# The Initial Functions in a Subdivision Scheme

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**Abstract.** In this paper we shall study the initial functions in a subdivision scheme in a Sobolev space. By investigating the mutual relations among the initial functions in a subdivision scheme, we are able to study in a relatively unified approach several questions related to a subdivision scheme in a Sobolev space such as convergence, error estimate and convergence rate of a subdivision scheme in a Sobolev space with a general dilation matrix. A generalized definition of convergence of subdivision schemes in Banach spaces is also introduced.

## §1. Introduction

Refinable functions play an important role in wavelet analysis. As widely used in computer aided geometric design, a subdivision scheme is a powerful tool in investigating various properties of refinable functions.

An $s \times s$ integer matrix $M$ is called a dilation matrix if $\lim_{k\to\infty} M^{-k} = 0$. We say that $a$ is a mask on $\mathbb{Z}^s$ if $a$ is a finitely supported sequence on $\mathbb{Z}^s$ such that $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 1$. Wavelets are derived from refinable functions via a standard multiresolution technique. A refinable function $\phi$ is a solution to the following refinement equation

$$\phi = |\det M| \sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi(M\cdot -\beta), \tag{1}$$

where $a$ is a mask and $M$ is a dilation matrix. For a mask $a$ on $\mathbb{Z}^s$ and an $s \times s$ dilation matrix $M$, it is known that there exists a unique compactly supported distributional solution, denoted by $\phi_a^M$ throughout the paper, to the refinement equation (1) such that $\hat{\phi}_a^M(0) = 1$, where the Fourier transform of $f \in L_1(\mathbb{R}^s)$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi}\,dx, \xi \in \mathbb{R}^s$ and can be naturally extended to tempered distributions.

Let $\mathbb{N}_0$ denote all the nonnegative integers. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, the length of $\mu$ is $|\mu| := \mu_1 + \cdots + \mu_s$, and $\xi^\mu := \xi_1^{\mu_1} \cdots \xi_s^{\mu_s}$ for $\xi = (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s$. The partial derivative of a differentiable function $f$ with respect to the $j$th

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coordinate is denoted by $D_j f$, $j = 1, \ldots, s$, and for $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, $D^\mu$ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. By $W_p^k(\mathbb{R}^s)$ we denote the Sobolev space that consists of all functions $f$ such that $D^\mu f \in L_p(\mathbb{R}^s)$ for all $\mu \in \mathbb{N}_0^s$ with $|\mu| \le k$, equipped with the norm defined by

$$\|f\|_{W_p^k(\mathbb{R}^s)} := \sum_{|\mu| \le k} \|D^\mu f\|_p.$$

An iteration scheme can be employed to solve the refinement equation (1). Start with some initial function $\phi_0 \in W_p^k(\mathbb{R}^s)$ such that $\hat{\phi}_0(0) = 1$. We employ the iteration scheme $Q_{a,M}^n \phi_0$, $n = 0, 1, 2, \ldots$, where $Q_{a,M}$ is the linear operator on $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ given by

$$Q_{a,M} f := |\det M| \sum_{\beta \in \mathbb{Z}^s} a(\beta) f(M \cdot - \beta), \qquad f \in L_p(\mathbb{R}^s).$$

This iteration scheme is called a subdivision scheme associated with mask $a$ and dilation $M$. When the sequence $Q_{a,M}^n \phi_0$ converges in the Sobolev space $W_p^k(\mathbb{R}^s)$, then the limit function must be $\phi_a^M$—the unique compactly supported distributional solution to the refinement equation (1) with $\hat{\phi}_a^M(0) = 1$.

It is the purpose of this paper to investigate the mutual relations among different initial functions in a subdivision scheme and then use such results to improve some results and to give alternative proofs to some known results on subdivision schemes in a Sobolev space.

The structure of the paper is as follows. In Section 2, we shall investigate the mutual relations of initial functions in a subdivision scheme in a Sobolev space. Then we shall give a proof of results in [6, 7, 8] on the characterization of convergence of a subdivision scheme. In Section 3, following the line developed in [3, 4], we shall investigate error estimates of a subdivision scheme in a Sobolev space with a perturbed mask. Therefore, we improve the results in [1, 2, 3, 4]. Finally, in Section 4, we shall discuss rate of convergence of a subdivision scheme in a Sobolev space and introduce a general definition of convergence of a subdivision scheme in a general Banach space.

## §2. Initial Functions in a Subdivision Scheme

In this section we shall investigate the mutual relations of initial functions in a subdivision scheme. By $\ell_0(\mathbb{Z}^s)$ we denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$. For $\alpha \in \mathbb{Z}^s$ and $y \in \mathbb{R}^s$, we define

$$\nabla_\alpha \lambda := \lambda - \lambda(\cdot - \alpha), \quad \nabla_y f := f - f(\cdot - y), \qquad \lambda \in \ell_0(\mathbb{Z}^s), f \in L_p(\mathbb{R}^s). \quad (2)$$

Define $[\lambda_1 * \lambda_2](\alpha) := \sum_{\beta \in \mathbb{Z}^s} \lambda_1(\beta)\lambda_2(\alpha - \beta)$ for $\lambda_1, \lambda_2 \in \ell_0(\mathbb{Z}^s)$. Define a semi-convolution as follows:

$$\lambda * f := \sum_{\beta \in \mathbb{Z}^s} \lambda(\beta) f(\cdot - \beta), \qquad \lambda \in \ell_0(\mathbb{Z}^s), f \in L_p(\mathbb{R}^s). \quad (3)$$

For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, $\nabla^\mu := \nabla_{e_1}^{\mu_1} \cdots \nabla_{e_s}^{\mu_s}$, where $e_j$ is the $j$th coordinate unit vector in $\mathbb{R}^s$. By $\delta$ we denote the Dirac sequence such that $\delta(0) = 1$ and $\delta(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$. It is easy to verify that $\nabla^\mu f = \nabla^\mu \delta * f$, $\nabla^\mu \lambda = \nabla^\mu \delta * \lambda$, $\nabla_\alpha \lambda = \nabla_\alpha \delta * \lambda$ and $\lambda_1 * (\lambda_2 * f) = (\lambda_1 * \lambda_2) * f$. Our discussion in this paper is based on the following fact.

**Theorem 1.** *Let $f$ be a compactly supported function in $L_p(\mathbb{R}^s)$, where $1 \le p \le \infty$. Then for $k \in \mathbb{N}$, the following statements are equivalent*

1) *$D^\mu \hat{f}(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s$ and $\mu \in \mathbb{N}_0^s$ with $|\mu| < k$;*
2) *$\sum_{\beta \in \mathbb{Z}^s} q(\beta) f(\cdot - \beta) = 0$ for all $q \in \Pi_{k-1}$, where $\Pi_{k-1}$ denotes the set of all polynomials of total degree less than $k$;*
3) *$f = \sum_{j=1}^r \lambda_j * g_j$ for some compactly supported functions $g_j \in L_p(\mathbb{R}^s)$ and some finitely supported sequences $\lambda_j$ in the linear space $V_k$ which consists of all $\lambda \in \ell_0(\mathbb{Z}^s)$ such that $\sum_{\beta \in \mathbb{Z}^s} q(\beta)\lambda(\beta) = 0$ for all $q \in \Pi_{k-1}$;*
4) *$f = \sum_{|\mu|=k} \nabla^\mu g_\mu$ for some compactly supported functions $g_\mu \in L_p(\mathbb{R}^s)$.*

**Proof:** By the Poisson summation formula, it is known that 1) $\iff$ 2). Note that $\{\nabla^\mu \delta(\cdot - \alpha) \;:\; \alpha \in \mathbb{Z}^s, |\mu| = k\}$ spans the linear space $V_k$. So 3) $\iff$ 4). Observe that $\lambda \in V_k$ if and only if $\lambda * q = 0$ for all $q \in \Pi_{k-1}$. When 3) holds, it is straightforward to see that

$$\sum_{\alpha \in \mathbb{Z}^s} q(\alpha) f(\cdot - \alpha) = \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} g_j(\cdot - \alpha) \sum_{\beta \in \mathbb{Z}^s} \lambda_j(\beta) q(\alpha - \beta) = 0 \qquad \forall\, q \in \Pi_{k-1}$$

since $\sum_{\beta \in \mathbb{Z}^s} \lambda_j(\beta) q(\alpha - \beta) = 0$. So 3) $\Rightarrow$ 2). To complete the proof, it suffices to show that 2) $\Rightarrow$ 3). Since $f$ is compactly supported, we can write

$$f = \sum_{\alpha \in \mathbb{Z}^s} f|_{\alpha + [0,1)^s} = \sum_{j=1}^n f_j(\cdot - \alpha_j),$$

where $\alpha_j \in \mathbb{Z}^s$ and all $f_j \in L_p(\mathbb{R}^s)$ are supported on $[0,1]^s$. Without loss of generality, we can assume that $f_1, \dots, f_r$ are linearly independent and for some numbers $c_j^m$,

$$f_m = \sum_{j=1}^r c_j^m f_j, \qquad m = r+1, \dots, n.$$

Let $g_j := f_j, j = 1, \dots, r$ and choose sequences $\lambda_j, j = 1, \dots, r$ as follows:

$$\lambda_j(\beta) := \delta(\beta - \alpha_j) + \sum_{m=r+1}^n c_j^m \delta(\beta - \alpha_m), \qquad \beta \in \mathbb{Z}^s,\, j = 1, \dots, r.$$

Obviously, all $\lambda_j$ are finitely supported sequences on $\mathbb{Z}^s$. All $g_j$ are compactly supported and are supported on $[0,1]^s$. By computation, we have $f = \sum_{j=1}^r \lambda_j * g_j$. To complete the proof, we prove $\lambda_j \in V_k$. By 2), we have

$$0 = \sum_{\alpha \in \mathbb{Z}^s} q(\alpha) f(\cdot - \alpha) = \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} g_j(\cdot - \alpha) \sum_{\beta \in \mathbb{Z}^s} \lambda_j(\beta) q(\alpha - \beta) \quad \forall\, q \in \Pi_{k-1}.$$

Since all $g_j$ are supported on $[0,1]^s$ and are linearly independent, it follows that $\sum_{\beta\in\mathbb{Z}^s}\lambda_j(\beta)q(\alpha-\beta)=0$ and therefore, all $\lambda_j$ are in $V_k$. □

When $s=1$, Theorem 1 is evident since $f=\nabla^k g$ with a compactly supported function $g\in L_p(\mathbb{R})$ given by $g=\sum_{j=1}^{\infty} j^{k-1}f(\cdot-j+1)/(k-1)!$.

It was proved in Jia, Jiang and Lee [8] that if a subdivision scheme with an initial function $f$ converges in $W_p^k(\mathbb{R}^s)$, then $f$ must satisfy the moment conditions of order $k+1$; that is, $\hat f(0)=1$ and

$$D^\mu \hat f(2\pi\beta)=0 \qquad \forall\, |\mu|<k+1,\ \beta\in\mathbb{Z}^s\backslash\{0\}.$$

As a consequence of Theorem 1, we have the following result.

**Corollary 2.** *Let $f\in W_p^k(\mathbb{R}^s)$ be compactly supported such that $f$ satisfies the moment conditions of order $k+1$. For any compactly supported function $g\in W_p^k(\mathbb{R}^s)$, $g$ satisfies the moment conditions of order $k+1$ if and only if*

$$D^\mu g = D^\mu f + \sum_{|\nu|=|\mu|+1} \nabla^\nu h_\nu^\mu, \qquad \forall\, |\mu|\le k,\ \mu\in\mathbb{N}_0^s \tag{4}$$

*for some compactly supported functions $h_\nu^\mu\in L_p(\mathbb{R}^s)$.*

**Proof:** For $|\mu|\le k$, let $h:=D^\mu g-D^\mu f$. Since $\hat h(\xi)=(-i\xi)^\mu[\hat g(\xi)-\hat f(\xi)]$, if both $f$ and $g$ satisfy the moment conditions of order $k+1$, then $D^\nu \hat h(2\pi\beta)=0$ for all $|\nu|<|\mu|+1$ and for all $\beta\in\mathbb{Z}^s$. Therefore, by Theorem 1, (4) holds.

Let us prove the converse statement. Note that $\widehat{\nabla^\nu h_\nu^\mu}(\xi_1,\ldots,\xi_s)=\widehat{h_\nu^\mu}(\xi_1,\ldots,\xi_s)\Pi_{j=1}^s(1-e^{-i\xi_j})^{\nu_j}$ for $\nu=(\nu_1,\ldots,\nu_s)\in\mathbb{N}_0^s$. Consequently, $D^\eta[\widehat{\nabla^\nu h_\nu^\mu}](2\pi\beta)=0$ for all $|\eta|<|\nu|$ and $\beta\in\mathbb{Z}^s$. When (4) holds, then in particular (4) holds for $|\mu|=k$. Let $h:=g-f$. Hence, $D^\eta(\widehat{D^\mu h})(2\pi\beta)=D^\eta[(-i\cdot)^\mu\hat h(\cdot)](2\pi\beta)=0$ for all $|\eta|\le|\mu|=k$ and $\beta\in\mathbb{Z}^s$. Therefore, $\hat h(0)=0$ and $D^\eta\hat h(2\pi\beta)=0$ for all $|\eta|<k+1$ and $\beta\in\mathbb{Z}^s\backslash\{0\}$. Since $f$ satisfies the moment conditions of order $k+1$, so does $g$ by $g=f+h$. □

Let $f$ be a compactly supported function in $L_p(\mathbb{R}^s)$. We say that the shifts of $f$ are stable if there exist two positive constants $A$ and $B$ such that $A\|\lambda\|_p\le\|\lambda*f\|_p\le B\|\lambda\|_p$ for all finitely supported sequences $\lambda$ on $\mathbb{Z}^s$.

The B-spline function $B_k$ of order $k$ is defined by the recursive formula: $B_0:=\chi_{[0,1]}$ and $B_k:=\int_0^1 B_{k-1}(\cdot-t)dt$. Define a function on $\mathbb{R}^s$ as follows

$$\varphi(x_1,\ldots,x_s):=B_{k+1}(x_1)\cdots B_{k+1}(x_s), \qquad x_1,\ldots,x_s\in\mathbb{R}. \tag{5}$$

Then $\varphi\in W_p^k(\mathbb{R}^s)$, the shifts of $\varphi$ are stable, and $\varphi$ satisfies the moment conditions of order $k+1$.

**Corollary 3.** *Let $f\in W_p^k(\mathbb{R}^s)$ be a compactly supported function such that $\hat f(0)=1$. If $f$ satisfies the moment conditions of order $k+1$, then*

$$D^\mu f=\sum_{|\nu|=|\mu|} \nabla^\nu h_\nu^\mu \qquad \forall\, |\mu|\le k,\ \mu\in\mathbb{N}_0^s \tag{6}$$

*for some compactly supported functions $h_\nu^\mu$ in $L_p(\mathbb{R}^s)$.*

**Proof:** Using a similar argument as in Corollary 2, (6) can be proved directly. On the other hand, take $g = \varphi$ in (5). Then it is obvious that $g \in W_p^k(\mathbb{R}^s)$ and satisfies the moment conditions of order $k+1$. Observe that $D^\mu\varphi = \nabla^\mu\psi$, where $\psi(x_1, \ldots, x_s) := B_{k+1-\mu_1}(x_1)\cdots B_{k+1-\mu_s}(x_s)$ for all $\mu = (\mu_1, \ldots, \mu_s)$ with $|\mu| \le k$. The claim in (6) now follows directly from (4). $\square$

A characterization of convergence of a subdivision scheme in the $L_p$ norm and in the Sobolev norm for $p = 2$ was given in Han and Jia [6] and Jia, Jiang and Lee [8], respectively. Also see Jia [7] for the univariate $L_p$ case. Here, we present another proof for such characterization using Theorem 1. To do so, we need the following facts.

**Proposition 4.** *Let $A$ be an $s \times s$ real-valued (or complex-valued) matrix. Then $A$ is isotropic (i.e., $A$ is similar to a diagonal matrix diag$(\sigma_1, \ldots, \sigma_s)$ with $|\sigma_1| = \cdots = |\sigma_s|$) if and only if there exists a norm $\|\cdot\|$ on $\mathbb{R}^s$ (or on $\mathbb{C}^s$) such that $\|Ax\| = |\det A|^{1/s}\|x\|$ for all $x \in \mathbb{R}^s$ (or for all $x \in \mathbb{C}^s$).*

**Proof:** When $A$ is isotropic, we can write $A = \Lambda^{-1} diag(\sigma_1, \ldots, \sigma_s)\Lambda$ for some nonsingular matrix $\Lambda$. Let $\|\cdot\|_E$ be the Euclidean norm on $\mathbb{C}^s$. Define an equivalent norm $\|x\| := \|\Lambda x\|_E, x \in \mathbb{C}^s$. Obviously, $\|Ax\| = |\det A|^{1/s}\|x\|$ for all $x \in \mathbb{C}^s$.

Conversely, when $A$ is real-valued and there is a norm on $\mathbb{R}^s$ such that $\|Ax\| = |\det A|^{1/s}\|x\|$ for all $x \in \mathbb{R}^s$. Then we can define a norm on $\mathbb{C}^s$ by $\|x + iy\| := \sqrt{\|x\|^2 + \|y\|^2}$ for all $x, y \in \mathbb{R}^s$, where $i$ is the imaginary unit such that $i^2 = -1$. Obviously, we have $\|A^n x\| = |\det A|^{n/s}\|x\|$ for all $x \in \mathbb{C}^s$ and $n \in \mathbb{N}$. Using the Jordan canonical form of $A$, it is easy to see that $A$ is isotropic. $\square$

When $A$ is an $r \times r$ isotropic matrix, throughout, we alway denote $\|\cdot\|$ to be the norm on $\mathbb{C}^r$ such that $\|Ax\| = |\det A|^{1/r}\|x\|$ for all $x \in \mathbb{C}^r$.

The following claim can be easily verified by a simple computation.

**Proposition 5.** *Let $D := [D_1, \ldots, D_s]^T$. Define a column vector of differential operators by $\otimes^k D := D \otimes \cdots \otimes D$ with $k$ copies of $D$, where $\otimes$ denotes the (right) Kronecker product. Let $A$ be an $s \times s$ real-valued matrix. Then*

$$\otimes^k D[f(A\cdot)] = (\otimes^k A^T)(\otimes^k Df)(A\cdot), \qquad k \in \mathbb{N},\ f \in C^\infty(\mathbb{R}^s).$$

Now, (4) can be rewritten as $\otimes^j D(g - f) = \sum_{|\nu|=j+1} \nabla^\nu h_\nu^j$ for all $j = 0, \cdots, k$ for some vectors $h_\nu^j$ of compactly supported functions in $L_p(\mathbb{R}^s)$. We can also rewrite (6) into a similar form. Define a norm on $W_p^k(\mathbb{R}^s)$ as follows: $\|f\|_* := \sum_{j=0}^k \|\otimes^j Df\|_p$, where $\|[f_1, \ldots, f_r]^T\|_p := \|[\|f_1\|_p, \ldots, \|f_r\|_p]^T\|$ and the norm $\|\cdot\|$ on $\mathbb{C}^r$ is determined in Proposition 4. It is evident that $\|\cdot\|_*$ is an equivalent norm to $\|\cdot\|_{W_p^k(\mathbb{R}^s)}$ in $W_p^k(\mathbb{R}^s)$.

Let $M$ be an $s \times s$ dilation matrix and $a$ be a finitely supported sequence on $\mathbb{Z}^s$. For $1 \le p \le \infty$ and $k \in \mathbb{N}_0$, we define

$$\rho_k(a, M; p) := \max\{\lim_{n\to\infty} \|\nabla^\mu S_{a,M}^n \delta\|_p^{1/n} \ : \mu \in \mathbb{N}_0^s, |\mu| = k\}, \tag{7}$$

$$\rho(a, M; p) := \inf\{\rho_k(a, M; p) \ : \ k \in \mathbb{N}_0\}, \tag{8}$$

where $\|\lambda\|_p := (\sum_{\beta\in\mathbb{Z}^s} |\lambda(\beta)|^p)^{1/p}$ and $S_{a,M}$ is the subdivision operator

$$[S_{a,M}\lambda](\alpha) := |\det M| \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)\lambda(\beta), \qquad \alpha\in\mathbb{Z}^s, \lambda\in\ell_0(\mathbb{Z}^s).$$

A mask $a$ satisfies the sum rules of order $k$ with respect to the lattice $M\mathbb{Z}^s$ if

$$\sum_{\beta\in M\mathbb{Z}^s} a(\alpha+\beta)q(\alpha+\beta) = \sum_{\beta\in M\mathbb{Z}^s} a(\beta)q(\beta) \qquad \forall\, \alpha\in\mathbb{Z}^s, q\in\Pi_{k-1}.$$

For simplicity, we omit the proof of the following result.

**Proposition 6.** *Let $M$ be an $s\times s$ isotropic dilation matrix. For any finitely supported sequence $a$ on $\mathbb{Z}^s$ such that $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 1$, then*

$$\rho_k(a,M;p) = \max\Big\{\, |\det M|^{1/p-k/s},\ \ \rho_{k+j}(a,M;p)\Big\} \qquad \forall\, j,k\in\mathbb{N}_0.$$

*Moreover, if $\rho(a,M;p) < |\det M|^{1/p-k/s}$ for some integer $k$, then $a$ must satisfy the sum rules of order $k+1$ with respect to the lattice $M\mathbb{Z}^s$.*

The following characterizes convergence of subdivision schemes in $W_p^k(\mathbb{R}^s)$.

**Theorem 7.** *Let $M$ be an $s\times s$ isotropic dilation matrix. Let $a$ be a finitely supported mask on $\mathbb{Z}^s$. Then the following statements are equivalent*

1) *For every compactly supported function $f\in W_p^k(\mathbb{R}^s)$ such that $f$ satisfies the moment conditions of order $k+1$, the subdivision scheme converges in $W_p^k(\mathbb{R}^s)$; that is, $\lim_{n\to\infty}\|Q^n_{a,M}f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$;*
2) *For some compactly supported function $f\in W_p^k(\mathbb{R}^s)$ such that $f$ satisfies the moment conditions of order $k+1$ and the shifts of $f$ are stable (one example of such $f$ is $\varphi$ in (5)), $\lim_{n\to\infty}\|Q^n_{a,M}f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$;*
3) *$\rho_{k+1}(a,M;p) < |\det M|^{1/p-k/s}$;*
4) *$\rho(a,M;p) < |\det M|^{1/p-k/s}$.*

**Proof:** Obviously, 1) $\Rightarrow$ 2). Suppose 2) holds. Let $f_n := Q^n_{a,M}f$. By induction, for $\mu=(\mu_1,\ldots,\mu_s)\in\mathbb{N}_0^s$, we have

$$\nabla^{\mu_1}_{M^{-n}e_1}\cdots\nabla^{\mu_s}_{M^{-n}e_s} f_n = \sum_{\beta\in\mathbb{Z}^s} [\nabla^\mu S^n_{a,M}\delta](\beta) f(M^n\cdot-\beta) \qquad \forall\, n\in\mathbb{N}_0. \quad (9)$$

Note that 2) implies that $\phi_a^M\in W_p^k(\mathbb{R}^s)$. Let $\nabla^{\mu,n} := \nabla^{\mu_1}_{M^{-n}e_1}\cdots\nabla^{\mu_s}_{M^{-n}e_s}$ and $m := |\det M|$ for short. Since the shifts of $f$ are stable, from (9), there exists a positive constant $C$ depending only of $f$ such that

$$m^{-n/p}\|\nabla^\mu S^n_{a,M}\delta\|_p \le C\|\nabla^{\mu,n}f_n\|_p \le C\|\nabla^{\mu,n}\phi_a^M\|_p + C\|\nabla^{\mu,n}(f_n-\phi_a^M)\|_p.$$

Note that all the functions $f_n$ and $\phi_a^M$ are supported on $[-N,N]^s$ for some integer $N$ independent of $n$. Since $M$ is isotropic, there is a constant $C_1$ independent of $n$ such that $\|\nabla^{\mu,n}(f_n-\phi_a^M)\|_p \le C_1 m^{-nk/s}\|f_n-\phi_a^M\|_{W_p^k(\mathbb{R}^s)}$ for all $|\mu| = k+1$. Since $\phi_a^M \in W_p^k(\mathbb{R}^s)$, we have $\lim_{n\to\infty} m^{-nk/s}\|\nabla^{\mu,n}\phi_a^M\|_p = 0$ for all $|\mu| = k+1$. Therefore, by $\lim_{n\to\infty}\|f_n - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$, we deduce that $\lim_{n\to\infty} m^{n(k/s-1/p)}\|\nabla^\mu S_{a,M}^n\delta\|_p = 0$ for all $|\mu| = k+1$. By joint spectral radius as in [6, Theorem 3.2], we have $\lim_{n\to\infty}\|\nabla^\mu S_{a,M}^n\delta\|_p^{1/n} < m^{1/p-k/s}$ and therefore, 3) holds. By Proposition 6, 3) $\iff$ 4).

Finally, we prove 3) $\Rightarrow$ 1). Let $f$ be a compactly supported function in $W_p^k(\mathbb{R}^s)$ such that $f$ satisfies the moment conditions of order $k+1$. Then 3) implies that $Q_{a,M}f$ also has such properties. By 3), there exist two constants $0 < \rho < 1$ and $C > 0$ such that

$$\|\nabla^\mu S_{a,M}^n\delta\|_p \le Cm^{n(1/p-k/s)}\rho^n \qquad \forall\, n \in \mathbb{N}, |\mu| = k+1. \tag{10}$$

Let $h = Q_{a,M}f - f$. By Corollary 2, $\otimes^k Dh = \sum_{|\nu|=k+1}\nabla^\nu h_\nu^k$ for some vectors $h_\nu^k$ of compactly supported functions in $L_p(\mathbb{R}^s)$. By the equation in (9) and Proposition 5, we have

$$\otimes^k D[f_{n+1} - f_n] = (\otimes^k M^T)\Big(\sum_{|\nu|=k+1}\sum_{\beta\in\mathbb{Z}^s}[\nabla^\nu S_{a,M}^n\delta](\beta)h_\nu^k(M^n\cdot - \beta)\Big).$$

Note that the isotropic matrix $\otimes^k(M^T)^n$ has the operator norm $\|\otimes^k(M^T)^n\| = m^{nk/s}$. Since all the components in $h_\nu^k$ are compactly supported functions in $L_p(\mathbb{R}^s)$, there exists a positive constant $C_1$ depending only on $h_\nu^k, |\nu| = k+1$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned}\big\|\otimes^k D[f_{n+1} - f_n]\big\|_p &\le C_1 m^{\frac{-n}{p}}\|\otimes^k(M^T)^n\|\max\{\|\nabla^\nu S_{a,M}^n\delta\|_p : |\nu| = k+1\}\\ &\le CC_1 m^{-n/p}m^{nk/s}m^{n(1/p-k/s)}\rho^n = CC_1\rho^n.\end{aligned}$$

Thus, $D^\mu f_n$ converges in the $L_p$ norm for all $|\mu| = k$. Therefore, we must have $\lim_{n\to\infty}\|Q_{a,M}^n f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = \lim_{n\to\infty}\|f_n - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$.

An alternative proof of 3) $\Rightarrow$ 1) is that 3) implies $\phi_a^M \in W_p^k(\mathbb{R}^s)$ and $\lim_{n\to\infty}\|Q_{a,M}^n(f - \phi_a^M)\|_{W_p^k(\mathbb{R}^s)} = 0$ by a similar argument. □

By a similar argument as in the proof of Theorem 7, we see that when a compactly supported refinable function $\phi_a^M \in W_p^k(\mathbb{R}^s)$ (for $p = \infty$, the $k$th derivatives of $\phi_a^M$ must be continuous) has stable shifts, then the subdivision scheme converges in $W_p^k(\mathbb{R}^s)$ and all the claims in Theorem 7 hold. Moreover, the quantities $\rho_k(a, M; 2)$ in Theorem 7 can be computed by [6, Theorem 4.1].

## §3. Error Estimate of a Subdivision Scheme in a Sobolev Space

In applications, when the coefficients in a mask (such as the Daubechies' orthogonal masks) are irrational numbers, one often needs to truncate such a mask. Daubechies and Huang [2] first studied how such truncation affects the associated refinable function in the univariate $L_\infty$ case. In [3, 4], Han first provided a sharp error estimate for multivariate refinable functions and their subdivision schemes with a perturbed mask in any $L_p$ norm. The main idea in Han [3, 4] was used in Han and Hogan [5] to obtain error estimate for vector subdivision schemes in the univariate $L_p$ case, and recently was generalized by Chen and Plonka [1] to establish error estimates for subdivision schemes in a Sobolev space with the particular initial function $\varphi$ defined in (5). The estimate in [1] essentially relies on the relation $D^\mu\varphi = \nabla^\mu\psi$, where $\psi(x_1,\ldots,x_s) = B_{k+1-\mu_1}(\xi_1)\cdots B_{k+1-\mu_s}(x_s)$ for all $\mu = (\mu_1,\ldots,\mu_s)$ with $|\mu| \le k$. In the following result, we shall remove such restriction on the initial functions with the help of Theorem 1 and Corollary 3.

Let $a$ be a finitely supported mask on $\mathbb{Z}^s$ such that $a$ vanishes outside a finite subset $\Omega$ of $\mathbb{Z}^s$. For any $\eta > 0$, we define the neighborhood $N_\eta(a;k,M,\Omega)$ to be the set of all sequences $b$ on $\mathbb{Z}^s$ such that $\sum_{\beta\in\mathbb{Z}^s} b(\beta) = 1$, $b$ vanishes outside $\Omega$, $b$ satisfies the sum rules of order $k+1$ with respect to the lattice $M\mathbb{Z}^s$, and $\|a-b\|_1 < \eta$.

Following the line developed in [3, 4], we have the following result.

**Theorem 8.** *Let $M$ be an $s\times s$ isotropic dilation matrix with $m := |\det M|$. Let $a$ be a mask on $\mathbb{Z}^s$ such that $a$ vanishes outside a finite subset $\Omega$ of $\mathbb{Z}^s$. Suppose that the subdivision scheme associated with mask $a$ and dilation $M$ converges in the Sobolev space $W_p^k(\mathbb{R}^s)$ (i.e., $\rho_{k+1}(a,M;p) < m^{1/p-k/s}$). Then there exists $\eta > 0$ such that for every $b \in N_\eta(a;k,M,\Omega)$, the subdivision scheme associated with mask $b$ and dilation $M$ converges in $W_p^k(\mathbb{R}^s)$. Moreover, for any compactly supported function $f$ in $W_p^k(\mathbb{R}^s)$ such that $f$ satisfies the moment conditions of order $k+1$, there is a constant $C > 0$ such that*

$$\|Q_{a,M}^n f - Q_{b,M}^n f\|_{W_p^k(\mathbb{R}^s)} \le C\|a-b\|_1 \qquad \forall\, n\in\mathbb{N},\ b\in N_\eta(a;k,M,\Omega),$$

*where the neighborhood $N_\eta(a;k,M,\Omega)$ of $a$ is defined above. Consequently, $\|\phi_a^M - \phi_b^M\|_{W_p^k(\mathbb{R}^s)} \le C\|a-b\|_1$ for all $b \in N_\eta(a;k,M,\Omega)$.*

**Proof:** By assumption and Theorem 7, we have $\rho_{k+1}(a,M;p) < m^{1/p-k/s}$. Using an argument of joint spectral radius, there exists a small enough $\eta > 0$ such that for any mask $b \in N_\eta(a;k,M,\Omega)$, $\rho_{k+1}(b,M;p) < m^{1/p-k/s}$ and therefore, the subdivision scheme associated with mask $b$ and dilation $M$ converges in $W_p^k(\mathbb{R}^s)$. Similarly, using joint spectral radius again (for detail, see [1, 3, 4]), there exists a constant $C > 0$ such that

$$\|\nabla^\mu S_{a,M}^n\delta - \nabla^\mu S_{b,M}^n\delta\|_p \le Cm^{n(1/p-|\mu|/s)}\|a-b\|_1 \qquad \forall\,|\mu|\le k. \tag{11}$$

By induction, we have

$$Q_{a,M}^n f - Q_{b,M}^n f = \sum_{\beta\in\mathbb{Z}^s}[S_{a,M}^n\delta(\beta) - S_{b,M}^n\delta(\beta)]f(M^n\cdot-\beta) \qquad \forall\, n\in\mathbb{N}.$$

By Proposition 5 and Corollary 3, for $j = 0,\ldots,k$, we have

$$\begin{aligned}
&\otimes^j D[Q_{a,M}^n f - Q_{b,M}^n f]\\
&\quad = \sum_{\beta\in\mathbb{Z}^s}[S_{a,M}^n\delta(\beta) - S_{b,M}^n\delta(\beta)](\otimes^j(M^T)^n)(\otimes^j Df)(M^n\cdot-\beta)\\
&\quad = \sum_{|\nu|=j}\sum_{\beta\in\mathbb{Z}^s}[\nabla^\nu S_{a,M}^n\delta(\beta) - \nabla^\nu S_{b,M}^n\delta(\beta)](\otimes^j(M^T)^n)h_\nu^j(M^n\cdot-\beta).
\end{aligned}$$

Since all the components of $h_\nu^j$ are compactly supported functions in $L_p(\mathbb{R}^s)$, there exists a constant $C_1 > 0$ depending only on $h_\nu^j$ such that

$$\begin{aligned}\big\|\otimes^j D[Q_{a,M}^n f - Q_{b,M}^n f]\big\|_p \\ &\le C_1 m^{n(j/s-1/p)} \max\{\|\nabla^\nu S_{a,M}^n\delta - \nabla^\nu S_{b,M}^n\delta\|_p \ : \ |\nu| = j\} \\ &\le C_1 C\|a-b\|_1,\end{aligned}$$

where we used (11) in the last inequality. This completes the proof. □

## §4. Convergence Rate and Generalized Definition of Convergence

In this section, we study the rate of convergence of a subdivision scheme. We also generalize the definition of convergence of a subdivision scheme.

**Theorem 9.** *Let $M$ be an $s\times s$ isotropic dilation matrix with $m := |\det M|$. Let $a$ be a finitely supported mask on $\mathbb{Z}^s$ and $k$ be a nonnegative integer such that $\rho(a,M;p) < m^{1/p-k/s}$. Let $K := \inf\{j \in \mathbb{N}_0 \ : \ \rho_{j+1}(a,M;p) = \rho(a,M;p)\}$ and $r := \rho(a,M;p)m^{k/s-1/p} < 1$. Suppose that $f$ is a compactly supported function in $W_p^k(\mathbb{R}^s)$ such that $D^\mu \hat f(0) = D^\mu \hat\phi_a^M(0)$ for all $|\mu| \le K-k$, and $D^\mu \hat f(2\pi\beta) = 0$ for all $|\mu| \le K$ and $\beta \in \mathbb{Z}^s\backslash\{0\}$. Then for any $0 < \epsilon < 1-r$, there exists a constant $C > 0$ such that*

$$\|Q_{a,M}^n f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} \le C(r+\epsilon)^n \qquad \forall\, n \in \mathbb{N}.$$

*Consequently, $\|Q_{a,M}^{n+1} f - Q_{a,M}^n f\|_{W_p^k(\mathbb{R}^s)} \le 2C(r+\epsilon)^n$ for all $n \in \mathbb{N}$.*

**Proof:** Let $h = f - \phi_a^M$. By assumption, $D^\mu \hat h(0) = 0$ for all $|\mu| \le K-k$ and $D^\mu \hat h(2\pi\beta) = 0$ for all $|\mu| \le K$ and $\beta \in \mathbb{Z}^s\backslash\{0\}$. According to Corollary 2 and Theorem 1, we have $D^\mu h = \sum_{|\nu|=|\mu|+K-k+1} \nabla^\nu h_\nu^\mu$ for all $|\mu| \le k$ and for some compactly supported functions $h_\nu^\mu$ in $L_p(\mathbb{R}^s)$. The rest of the proof is the same as that in Theorem 7. □

By definition, $\rho(a,M;p)$ is isotropic, but each $\lim_{n\to\infty} \|\nabla^\mu S_{a,M}^n\delta\|_p$ depends on $\mu \in \mathbb{N}_0^s$. Therefore, better convergence rate than that in Theorem 9 can be easily achieved as long as the initial function matches the refinable functions well in certain directions. Also $D^\mu \hat\phi_a^M(0), |\mu| < k$ can be easily obtained by a recursive formula derived from the refinement equation (1).

Finally, the discussion in this paper motivates us to give a general definition about the convergence of a subdivision scheme. Let $(X, \|\cdot\|)$ be a Banach space with $X \subseteq L_1(\mathbb{R}^s)$. Let $M$ be a dilation matrix and $a$ be a finitely supported mask on $\mathbb{Z}^s$. We say that a linear space $X_0$ is an admissible space if for every $f \in X_0$, $f$ is a compactly supported function in $X$ and $Q_{a,M} f \in X_0$. We say that a subdivision scheme associated with mask $a$ and dilation $M$ converges in $(X, \|\cdot\|)$ with the admissible space $X_0$ if $\phi_a^M \in X$ and $\lim_{n\to\infty} \|Q_{a,M}^n f\| = 0$ for all $f \in X_0$. This definition implies that $\lim_{n\to\infty} \|Q_{a,M}^n (f + \phi_a^M) - \phi_a^M\| = 0$ for all $f \in X_0$.

Let $X_0$ denote the set of all compactly supported functions $f \in W_p^k(\mathbb{R}^s)$ such that $\hat{f}(0) = 0$ and $D^\mu \hat{f}(2\pi\beta) = 0$ for all $|\mu| \le k$ and $\beta \in \mathbb{Z}^s \backslash \{0\}$. Then the definition of convergence of subdivision schemes in Theorem 7 can be restated as that the subdivision scheme converges in $(W_p^k(\mathbb{R}^s), \|\cdot\|_{W_p^k(\mathbb{R}^s)})$ with the admissible space $X_0$.

**Example.** Let $a(0) = a(2) = 1/2$ and $a(\beta) = 0$ for all $\beta \in \mathbb{Z}\backslash\{0,2\}$. Let $X_0$ denote the set of all compactly supported functions $f \in L_p(\mathbb{R})$ such that $\hat{f}(\pi\beta) = 0$ for all $\beta \in \mathbb{Z}$. Then it is not difficult to verify that for any $1 \le p < \infty$, the subdivision scheme associated with mask $a$ and dilation 2 converges in $(L_p(\mathbb{R}), \|\cdot\|_p)$ with the admissible space $X_0$. However, the subdivision scheme with mask $a$ and dilation 2 does not converge in the $L_p$ norm in the sense of Theorem 7 with $k = 0$.

From the proofs, all results hold for general dilation matrices when $k = 0$.

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