# Optimal Blowup Rates for the Minimal Energy Null Control for the Structurally Damped Abstract Wave Equation

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**Abstract**

The null controllability problem for a structurally damped abstract wave equation–a so-called elastic model–is considered with a view towards obtain optimal rates of blowup for the associated minimal energy function $\mathcal{E}_{\min}(T)$, as terminal time $T \downarrow 0$. Key use is made of the underlying analyticity of the elastic generator $\mathcal{A}$, as well as of the explicit characterization of its domain of definition. We ultimately find that the blowup rate for $\mathcal{E}_{\min}(T)$, as $T$ goes to zero, depends on the extent of structural damping.

## 1 Introduction

Let $\mathring{\mathbf{A}}$: $D(\mathring{\mathbf{A}}) \subset H \to H$ be a strictly positive, self-adjoint operator; of course, $H$ is Hilbert. Therewith, we consider the structurally damped and controlled abstract model

$$\begin{cases} v_{tt} + \mathring{\mathbf{A}}v + \rho\mathring{\mathbf{A}}^{\alpha}v_t = u \text{ on } (0,T) \\ [v(0), v_t(0)] = [v_0, v_1] \in D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times H \end{cases} \tag{1}$$

where here, $0 \leq \alpha < 1$, and $\rho > 0$. Also, the "control" $u(t)$ is a function in $L^2(0,T;H)$. So as it appears, this model constitutes an abstract wave equation, under the influence of the structural damping term $\rho\mathring{\mathbf{A}}^{\alpha}v_t$. (This form of interior damping is referred to as being of *Kelvin-Voight* type.) It is now wellknown that for damping parameter $\alpha$ in the range $\frac{1}{2} \leq \alpha \leq 1$, the system's underlying generator $\mathcal{A} : D(\mathcal{A}) \subset D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times H \to D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times H$ is of analytic character (see [3]). Consequently, those controlled partial differential equations which can be described by the abstract system (1) manifest parabolic-like dynamics.

For this model, we wish to consider the *null controllability problem*. This problem can be broadly stated as that of finding a control function $u$, such that the corresponding solution of (1) is brought from the initial state to rest at terminal time $T$. Because the abstract system (1) models parabolic-like behaviour, including an infinite speed of propagation, one should expect that if this system is indeed null controllable within the given class of control inputs $u$, the property should hold true in arbitrary short time $T > 0$. This expectation is fully in line with what is known about the canonical parabolic controllability problem; namely the problem of controlling the heat equation, be it via boundary or interior control (see e.g., [2], [12], [15]). Denoting

$$\mathcal{X} = D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times H, \tag{2}$$

we are accordingly led to our working definition of null controllability:

**Definition 1** *The abstract system (1) is said to be be null controllable, if for any time $T > 0$ and arbitrary initial data $[v_0, v_1] \in \mathcal{X}$, there exists a control function $u \in L^2(0,T;H)$ such that the corresponding solution $[v, v_t]$ to (1) satisfies $[v(T), v_t(T)] = [0,0]$.*

The null controllability problem for the system (1) has in fact been successfully addressed in [9]. Indeed, in [9] (Theorem 1.1.1 therein), it was shown that for $\frac{1}{2} \leq \alpha < 1$, the system (1) is null controllable within the class of controls $L^2(0,T;H)$, *provided* that the damping parameter $\rho$ meets a certain compatibility condition with respect to the eigenvalues of $\mathring{\mathbf{A}}$. This restriction on $\rho$ is essentially fallout from the spectral analytical approach employed in [9]. So one aim of this present paper is to remove altogether the restriction placed upon $\rho$, so as to assure the null controllability of (1) for any positive value of this damping parameter. To this end, a multiplier method is invoked with respect to the dual (homogeneous) version of (1).

However, the principle aim of this paper is not merely a reproof of the null controllability problem. Rather, the main intent here is to obtain a precise and optimal estimate for the norm of the "minimal norm steering control", as $T \downarrow 0$. In turn, it is known that the rate of blowup for the minimal norm control is directly related to the "sharpest" constant $C_T$ appearing in the "observability" inequality which is associated with null controllability (see (5) below). (Since the primary intent of [9] was to first and foremost establish the null controllability for abstract analytic systems such as (1), the issue of blowup rates of the minimal norn control was not intended to be addressed therein). Accordingly, we will primarily concern ourselves with the problem of deriving those "sharp" observability estimates which give rise to the null controllability property stated in Definition 1.

We now briefly explain our task of obtaining the optimal blowup rate for the minimal norm steering control. Assume for the time being that the null controllability property given in Definition 1 holds true for the abstract system (1), for arbitrary $T > 0$ (and arbitrary $\rho > 0$). Then for each fixed $T$ and given initial data $[v_0, v_1] \in \mathcal{X}$, one can proceed to solve the associated optimization problem of finding a control $u$ such that the corresponding solution $[v, v_t]$ satisfies $[v(T), v_t(T)] = [0,0]$, and moreover has its $L^2(0,T;H)$–measurement being minimized over all $L^2(0,T;H)$-controls which steer the solution to zero. Assuming the null controllability property to hold true, this optimization problem has a wellknown method of solution (see e.g., Appendix B of [8] and [10]). We denote this minimizer, or minimal norm control, as $u_T^0(v_0, v_1)$. With this minimizer in hand for each fixed $T > 0$, have the following:

**Definition 2** *The* minimal energy function $\mathcal{E}_{\min}(T)$ *is defined as*

$$\mathcal{E}_{\min}(T) \equiv \sup_{\|[v_0,v_1]\|_{\mathcal{X}}=1} \left\| u_T^0(v_0, v_1) \right\|_{L^2(0,T;H)} . \tag{3}$$

Given the presumed null controllability of the system (1), this function $\mathcal{E}_{\min}(T)$ is evidently bounded on $(0, \infty)$. Moreover, it seems clear that this function should tend to blowup as $T \downarrow 0$. Capturing the precise estimate of this blowup is the very objective of this paper. The problem of studying the order of the singularity for the minimal energy function is a rather classical one, and indeed is now well understood for finite dimensions (see [16], [18]). Concerning infinite dimensions, the recent paper [1] has addressed the null controllability problem and the related question of blowup for $\mathcal{E}_{\min}(T)$, in the case of 2-dimensional linear thermoelastic systems. (See also [17], wherein the estimate $e^{1/T}$ is shown for the heat equation under boundary control.) As we have already noted, the key to determining the rate of blowup of $\mathcal{E}_{\min}(T)$ as $T \downarrow 0$, is ascertaining the "best" constant $C_T$ possible for the observability inequality associated to null controllability. Our proof below is accordingly geared toward finding such $C_T$.

In contrast to the spectral approach adopted in [9], in order to obtain the observability inequality requisite for null controllability (see (5) below), we will start by invoking a relatively user-friendly multiplier method. However, in the course of the proof, absolutely critical use is made of intermediate results which are built, not only on the underlying analyticity of the system (1), but also on the characterization of the domain of the underlying generator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ (as defined in (8) below).

Our main result is as follows:

**Theorem 3** *With $\alpha$ in the range $0 \le \alpha < 1$, the abstract system (1) is null controllable (for any value of $\rho$) within the class of controls $L^2(0,T;H)$. The minimal energy function $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{\mu_\alpha}{2}})$, where*

$$\mu_\alpha = \begin{cases} 3, & \text{if } 0 \le \alpha \le \frac{3}{4}; \\ \frac{\alpha}{1-\alpha}, & \text{if } \frac{3}{4} < \alpha < 1. \end{cases} \tag{4}$$

**Remark 4** *We point out that Theorem 3 is optimal for $0 \le \alpha \le \frac{3}{4}$, in view of Seidman's finite dimensional result in [16]. In fact, [16] provides, for the finite dimensional case, an explicit formula for computing the growth of the minimal norm. However, for $\alpha > \frac{3}{4}$ the controllability problem is of a purely infinite dimensional nature, with rates for $\mathcal{E}_{\min}(T)$ which will be arbitrarily large as $\alpha$ increases. In short, for $\alpha \le \frac{3}{4}$ the problem is governed and predicted by the known finite dimensional theory, but for $\alpha > \frac{3}{4}$ the infinite dimensional character of the problem dominates.*

**Remark 5** *The proof of Theorem 3 is carried out for the most demanding case $\alpha$; namely, $\alpha$ in the range $\frac{1}{2} \le \alpha < 1$. In fact, for this range of $\alpha$ the semigroup is analytic, hence nullcontrollability is a delicate issue. However, the method of proof below clearly shows that the statement is also true for $0 \le \alpha < \frac{1}{2}$. In this case,as for $\frac{1}{2} \le \alpha \le \frac{3}{4}$, the minimal energy function $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{3}{2}})$. We will point out below where an amendment in the proof can be made so as to easily handle $0 \le \alpha < \frac{1}{2}$. In addition, the explicit estimate (4) gives the inference that the system (1) is not null controllable for $\alpha = 1$, as was shown outright in [9].*

**Remark 6** *Since the preliminary results and estimates below are wholly independent of the value of the parameter $\rho$, for the sake of aesthetics we will then take throughout $\rho \equiv 1$.*

By way of further motivating the present paper, we note that those null controllability studies of infinite dimensional systems, which consider the issue of obtaining precise estimates on the norm measurements of minimal steering controls, are closely connected to current problems arising in the field of stochastic differential equations. For example, null controllability is tied to the analysis involved in deriving regularity properties for the so-called Bellman's function, a quantity associated with the minimal time control problem. In addition, null controllability is closely related to the regularity of several Markov semigroups such those which deal with Orstein-Uhlenbeck processes and related Kolmolgorov equations. In fact, it can be shown in some cases (see e.g., [5], Theorem 8.3.3) that null controllability is equivalent to the differentiability and regularizing effect of the Orstein-Uhlenbeck process. Moreover, the regularity of solutions to the Kolmogorov equation depends on the singularity of the minimal energy function as $T \downarrow 0$. In addition, for some special examples of Orstein-Uhlenbeck semigroups, it is shown that null controllability is equivalent to the hypoellipticity condition of Hörmander (see [5], p. 112 and [11]). Also, as shown in [5], optimal estimates for the norms of controls are critical in being able to prove Liouville's property for harmonic functions of Markov processes (see p. 108 of [5]).

We note furthermore that in the deterministic case, the connection between the asymptotic behavior of the minimal energy function and the regularity of the Bellman's function (which describes the minimal time control for the given control process) is made very clear in the recent paper [6]. It is shown there that the Holderian regularity of Bellman's function, and its modulus of continuity, are determined by the singularity of the minimal energy function when $T \downarrow 0$. In sum, the issue of obtaining optimal estimates of the singularity of $\mathcal{E}_{\min}(T)$ is not only a problem of interest in the specific context of null controllability, but is also key in the solution of problems drawn from several areas of deterministic and stochastic PDE's.

Finally, we note that the question of the asymptotic behavior of the minimal energy function as $T \to \infty$ has very recently drawn considerable attention (see [14]). In fact, this asymptotic behavior (i.e., the vanishing energy at infinity) is shown in [14] to be connected with the validity of Liouville's Theorem for Ornstein-Uhlenbeck operators.

## 2 The Needed Observability Inequality

By the standard duality argument, the validity of the given null controllability statement is equivalent to the existence of the inequality

$$\|[\upsilon(T), \upsilon_t(T)]\|_{\mathcal{X}} \leq C_T \|\upsilon_t\|_{L^2(0,T;H)}, \tag{5}$$

where $[\upsilon, \upsilon_t] \in C([0,T]; \mathcal{X})$ is the solution to the (adjoint) homogeneous problem

$$\begin{cases} \upsilon_{tt} + \mathring{\mathbf{A}}\upsilon + \mathring{\mathbf{A}}^{\alpha}\upsilon_t = 0 \text{ on } (0,T) \\ [\upsilon(0), \upsilon_t(0)] = [\upsilon_0, \upsilon_1] \in \mathcal{X}. \end{cases} \tag{6}$$

Accordingly, we will work towards the attainment of the inequality (5). Associated with this adjoint problem is the so-called energy of the system, given by

$$\mathcal{E}(t) = \frac{1}{2}\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\upsilon(t)\right\|_H^2 + \frac{1}{2}\|\upsilon_t(t)\|_H^2. \tag{7}$$

In addition, on the Hilbert space $\mathcal{X}$ we denote $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ to be

$$\begin{aligned} \mathcal{A} &\equiv \begin{bmatrix} 0 & I \\ -\mathring{\mathbf{A}} & -\mathring{\mathbf{A}}^{\alpha} \end{bmatrix}; \\ D(\mathcal{A}) &= \left\{[\upsilon_0, \upsilon_1] \in D(\mathring{\mathbf{A}}^{\frac{3}{2}-\alpha}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) : \mathring{\mathbf{A}}^{1-\alpha}\upsilon_0 + \upsilon_1 \in D(\mathring{\mathbf{A}}^{\alpha})\right\}. \end{aligned} \tag{8}$$

For $\frac{1}{2} \leq \alpha \leq 1$, it is wellknown that $\mathcal{A}$ generates an *analytic* contraction semigroup $\left\{e^{\mathcal{A}t}\right\}_{t \geq 0}$ on $\mathcal{X}$ (see [3]). In consequence of this analyticity, we have the estimate (see e.g., p. 70 of [13])

$$\left\|\mathcal{A}^{\eta}e^{\mathcal{A}t}\right\|_{\mathcal{L}(\mathcal{X})} \leq \frac{C_\eta}{t^\eta} \text{ for all } t > 0. \tag{9}$$

Upon application of these dynamics to the initial data $[\upsilon_0, \upsilon_1]$, the solution $[\upsilon, \upsilon_t]$ of (6) may accordingly be written as

$$\begin{bmatrix} \upsilon(t) \\ \upsilon_t(t) \end{bmatrix} = e^{\mathcal{A}t}\begin{bmatrix} \upsilon_0 \\ \upsilon_1 \end{bmatrix}. \tag{10}$$

## 3 Some Preliminary Results

Using these operator theoretic notions, we first prove some supporting results, which will be key in what follows.

**Lemma 7** *Let $\alpha \geq \frac{1}{2}$. Then for any $\theta \in [0,1]$ and integer $k = 1, 2, 3 \ldots$, we have the continuous inclusion*

$$D(\mathcal{A}^{k+\theta}) \subset D(\mathring{\mathbf{A}}^{k+\frac{1}{2}-k\alpha+\theta(1-\alpha)}) \times D(\mathring{\mathbf{A}}^{k-\frac{1}{2}-(k-1)\alpha+\theta(1-\alpha)}). \tag{11}$$

**Proof of Lemma 7:** We will first show the following containment for all integer $k = 1, 2, ...$:

$$D(\mathcal{A}^k) \subset D(\mathring{\mathbf{A}}^{k+\frac{1}{2}-k\alpha}) \times D(\mathring{\mathbf{A}}^{k-\frac{1}{2}-(k-1)\alpha}), \tag{12}$$

from which the estimate (11) will readily follow by interpolation.

To this end, we have by definition that for all $k = 1, 2, ...,$

$$D(\mathcal{A}^k) = \left\{ [\upsilon_0, \upsilon_1] \in D(\mathcal{A}^{k-1}) : \mathcal{A} \begin{bmatrix} \upsilon_0 \\ \upsilon_1 \end{bmatrix} \in D(\mathcal{A}^{k-1}) \right\}. \tag{13}$$

In particular, if $[\upsilon_0, \upsilon_1] \in D(\mathcal{A})$, we have from its definition in (8) that

$$\begin{array}{l} \upsilon_1 \in D(\mathring{\mathbf{A}}^{\frac{1}{2}}); \\ \mathring{\mathbf{A}}\upsilon_0 + \mathring{\mathbf{A}}^{\alpha}\upsilon_1 = g \in H; \end{array}$$

whence we obtain

$$\mathring{\mathbf{A}}^{\frac{3}{2}-\alpha}\upsilon_0 = \mathring{\mathbf{A}}^{\frac{1}{2}-\alpha} g - \mathring{\mathbf{A}}^{\frac{1}{2}}\upsilon_1 \in H,$$

given that $\alpha \geq \frac{1}{2}$. We conclude then that the containment (12) is true for $k = 1$.

Assume now that the containment (12) is valid for $k = n$, $n \geq 1$. Then if $[\upsilon_0, \upsilon_1] \in D(\mathcal{A}^{n+1})$, we have

$$\begin{bmatrix} \upsilon_1 \\ \mathring{\mathbf{A}}\upsilon_0 + \mathring{\mathbf{A}}^{\alpha}\upsilon_1 \end{bmatrix} = \mathcal{A} \begin{bmatrix} \upsilon_0 \\ \upsilon_1 \end{bmatrix} \in D(\mathcal{A}^n).$$

In other words,

$$\begin{array}{c} \upsilon_1 \in D(\mathring{\mathbf{A}}^{n+\frac{1}{2}-n\alpha}) \\ \mathring{\mathbf{A}}\upsilon_0 + \mathring{\mathbf{A}}^{\alpha}\upsilon_1 = g \in D(\mathring{\mathbf{A}}^{n-\frac{1}{2}-(n-1)\alpha}). \end{array}$$

We have then

$$\mathring{\mathbf{A}}^{(n+1)+\frac{1}{2}-(n+1)\alpha}\upsilon_0 = \mathring{\mathbf{A}}^{n+\frac{1}{2}-(n+1)\alpha} g - \mathring{\mathbf{A}}^{n+\frac{1}{2}-n\alpha}\upsilon_1 \in H,$$

since $n + \frac{1}{2} - (n+1)\alpha \leq n - \frac{1}{2} - (n-1)\alpha$ for $\alpha \geq \frac{1}{2}$. Hence the validity of the containment (12) is established. Interpolating now between $k$ and $k+1$ gives the asserted result. $\square$

In turn, we can use this Lemma to prove

**Lemma 8** *Let $\alpha \geq \frac{1}{2}$. Then for any $\theta \in [0, 1]$ and integer $k = 1, 2, 3, ...$, we have that the solution $[\upsilon, \upsilon_t]$ of (6) satisfies the following estimate for all $t > 0$:*

$$\left\| \mathring{\mathbf{A}}^{k+\frac{1}{2}-k\alpha+\theta(1-\alpha)}\upsilon(t) \right\|_H \leq C_{k,\theta} \frac{1}{t^{k+\theta}} \sqrt{\mathcal{E}\left(\frac{t}{k+2}\right)}. \tag{14}$$

**Proof of Lemma 8:** Using Lemma 7, the semigroup representation of $[\upsilon(t), \upsilon_t(t)]$ in (10), and the fact that the solution $[\upsilon(t), \upsilon_t(t)] \in D(\mathcal{A}^{k+\theta})$ for $t > 0$ (by virtue of the analyticity of $\{e^{\mathcal{A}t}\}_{t\geq 0}$), we have

$$\left\| \mathring{\mathbf{A}}^{k+\frac{1}{2}-k\alpha+\theta(1-\alpha)}\upsilon(t) \right\|_H \leq \left\| \mathcal{A}^{k+\theta} e^{\mathcal{A}t} \begin{bmatrix} \upsilon_0 \\ \upsilon_1 \end{bmatrix} \right\|_H. \tag{15}$$

Now one can use the commutativity property of semigroups and their generators to rewrite the right hand side of (15) as

$$\mathcal{A}^{k+\theta} e^{\mathcal{A}t} \begin{bmatrix} \upsilon_0 \\ \upsilon_1 \end{bmatrix} = \left( \mathcal{A} e^{\mathcal{A}\frac{t}{k+2}} \right)^k \mathcal{A}^{\theta} e^{\mathcal{A}\frac{t}{k+2}} e^{\mathcal{A}\frac{t}{k+2}} \begin{bmatrix} \upsilon_0 \\ \upsilon_1 \end{bmatrix}.$$

Combining this relation with the analytic estimate (9) gives now

$$\begin{aligned}
&\left\| \mathcal{A}^{k+\theta} e^{\mathcal{A}t} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_H \\
&\left\| \mathcal{A} e^{\mathcal{A}\frac{t}{k+2}} \right\|_{\mathcal{L}(H)}^k \left\| \mathcal{A}^\theta e^{\mathcal{A}\frac{t}{k+2}} \right\|_{\mathcal{L}(H)} \left\| e^{\mathcal{A}\frac{t}{k+2}} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_H \\
\leq \quad & C_{k,\theta} \frac{(k+2)^{k+\theta}}{t^{k+\theta}} \left\| e^{\mathcal{A}\frac{t}{k+2}} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_H .
\end{aligned}$$

Combining this inequality with (15) now establishes the assertion. □

Given $1 > \alpha \geq \frac{1}{2}$, we now choose nonnegative integer $k$ and $\theta \in [0,1]$ so that

$$\alpha = \frac{1/2 + k + \theta}{1 + k + \theta},$$

or

$$k + \theta = \frac{\alpha - \frac{1}{2}}{1 - \alpha}.$$

In other words, we take

$$\begin{aligned}
k &= \left[ \frac{\alpha - \frac{1}{2}}{1 - \alpha} \right]; \\
\theta &= \frac{\alpha - \frac{1}{2}}{1 - \alpha} - k
\end{aligned} \tag{16}$$

(where $[\cdot]$ above denotes the integral part of a real number). Combining this choice of $(k, \theta)$ with Lemma 8 gives now,

**Corollary 9** *Let $k = \left[ \frac{\alpha - \frac{1}{2}}{1-\alpha} \right]$ and $\theta = \frac{\alpha - \frac{1}{2}}{1-\alpha} - k$. Then for $\alpha \in [\frac{1}{2}, 1)$, the solution $[v, v_t]$ of (6) obeys the following estimate for all $t > 0$:*

$$\left\| \mathring{\mathbf{A}}^\alpha v(t) \right\|_H \leq \frac{C_\alpha}{t^{\frac{\alpha - 1/2}{1-\alpha}}} \sqrt{\mathcal{E}\left( \frac{t}{k+2} \right)}.$$

# 4 Proof Proper of Theorem 3

In what follows, we will have need of the polynomial

$$h(t) \equiv t^s (T - t)^s, \tag{17}$$

where

$$s = \begin{cases} 2, & \text{if } \frac{1}{2} \leq \alpha \leq \frac{3}{4} \\ \frac{2\alpha - 1}{1 - \alpha}, & \text{if } \frac{3}{4} < \alpha < 1 \end{cases} \tag{18}$$

(so in particular, $s \geq 2$ for given $\alpha \in \left[\frac{1}{2}, 1\right)$). This function to be used in a multiplier method.

To start, we multiply the equation (6) by $h(t)v$, and integrate in time and space, so as to have

$$\int_0^T h(t) \left( v_{tt} + \mathring{\mathbf{A}} v + \mathring{\mathbf{A}}^\alpha v_t, v \right)_H dt = 0.$$

An integration of parts with respect to this expression (using implicitly $h(0)=h(T)=0$) yields now the following:

$$\begin{aligned}\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\upsilon\right\|_H^2 &= \int_0^T h(t)\left(\mathring{\mathbf{A}}^{\alpha}\upsilon,\upsilon_t\right)_H dt+\int_0^T h(t)\left\|\upsilon_t\right\|_H^2 dt\\ &\quad+\int_0^T h'(t)\left(\upsilon_t,\upsilon\right)_H dt. \end{aligned} \tag{19}$$

*(i)* Now, concerning the first term on the right hand side of (19),

$$\begin{aligned}&\int_0^T h(t)\left(\mathring{\mathbf{A}}^{\alpha}\upsilon,\upsilon_t\right)_H dt\\ \le&\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\alpha}\upsilon\right\|_H\left\|\upsilon_t\right\|_H dt\\ \le&\int_0^T h(t)\frac{C}{t^{\frac{\alpha-1/2}{1-\alpha}}}\sqrt{\mathcal{E}\left(\frac{t}{k+2}\right)}\left\|\upsilon_t\right\|_H dt,\end{aligned}$$

after using Corollary 9, where positive integer $k$ is as prescribed therein[1]. This gives then

$$\begin{aligned}&\int_0^T h(t)\left(\mathring{\mathbf{A}}^{\alpha}\upsilon,\upsilon_t\right)_H dt\\ \le&\frac{\epsilon}{k+2}\int_0^T h(t)\mathcal{E}\left(\frac{t}{k+2}\right)dt+C_{\epsilon,\alpha}\int_0^T\frac{h(t)}{t^{\frac{2\alpha-1}{1-\alpha}}}\left\|\upsilon_t\right\|_H^2 dt\\ \le&\frac{\epsilon}{k+2}\int_0^T h(t)\mathcal{E}\left(\frac{t}{k+2}\right)dt+C_{\epsilon,\alpha}T^{2s-\frac{2\alpha-1}{1-\alpha}}\int_0^T\left\|\upsilon_t\right\|_H^2 dt.\end{aligned} \tag{20}$$

*(ii)* Moreover, concerning the third term on the right hand side of (19),

$$\begin{aligned}&\int_0^T h'(t)\left(\upsilon_t,\upsilon\right)_H dt\\ \le&\int_0^T h'(t)\left\|\mathring{\mathbf{A}}^{-\frac{1}{2}}\right\|_{\mathcal{L}(H)}\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\upsilon\right\|_H\left\|\upsilon_t\right\|_H\frac{\sqrt{h(t)}}{\sqrt{h(t)}}dt\\ \le&\frac{\epsilon}{2}\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\upsilon\right\|_H^2 dt+C_\epsilon\int_0^T\frac{[h'(t)]^2}{h(t)}\left\|\upsilon_t\right\|_H^2 dt\\ \le&\epsilon\int_0^T h(t)\mathcal{E}\left(t\right)dt+C_\epsilon T^{2s-2}\int_0^T\left\|\upsilon_t\right\|_H^2 dt.\end{aligned} \tag{21}$$

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[1]Note that if $\alpha\in[0,\frac{1}{2})$, then the dynamical operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{X}\to\mathcal{X}$ is no longer of analytic character. But on the other hand, for such values of $\alpha$, $\mathring{\mathbf{A}}^{\alpha}\upsilon$ is strictly below the level of energy, and so the Lemma 8, essentially a product of analyticity, is not needed at all for $\alpha\in\left[0,\frac{1}{2}\right]$. Indeed, to estimate the first term on the right hand side of (19) for $\alpha<\frac{1}{2}$, we proceed as follows:

$$\begin{aligned}\int_0^T h(t)\left(\mathring{\mathbf{A}}^{\alpha}\upsilon,\upsilon_t\right)_H dt &\le \frac{\epsilon}{2}\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\alpha}\upsilon\right\|_H^2 dt+C_\epsilon\left\|\upsilon_t\right\|_H^2 dt\\ &\le \epsilon\left\|\mathring{\mathbf{A}}^{\alpha-\frac{1}{2}}\right\|_{\mathcal{L}(H)}^2\int_0^T h(t)\mathcal{E}(t)dt+C_\epsilon\left\|\upsilon_t\right\|_H^2 dt.\end{aligned}$$

Continuing the rest of this proof in exactly the same fashion as in our present analytic case, we subsequently find that the corresponding observability inequality (and so too then $\mathcal{E}_{\min}(T)$) will be $\mathcal{O}(T^{-\frac{3}{2}})$.

Incorporating (20) and (21) into (19) yields now for $T < 1$,

$$\begin{aligned}
&\int_0^T h(t) \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \upsilon \right\|_H^2 \\
\leq\ & \epsilon \int_0^T h(t)\mathcal{E}(t)\,dt + \frac{\epsilon}{k+2}\int_0^T h(t)\mathcal{E}\left(\frac{t}{k+2}\right)dt \\
&+ C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|\upsilon_t\|_H^2\,dt + C_\epsilon T^{2s-2}\int_0^T \|\upsilon_t\|_H^2\,dt.
\end{aligned}$$

This gives in turn,

$$\begin{aligned}
&\int_0^T (1-\epsilon)h(t)\mathcal{E}(t)\,dt \\
\leq\ & \frac{\epsilon}{k+2}\int_0^T h(t)\mathcal{E}\left(\frac{t}{k+2}\right)dt \\
&+ C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|\upsilon_t\|_H^2\,dt + C_\epsilon T^{2s-2}\int_0^T \|\upsilon_t\|_H^2\,dt,
\end{aligned}$$

or

$$\begin{aligned}
&\int_0^{\frac{T}{k+2}} \left[(1-\epsilon)h(t) - \epsilon h((k+2)t)\right]\mathcal{E}(t)\,dt + (1-\epsilon)\int_{\frac{T}{k+2}}^T h(t)\mathcal{E}(t)\,dt \\
\leq\ & C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|\upsilon_t\|_H^2\,dt + C_\epsilon T^{2s-2}\int_0^T \|\upsilon_t\|_H^2\,dt. \qquad (22)
\end{aligned}$$

Choosing $\epsilon > 0$ small enough so that

$$(1-\epsilon)h(t) - \epsilon h((k+2)t) > 0 \text{ on } (0, \frac{T}{k+2}) \ \text{ and } (1-\epsilon) > 0,$$

then the estimate (22) and the inherent dissipativity of the structurally damped system (6) give in combination,

$$\mathcal{E}(T)\int_{\frac{T}{k+2}}^T h(t)dt \leq C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|\upsilon_t\|_H^2\,dt + C_\epsilon T^{2s-2}\int_0^T \|\upsilon_t\|_H^2\,dt. \qquad (23)$$

Since

$$\begin{aligned}
&\int_{\frac{T}{k+2}}^T h(t)dt \\
\geq\ & \int_{\frac{T}{k+2}}^T \left(t - \frac{T}{k+2}\right)^s (T-t)^s\,dt \\
=\ & \left(\frac{k+1}{k+2}T\right)^{2s+1} B(s+1, s+1) \qquad (24)
\end{aligned}$$

(see e.g., [7], p. 285), where $B(\cdot,\cdot)$ denotes the Beta function; then combining this inequality with (23) gives finally

$$\mathcal{E}(T) \leq C_\alpha T^{-\mu_\alpha}\int_0^T \|\upsilon_t\|_H^2\,dt,$$

where

$$\mu_\alpha = \begin{cases} 3, & \text{if } \frac{1}{2} \le \alpha \le \frac{3}{4}; \\ \frac{\alpha}{1-\alpha}, & \text{if } \frac{3}{4} < \alpha < 1. \end{cases}$$

We conclude therefore, that the abstract system (1) is null controllable, with the associated observability inequality $C_T$ of (5) being $\mathcal{O}(T^{-\frac{\mu_\alpha}{2}})$. Subsequently, a standard argument (see e.g., [10], [1]) gives now that likewise, the minimal energy function $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{\mu_\alpha}{2}})$. This completes the proof of Theorem 3.

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