1}^{p-p_1}\left[(m+n+a_{p_1+j})(a_{p_1+j})_{m+n}\right]\prod_{i=1}^{p_1}(a'_i)_n\right.$$

$$\left.\prod_{i=1}^{r_1}(c_i)_{-m}\prod_{i=1}^{r_1}(c')_{-n}\prod_{j=1}^{r-r_1}(c_{r_1+j})_{-m-n}\;x^m\;y^n\right\}$$

&

$$S_1=\left\{\prod_{i=1}^{q_1}(b_i)_m\prod_{i=1}^{q_1}(b'_i)_n\prod_{j=1}^{q-q_1}(b_{q_1+j})_{m+n}\prod_{i=1}^{s_1}\left(\frac{(d_i)_{-m}}{(d_i-m-1)}\right)\right.$$

$$\left.\prod_{i=1}^{s_1}(d'_i)_{-n}\prod_{j=1}^{s-s_1}\left(\frac{(d_{s_1+j})_{-m-n}}{(d_{s_1+j}-m-n-1)}\right)\;m!\;\;n!\right\}$$

or

$$\sum_{m,n=0}^{\infty} \frac{\left\{ \prod_{i=1}^{p_1} (a_i)_{m+1} \prod_{j=1}^{p-p_1} (a_{p_1+j})_{m+n+1} \prod_{i=1}^{p_1} (a'_i)_n \prod_{i=1}^{r_1} (c_i)_{-m} \prod_{i=1}^{r_1} (c'_i)_{-n} \prod_{j=1}^{r-r_1} (c_{r_1+j})_{-m-n} \; x^m \; y^n \right\}}{\left\{ \prod_{i=1}^{q_1} (b_i)_m \prod_{i=1}^{q_1} (b'_i)_n \prod_{j=1}^{q-q_1} (b_{q_1+j})_{m+n} \prod_{i=1}^{s_1} (d_i)_{-m-1} \prod_{i=1}^{s_1} (d'_i)_{-n} \prod_{j=1}^{s-s_1} (d_{s_1+j})_{-m-n-1} \; m! \; n! \right\}} \qquad \ldots\ldots\ldots\ldots(11)$$

Now replacing m by m-1, we get

$$\sum_{m,n=0}^{\infty} \frac{\left\{ \prod_{i=1}^{p_1} (a_i)_m \prod_{j=1}^{p-p_1} (a_{p_1+j})_{m+n} \prod_{i=1}^{p_1} (a'_i)_n \prod_{i=1}^{r_1} (c_i)_{-m+1} \prod_{i=1}^{r_1} (c'_i)_{-n} \prod_{j=1}^{r-r_1} (c_{r_1+j})_{-m-n+1} \; x^{m-1} \; y^n \right\}}{\left\{ \prod_{i=1}^{q_1} (b_i)_{m-1} \prod_{i=1}^{q_1} (b'_i)_n \prod_{j=1}^{q-q_1} (b_{q_1+j})_{m+n-1} \prod_{i=1}^{s_1} (d_i)_{-m} \prod_{i=1}^{s_1} (d'_i)_{-n} \prod_{j=1}^{s-s_1} (d_{s_1+j})_{-m-n} \; (m-1)! \; n! \right\}} \qquad \ldots\ldots\ldots\ldots(12)$$

or

$$\sum_{m,n=0}^{\infty} \frac{\left\{ \prod_{i=1}^{p_1} \left[(a_i)_m (a'_i)_n \right] \prod_{j=1}^{p-p_1} (a_{p_1+j})_{m+n} \prod_{i=1}^{r_1} \left[(c_i-m)(c_i)_{-m} (c'_i)_{-n} \right] \prod_{j=1}^{r-r_1} \left[(c_{r_1+j})_{-m-n} (c_{r_1+j}-m-n) \right] \; x^{m-1} \; y^n \right\}}{\left\{ \prod_{i=1}^{q_1} \left(\frac{(b_i)_m (b'_i)_n}{(b_i+m-1)} \right) \prod_{j=1}^{q-q_1} \left(\frac{(b_{q_1+j})_{m+n}}{(b_{q_1+j}+m+n-1)} \right) \prod_{i=1}^{s_1} (d_i)_{-m} \prod_{i=1}^{s_1} (d'_i)_{-n} \prod_{j=1}^{s-s_1} (d_{s_1+j})_{-m-n} \left(\frac{m!}{m} \right) n! \right\}} \qquad \ldots\ldots\ldots\ldots(13)$$

$$=\frac{1}{x}\sum_{m,n=0}^{\propto}\frac{R_2}{S_2}\quad\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

where

$$R_2=\prod_{i=1}^{p_1}[(a_i)_m(a'_i)_n]\prod_{j=1}^{p-p_1}(a_{p_1+j})_{m+n}\prod_{i=1}^{r_1}[(c_i)_{-m}(c'_i)_{-n}]\prod_{j=1}^{r-r_1}(c_{r_1+j})_{-m-n}$$

$$.\quad m\prod_{i=1}^{r_1}(c_i-m)\prod_{j=1}^{r-r_1}(c_{r_1+j}-m-n)\prod_{i=1}^{q_1}(b_i+m-1)\prod_{j=1}^{q-q_1}(b_{q_1+j}+m+n-1)\;x^m\,y^n$$

&

$$S_2=\prod_{i=1}^{q_1}[(b_i)_m(b'_i)_n]\prod_{j=1}^{q-q_1}(b_{q_1+j})_{m+n}\prod_{i=1}^{s_1}[(d_i)_{-m}(d'_i)_{-n}]\prod_{j=1}^{s-s_1}(d_{s_1+j})_{-m-n}\;m!\,n!$$

which by using the notation given by relation (4), becomes

$$=\frac{1}{x}\sum_{m,n=0}^{\propto}A_{mn}\,m\prod_{i=1}^{r_1}(c_i-m)\prod_{j=1}^{r-r_1}(c_{r_1+j}-m-n)\prod_{i=1}^{q_1}(b_i+m-1)\prod_{j=1}^{q-q_1}(b_{q_1+j}+m+n-1)\;x^m\,y^n\quad\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

0r

$$\frac{\theta}{x}\prod_{i=1}^{r_1}(c_i-\theta)\prod_{j=1}^{r-r_1}(c_{r_1+j}-\theta-\phi)\prod_{j=1}^{q_1}(b_i+\theta-1)\prod_{j=1}^{q-q_1}(b_{q_1+j}+\theta+\phi-1)\sum_{m,n=0}^{\propto}A_{mn}\,x^m\,y^n\quad\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

$$=\left\{\frac{\theta}{x}\prod_{i=1}^{r_1}(c_i-\theta)\prod_{j=1}^{r-r_1}(c_{r_1+j}-\theta-\phi)\prod_{j=1}^{q_1}(b_i+\theta-1)\prod_{j=1}^{q-q_1}(b_{q_1+j}+\theta+\phi-1)\right\}B\ldots\ldots\ldots(17)$$

writing Z for B - function, the differential equation assumes the form

$$\left\{\prod_{i=1}^{p_1}(\theta+a_i)\prod_{j=1}^{p-p_1}(\theta+\phi+a_{p_1+j})\prod_{i=1}^{s_1}(d_i-\theta-1)\prod_{j=1}^{s-s_1}(d_{s_1+j}-\theta-\phi-1)\right.$$

$$\left.-\frac{\theta}{x}\prod_{i=1}^{r_1}(c_i-\theta)\prod_{j=1}^{r-r_1}(c_{r_1+j}-\theta-\phi)\prod_{j=1}^{q_1}(b_i+\theta-1)\prod_{j=1}^{q-q_1}(b_{q_1+j}+\theta+\phi-1)\right\}Z=0\ldots\ldots\ldots(18)$$

Equation (18) can also be put in the form

$$\left\{ \prod_{i=1}^{p_1} (\phi+a'_i) \prod_{j=1}^{p-p_1} (\theta+\phi+a_{p_1+j}) \prod_{i=1}^{s_1}(d'_i-\phi-1) \prod_{j=1}^{s-s_1} (d_{s_1+j}-\theta-\phi-1) \right.$$

$$\left. -\frac{\phi}{y} \prod_{i=1}^{r_1}(c'_i-\phi) \prod_{j=1}^{r-r_1} (c_{r_1+j}-\theta-\phi) \prod_{j=1}^{q_1}(b'_j+\phi-1) \prod_{j=1}^{q-q_1} (b_{q_1+j}+\theta+\phi-1) \right\} Z=0 \qquad \text{..................(19)}$$

4. *Special Cases of the Generalized Multiple Hypergeometric Function*

Putting

$$\begin{aligned} &p=2,\ q=1,\ r=0,\ s=0\\ &p_1=1,\ q_1=0\ ,r_1=0,\ s_1=0\\ &a_1=b,\ a'_1=b_1,\ a_2=a,\ b_1=c \end{aligned}$$

So that

$$B^{(p,\,p_1)\ (r,\,r_1)}_{(q,\,q_1)\ (s,\,s_1)} = B^{(2,1)\ \ (0,0)}_{(1,0)\ \ (0,0)}$$

we can speculate

$$B^{(2,1)\ \ (0,0)}_{(1,0)\ \ (0,0)}\left[\begin{array}{ccccc} a, & (b,b_1) & & - & \\ & & ; & & ;\ x,\ y \\ & c & & - & \end{array}\right] = F \quad [a,\ b,\ b';\ c;\ x,\ y\ ] \text{..........(20)}$$

which is Appell's function F_1.

and

$$B^{(2,0)\ \ (0,0)}_{(1,0)\ \ (0,0)}\left[\begin{array}{ccccc} a,b & & - & \\ & ; & & ;\ x \\ c, & & - & \end{array}\right] = {}_2F_1\left[\begin{array}{ccc} a, & b & \\ & & ;\ x \\ & c & \end{array}\right] \text{................(21)}$$

which is Gauss hypergeometric function.

The differential equation for Appell's function F_1 can be obtained as special case of result (18) or (19), which shall be

$$[\,(\theta+b)\,(\theta+\phi+a)\,-\,(\theta/x)\,(c+\theta+\phi-1)\,]\,Z=0 \qquad \text{.............(22)}$$

Appell function of second kind can be obtained as special case of generalized B-function when we impose the conditions

$$p=2,\ q=1,\ r=0,\ s=0$$
$$p_1=1,\ q_1=1,\ r_1=0,\ s_1=0$$
$$a_1=b,\ a'_1=b_1,\ a_2=a,\ b_1=c,\ b'_1=c_1$$

Differential equation for the function F_2 shall be

$[(\theta+b)(\theta+\phi+a) - (\theta / x)(c+\theta-1)]\, Z = 0$ ....(23)

Which can be easily transformed in the form

$x(1-x)r - xys + \{c-(a+b+1)x\}p - syq - abz = 0$ ....(24)

which is same as given for F_2 in the book of Slater[1966] and thus evinces the validity of the results obtained in the present paper.

In a similar manner we can obtain all other multiple hypergeometric functions like Appell functions F_3, F_4 ; Horn functions and particular cases for Srivastava functions & Pandey functions involving two variables etc., given in the book of Exton [1976], for different values of the p, q, r, s and p_1, q_1, r_1 & s_1.

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