# Global existence of solutions to slightly compressible Navier-Stokes equations in two dimension

Hi Jun Choe and Bum Ja Jin
Department of Mathematics, KAIST
Yousung, Taejon, 305-701
Republic of Korea

**Abstract**

It has been known that a strong solution of 2 dimensional almost incompressible Navier Stokes equations with zero external force exists for all time without any restriction of the size of initial data. In this paper, we prove the global existence of strong solutions to almost imcompressible Navier-Stokes equations when a sufficient regular external forces are given and the initial data are almost incompressible. Our results hold independent of the size of forces and the size of intial data if the Mach number is sufficiently small and the initial data is sufficiently close to incompressible flow.

## 1 introduction

To avoid complicated boundary estimate, we consider 2 dimensional periodic domain $T_2$ with periodicity 1 on both $x$ and $y$ directions. The isentropic compressible Navier-Stokes equations of a polytropic gas are

$$\begin{aligned} \rho_t + u\cdot\nabla\rho + \rho\text{div}u &= 0 \text{ in } T_2 \\ \rho(u^j_t + u\cdot\nabla u^j) + \nabla p &= \mu\Delta u^j + (\mu+\lambda)\frac{\partial}{\partial x_j}\text{div} + \rho f^j \text{ in } T_2, \end{aligned}$$

where $u = (u^1, u^2)$ and $\rho$ denote velocity field and density function, respectively. Also, the pressure $p$ is a function of density, $p = R\rho^\gamma$ for $\gamma \geq 1$. The constants $\mu$ and $\lambda$ denote the coefficients of viscosity. In terms of physical view points, we consider the case $\mu > 0$ and $\mu + \lambda > 0$. We let $\varepsilon$ be the Mach number. With the scaling, we can write the compressible Navier-Stokes equations into dimensionless form

$$
\begin{aligned}
\rho u_t + \rho(u\cdot\nabla)u + \frac{1}{\varepsilon^2}\rho^{\gamma-1}\nabla\rho &= \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u + \rho f \quad \text{in } T_2 \\
\rho_t + \mathrm{div}\rho u &= 0 \quad \text{in } T_2.
\end{aligned}
$$

If the initial density $\rho_0$ satisfies $\int_{T_2}\rho_0 dx = \bar{\rho}$, then the conservation of mass implies $\int_{T_2}\rho dx(t) = \bar{\rho}$ for all $t \geq 0$ if $\rho$ is sufficiently regular. Since we deal with almost incompressible flows, we assume $\bar{\rho} > 0$. In fact, if $\bar{\rho} = 0$, there is nothing to prove. Furthermore, with suitable changes of various coefficients, we assume $\bar{\rho} = 1$ without loss of generality. Now, by perturbing $\rho$ near $\bar{\rho} = 1$, we replace $\rho$ by $1+\varepsilon^2\rho$. Then, introducing initial velocity $u_0$, we obtain the following perturbed equations

$$
\begin{aligned}
& u_t + u\cdot\nabla u + (1+\varepsilon^2\rho)^{\gamma-2}\nabla\rho = f \qquad (1)\\
& \quad + (1+\varepsilon^2\rho)^{-1}(\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div}\ u) \quad \text{in } T_2 \\
& \varepsilon^2\rho_t + \varepsilon^2 \mathrm{div}\ \rho u + \ \mathrm{div}\ u = 0 \quad \text{in } T_2 \qquad (2)
\end{aligned}
$$

for the initial data

$$
u(x,0) = u_0(x) \quad \text{and} \quad \rho(x,0) = \rho_0(x). \qquad (3)
$$

From the definition of $\bar{\rho}$, we assume naturally that

$$
\int_{T_2}\rho_0 dx = 0.
$$

and we also assume all functions are periodic.

In a view of real phenomena, fluid behaves similar to incompressible flows when the Mach number is small enough. It has been known that nonstationary incompressible Navier-stokes equations in two dimensional domains have strong solutions for all time if the given external forces are sufficiently regular. Moreover, if there are no external forces, then the exponential decay (in time) of the solutions can be proved. For the compressible equations, it is plausible that if the initial datas are near incompressible flows and the Mach numbers are sufficiently small, then there exist strong solutions of compressible Navier-Stokes equations close to the solutions of incompressible equations. In 1998, Hagstrom and Lorenz [2] proved the global existence of the strong solutions of 2 dimensional compressible Navier-Stokes equations without force terms near incompressible flows if the Mach number are sufficiently small and the initial data are almost incompressible. The essential ingredient of their proof is the fact that two dimensional incompressible flows decay exponentially if there are no external forces.

We let $L^p(T_2)$ be the usal $L^p$-integrable Lebesgue space in $T_2$ and Sobolev space $W^{k,p}(T_2)$ be the set of $L^p(T_2)$ functions whose $k$-th derivatives are also $L^p$-integrable. Naturally we define the norm of $W^{k,p}$ by

$$\| u \|_{k,p}= \left( \sum_{|\alpha|\le k} \int_{T_2} |\nabla^\alpha u|^p dx \right)^{\frac{1}{p}}.$$

For simplicity, we also denote $H^k = W^{k,2}$ and $\| \cdot \|_{k,2}=\| \cdot \|_k$. We let $J(T_2) = \{h \in L^2(T_2) : \text{div} h = 0 \quad \text{in the sense of distributions}\}$ and $G(T_2) = \{g \in L^2(T_2) : g = \nabla p, p \in H^1(T_2)\}$. Then from Helmholtz-Weyl theorem we have

$$L^2(T_2) = J(T_2) \oplus G(T_2)$$

and we define $\mathcal{P}$ as the projection of $L^2(T_2)$ to the divergence free space $J(T_2)$.

We let $U_0 = \mathcal{P}(u_0)$ and $f \in L^2((0,\infty)\times T_2)$. We suppose $U \in L^\infty(0,\infty; L^2(T_2)) \cap L^2(0,\infty; H^1(T_2))$ be the solution to the 2 dimensional incompressible Navier-Stokes equations

$$U_t - \mu\Delta U + U \cdot \nabla U + \nabla P = f \tag{4}$$

$$\text{div} U = 0$$

satisfying the initial conidion $U(x,0) = U_0(x)$. From Galerkin approximations and 2 dimensional regularity theory, we know there is a unique solution

$$U \in L^\infty(0,\infty; L^2(T_2)) \cap L^2(0,\infty; H^1(T_2)).$$

Moreover we can assume that $P(\cdot,t) \in L^2(T_2)$ is average free, that is,

$$\int_{T_2} P(x,t)dx = 0. \tag{5}$$

We show the existence of unique solution $(u,\rho)$ to (1) and (2) if $f$ satisfies a smoothness condition and the initial data $u_0$ is almost incompressible. We emphasize that any smallness conditions are imposed on $f$ and $u_0$. The novelty in our paper lies on the perturbation method of the incompressible Navier-Stokes flow with large force and the corresponding Gronwall type inequality. To be more precise, we state our main theorem.

**Theorem 1.1** *Suppose that an external force $f$ is given to satisfy*

$$f \in L^2([0,\infty); H^3(T_2)) \cap L^\infty([0,\infty); H^2(T_2))$$

*and*

$$f_t \in L^2([0,\infty); H^3(T_2)).$$

*There are positive constants* $\varepsilon_0(\mu, \lambda, u_0, f)$ *and* $\delta_0(\varepsilon_0, \mu, \lambda, u_0, f)$ *such that if* $0 < \varepsilon \leq \varepsilon_0$ *and for* $U_0(x) = \mathcal{P}(u_0)(x)$ *and* $P_0(x) = P(x, 0)$

$$\| u_0 - U_0 \|_3^2 + \varepsilon^2 \| \rho_0 - P_0 \|_3^2 \leq \delta_0^2,$$

*then there is a solution* $(u, \rho)$ *to the isentropic compressible Navier-Stokes equations (1) and (2) for all time* $t > 0$ *such that*

$$(u, \rho) \in C([0,\infty); H^3(T_2)) \times C([0,\infty); H^3(T_2))$$

*and*

$$\sup_{t<\infty} \| (u-U)(\cdot, t) \|_3 + \sup_{t<\infty} \varepsilon^2 \| (\rho - P)(\cdot, t) \|_3 + \varepsilon^2 \int_0^\infty \| \nabla(u-U) \|_3^2 (t) dt \leq M$$

*for some* $M$ *depending only on* $\varepsilon_0, \mu, \lambda, u_0, f$.

# 2 Preliminaries

Set $v = u - U$ and $\sigma = \rho - P$. Now, we get the following equations by perturbing the compressible equations (1), (2) and (3) near the solutions $(U, P)$ of the incompressible Navier-Stokes equations (4).

$$\begin{aligned} v_t + (v+U)\cdot\nabla v + v\cdot\nabla U + \nabla\sigma = \mu\Delta v + (\mu+\lambda)\nabla\text{div } v \\ + (1 - (1+\varepsilon^2(\sigma+P))^{\gamma-2})\nabla(\sigma+P) \\ + ((1+\varepsilon^2(\sigma+P))^{-1} - 1)(\mu\Delta v + (\mu+\lambda)\nabla\text{div } v + \mu\Delta U) \end{aligned} \tag{6}$$

$$\begin{aligned} \varepsilon^2\sigma_t + \varepsilon^2(v+U)\cdot\nabla\sigma + \varepsilon^2\sigma\text{div } v + \text{div } v = -\varepsilon^2 P_t \\ - \varepsilon^2(v+U)\cdot\nabla P - \varepsilon^2 P\text{div } v. \end{aligned} \tag{7}$$

To make our equations symmetric, we set $w = (v, \varepsilon\sigma)^t$. Then we can write our equations as the following system from (6) and (7)

$$w_t + (v+U)\cdot\nabla w - A_\varepsilon w = F + \varepsilon G, \tag{8}$$

where $A_\varepsilon, F = F_1 + F_2 + F_3$ and $G$ are

$$A_\varepsilon = \begin{bmatrix} \mu\Delta + (\mu+\lambda)\frac{\partial^2}{\partial x_1^2} & (\mu+\lambda)\frac{\partial^2}{\partial x_1 \partial x_2} & 0 \\ (\mu+\lambda)\frac{\partial^2}{\partial x_1 \partial x_2} & \mu\Delta + (\mu+\lambda)\frac{\partial^2}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{\varepsilon}\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} J + ((1+\varepsilon^2(\sigma+P))^{-1} - 1)(\mu\Delta v + (\mu+\lambda)\nabla \mathrm{div} v) \\ -\varepsilon\sigma \mathrm{div}\ v \end{bmatrix}$$

$$F_2 = \varepsilon \begin{bmatrix} 0 \\ -P_t - U\cdot\nabla P \end{bmatrix}, \qquad F_3 = \begin{bmatrix} -v\cdot\nabla U \\ 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{\mu}{\varepsilon}((1+\varepsilon^2(\sigma+P))^{-1} - 1)\Delta U \\ -v\cdot\nabla P - P\mathrm{div}\ v \end{bmatrix}.$$

Here $J$ denotes the function

$$J = (1 - (1+\varepsilon^2(\sigma+P))^{\gamma-2})\nabla(\sigma+P).$$

The defferential operator $A_\varepsilon$ is not strictly elliptic, but we can show $A_\varepsilon$ induces a coercive map for $H^1$ modulo constant. This observation is crucial in [2] to show global existence in the homogeneous cases. We follow the same idea for diffusion parts. We interpret in terms of Fourier series. For $u,\ v \in L^2(T_2, \mathsf{R}^2)$, we define the $L^2$ inner product and norm by

$$(u,v) = \sum_{j=1}^{n} \int_{T_2} \bar{u}_j v_j dx \text{ and } \parallel u \parallel_0 = (u,u)^{\frac{1}{2}}$$

and the Sobolev inner products and norms on $H^k$ are

$$(u,v)_k = \sum_{|\alpha|\le k} (D^\alpha \bar{u}, D^\alpha v) \text{ and } \parallel u \parallel_k = (u,u)_k^{\frac{1}{2}}, k = 1, 2, \cdots,$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$. We recall Parseval's equality

$$(u,v) = \sum_{\mathsf{k}\in\mathcal{Z}^2} \sum_{j=1}^{2} \bar{\hat{u}}_j(\mathsf{k}) \hat{v}_j(\mathsf{k}),$$

where

$$\hat{u}(\mathsf{k}) = \frac{1}{2\pi}\int_{T_2} exp^{-i(k_1x_1+k_2x_2)}u(x)dx.$$

In the frequency domain, our differential operator $A_\varepsilon$ will be a Fourier multiplier matrix such that

$$\hat{A}_\varepsilon(\mathsf{k}) = -\frac{i}{\varepsilon}\begin{bmatrix} 0 & 0 & k_1 \\ 0 & 0 & k_2 \\ k_1 & k_2 & 0 \end{bmatrix} - \begin{bmatrix} \mu|\mathsf{k}|^2 + (\mu+\lambda)k_1^2 & (\mu+\lambda)k_1k_2 & 0 \\ (\mu+\lambda)k_1k_2 & \mu|\mathsf{k}|^2 + (\mu+\lambda)k_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $\hat{A}_\varepsilon(\mathsf{k})$ are

$$\lambda_1 = -\mu|\mathsf{k}|^2,\ \lambda_{2,3} = -\frac{(\mu+\lambda)}{2}|\mathsf{k}|^2\left(1 \pm \sqrt{1 - \frac{4}{(\mu+\lambda)^2\varepsilon^2|\mathsf{k}|^2}}\right)$$

and all of them have negative real part, more precisely,

$$Re_{j=1,2,3}\lambda_j(\mathsf{k}) \le \max\{-\mu, -\frac{\mu+\lambda}{2}, -\frac{1}{(\mu+\lambda)\varepsilon^2}\} \text{ for } \mathsf{k} \in \mathcal{Z}^2, \mathsf{k} \ne 0.$$

We let $\varepsilon_0$ so small that

$$\frac{1}{(\mu+\lambda)\varepsilon_0^2} > \max\{\mu, \frac{\mu+\lambda}{2}\}. \tag{9}$$

Then, if $0 < \varepsilon \le \varepsilon_0$, $Re_{j=1,2,3}\lambda_j(\mathsf{k}) \le -c_0$, where $c_0 = \max\{\mu, \frac{\mu+\lambda}{2}\}$. As in the case of symmetric hyperbolic system, we can find a symmetrizer $H$ for $A_\varepsilon$. We define a symmetrizer matrix $H$ for $A_\varepsilon$ by

$$H = S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i\frac{\varepsilon(\mu+\lambda)}{8|\mathsf{k}|} \\ 0 & -i\frac{\varepsilon(\mu+\lambda)}{8|\mathsf{k}|} & 1 \end{bmatrix} S^*,$$

where

$$S = \frac{1}{|\mathsf{k}|}\begin{bmatrix} k_2 & k_1 & 0 \\ k_1 & k_2 & 0 \\ 0 & 0 & |\mathsf{k}| \end{bmatrix}.$$

The following lemma has been stated in [2](see Lemma 4.2 in [2]).

**Lemma 2.1** *There are positive constants $c_0$, $c_1$, $C_1$, $C_2$, $\varepsilon_0$, depending only on $\mu$ and $\mu+\lambda$ such that for $0<\varepsilon\le\varepsilon_0$, $\mathsf{k}\in\mathcal{Z}^2$, $\mathsf{k}\neq 0$, there are Hermitian matrices $H=H(\mu,\lambda,\varepsilon,\mathsf{k})\in\mathcal{C}^{3\times 3}$ with the following properties hold.*

$$
\begin{aligned}
0<(1-C_1\varepsilon)I\le H &\le (1+C_1\varepsilon)I\\
q^*(H\hat{A}_\varepsilon+\hat{A}_\varepsilon^*H)q &\le -c_0q^*Hq-c_1|\mathsf{k}|^2(|q_1|^2+|q_2|^2), \quad \textit{for all } q\in\mathcal{C}^3\\
|H-I| &\le \frac{C_2\varepsilon}{|\mathsf{k}|}
\end{aligned}
$$

To obtain the energy estimate, we define a new inner product on $L^2=L^2(T_2,\mathsf{R}^3)$ by

$$
(u,v)_H=\sum_{\mathsf{k}\in\mathcal{Z}^2}\hat{u}(\mathsf{k})^*H(\mu,\lambda,\varepsilon,\mathsf{k})\hat{v}(\mathsf{k}) \text{ for } 0<\varepsilon\le\varepsilon_0,\mathsf{k}\in\mathcal{Z}^2,\mathsf{k}\neq 0.
$$

Then we deduce the following results from Lemma 2.1.

**Lemma 2.2** *a) For all $w=(v,w^3)^t\in L^2$,*

$$
(1-C_1\varepsilon)\parallel w\parallel^2\le\parallel w\parallel_H^2\le(1+C_1\varepsilon)\parallel w\parallel^2,
$$

*b) If $w=w(x)$ is sufficiently regular (e.g., $w\in H^2$) and $\hat{w}(0)=0$ (i.e., the spatial averages of the components of $w$ are zero), then*

$$
(w,A_\varepsilon w)_H+(A_\varepsilon w,w)_H\le -c_0\parallel w\parallel_H^2-c_1\parallel Dv\parallel^2 .
$$

*c) If $w_1\in L^2, w_2\in H^1$, then*

$$
|(w_1,Dw_2)_H-(w_1,Dw_2)|\le\varepsilon C_2\parallel w_1\parallel_0\parallel w_2\parallel_0 .
$$

*Here $D=\frac{\partial}{\partial x_1}$ or $D=\frac{\partial}{\partial x_2}$.*

*d) If $w_1,w_2\in H^1$ then*

$$
(w_1,Dw_2)_H=-(Dw_1,w_2)_H.
$$

## 3 A priori estimates for the compressible equations

The short time existence of the strong solutions of (1) and (2) has been proved by several authors- Ebin [1], Klainerman and Majda [4], Beiraó da Veiga [6]- when there are no external forces and the initial data are slightly compressible. The presence of external forces makes the proof of the short time existence theorem a little complicated. In this section, we suppose the short time existence of the solutions of (1) and (2). The short time existence of strong solutions will be proved later. Consequently, there is a maximal time $T^* > 0$ such that there is a $v \in C([0,T^*); H^3(T_2)) \cap L^2([0,T^*); H^4(T_2))$ and $\sigma \in C([0,T^*); H^3(T_2)) \cap L^2([0,T^*); H^4(T_2))$. In this section, we will estimate the $H^3$ norm of $w = (v, \varepsilon\sigma)$ from the equation (8).

Set

$$\Phi(t)^2 = \sum_{|\alpha|\leq 3} (D^\alpha w, D^\alpha w)_H$$

and

$$\Psi(t)^2 = \sum_{|\alpha|=4} (D^\alpha v, D^\alpha v).$$

If we take derivative $D^\alpha$ to the system (8), we get the equation

$$(D^\alpha w)_t - A_\varepsilon D^\alpha w + D^\alpha((v+U)\cdot\nabla w) = D^\alpha F + \varepsilon D^\alpha G. \tag{10}$$

Thus, applying $D^\alpha w$ to (10), we obtain our basic inequality

$$\begin{aligned}
\frac{d}{dt}\Phi(t)^2 =& 2\mathrm{Re} \sum_{|\alpha|\leq 3} (D^\alpha w, D^\alpha w_t)_H \qquad (11)\\
=& \sum_{|\alpha|\leq 3} ((D^\alpha w, A_\varepsilon D^\alpha w)_H + (A_\varepsilon D^\alpha w, D^\alpha w)_H)\\
&- 2\mathrm{Re} \sum_{|\alpha|\leq 3} (D^\alpha w, D^\alpha((v+U)\cdot\nabla w))_H\\
&+ 2\mathrm{Re} \sum_{|\alpha|\leq 3} (D^\alpha F, D^\alpha w)_H + 2\varepsilon\mathrm{Re} \sum_{|\alpha|\leq 3} (D^\alpha G, D^\alpha w)\\
=& I + II + III + IV.
\end{aligned}$$

## 3.1 Estimate for $I$

From Lemma 2.2, we observe that

$$\sum_{|\alpha|\le 3}((D^\alpha w, A_\varepsilon D^\alpha w)_H + (A_\varepsilon D^\alpha w, D^\alpha w)_H) \tag{12}$$
$$\le -\sum_{0<|\alpha|\le 3}(c_0 \parallel D^\alpha w \parallel_H^2 + c_1 \parallel D^\alpha \nabla v \parallel^2)$$
$$-((w, A_\varepsilon w)_H + (A_\varepsilon w, w)_H).$$

Let $w^c = w - (\bar{v}, 0)$. Here $\bar{v} = \int_{T_2} v dx$. We observe that the mean of $w^c$ is zero, since

$$\int_{T_2} (v - \bar{v}) dx = 0 \text{ by definition of } \bar{v}$$

and

$$\int_{T_2} \varepsilon\sigma dx = \varepsilon\int_{T_2}(\sigma + P)dx - \varepsilon\int_{T_2} P dx = 0,$$

by (2) and (5). Now we estimate $|\bar{v}|$. We find that

$$\bar{v} = \varepsilon^2\int_{T_2}(\sigma_0 + P_0)(v_0 + U_0)dx - \varepsilon^2\int_{T_2}(\sigma + P)(v + U)dx \tag{13}$$
$$+\varepsilon^2\int_0^t(\sigma + P)f dxdt$$

since

$$\begin{aligned}
\bar{v} &= \int_{T_2} v dx \\
&= \int_{T_2}(1 + \varepsilon^2(\sigma + P))(v + U)dx - \int_{T_2} U dx - \varepsilon^2\int_{T_2}(\sigma + P)(v + U)dx \\
&= \int_{T_2}(1 + \varepsilon^2(\sigma_0 + P_0))(v_0 + U_0)dx + \int_0^t\int_{T_2}(1 + \varepsilon^2(\sigma + P))f \\
&\quad -\int_{T_2} U dx - \varepsilon^2\int_{T_2}(\sigma + P)(v + U)dx \\
&= \varepsilon^2\int_{T_2}(\sigma_0 + P_0)(v_0 + U_0)dx - \varepsilon^2\int_{T_2}(\sigma + P)(v + U)dx \\
&\quad +\varepsilon^2\int_0^t\int_{T_2}(\sigma + P)f dx.
\end{aligned}$$

by (1) and (4). We suppose

$$\| v_0 \|_3^2 + \varepsilon^2 \| \sigma_0 \|_3^2 \leq \delta_0^2.$$

Note $w^3 = \varepsilon\sigma$. From (13), we get the following estimate for $|\bar{v}|$ such that

$$\begin{aligned} |\bar{v}| \leq & c\varepsilon\delta_0^2 + c\varepsilon(\| U_0 \|_3^2 + \| P_0 \|_3^2 + \int_0^t \| f \|_0^2 \, dt) \\ & + c\varepsilon(\Phi(t)^2 + \Psi(t)^2) + c\varepsilon(\| U \|_3^2 + \| P \|_3^2). \end{aligned}$$

We obtain the estimate for the second term of the right hand side of (12):

$$\begin{aligned} (w, A_\varepsilon w) + (A_\varepsilon w, w) & \leq -c_0 \| w^c \|_H^2 - c_1 \| \nabla v \|_0^2 \\ & \leq -c_0 \| w \|_H^2 - c_1 \| \nabla v \|_0^2 + c|\bar{v}|^2. \end{aligned} \tag{14}$$

Therefore, from (12) and (14), we get the following estimate for $I$:

$$\begin{aligned} |I| \leq & - c_0\Phi(t)^2 - c_1\Psi(t)^2 + c\varepsilon\delta_0^2 + c\varepsilon(\| U_0 \|_3^2 + \| P_0 \|_3^2 + \int_0^t \| f \|_0^2 \, dt) \\ & + c\varepsilon(\Phi(t)^2 + \Psi(t)^2) + c\varepsilon(\| U \|_3^2 + \| P \|_3^2). \end{aligned} \tag{15}$$

## 3.2 Estimate for $II$

Note that $(v + U) \cdot \nabla w^j = \frac{\partial}{\partial x_i}((v^i + U^i)w^j) - \mathrm{div}(v)w^j$. From Lemma 2.2, we know that

$$\begin{aligned} \sum_{|\alpha|\leq 3} & |(D^\alpha w, D^\alpha((v+U) \cdot \nabla w))_H - (D^\alpha w, D^\alpha((v + U) \cdot \nabla w))| \\ & \leq \varepsilon C_2 \| w \|_3 \| (v^i + U^i) \cdot \nabla w^j \|_2 \\ & \leq c\varepsilon(\Phi(t)^3 + \| U \|_3 \, \Phi(t)^2). \end{aligned} \tag{16}$$

Note that

$$\begin{aligned} (D^\alpha w, D^\alpha((v + U) \cdot \nabla w)) = & (D^\alpha w, D^\alpha((v + U) \cdot \nabla w) \\ & -(v + U) \cdot \nabla D^\alpha w) - \frac{1}{2} \int_{T_2} \mathrm{div}\ v |D^\alpha w|^2 dx. \end{aligned} \tag{17}$$

By Gagliardo-Nirenberg inequality, we can estimate (17) such that

$$\sum_{|\alpha|\leq 3} |(D^\alpha w, D^\alpha((v + U) \cdot \nabla w))| \leq c\Phi(t)^2(\Psi(t) + \| \nabla U \|_3).$$

Therefore, from the stimates (16) and (17), we estimate $II$ so that

$$|II| \leq c(\varepsilon(\Phi(t)^3 + \| U \|_3 \, \Phi(t)^2) + \Psi(t)\Phi(t)^2 + \| \nabla U \|_3 \, \Phi(t)^2). \tag{18}$$

### 3.3 Estimate for $III$

Since $F$ is defined by $F = F_1 + F_2 + F_3$, we write $III$ as

$$III = 2\mathrm{Re}\sum_{|\alpha|\le 3}(D^\alpha F_1, D^\alpha w)_H + 2\mathrm{Re}\sum_{|\alpha|\le 3}(D^\alpha F_2, D^\alpha w)_H + 2\mathrm{Re}\sum_{|\alpha|\le 3}(D^\alpha F_3, D^\alpha w)_H$$

$$= (III-1) + (III-2) + (III-3).$$

(i) Estimate for $III - 1$

If we define the function $h(s)$ by

$$h(s) = \begin{cases} \frac{1}{\gamma-1}(1+s)^{\gamma-1} - \frac{1}{\gamma-1} - s & \text{if } \gamma \ne 1 \\ \log(1+s) - s & \text{if } \gamma = 1 \end{cases},$$

then we find that

$$(1-(1+\varepsilon^2(\sigma+P))^{\gamma-2})\nabla(\sigma+P) = -\frac{1}{\varepsilon^2}\nabla h(\varepsilon^2(\sigma+P)).$$

Note that

$$h(s) = s^2 g(s),$$

where $g(s)$ is defined by

$$g(s) = \begin{cases} (\gamma-2)\int_0^1\int_0^1(1+r_1r_2s)^{\gamma-3}r_2dr_1dr_2 & \text{if } \gamma \ne 1 \\ \int_0^1\int_0^1 \frac{1}{(1+r_1r_2)^2}r_2dr_2dr_2 & \text{if } \gamma = 1. \end{cases}$$

So we can write $J$ in $F_1$ such that

$$(1-(1+\varepsilon^2(\sigma+P))^{\gamma-2})\nabla(\sigma+P) = -\varepsilon^2\nabla((\sigma+P)^2 g(\varepsilon^2(\sigma+P))). \qquad (19)$$

Consequently, from direct calculations with Sobolev inequality in mind, we obtain

$$\begin{aligned} \| F_1 \|_2 \le c(&\varepsilon^3 \| g''' \|_{L^\infty} \| w^3 + \varepsilon P \|_3^5 + \varepsilon^2 \| g'' \|_{L^\infty} \| w^3 + \varepsilon P \|_3^4 \qquad (20)\\ &+ \varepsilon \| g' \|_{L^\infty} \| w^3 + \varepsilon P \|_3^3 + \| g \|_{L^\infty} \| w^3 + \varepsilon P \|_3^2 \\ &+ \varepsilon \| w^3 + \varepsilon P \|_3 \| \nabla v \|_3 \\ &+ \varepsilon \left\| \frac{1}{1+\varepsilon^2(\sigma+P)} \right\|_{L^\infty}^3 \| w^3 + \varepsilon P \|_3 \| \nabla v \|_3 \\ &+ \| w^3 \|_3 \| v \|_3) \end{aligned}$$

Set $a(t) = \| \frac{1}{1+\varepsilon w^3+\varepsilon^2 P} \|_{L^\infty}$, $b_1(t) = \| g(\varepsilon w^3 + \varepsilon^2 P) \|_{L^\infty}$ , $b_2(t) = \| g'(\varepsilon w^3 + \varepsilon^2 P) \|_{L^\infty}$, $b_3(t) = \| g''(\varepsilon w^3 + \varepsilon^2 P) \|_{L^\infty}$ and $b_4(t) = \| g'''(\varepsilon w^3 + \varepsilon^2 P) \|_{L^\infty}$. If we apply Lemma 2.2, (19) and (20) and integrate by parts, we can see the followings(we move one derivative of $(D^\alpha F_1^1, D^\alpha F_1^2)$ to $v^t = (w^1, w^2)$):

$$
\begin{aligned}
&\sum_{0<|\alpha|\le 3} |(D^\alpha w, D^\alpha F_1)_H - (D^\alpha w, D^\alpha F_1)| \qquad (21)\\
&\le C_2 \sum_{0<|\alpha|\le 3} \varepsilon(\| D^\alpha w \|_0 + \| D^\alpha \nabla v \|_0) \| F_1 \|_2 \\
&\le c\varepsilon(\Phi(t)+\Psi(t))(\Phi(t)^2 + \varepsilon a(t)^3(\Phi(t)+\varepsilon \| P \|_3)\Psi(t)) \\
&\quad + c\varepsilon^2(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)\Psi(t) \\
&\quad + c\varepsilon b_1(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^2 \\
&\quad + c\varepsilon^2 b_2(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^3 \\
&\quad + c\varepsilon^3 b_3(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^4 \\
&\quad + c\varepsilon^4 b_4(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^5.
\end{aligned}
$$

Integrating by parts, we have the inequality

$$
\begin{aligned}
|\sum_{0<|\alpha|\le 3} (D^\alpha w, D^\alpha F_1)| \le \sum_{0<|\alpha|\le 3} (&\| D^\alpha((w^3+\varepsilon P)^2 g(\varepsilon w^3+\varepsilon^2 P)) \|_0 \| D^\alpha \text{ div } v \|_0 \\
&+ \| \frac{\varepsilon w^3+\varepsilon^2 P}{1+\varepsilon w^3+\varepsilon^2 P}(\mu\Delta v + (\mu+\lambda)\nabla \text{div } v) \|_{|\alpha|-1} \| \nabla v \|_{|\alpha|+1} \\
&+ \| D^\alpha(w^3 \text{div } v) \|_0 \| D^\alpha w^3 \|_0).
\end{aligned}
$$

Hence, with (21), we obtain the inequality

$$
\begin{aligned}
\sum_{0<|\alpha|\le 3} |(D^\alpha w, D^\alpha F_1)| \le\ & c(\Phi(t)+\Psi(t))\Phi(t)^2 \\
&+ c\varepsilon a(t)^3(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)\Psi(t) \\
&+ c b_1(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^2 \\
&+ c\varepsilon b_2(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^3 \\
&+ c\varepsilon^2 b_3(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^4 \\
&+ c\varepsilon^3 b_4(t)(\Phi(t)+\Psi(t))(\Phi(t)+\varepsilon \| P \|_3)^5.
\end{aligned}
$$

By Lemma 2.1, we know $|H| \leq (1 + C_1\varepsilon)$, where $|H|$ means the usual matrix norm. We also have the inequality for the case $\alpha = 0$ :

$$
\begin{aligned}
|(F_1, w)|_H &\leq \sum_{\mathsf{k}} |\hat{F}_1(\mathsf{k})||H(\mathsf{k})||\hat{w}| \\
&\leq (1 + C_1\varepsilon) \sum_{\mathsf{k}} (|\hat{F}_1(\mathsf{k})|^2)^{\frac{1}{2}} \sum_{\mathsf{k}} (|\hat{w}(\mathsf{k})|^2)^{\frac{1}{2}} \\
&\leq (1 + C_1\varepsilon) \parallel F_1 \parallel_0 \parallel w \parallel_0,
\end{aligned}
$$

by Schwarz inequality and Plancherel identity. Hence, from Hölder inequality and Sobolev inequality, we obtain the estimate for $III - 1$ as follows:

$$
\begin{aligned}
|III - 1| \leq\ & c(\Phi(t) + \Psi(t))\Phi(t)^2 \qquad (22) \\
&+ c\varepsilon a(t)^2 (\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_3)\Psi(t) \\
&+ cb_1(t)(\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_3)^2 \\
&+ c\varepsilon b_2(t)(\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_3)^3 \\
&+ c\varepsilon^2 b_3(t)(\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_3)^4 \\
&+ c\varepsilon^3 b_4(t)(\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_3)^5
\end{aligned}
$$

(23)

(ii) Estimates for $III - 2$ and $III - 3$

We get the following estimates for $III - 2$ and $III - 3$ by applying Lemma 2.1 and Lemma 2.2:

$$
\sum_{|\alpha| \leq 3} |(D^\alpha w, D^\alpha F_2)_H| \leq c\varepsilon\Phi(t)(\parallel P_t \parallel_3 + \parallel U \parallel_3 \parallel \nabla P \parallel_3) \qquad (24)
$$

and

$$
\sum_{|\alpha| \leq 3} |(D^\alpha w, D^\alpha F_3)_H| \leq c\varepsilon \parallel \nabla U \parallel_2 \Phi(t)^2 + c \parallel \nabla U \parallel_3 \Phi(t)^2. \qquad (25)
$$

## 3.4 Estimate for $IV$

By the same reasoning as $(iii)$ in subsection 3.3, we have the following estimate for the fourth term in (11):

$$
\begin{aligned}
|IV| \leq\ & c\varepsilon a(t) \parallel \nabla U \parallel_3 (\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_2) \qquad (26) \\
&+ c\varepsilon^2 a(t)^2 \parallel \nabla U \parallel_3 (\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_3)^2 \\
&+ c\varepsilon^3 a(t)^3 \parallel \nabla U \parallel_3 (\Phi(t) + \Psi(t))(\Phi(t) + \varepsilon \parallel P \parallel_3)^3 \\
&+ c\varepsilon\Phi(t)(\Phi(t) + \Psi(t)) \parallel P \parallel_3
\end{aligned}
$$

# 4 Main result

In this section, we will derive the global time existence of the solutions to (6) and (7) by establishing Gronwall inequality using the results from subsections 3.1, 3.2, 3.3 and 3.4. The short time existence is essenstial to continue indirect argument. Although, the short time existence for homogeneous cases have been considered in many papers(see [1], [4], [5], [6]), we do not know local existence theorem when there is a large force. Following a similar argument to [4] or [6], we will find a contraction map in short time and this implies the local existence via fixed point theorem.

First, we need global energy estimates for higher norms of $(U, P)$ for incompressible parts. The estimates will be useful in the proof for short time existence and global time existence. In fact, the $L^1$ integrability in time of various space norms are essential to find the global estimate. We suppose that

$$f \in L^2([0,\infty); H^3(T_2)) \cap L^\infty([0,\infty); H^2(T_2))$$

and

$$f_t \in L^2([0,\infty); H^3(T_2)).$$

We also assume that $U_0 \in H^3(T_2)$. Then we see the following facts from standard energy estimates;

$$\begin{aligned}
\sup_{0\le t<\infty} \| U(\cdot,s) \|_3 &\le M_1(\int_0^\infty \| f(\cdot,s) \|_2^2 \, ds; \| U_0 \|_3) \\
\int_0^\infty \| \nabla U(\cdot,s) \|_3^2 &\le M_1(\int_0^\infty \| f(\cdot,s) \|_2^2 \, ds; \| U_0 \|_3) \\
\sup_{0\le t<\infty} \| P(\cdot,s) \|_3 &\le M_2(\sup_{0<t<\infty} \| f \|_2 \\
&\quad ;\int_0^\infty \| f(\cdot,s) \|_2^2 \, ds; \| U_0 \|_3) \\
\int_0^\infty \| \nabla P(\cdot,s) \|_3^2 \, ds &\le M_3(\int_0^\infty \| f(\cdot,s) \|_3^2 \, ds; \| U_0 \|_3) \\
\int_0^\infty \| P(\cdot,s)_t \|_3^2 &\le M_4(\int_0^\infty \| f(\cdot,s) \|_3^2 + \| f_s(\cdot,s) \|_3^2 \, 2ds; \| U_0 \|_3 \, 2).
\end{aligned}$$

Here $M_i(s_1; s_2; s_3)$ means that the positive constant $M_i$ depending only on $s_1, s_2, s_3$. Since the proofs are rather obvious, we omit them.

Once we found the energy estimates for $(U, P)$, we proceed to show the short time existence of compressible parts. As far as we know, there seems no known explicit proof of short time existence independent of Mach number when external force presents. We have to deal with the perturbation terms involving with

$(U,P)$ since $(U,P)$ remains large. We fix an arbitrary time $T$. For given $(v,\sigma) \in C([0,T);H^3(T_2)) \times C([0,T);H^3(T_2))$, we suppose $(w,\eta)$ is the solution to

$$\begin{aligned} w_t + &(v+U)\cdot\nabla w - \mu\Delta w - (\mu+\lambda)\nabla \mathrm{div} w + \nabla\eta = -v\cdot\nabla U \\ &+(1-(1+\varepsilon^2(\sigma+P))^{\gamma-2})\nabla(\sigma+P) \\ &+((1+\varepsilon^2(\sigma+P))^{-1}-1)(\mu\Delta v + (\mu+\lambda)\nabla\mathrm{div} v + \mu\Delta U) \end{aligned} \tag{27}$$

$$\begin{aligned} \varepsilon^2\eta_t + &\varepsilon^2(v+U)\cdot\nabla\eta + \mathrm{div} w = -\varepsilon^2\sigma\mathrm{div} v - \varepsilon^2 P_t \\ &-\varepsilon^2(v+U)\cdot\nabla P - \varepsilon^2 P\mathrm{div} v. \end{aligned} \tag{28}$$

in $T_2 \times (0,T)$ satisfying the initial condition

$$w(x,0) = u_0(x) - U_0(x) \quad \text{and} \quad \eta(x,0) = \rho_0(x).$$

We will prove the global existence of solutions

$$(w,\eta) \in C([0,T);H^3(T_2)) \times C([0,T);H^3(T_2))$$

to (27) and (28) from Galerkin approximations. Since $U$ is divergence free, we do not need to impose any smallness condition on the norms of $U$. But, to obtain uniform bound in time of the $H^3$ norm of $(w,\eta)$, we need to assume that

$$\sup_{0<t<T} \| v \|_3 \leq c_0 \min(\mu, \mu+\lambda) \tag{29}$$

for some small $c_0$ depending only on $\lambda, \mu$. Since $\varepsilon$ is small, we can assume that $\sup_{0<t<T} \| v \|_3 + \sup_{0<t<T} \| \sigma \|_3$ is small compared to $\min(\mu, \mu+\lambda)$. Let

$$\begin{aligned} K = \{ &(v,\sigma) \in C([0,T);H^3(T_2)) \times C([0,T);H^3(T_2)) : \\ &(v(x,0),\sigma(x,0)) = (u_0(x) - U_0(x), \rho_0(x)), \\ &\sup_{0<t<T} \| v \|_3 + \varepsilon \| \sigma \|_3 \leq c_0 \min(\mu,\mu+\lambda)\}. \end{aligned}$$

We define the map $M_T; K \to C([0,T);H^3(T_2)) \times C([0,T);H^3(T_2))$ by

$$M_T(v,\sigma) = (w,\eta).$$

**Lemma 4.1** *We fix a small positive constant* $\varepsilon_0 \leq \frac{1}{2}\min(\mu, \lambda+\mu)$ *and suppose that*

$$\| u_0 - U_0 \|_3 + \varepsilon \| \rho_0 \|_3 \leq c_0\varepsilon_0$$

*for some small small constant* $c_0$. *Then, for each* $0 < \varepsilon < \varepsilon_0$, *there is* $T_0$ *independent of* $\varepsilon$ *(as* $\varepsilon$ *goes to zero) such that the map* $M_{T_0}$ *is a contraction map in* $K$. *Consequently there is a unique solution* $(v,\sigma) \in C([0,T_0);H^3(T_2)) \times C([0,T_0);H^3(T_2))$ *to (6) and (7) if* $0 < \varepsilon < \varepsilon_0$.

We only skecth the proof of Lemma 4.1 since we are following a similar argument in [4] for hyperbolic systems. In fact they applied the contraction mapping argument for compressible Navier-Stokes equations when there is no external force. Therefore, we need only to take care of the perturbation terms involving with $(U, P)$.
*Proof.* We let $\tilde{w} = (w^1, w^2, \varepsilon\eta)^t$ and $\tilde{v} = (v^1, v^2, \varepsilon\sigma)^t$. Then we can write (27) and (28) in matrix form as in (8) such that

$$\tilde{w}_t + (v+U)\cdot\nabla\tilde{w} - A_\varepsilon\tilde{w} = \tilde{F} + \varepsilon\tilde{G} \tag{30}$$

with the initial condition

$$\tilde{w}(x,0) = (u_0(x) - U_0(x), \varepsilon\rho_0(x))^t,$$

where $A_\varepsilon, \tilde{F} = \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3$ and $\tilde{G}$ are

$$A_\varepsilon = \begin{bmatrix} \mu\Delta + (\mu+\lambda)\frac{\partial^2}{\partial x_1^2} & (\mu+\lambda)\frac{\partial^2}{\partial x_1\partial x_2} & 0 \\ (\mu+\lambda)\frac{\partial^2}{\partial x_1\partial x_2} & \mu\Delta + (\mu+\lambda)\frac{\partial^2}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{\varepsilon}\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} J + ((1+\varepsilon^2(\sigma+P))^{-1} - 1)(\mu\Delta v + (\mu+\lambda)\nabla\mathrm{div}v \\ -\varepsilon\sigma\mathrm{div}\ v \end{bmatrix}$$

$$F_2 = \varepsilon\begin{bmatrix} 0 \\ -P_t - U\cdot\nabla P \end{bmatrix}, \qquad F_3 = \begin{bmatrix} -v\cdot\nabla U \\ 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{\mu}{\varepsilon}((1+\varepsilon^2(\sigma+P))^{-1} - 1)\Delta U \\ -v\cdot\nabla P - P\mathrm{div}\ v \end{bmatrix}.$$

Here $J$ denotes the function

$$J = (1 - (1+\varepsilon^2(\sigma+P)))^{\gamma-2})\nabla(\sigma+P).$$

We let $\{\phi_k(x) : k = 1, 2, \cdots\}$ be the smooth orthonormal $L^2$ basis and we approximate $\tilde{w}$ by

$$\tilde{w}^N(x,t) = \sum_{k=1}^{N} c_k^N(t)\phi_k(x),$$

with the initial condition

$$c_k^N(0) = \int_{T_2} \phi_k(x) \cdot \tilde{w}(x,0)dx.$$

As usual, $c_k^N$ are solutions to ordinary differential system

$$\frac{d}{dt}\int_{T_2} w^N(x,t)\phi_n(x)dx + \int_{T_2}(v+U)\cdot\nabla w^N(x,t)\phi_n(x)dx - \int_{T_2} A_\varepsilon w^N(x,t)\phi_n(x)dx$$

$$= \int_{T_2} \tilde{F}\phi_n(x)dx + \varepsilon\int_{T_2}\tilde{G}\phi_n(x)dx$$

for $n = 1, \cdots, N$. The local existence is proved by Picard method. Then multiplying $c_n^K$ and summing up, we obtain the energy estimate. In fact, since $U$ is divergence free, we get

$$\int_{T_2}(v+U)\cdot\nabla w^N(x,t)w^N(x,t)dx = -\frac{1}{2}\int_{T_2}\mathrm{div}v|w^N(x,t)|^2dx$$

and then the smallness condition (29) of $v$ implies $\sup_t \| w^N(\cdot,t) \|_3$ is uniformly bounded as long as $\tilde{F},\tilde{G} \in L^2(0,\infty : H^3)$. This implies the global existence of $C(0,\infty : H^3)$ solution to (30). From subsection 3.3 and 3.4 with $(v,\sigma)$ for $w$, we show that there exists $T_0$ independent of $\varepsilon$ such that $\tilde{w}$ satisfies

$$\sup_{0<t<T_0} \| \tilde{w} \|_3 \leq c_0 \min(\mu, \mu+\lambda).$$

This implies that $M_{T_0}(v,\sigma) \in K$ and there is a fixed point for $M_{T_0}$ in $C(0,\infty : H^3)$. □

Now we are ready to prove our main theorem. Let $K_0 < 1$ be a small positive constant that will be determined later. If we suppose that the maximal time $T^*$ defined in section 3 is finite, that is, $T^* < \infty$, there is a time $T_1$ such that

$$\Phi^2(t) < K_0 \text{ if } t < T_1 \text{ and } \Phi^2(T_1) = K_0.$$

Let $\varepsilon_0 < 1$ be a small positive constant that will be determined later. Let $\varepsilon \leq \varepsilon_0$.

By Sobolev inequality, we note that

$$\| w^3 \|_\infty \leq \beta_1 \| w^3 \|_3 .$$

If we take $\varepsilon_0$ such that

$$\varepsilon_0^2 M_2 \leq \frac{1}{4}$$

and

$$\varepsilon_0\beta_1 K_0^{\frac{1}{2}} \le \frac{1}{4},$$

then there is a positive constant $c$ independent of $U, P, \varepsilon_0$, etc,such that

$$a(t), b_1(t), b_2(t), b_3(t) \le c.$$

Using the estimates (15), (18), (22), (25) and (26) in section 3 and applying Young's inequalities, the following Gronwall inequality is deriveded from (11) for the time $0 \le t < T_1$:

$$\begin{aligned}\frac{d}{dt}\Phi(t)^2 + c_0\Phi(t)^2 + c_1\Psi(t)^2 &\le \varepsilon c(\delta_0, M_1, M_2, M_3, M_4) \\ &+ c(\varepsilon + \Phi(t) + \varepsilon(M_1+M_2)^2 + (\| \nabla U \|_3^2 + \| P \|_3^2))\Phi(t)^2 \\ &+ c(\varepsilon + \Phi(t) + \varepsilon(M_1+M_2)^2)\Psi(t)^2,\end{aligned} \tag{31}$$

where $c(\delta_0, M_1, M_2, M_3, M_4)$ is a constant depending on $\delta_0, M_1, M_2, M_3, M_4$. Take $K_0$ and $\varepsilon_0$ so small that

$$c(\varepsilon_0 + K_0 + \varepsilon_0(M_1+M_2)^2) \le \frac{c_1}{2}$$

and

$$c(\varepsilon_0 + K_0 + \varepsilon_0(M_1+M_2)^2) \le \frac{c_0}{2}.$$

Then from (31) we obtain the following:

$$\begin{aligned}\frac{d}{dt}\Phi(t)^2 + \frac{c_0}{2}\Phi(t)^2 + \frac{c_1}{2}\Psi(t)^2 &\le \varepsilon c(\delta_0, M_1, M_2, M_3, M_4) \\ &+ c(\| \nabla U \|_3^2 + \| P \|_3^2)\Phi(t)^2\end{aligned} \tag{32}$$

for all $0 < t < T_1$. We know that $\| \nabla U \|_3^2 + \| P \|_3^2\ (t)$ is integrable as a function of $t$ and

$$\int_0^\infty \| \nabla U \|_3^2 + \| P \|_3^2\ (t)dt \le M_1 + M_3.$$

Hence, if we solve the inequality (32), then we get the following inequality:

$$\begin{aligned}&\Phi(t)^2 + \int_0^t \Psi(s)^2 ds \\ &\qquad \le \varepsilon c(\delta_0, M_1, M_2, M_3, M_4)\int_0^t \exp(-\frac{c_0}{2}(t-\tau))C_0(t,\tau)d\tau \\ &+\Phi(0)^2\exp(-\frac{c_0}{2}t)C_0(t,0)\end{aligned} \tag{33}$$

for all $t \in [0, T_1]$, where $C_0(t,\tau) = exp(c\int_\tau^t \parallel \nabla U \parallel_3^2 + \parallel P \parallel_3^2 ds)$. Thus if we take $\varepsilon_0$ so small that

$$\varepsilon c(\delta_0, M_1, M_2, M_3, M_4) \exp(c(M_1 + M_3)) \leq \frac{K_0}{8},$$

and

$$\Phi(0)^2 \exp(c(M_1 + M_3)) \leq c_0^2 \varepsilon_0^2 \leq \frac{K_0}{8}$$

then we get the inequality

$$\Phi(t)^2 + \int_0^t \Psi(s)^2 ds \leq \frac{K_0}{2}, \quad \text{for all} \quad 0 \leq t \leq T_1.$$

This implies that $\Phi(T_1)^2 < K_0$, and leads to the contradiction to our assumption. Thus the solution must exist globally if the Mach number $\varepsilon$ is sufficiently small.

**Acknowledgement.** The authors were supported by GARC, KOSEF and BSRI-prg at POSTECH.

# References

[1] D. G. Ebin, *The motion of slightly compressible fluids viewed as motion with a strong constraining force,* Annals of Math., Vol. 150, 141-200(1977).

[2] T. Hagstrom and J. Lorenz, *All-time existence of classical solutions for slightly compressible flow,* SIAM J. Math. Anal., Vol. 29, 652-672(1998).

[3] P. L. Lions and N. Masmoudi, *Incompressible limit for a viscous compressible fluid,* J. Math. Pures Appl., Vol 77, 585-627(1998).

[4] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids,* Comm. Pure Appl. Math., Vol. 34, 481-524(1981).

[5] S. Klainerman and A. Majda, *Compressible and incompressible fluids,* Comm. Pure Appl. Math., Vol. 35, 629-652(1982).

[6] H. Beiraó da Veiga, *Singular limits in compressible fluid dynamics,* Arch. Rational Mech. Anal., Vol. 128, 313-327(1994).