A Stefan problem for a protocell model with symmetry-breaking bifurcations of analytic solutions

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Abstract

A simple model of a living cell which undergoes processes of growth and dissolution is described as a free boundary problem for a system of two reaction-diffusion equations; the condition on the free boundary is of the Stefan type. The special case of radially symmetric cells was studied in earlier work. The present paper is concerned with the existence of symmetry-breaking stationary solutions, i.e., with solutions which are not radially symmetric. It is proved, in the 2-dimensional case, that there exist branches of non-radial stationary solutions bifurcating from radially symmetric solutions; indeed, for any mode l, $l \geqslant 2$, there exists a unique bifurcation branch whose free boundary has the form $r = R_l + \varepsilon \cos l\theta + \sum_{n\geqslant 2} \varepsilon^n \lambda_n(\theta)$, $|\varepsilon|$ small, with $\lambda_n(\theta)$ orthogonal to $\cos l\theta$.

1 The model

We denote a variable point in $\mathbb{R}^2$ by $x = (x_1, x_2)$ or, in polar coordinates, by (r, θ). Consider the following free boundary problem: Find a 2-dimensional bounded domain Ω and functions μ and u defined, respectively, in $\mathbb{R}^2$ and Ω, such that

$$\Delta\mu = \chi_\Omega \mu \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

$$\mu = \log r(1 + o(1)) \quad \text{as } r = |x| \to \infty, \tag{1.2}$$

$$\mu \text{ is continuously differentiable across } \partial\Omega, \tag{1.3}$$

$$-\Delta u = \mu \quad \text{in } \Omega, \tag{1.4}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

$$\frac{\partial u}{\partial n} + \beta = 0 \quad \text{on } \partial\Omega \quad (\beta > 0); \tag{1.6}$$

here χ_Ω denotes the characteristic function of Ω; we shall sometimes use the notation

$$\mu^+ = \mu\Big|_{\mathbb{R}^2\setminus\Omega}, \quad \mu^- = \mu\Big|_{\overline{\Omega}}.$$

The above system is a 2-dimensional version of a stationary "protocell;" the concept of a protocell was introduced in [7] [8] and it attempts to capture some of the average physical and chemical properties of growth and dissolution of a living cell. The function μ represents

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the nutrient concentration in the entire space with a given source at ∞, and the function u is defined as $C - C^*$, where C is the concentration of the fluid-like building material of the cell, and C^* is the equilibrium concentration of the building material, so that u vanishes at the cell's boundary $\partial\Omega$. The cell feeds on the nutrients in accordance with (1.4). Dissolution of the cell, at the rate β across the boundary, represents waste removal through the cell. Actually the model studied in [7] [8] is somewhat different, and equally motivated, but the present version is mathematically a little simpler.

It is easy to check that for any $\beta > 0$ there exists a unique radial solution

$$\mu = \mu(r), u = u(r), \Omega = \{r < R\}$$

and, in fact, $R = 1/\beta$. The purpose of this paper is to prove that there exist also non-radial solutions. we shall prove that for any integer $l \geqslant 2$ there exists a unique radius $r = R_{0l}$ which is a bifurcation point of a family of analytic symmetry-breaking solutions with free boundary

$$r = R_{0l} + \varepsilon \cos l\theta + \sum_{n=2}^{\infty} \varepsilon^n \lambda_{ln}(\theta) \tag{1.7}$$

and

$$\beta = \beta_{l0} + \sum_{n=2}^{\infty} \varepsilon^n \beta_{ln} \qquad (\beta_{l0} = \frac{1}{R_{0l}}). \tag{1.8}$$

The corresponding solution u_l, μ_l^+, μ_l^- of (1.1)–(1.6) are also analytic in (r, θ, ε) in their respect regions, up to the boundary. Each bifurcation branch is uniquely determined under the following assumptions:

$$u \text{ and } \mu \text{ are even functions in } \theta, \tag{1.9}$$

$$\int_0^{2\pi} u(r, \theta, \varepsilon) \cos\theta d\theta \equiv 0, \qquad \int_0^{2\pi} \mu(r, \theta, \varepsilon) \cos\theta d\theta \equiv 0, \tag{1.10}$$

$$\int_0^{2\pi} \lambda_{ln}(\theta) \cos m\theta d\theta = 0 \qquad \text{for } n \geqslant 2 \text{ and } m = 1, l. \tag{1.11}$$

Note that the orthogonality of $\lambda_{ln}(\theta)$ to $\cos l\theta$ for $n \geqslant 2$ is achieved by a choice of the bifurcation parameter: If this condition is not satisfied for the expansions (1.7), (1.8), then by setting

$$\varepsilon' = \varepsilon + \sum_{n=2}^{\infty} \varepsilon^n \lambda_{ln}^0 \quad \text{where } \lambda_{ln}^0 = \int_0^{2\pi} \lambda_{ln}(\theta) \cos\theta d\theta,$$

we obtain a new family of solutions with

$$r = R_0 + \varepsilon' \cos l\theta + \sum_{n=2}^{\infty} (\varepsilon')^n \widehat{\lambda}_{ln}(\theta), \quad \beta = \beta_{l0} + \sum_{n=2}^{\infty} (\varepsilon')^n \widehat{\beta}_{ln}$$

and $\widehat{\mu}(r, \theta, \varepsilon') = \mu(r, \theta, \varepsilon)$, $\widehat{u}(r, \theta, \varepsilon') = u(r, \theta, \varepsilon)$ for which all the conditions in (1.9)–(1.11) are satisfied.

The general theory of bifurcation (see, for instance, [5] [6]) deals with problems of the form $F(\lambda, v) = 0$ where λ is a real parameter (like the parameter β in our case) and v varies in a fixed space (finite or infinite dimensional). The bifurcation problem considered in this paper does not fall within this theory because there is no such fixed space; indeed, the space in which the solution is sought is itself one of the unknowns of the problem, due to the free boundary nature of the system. On the other hand, the orthogonality condition (1.9)–(1.11) which ensure uniqueness are reminiscent of the orthogonality conditions imposed in the Lyapounov-Schmidt method for constructing bifurcation branches.

The method we shall use to establish the existence of analytic bifurcation branches is based on the recent approach by Friedman and Reitich [4] who established the existence of analytic bifurcation branches for a free boundary problem modeling tumor growth. However the system (1.1)–(1.6) presents new difficulties due to the following facts: (A) The present system of three PDEs (for μ^+, μ^-, u) is more complicated than the system of two PDEs studied in [4]; (B) The present set of free boundary conditions does not allow as strong a priori bounds as in [4] needed to estimate, by induction, the coefficients in the series expansion for the solution.

Because of (A) we shall need to analyze very carefully properties of zeros of quotients of Bessel functions; some of these results are actually of intrinsic interest. Because of both (A) and (B), the inductive process is far more delicate than in [4].

The structure of the paper is as follows: In §2 we write down the radial solution of (1.1)–(1.6).

In §3 we consider the linearized (stationary) problem about the radial solution of (1.1)–(1.6) with free boundary $r = R + \varepsilon \cos l\theta$ ($l \geqslant 2$) and prove that it has a non-trivial solution if and only if $R = R_{0l}$ where R_{0l} is the solution of

$$\frac{I_0(R_{0l})}{I_1(R_{0l})} \frac{I_l(R_{0l})}{I_{l-1}(R_{0l})} = \frac{l+1}{2l}, \tag{1.12}$$

and where $I_m(r)$ are the Bessel functions. Some known and some new facts about the Bessel functions are given in the Appendix A. In particular it is proved that for any $l \geqslant 2$ there exists a unique solution R_l of (1.12), and that $R_{0l} < R_{0m}$ if $2 \leqslant l < m$. One of the auxiliary but also intrinsically interesting results proved is that the function $I_1(x)/I_0(x)$ is concave.

In §4 we formally expand the solution of (1.1)–(1.6) into series

$$\mu = \mu_0(r) + \sum_{n\geqslant 1} \varepsilon^n \mu_n(r, \theta), \tag{1.13}$$

$$u = u_0(r) + \sum_{n\geqslant 1} \varepsilon^n u_n(r, \theta), \tag{1.14}$$

with the free boundary and β as in (1.7), (1.8), where $(\mu_0(r), u_0(r), \{r < R_{0l}\})$ is a radial solution of (1.1)–(1.6). We prove that, subject to the complimentary conditions (1.9)–(1.11), the $\mu_n, u_n, \lambda_{ln}, \beta_{l,n-1}$ are uniquely determined by induction. Here again we require some results on the Bessel functions that are proved in Appendix A. To prove convergence of the formal series, however, we use another approach (As shown in [4], the standard majorization approach one uses to establish convergence of the formal series does not work even for much simpler problems).

In §5 we transform the free boundary bifurcation problem to one with fixed boundary by the change of variables

$$r' = \frac{r}{R_{0l} + \varepsilon \cos l\theta + \sum_{n\geqslant 2} \varepsilon^n \lambda_{ln}(\theta)}.$$

For the new system we write down formal series

$$\mu = \widetilde{\mu}_0(r') + \sum_{n\geqslant 1} \varepsilon^n \widetilde{\mu}_n(r', \theta),$$

$$u = u_0(r') + \sum_{n\geqslant 1} \varepsilon^n \widetilde{u}_n(r',\theta),$$

with the free boundary and β as in (1.7), (1.8). Here again the coefficients $\widetilde{\mu}_n, \widetilde{u}_n, \lambda_{ln}, \beta_{l,n-1}$ are uniquely determined by induction. However, in contrast with the situation in §4, the nonlinear structure of the inductive formulas which define the $\widetilde{\mu}_n$, $\widetilde{u}_n$ and $\lambda_{l,n}$, $\beta_{l,n-1}$ is much simpler. This fact enables us to derive good enough estimates on $\widetilde{\mu}_n$, $\widetilde{u}_n$, λ_{ln}, $\beta_{l,n-1}$ for proving convergence. The proof of convergence is derived in §7. Several fundamental estimates needed for this proof are given in §6. The proof in §7 requires also some calculus type estimates on the derivatives of a composite function $\Phi(f_1(x), \cdots, f_k(x))$ in terms of derivatives of both $\Phi(u_1, \cdots, u_k)$ (in the u's) and the $f_j(x)$ (in x). These results are proved in Appendix B. In §8 we state the main theorem asserting the existence, uniqueness and analyticity (in (x,ε)) of the symmetry-breaking bifurcation branches of solutions of (1.1)–(1.6).

2 Radial solutions

We seek stationary solutions $(\mu(r), u(r), R)$ of the system

$$\Delta\mu = \chi_{\{r<R\}}\mu \quad \text{for } 0 < r < \infty, \tag{2.1}$$
$$\mu(r) = \log r + const. + O(1/r), \quad r \to \infty, \tag{2.2}$$
$$\mu \text{ is continuously differentiable across } r = R, \tag{2.3}$$
$$\Delta u = -\mu \quad \text{if } r < R, \tag{2.4}$$
$$u(R) = 0, \tag{2.5}$$
$$u'(R) + \beta = 0. \tag{2.6}$$

The general solution of this system is given by

$$\mu(r) = \begin{cases} \log r + \lambda I_0(R) - \log R, & r > R \\ \lambda I_0(r), & r \leqslant R \end{cases}$$

where λ is a constant. Since $\mu'(r)$ is continuous across $r = R$, $1/R = \lambda I_0'(R) = \lambda I_1(R)$, so that $\lambda = 1/(RI_1(R))$. By (2.4), $u(r) + \mu(r)$ is harmonic in $\{r < R\}$, so that $u + \mu = const.$, or

$$u_r = -\mu_r = -\lambda I_0'(r) = -\lambda I_1(r),$$

which implies that $u(r) = \lambda \int_r^R I_1(\xi)d\xi$ and $\beta = -u'(R) = 1/R$. We summarize:

Theorem 2.1 *For any $\beta > 0$ there exists a unique radially symmetric solution of (2.1)–(2.6), given by*

$$\mu(r) = \begin{cases} \log r + \dfrac{I_0(R)}{RI_1(R)} - \log R, & r > R \\ \dfrac{I_0(r)}{RI_1(R)}, & r \leqslant R, \end{cases} \tag{2.7}$$

$$u(r) = \frac{1}{RI_1(R)} \int_r^R I_1(\xi)d\xi = \frac{I_0(R) - I_0(r)}{RI_1(R)}, \tag{2.8}$$

$$R = \frac{1}{\beta}. \tag{2.9}$$

In the sequel we use the notation

$$[v]_{r=f(\theta)}$$

to denote the jump of a function $v(r,\theta)$ across $r = f(\theta)$, i.e.,

$$[v]_{r=f(\theta)} = v(f(\theta+0,\theta) - v(f(\theta)-0,\theta).$$

We also write

$$[v]_{r=R} = [v](R) = v(R+0) - v(R-0).$$

Remark 2.1. In the time-dependent version of the protocell model [3], the differential equation for u is

$$cu_t - \Delta u = \mu$$

where c is a positive number, and the Stefan free boundary condition is

$$V_n = -\frac{\partial u}{\partial n} - \beta.$$

By the method developed in [3] one can prove that for any initial data for u, there exists a unique radially symmetric solution $\mu(r,t), u(r,t), \{r < R(t)\}$ to this problem for all $t > 0$, and, if c is sufficiently small,

$$\left|R(t) - \frac{1}{\beta}\right| < Ce^{-\alpha t} \qquad \forall t > 0,$$

where C, α are some positive constants.

3 The linearized problem

We want to construct non-radially symmetric stationary solutions of (1.1)–(1.6). To do this we first need to find the bifurcation points $R = R_0$, i.e., the values $R = R_0$ for which the linearized system of (1.1)–(1.6) about the radially symmetric solution

$$(\mu_0(r), u_0(r), R_0)$$

has a non-trivial solution. The linearization can be made for any mode $l \geqslant 2$ and it corresponds to a perturbation of the free boundary of the form $r = R_0 + \varepsilon(a_1 \cos l\theta + a_2 \sin l\theta)$. By a translation $\theta \to \theta + \theta_0$ and a scaling of ε we may take, without loss of generality,

$$r = R_0 + \varepsilon\lambda_1(\theta), \qquad \lambda_1(\theta) = \cos l\theta. \tag{3.1}$$

Writing

$$\mu(r,\theta) = \mu_0(r) + \varepsilon\mu_1(r)\lambda_1(\theta),$$
$$u(r,\theta) = u_0(r) + \varepsilon u_1(r)\lambda_1(\theta),$$

we easily derive a system of equations and boundary conditions for μ_1, u_1:

$$\Delta\mu_1 - \frac{l^2}{r^2}\mu_1 = \chi_{\{r<R_0\}}\mu_1, \qquad 0 < r < \infty, \tag{3.2}$$

$$\mu_1 = O(1) \quad \text{as } r \to \infty, \tag{3.3}$$

$$[\mu_1](R_0) = 0, \tag{3.4}$$

$$\left[\frac{\partial \mu_1}{\partial r}\right](R_0) + \left[\frac{\partial^2 \mu_0}{\partial r^2}\right](R_0) = 0 \tag{3.5}$$

and

$$-\Delta u_1 + \frac{l^2}{r^2} u_1 = \mu_1 \quad \text{if } r < R_0, \tag{3.6}$$

$$u_1 + \frac{\partial u_0}{\partial r}(R_0) = 0 \quad \text{if } r = R_0, \tag{3.7}$$

$$\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_0}{\partial r^2}(R_0) = 0 \quad \text{if } r = R_0; \tag{3.8}$$

recall that

$$\frac{\partial u_0}{\partial r}(R_0) = -\beta = -\frac{1}{R_0}.$$

From (3.2), (3.3) we get

$$\mu_1(r) = AI_l(r) \quad \text{if } r < R_0,$$
$$\mu_1(r) = Br^{-l} \quad \text{if } r > R_0$$

where A, B are constants. From (3.2), (3.6) we see that the function $T = \mu_1(r) + u_1(r)$ satisfies:

$$T_{rr} + \frac{1}{r} T_r - \frac{l^2}{r^2} T = 0 \quad \text{if } r < R_0,$$

and therefore $T = Cr^l$, where C is a constant. Thus

$$u_1(r) = Cr^l - AI_l(r).$$

The boundary conditions (3.4), (3.5), (3.7), (3.8) then become

$$BR_0^{-l} - AI_l(R_0) = 0, \tag{3.9}$$

$$-BlR_0^{-l-1} - AI_l'(R_0) = \frac{I_0(R_0)}{R_0 I_1(R_0)}, \tag{3.10}$$

$$-AI_l(R_0) + CR_0^l = \frac{1}{R_0}, \tag{3.11}$$

$$-AI_l'(R_0) + ClR_0^{l-1} = \frac{I_0(R_0)}{R_0 I_1(R_0)} - \frac{1}{R_0^2} \tag{3.12}$$

and it remains to find R_0 such that this system has a non-trivial solution (A, B, C).

Multiplying (3.9) by l/R_0 and adding to (3.10), we get

$$\Big[\frac{l}{R_0} I_l(R_0) + I_l'(R_0)\Big] A = -\frac{I_0(R_0)}{R_0 I_1(R_0)}$$

or, by (A.3),

$$A = -\frac{I_0(R_0)}{R_0 I_1(R_0) I_{l-1}(R_0)}. \tag{3.13}$$

Multiplying (3.11) by $-l/R_0$ and adding to (3.12), we find that

$$\Big[\frac{l}{R_0} I_l(R_0) - I_l'(R_0)\Big] A + \frac{l}{R_0^2} = \frac{I_0(R_0)}{R_0 I_1(R_0)} - \frac{1}{R_0^2}.$$

Hence, by (A.4),

$$A = \Big[\frac{l+1}{R_0^2} - \frac{I_0(R_0)}{R_0 I_1(R_0)}\Big]/I_{l+1}(R_0),$$

and the expression in brackets can also be written as

$$[(l+1)I_1(R_0) - R_0 I_0(R_0)]/[R_0^2 I_1(R_0)].$$

Comparing this value of A with the value of A in (3.13), we conclude that a non-trivial solution to (3.9)–(3.12) exists if and only if R_0 satisfies:

$$R_0 I_0(R_0) - (l+1)I_1(R_0) = R_0 I_0(R_0)\frac{I_{l+1}(R_0)}{I_{l-1}(R_0)},$$

or,

$$R_0 I_0(R_0)\Big[1 - \frac{I_{l+1}(R_0)}{I_{l-1}(R_0)}\Big] = (l+1)I_1(R_0).$$

Using (A.5) with $m = l$ results in the simpler condition

$$\frac{I_0(R_0)}{I_1(R_0)}\frac{I_l(R_0)}{I_{l-1}(R_0)} = \frac{l+1}{2l}. \tag{3.14}$$

We now apply Corollary A.4 to conclude:

Theorem 3.1 *The linearized system for mode l has a non-trivial solution if and only if R_0 is the unique solution $R_0 = R_{0l}$ of (3.14)*

From the previous formulas we find that the solution $\mu_1(r)$, $u_1(r)$ is given by

$$\mu_1(r) = \begin{cases} -\dfrac{l+1}{2l}R_0^{l-1}r^{-l} & \text{if } r > R_0 \\ -\dfrac{l+1}{2l}\dfrac{I_l(r)}{R_0 I_l(R_0)} & \text{if } r < R_0 \end{cases}, \tag{3.15}$$

$$u_1(r) = \frac{l-1}{2l}\frac{r^l}{R_0^{l+1}} + \frac{l+1}{2l}\frac{I_l(r)}{R_0 I_l(R_0)} \qquad \text{if } r < R_0. \tag{3.16}$$

Remark 3.1. For $l = 1$, (3.14) holds for all $0 < R_0 < \infty$. The reason is that the problem (1.1)–(1.6) is invariant under translation. Then, given any radial solution of (1.1)–(1.6), we can obtain other radial solutions centered at any point. By translating the center of the free boundary from $x = (0,0)$ to $x = (0,\varepsilon)$, the solution changes according to mode $\cos\theta$; more precisely, the new free boundary satisfies:

$$r = R_0 + \varepsilon\cos l\theta + O(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

Since the change experienced by the solution is then trivial, we will not consider the bifurcation associated to $l = 1$. On the other hand, if $l \geqslant 2$, the solutions with free boundary given by (1.7), as $\varepsilon \to 0$, are not radially symmetric with respect to any center.

Our goal is to prove that about each bifurcation point (μ_1, u_1, R_{0l}), there is a bifurcation branch of (symmetry-breaking) analytic solutions with free boundary given by (1.7) and $\beta = \beta(\varepsilon)$ given by (1.8).

4 Formal solution

We seek to find symmetry-breaking solutions initiating at R_{0l} with free boundary

$$r = R_0 + f(\theta) \equiv \widetilde{f}(\theta) \quad \text{where } R_0 = R_{0l},\ l \geqslant 2. \tag{4.1}$$

Note that the boundary condition $\partial u/\partial n + \beta = 0$ can be written in the form

$$\widetilde{f}(\theta)\frac{\partial u}{\partial r} - \frac{\widetilde{f}'(\theta)}{\widetilde{f}(\theta)}\frac{\partial u}{\partial \theta} + \beta\sqrt{\widetilde{f}^2(\theta) + (\widetilde{f}'(\theta))^2} = 0. \tag{4.2}$$

Since $u(\widetilde{f}(\theta), \theta) = 0$, we have $u_r\widetilde{f}' + u_\theta = 0$ on $r = \widetilde{f}(\theta)$ and, consequently, the boundary condition (4.2) can also be written in the form

$$\frac{1}{\widetilde{f}(\theta)}[\widetilde{f}^2(\theta) + (\widetilde{f}'(\theta))^2]\frac{\partial u}{\partial r} + \beta\sqrt{\widetilde{f}^2(\theta) + (\widetilde{f}'(\theta))^2} = 0,$$

or

$$\frac{\partial u}{\partial r} + \beta\frac{R_0 + f}{\sqrt{(R_0 + f)^2 + f_\theta^2}} = 0. \tag{4.3}$$

In this section we describe a natural formal approach to computing the coefficients in the power series expansions of the bifurcation solutions given by (1.7), (1.8) and (1.13), (1.14). As was shown in [4], this approach, however, surprising enough, is not a good one for actually proving convergence, even for very simple elliptic problems with prescribed boundary that depends analytically on a parameter ε. Therefore, in the subsequent sections we shall use another scheme for proving convergence; this latter scheme looks more complicated but is nevertheless much easier to work with.

The reason we have included the formal approach of this section in our paper is that it enables us to reduce the compatibility condition which determines the β_n to a much simpler formula than does the scheme of the subsequent sections. This compatibility condition is based on Lemma 4.2 and is established in Theorem 4.3; it will be used to prove the assertion (7.1) of Lemma 7.1.

In determining (inductively) the coefficients λ_{lm}, μ_m, u_m in the expansions (1.7), (1.13), (1.14), only powers of $\cos l\theta$ to order $\leqslant m$ will occur. Hence these coefficients will be finite linear combinations of $\cos jl\theta$ with $j \leqslant m$. We can therefore write (1.7) and (1.13), (1.14) more explicitly in the form

$$r = R_0 + \sum_{m=1}^{\infty}\varepsilon^m\sum_{j=0}^{m}\tau_{mj}\cos jl\theta \equiv R_0 + f(\theta, \varepsilon), \tag{4.4}$$

$$\mu = \mu_0(r) + \sum_{m=1}^{\infty}\varepsilon^m\sum_{j=0}^{m}\mu_{mj}(r)\cos jl\theta, \tag{4.5}$$

$$u = u_0(r) + \sum_{m=1}^{\infty}\varepsilon^m\sum_{j=0}^{m}u_{mj}(r)\cos jl\theta. \tag{4.6}$$

From the differential equation (1.3) we get, for $m \geqslant 1$,

$$\Delta\mu_{mj} - \frac{j^2l^2}{r^2}\mu_{mj} = \chi_{\{r<R_0\}}\mu_{mj}, \tag{4.7}$$

so that

$$\mu_{mj}(r) = \begin{cases} A_{mj} I_{jl}(r) & \text{if } r < R_0, \\ B_{mj} r^{-jl} & \text{if } r > R_0, \end{cases} \tag{4.8}$$

where A_{mj}, B_{mj} are constants.

From the differential equations (1.1), (1.4) we find that $(\mu_{mj} + u_{mj}) \cos jl\theta$ is harmonic in $\{r < R_0\}$ so that

$$u_{mj}(r) = C_{mj} r^{jl} - A_{mj} I_{jl}(r) \quad \text{if } r < R_0, \tag{4.9}$$

where C_{mj} is a constant.

We now wish to determine the constants $A_{mj}, B_{mj}, C_{mj}, \tau_{mj}$ and β_{m-1} so that the boundary conditions (1.3), (1.5), (1.6) are satisfied.

Proceeding by induction we assume that these coefficients have already been determined for $m < n$, and we shall proceed to determine them for $m = n$.

Setting

$$f_n(\theta) = R_0 + \sum_{m=1}^{n} \varepsilon^m \sum_{j=0}^{m} \tau_{mj} \cos jl\theta, \tag{4.10}$$

the condition (1.3) becomes:

$$[\mu_0]_{r=f_n(\theta)} + \sum_{m=1}^{n} \varepsilon^m \sum_{j=0}^{m} \Big\{ B_{mj} r^{-jl} - A_{mj} I_{jl}(r) \Big\}_{r=f_n(\theta)} \cos jl\theta = 0, \tag{4.11}$$

$$\Big[\frac{\partial \mu_0}{\partial r}\Big]_{r=f_n(\theta)} + \sum_{m=1}^{n} \varepsilon^m \sum_{j=0}^{m} \Big\{ -(jl) B_{mj} r^{-jl-1} - A_{mj} I'_{jl}(r) \Big\}_{r=f_n(\theta)} \cos jl\theta = 0. \tag{4.12}$$

The boundary condition (1.5) gives

$$u_0(r)\Big|_{r=f_n(\theta)} + \sum_{m=1}^{n} \varepsilon^m \sum_{j=0}^{m} \Big\{ C_{mj} r^{jl} - A_{mj} I_{jl}(r) \Big\}_{r=f_n(\theta)} \cos jl\theta = 0. \tag{4.13}$$

Finally, the boundary condition (1.6), written in the form (4.3), becomes

$$\begin{aligned} \frac{\partial u_0(r)}{\partial r}\Big|_{r=f_n(\theta)} &+ \sum_{m=1}^{n} \varepsilon^m \sum_{j=0}^{m} \Big\{ (jl) C_{mj} r^{jl-1} - A_{mj} I'_{jl}(r) \Big\}_{r=f_n(\theta)} \cos jl\theta \\ &+ \Big(\beta_0 + \sum_{m=2}^{n} \varepsilon^m \beta_m \Big) \sqrt{\frac{1}{1 + \{f'_n(\theta)/f_n(\theta)\}^2}} = 0. \end{aligned} \tag{4.14}$$

Using the fact that $[\mu_0](R_0) = [\frac{\partial \mu_0}{\partial r}](R_0) = 0$, we find from (4.11), (4.12), by equating the coefficients of $\varepsilon^n \cos jl\theta$, that

$$B_{nj} R_0^{-jl} - A_{nj} I_{jl}(R_0) = F^1_{nj}, \tag{4.15}$$

$$-(jl) B_{nj} R_0^{-jl-1} - A_{nj} I'_{jl}(R_0) + \tau_{nj} \Big[\frac{\partial^2 \mu_0}{\partial r^2}\Big](R_0) = F^2_{nj}, \tag{4.16}$$

where F^1_{nj}, F^2_{nj} are determined by the inductive assumption, i.e., they are given in terms of the $A_{mk}, B_{mk}, C_{mk}, \tau_{mk}, \beta_{m-1}$ for $m \leqslant n-1$; the parameter β_{n-1} however has not yet been determined.

Similarly, from (4.13) we obtain, using the relations $u_0(R_0) = 0, \partial u_0/\partial r(R_0) = -\beta_0 = -1/R_0$,

$$-A_{nj}I_{jl}(R_0) + C_{nj}R_0^{jl} - \frac{1}{R_0}\tau_{nj} = F_{nj}^3, \tag{4.17}$$

where F_{nj}^3 is determined by the inductive assumption, i.e., it is given in terms of A_{mk}, B_{mk}, C_{mk}, τ_{mk} and β_{m-1} for $m \leqslant n-1$.

Finally, from (4.14) we get

$$-A_{nj}I_{jl}(R_0) + (jl)C_{nj}R_0^{jl-1} + \tau_{nj}\frac{\partial^2 u_0(R_0)}{\partial r^2} = \widetilde{F}_{nj}^4 - \beta_n\delta_{j0} \equiv F_{nj}^4, \tag{4.18}$$

where $\widetilde{F}_{nj}^4$ is given by the inductive assumption as the preceding F_{nj}^i. The coefficients matrix of the linear system (4.15)–(4.18), for fixed j, is

$$T_{jl} = \begin{pmatrix} -I_{jl}(R_0) & R_0^{-jl} & 0 & 0 \\ -I'_{jl}(R_0) & (-jl)R_0^{-jl-1} & 0 & \left[\frac{\partial^2 \mu_0}{\partial r^2}\right](R_0) \\ -I_{jl}(R_0) & 0 & R_0^{jl} & -\frac{1}{R_0} \\ -I'_{jl}(R_0) & 0 & (jl)R_0^{jl-1} & \frac{\partial^2 u_0(R_0)}{\partial r^2} \end{pmatrix}.$$

Adding jl/R_0 times the first row in T_{jl} to the second row we find, after using (A.3), that

$$\det T_{jl} = R_0^{-jl}\det\begin{pmatrix} -I_{jl-1}(R_0) & 0 & -\alpha \\ -I_{jl}(R_0) & R_0^{jl} & -\frac{1}{R_0} \\ -I'_{jl}(R_0) & (jl)R_0^{jl-1} & \gamma \end{pmatrix},$$

where α and γ are defined as:

$$\alpha = -\left[\frac{\partial^2 \mu_0}{\partial r^2}\right](R_0) = \frac{I_0(R_0)}{R_0 I_1(R_0)}, \quad \gamma = \frac{\partial^2 u_0(R_0)}{\partial r^2} = \frac{1}{R_0^2} - \alpha. \tag{4.19}$$

Adding $(-jl)/R_0$ times the second row to the third row and using (A.4), we get

$$\det T_{jl} = \det\begin{pmatrix} -I_{jl-1}(R_0) & -\alpha \\ -I_{jl+1}(R_0) & \frac{jl}{R_0^2} + \gamma. \end{pmatrix}$$

Setting $m = jl$, we obtain

$$\begin{aligned} \det T_m &= -I_{m-1}(R_0)\Big(\frac{m}{R_0^2} + \gamma\Big) - \alpha I_{m+1}(R_0) \\ &= -I_{m-1}(R_0)\frac{m+1}{R_0^2} + [I_{m-1}(R_0) - I_{m+1}(R_0)]\alpha \\ &= -I_{m-1}(R_0)\frac{m+1}{R_0^2} + \frac{2m}{R_0}I_m(R_0)\frac{I_0(R_0)}{R_0 I_1(R_0)} \\ &= 2m\frac{I_{m-1}(R_0)}{R_0^2}\Big(\frac{I_0(R_0)}{I_1(R_0)}\frac{I_m(R_0)}{I_{m-1}(R_0)} - \frac{m+1}{2m}\Big). \end{aligned}$$

Recalling Theorem A.5 we conclude:

Theorem 4.1 *There holds:*

$$\det T_{jl} \neq 0 \quad \text{if } j \neq 1; \qquad \det T_l = 0.$$

The condition $\det T_l = 0$ is just (3.14), the solvability condition for the linearized problem. The condition $\det T_{jl} \neq 0$ for $j \neq 1$ implies the solvability of the equations for the coefficients in (4.4)–(4.6) for any $j \neq 1$. Thus it remains to solve the system (4.15)–(4.18) in case $j = 1$. Here we shall need to use the parameter β_{n-1} to ensure solvability.

Remark 4.1. For $n = 2$ model l terms do not appear in the F^0_{nj} and thus the system (4.15)–(4.18) has a unique solution. However, in this special case, we necessarily have that $\beta_1 = 0$. Indeed, if $n = 1$ then $F^1_{nj} = 0, F^2_{nj} = 0, F^3_{nj} = 0$ and $\widetilde{F}^4_{nj} = 0$, and since the linearized solution is of mode l, also $A_{1j} = B_{1j} = C_{1j} = \tau_{1j} = 0$ if $j = 0$. Equation (4.18) then implies that $\beta_1 = 0$. This fact can also be proved in another way. Denote by Ω_ε the domain bounded by $r = R_0 + \varepsilon \cos l\theta + O(\varepsilon^2)$. Then

$$\beta \int_{\partial\Omega_\varepsilon} 1 = -\int_{\partial\Omega_\varepsilon} \frac{\partial u}{\partial n} = -\int_{\Omega_\varepsilon} \Delta u = \int_{\Omega_\varepsilon} \mu$$

where $\mu = \mu_0(r) + \varepsilon\mu_1(r)\cos l\theta + O(\varepsilon^2)$. Since

$$\int_{\partial\Omega_\varepsilon} 1 = 2\pi R_0 + O(\varepsilon^2), \qquad \int_{\Omega_\varepsilon} \mu(r) = \int_{\Omega_0} \mu_0(r) + O(\varepsilon^2),$$

writing $\beta = \beta_0 + \varepsilon\beta_1 + O(\varepsilon^2)$ we get

$$(\beta_0 + \varepsilon\beta_1 + O(\varepsilon^2))(2\pi R_0 + O(\varepsilon^2)) = \int_{\Omega_0} \mu_0(r) + O(\varepsilon^2)$$

which implies that $\beta_1 = 0$.

Introducing the vector notation

$$\mathbf{X}_{nj} = \begin{pmatrix} A_{nj} \\ B_{nj} \\ C_{nj} \\ \tau_{nj} \end{pmatrix}, \quad \mathbf{F}_{nj} = \begin{pmatrix} F^1_{nj} \\ F^2_{nj} \\ F^3_{nj} \\ F^4_{nj} \end{pmatrix},$$

we can write the system (4.15)–(4.18) for $j = 1$ in the form

$$T_l \mathbf{X}_{n1} = \mathbf{F}_{n1}. \tag{4.20}$$

This system is solvable if and only if the augmented matrix has the same rank as T_l, i.e., if and only if

$$\det \begin{pmatrix} F^1_{n1} & R_0^{-l} & 0 & 0 \\ F^2_{n1} & -lR_0^{-l-1} & 0 & -\alpha \\ F^3_{n1} & 0 & R_0^l & -\frac{1}{R_0} \\ F^4_{n1} & 0 & lR_0^l & \gamma \end{pmatrix} = 0.$$

Adding l/R_0 times the first row to the second row and $(-l)/R_0$ times the third row to the fourth row, the above condition reduces to

$$\det \begin{pmatrix} \frac{l}{R_0} F^1_{n1} + F^2_{n1} & -\alpha \\ -\frac{l}{R_0} F^3_{n1} + F^4_{n1} & \frac{l+1}{R_0^2} - \alpha \end{pmatrix} = 0$$

(here we used the definitions of α, γ in (4.19)). Thus we have:

Lemma 4.2 *The system (4.20) has a solution if and only if*

$$\Big(\frac{l+1}{R_0^2}-\alpha\Big)\Big(\frac{l}{R_0}F_{n1}^1+F_{n1}^2\Big)+\alpha\Big(-\frac{l}{R_0}F_{n1}^3+F_{n1}^4\Big)=0. \tag{4.21}$$

We want to prove that β_{n-1} can be uniquely determined so that (4.21) holds. We first have to examine how β_{n-1} enters into $X_{n-1,j}$. It clearly appears only in the last equation for $X_{n-1,0}$ (see (4.18) with n replaced by $n-1$):

$$T_0\mathbf{X}_{n-1,0}=\mathbf{F}_{n-1,0}-\beta_{n-1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

We can therefore write

$$\mathbf{X}_{n-1,0}=\widehat{\mathbf{X}}_{n-1,0}+\overline{\mathbf{X}}_{n-1,0} \tag{4.22}$$

where $\widehat{\mathbf{X}}_{n-1,0}$ does not depend on β_{n-1}, and, setting

$$\overline{\mathbf{X}}_{n-1,0}=\begin{pmatrix}\overline{A}_{n-1}\\\overline{B}_{n-1}\\\overline{C}_{n-1}\\\overline{\tau}_{n-1}\end{pmatrix}=\beta_{n-1}\overline{\mathbf{X}},\quad \overline{\mathbf{X}}=\begin{pmatrix}\overline{A}\\\overline{B}\\\overline{C}\\\overline{\tau}\end{pmatrix},$$

there holds:

$$T_0\overline{\mathbf{X}}=-\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

One can easily compute that

$$\overline{A}=R_0\frac{I_0(R_0)}{I_1^2(R_0)},\quad \overline{B}=R_0\frac{I_0^2(R_0)}{I_1^2(R_0)},\quad \overline{C}=R_0\Big(\frac{I_0^2(R_0)}{I_1^2(R_0)}-1\Big),\quad \overline{\tau}=-R_0^2. \tag{4.23}$$

We now return to (4.20) and write

$$\mathbf{F}_{n1}=\widehat{\mathbf{F}}_{n1}+\overline{\mathbf{F}}_{n1}\beta_{n-1}, \tag{4.24}$$

where $\widehat{\mathbf{F}}_{n1}$ and $\widehat{\mathbf{F}}_{n2}$ are independent of β_{n-1}. We need to compute $\overline{\mathbf{F}}_{n1}=(\overline{F}_1,\overline{F}_2,\overline{F}_3,\overline{F}_4)^T$.

To do that we write

$$r=f_n(\theta)=R_0+g(\theta)$$

so that

$$f_n(\theta)=R_0+\varepsilon\cos l\theta+\varepsilon^{n-1}\overline{\tau}_{n-1}+O(\varepsilon^2) \tag{4.25}$$

and

$$\begin{aligned}g&=\varepsilon\cos l\theta+\varepsilon^{n-1}\overline{\tau}_{n-1}+O(\varepsilon^2),\\(g)^2&=2\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1}+O(\varepsilon^2),\\(g)^k&=O(\varepsilon^2)\qquad(k\geqslant 3).\end{aligned} \tag{4.26}$$

Here $O(\varepsilon^2)$ does not depend explicitly on $\overline{\tau}_{n-1}$ and terms of order ε^{n+1} have been discarded; these conventions will be used also in the sequel.

We also have, for $r = f_n(\theta)$,

$$(f_n)^k = R_0^k + kR_0^{k-1}\varepsilon^{n-1}\overline{\tau}_{n-1} + kR_0^{k-1}\varepsilon\cos l\theta + O(\varepsilon^2) + O_{\overline{\tau}}(\varepsilon^n), \tag{4.27}$$
$$(f_n)^{-k} = R_0^{-k} - kR_0^{-k-1}\varepsilon^{n-1}\overline{\tau}_{n-1} - kR_0^{-k-1}\varepsilon\cos l\theta + O(\varepsilon^2) + O_{\overline{\tau}}(\varepsilon^n), \tag{4.28}$$

where $O_{\overline{\tau}}(\varepsilon^n)$ may depend on $\overline{\tau}_{n-1}$.

From (4.26) we obtain

$$I_m(r)\Big|_{r=f_n(\theta)} = I_m(R_0) + I'_m(R_0)\varepsilon^{n-1}\overline{\tau}_{n-1} + I'_m(R_0)\varepsilon\cos l\theta + O(\varepsilon^2) + O_{\overline{\tau}}(\varepsilon^n), \tag{4.29}$$
$$I'_m(r)\Big|_{r=f_n(\theta)} = I'_m(R_0) + I''_m(R_0)\varepsilon^{n-1}\overline{\tau}_{n-1} + I'_m(R_0)\varepsilon\cos l\theta + O(\varepsilon^2) + O_{\overline{\tau}}(\varepsilon^n). \tag{4.30}$$

We proceed to compute the first component $\overline{F}^1$ of $\overline{\mathbf{F}}_{n1}$. For this we need to identify in (4.11) the terms of the form $\lambda\varepsilon^n\cos l\theta$ where λ depends only on $\overline{A}_{n-1}$, $\overline{B}_{n-1}$, $\overline{C}_{n-1}$, $\overline{\tau}_{n-1}$; all other terms are irrelevant and will be collectively designated by "$\cdots$".

Using (4.26) we find that the first term on the left-hand side of (4.11) is of the form

$$[\mu_0]_{r=f_n(\theta)} = \frac{1}{2}g^2\Big[\frac{\partial^2\mu_0}{\partial r^2}\Big](R_0) + \cdots = \Big[\frac{\partial^2\mu_0}{\partial r^2}\Big](R_0)\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1} + \cdots \tag{4.31a}$$

Using (4.28) we see that from the series $\sum\varepsilon^m\sum B_{mj}r^{-jl}\cos jl\theta$ we can get $\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1}$ only if $m = 1$ and (necessarily) $j = 1$. We thus get

$$-\frac{l}{R_0^{l+1}}B_{11}\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1}. \tag{4.31b}$$

The coefficient $\overline{B}_{n-1}$ does not come with $\varepsilon^n\cos l\theta$. Similarly, using (4.29) we get from $-\sum\varepsilon^m\sum A_{mj}I_{jl}(r)\cos jl$ the term

$$-I'_l(R_0)A_{11}\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1}; \tag{4.31c}$$

we also observe that $\overline{A}_{n-1}$ appears when $m = n-1$, $j = 0$ in the form

$$-\varepsilon^n\cos l\theta\cdot\overline{A}_{n-1}I'_0(R_0). \tag{4.31d}$$

Combining (4.31a)–(4.31d) we conclude that

$$-\overline{F}^1 = \overline{\tau}_{n-1}\Big(-\alpha - \frac{l}{R_0^{l+1}}B_{11} - I'_l(R_0)A_{11}\Big) - \overline{A}_{n-1}I_1(R_0). \tag{4.32}$$

Next we consider (4.12). From the first term or the left-hand side we get

$$\Big[\frac{\partial^3\mu_0}{\partial r^3}\Big](R_0)\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1}. \tag{4.33a}$$

From the sum of the B_{mj} we can get $\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1}$ only if $m = 1$ and $j = 1$; the resulting term is

$$\frac{l+1}{R_0^{l+2}}lB_{11}\varepsilon^n\cos l\theta\cdot\overline{\tau}_{n-1}. \tag{4.33b}$$

From the sum of the A_{mj} we get $\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1}$ only if $m = 1$ and $j = 1$. Using (4.30) we find that this gives the term

$$-I_l''(R_0)A_{11}\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1}. \tag{4.33c}$$

Finally, $\overline{A}_{n-1}$ appears (when $m = n-1, j = 0$) in the form

$$-\varepsilon^n \cos l\theta \cdot \overline{A}_{n-1} \cdot I_1'(R_0). \tag{4.33d}$$

Combining (4.33a)–(4.33d) we see that

$$-\overline{F}^2 = \overline{\tau}_{n-1}\Big\{\Big[\frac{\partial^3 \mu_0}{\partial r^3}\Big](R_0) + \frac{l(l+1)}{R_0^{l+2}}B_{11} - I_l''(R_0)A_{11}\Big\} - \overline{A}_{n-1}I_1'(R_0). \tag{4.34}$$

Next, by the same analysis as for (4.11), from the first term in (4.13) we obtain

$$\frac{\partial^2 u_0}{\partial r^2}(R_0)\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1},$$

and from the remaining terms we obtain

$$C_{11} l R_0^{l-1} \varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1},$$
$$-I_l'(R_0)A_{11}\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1}$$

and

$$-\varepsilon^n \cos l\theta \cdot \overline{A}_{n-1} I_0'(R_0),$$

so that

$$-\overline{F}^3 = \overline{\tau}_{n-1}[\gamma + C_{11} l R_0^{l-1} - I_l'(R_0)A_{11}] - \overline{A}_{n-1}I_1(R_0). \tag{4.35}$$

Finally we consider (4.14). Computations as above show that we collect the terms

$$\frac{\partial^3 u_0}{\partial r^3}(R_0)\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1} \tag{4.36a}$$

from $\partial u_0/\partial r$ and

$$l(l-1)C_{11}R_0^{l-2}\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1}, \tag{4.36b}$$

$$-I_l''(R_0)A_{11}\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1}, \tag{4.36c}$$

$$-I_1'(R_0)\varepsilon^n \cos l\theta \cdot \overline{A}_{n-1} \tag{4.36d}$$

from the double sum in (4.14).

Considering the last term on the left-hand side of (4.14), we note that $f'(\theta)$ does not contain $\overline{\tau}_{n-1}$. Therefore $\{1+(f'(\theta)/f(\theta))^2\}^{-1/2}$ does not contain terms of the form const $.\varepsilon^n \cos l\theta \cdot \overline{\tau}_{n-1}$. It follows that

$$-\overline{F}^4 = \overline{\tau}_{n-1}\Big\{\frac{\partial^3 u_0}{\partial r^3}(R_0) + l(l-1)C_{11}R_0^{l-2} - I_l''(R_0)A_{11}\Big\} - \overline{A}_{n-1}I_1'(R_0). \tag{4.37}$$

We shall next compute more explicitly the coefficients $\overline{F}^j$, making use of the formulas (see (3.15), (3.16))

$$A_{11} = -\frac{l+1}{2l}\frac{1}{R_0 I_l(R_0)},$$
$$B_{11} = -\frac{l+1}{2l}R_0^{l-1},$$
$$C_{11} = \frac{l-1}{2l}\frac{1}{R_0^{l+1}}$$

which follow from §4 (after using also (3.14)).

Dropping the independent variable R_0 in the various Bessel functions, we find that the coefficient of $\overline{\tau}_{n-1}$ in $\overline{F}^1$ is equal to

$$\begin{aligned}
&\frac{I_0}{R_0 I_1} + I_l' A_{11} + \frac{l}{R_0^{l+1}} B_{11} \\
&= \frac{I_0}{R_0 I_1} - I_l' \frac{l+1}{2l}\frac{1}{R_0 I_l} - \frac{l}{R_0^{l+1}}\frac{l+1}{2l}R_0^{l-1} \\
&= \frac{I_0}{R_0 I_1} - \frac{l+1}{2l R_0 I_l}[I_l' + \frac{l}{R_0} I_l] \\
&= \frac{I_0}{R_0 I_1} - \frac{l+1}{2l R_0}\frac{I_{l-1}}{I_l} \qquad \text{(by (A.3))} \\
&= 0 \qquad \text{(by (3.14))}.
\end{aligned}$$

Thus, by (3.14)

$$\overline{F}^1 = I_1(R_0)\overline{A}_{n-1}. \tag{4.38}$$

The coefficient of $\overline{\tau}_{n-1}$ in $-\overline{F}_2$ is equal to

$$\begin{aligned}
&\left[\frac{\partial^3 \mu_0}{\partial r^3}\right](R_0) + \frac{l(l+1)}{R_0^{l+2}} B_{11} - A_{11} I_l'' \\
&= \frac{I_0}{R_0^2 I_1} - \frac{1}{R_0} - \frac{l(l+1)}{R_0^{l+2}}\frac{l+1}{2l}R_0^{l-1} + \frac{l+1}{2l}\frac{I_l''}{R_0 I_l} \qquad \text{(by direct computation)} \\
&= \frac{I_0}{R_0^2 I_1} - \frac{1}{R_0} - \frac{(l+1)^2}{2}\frac{1}{R_0^3} - \frac{l+1}{2l}\frac{I_l'}{R_0^2 I_l} + \frac{l+1}{2l R_0}\left(1 + \frac{l^2}{R_0^2}\right) \qquad \text{(by (A.1))} \\
&= \frac{I_0}{R_0^2 I_1} - \frac{l-1}{2l}\frac{1}{R_0} - \frac{l+1}{2R_0^2 l I_l}\left(I_l' + \frac{l}{R_0} I_l\right) \\
&= -\frac{l-1}{2l}\frac{1}{R_0} + \frac{I_0}{R_0^2 I_1} - \frac{l+1}{2R_0^2 l I_l} I_{l-1} \qquad \text{(by (A.3))} \\
&= -\frac{l-1}{2l}\frac{1}{R_0}, \qquad \text{(by (3.14))}.
\end{aligned}$$

Hence

$$\overline{F}^2 = \frac{l-1}{2l}\frac{1}{R_0}\overline{\tau}_{n-1} + I_1'(R_0)\overline{A}_{n-1}. \tag{4.39}$$

Next, the coefficient of $\overline{\tau}_{n-1}$ in $-\overline{F}^3$ is equal to

$$\left(\frac{1}{R_0^2} - \frac{I_0}{R_0 I_1}\right) + \frac{1}{R_0^{l+1}}\frac{l-1}{2l} l R_0^{l-1} + \frac{I_l'}{I_l}\frac{l+1}{2l R_0}$$

$$
\begin{aligned}
&= -\frac{I_0}{R_0 I_1} + \frac{l+1}{2lR_0 I_l}\Big(I_l' + \frac{l}{R_0} I_l\Big) \\
&= -\frac{I_0}{R_0 I_1} + \frac{l+1}{2lR_0 I_l} I_{l-1} \qquad \text{(by (A.3))} \\
&= 0 \qquad \text{(by (3.14))}.
\end{aligned}
$$

Hence

$$
\overline{F}^3 = I_1(R_0)\overline{A}_{n-1}. \tag{4.40}
$$

Finally the coefficient of $\overline{\tau}_{n-1}$ in $-\overline{F}^4$ is equal to

$$
\begin{aligned}
&\frac{\partial^3 u_0}{\partial r^3}(R_0) + l(l-1)C_{11}R_0^{l-2} - A_{11}I_l'' \\
&= \frac{\partial^3 u_0}{\partial r^3}(R_0) + l(l-1)\frac{l-1}{2l}\frac{1}{R_0^{l+1}}R_0^{l-2} + I_l''\frac{l+1}{2l}\frac{1}{R_0 I_l} \\
&= -\frac{2}{R_0^3} - \frac{1}{R_0} + \frac{I_0}{R_0^2 I_1} + \frac{(l-1)^2}{2}\frac{1}{R_0^3} + \frac{l+1}{2lR_0}\Big[\Big(1+\frac{l^2}{R_0^2}\Big) - \frac{I_l'}{R_0 I_l}\Big] \qquad \text{(by (A.1))} \\
&= \frac{1}{R_0^3}\Big(-2 + \frac{(l-1)^2}{2} + \frac{l(l+1)}{2}\Big) - \frac{l-1}{2lR_0} + \frac{I_0}{R_0^2 I_1} - \frac{l+1}{2lR_0^2}\Big(\frac{I_{l-1}}{I_l} - \frac{l}{R_0}\Big) \qquad \text{(by (A.3))} \\
&= \frac{1}{R_0^3}\Big(-2 + \frac{(l-1)^2}{2} + \frac{l(l+1)}{2} + \frac{l+1}{2}\Big) - \frac{l-1}{2lR_0} \qquad \text{(by (3.14))} \\
&= \frac{l^2-1}{R_0^3} - \frac{l-1}{2lR_0}.
\end{aligned}
$$

We conclude that

$$
\overline{F}^4 = -\Big[\frac{l^2-1}{R_0^3} - \frac{l-1}{2lR_0}\Big]\overline{\tau}_{n-1} + \overline{A}_{n-1}I_1'(R_0). \tag{4.41}
$$

Since, by (4.23),

$$
\overline{\tau}_{n-1} = -\frac{R_0 I_1^2(R_0)}{I_0(R_0)}\overline{A}_{n-1},
$$

we can rewrite (4.39) and (4.41) in the form

$$
\overline{F}^2 = \Big[-\frac{l-1}{2l}\cdot\frac{I_1^2(R_0)}{I_0(R_0)} + I_1'(R_0)\Big]\overline{A}_{n-1}, \tag{4.42}
$$

$$
\overline{F}^4 = \Big\{\Big[\frac{l^2-1}{R_0^3} - \frac{l-1}{2lR_0}\Big]\frac{R_0 I_1^2(R_0)}{I_0(R_0)} + I_1'(R_0)\Big\}\overline{A}_{n-1}. \tag{4.43}
$$

Substituting (4.38), (4.42), (4.40) and (4.43) into the solvability condition (4.21), we get

$$
\Lambda(R_0)\beta_{n-1} = \widetilde{F}_n \tag{4.44}
$$

where $\widetilde{F}_n$ depends only on $A_{mj}, B_{mj}, C_{mj}, \tau_{mj}, \beta_{m-1}$ for $m \leqslant n-1$, and

$$
\begin{aligned}
\Lambda(R) = {} & \Big(\frac{l+1}{R^2} - \frac{I_0}{RI_1}\Big)\Big(\frac{l}{R}I_1 - \frac{l-1}{2l}\frac{I_1^2}{I_0} + I_1'\Big) + \frac{I_0}{RI_1}\Big(-\frac{l}{R}I_1\Big) \\
& + \frac{I_0}{RI_1}\Big\{\Big[\frac{l^2-1}{R^3} - \frac{l-1}{2lR}\Big]\frac{RI_1^2(R)}{I_0(R)} + I_1'\Big\}
\end{aligned} \tag{4.45}
$$

where $I_0 = I_0(R), I_1 = I_1(R)$. Using the relation $I_1' = I_0 - I_1/R$, we easily obtain

$$\Lambda(R) \quad = \quad \frac{l-1}{R} I_0(R) \Big\{ \frac{2(l+1)}{R} \Big(\frac{I_1(R)}{I_0(R)} \Big) - \frac{l+1}{2l} \Big(\frac{I_1(R)}{I_0(R)} \Big)^2 - 1 \Big\}. \tag{4.46}$$

We shall prove

Theorem 4.3

$$\Lambda(R_0) \neq 0. \tag{4.47}$$

Consequently the equation (4.44) defines β_{n-1} uniquely in terms of A_{mj}, B_{mj}, C_{mj}, τ_{mj} and β_{m-1} for all $m \leqslant n-1$. Thus the asymptotic expansion can be developed for all n, and it is easy to see that it formally defines a solution to the free boundary problem up to any order of precision ε^n.

Proof of Theorem 4.3 Let $m \geqslant 2$ and denote by x_m the unique solution to

$$\frac{I_0(x_m)}{I_1(x_m)} \frac{I_m(x_m)}{I_{m-1}(x_m)} = \frac{m+1}{2m}. \tag{4.48}$$

The function

$$G(x) = \frac{I_1(x)}{I_0(x)}$$

satisfies

$$G' + G^2 + \frac{1}{x} G = 1. \tag{4.49}$$

The function

$$V(x) = \frac{I_m(x)}{I_{m-1}(x)}$$

satisfies (see (A.45))

$$V' + V^2 + \frac{2m-1}{x} V = 1, \tag{4.50}$$

and (4.48) can be rewritten as

$$V(x_m) = \frac{m+1}{2m} G(x_m). \tag{4.51}$$

Using (A.20) we have

$$\frac{V'(x)}{V(x)} - \frac{G'(x)}{G(x)} = \frac{G(x)}{V(x)} \frac{d}{dx} \Big(\frac{V(x)}{G(x)} \Big) > 0. \tag{4.52}$$

Substituting (4.49), (4.50), (4.51) into (4.52), we obtain

$$\begin{aligned} 0 \quad &< \quad \Big(\frac{1}{V} - V - \frac{2m-1}{x} \Big) - \Big(\frac{1}{G} - G - \frac{1}{x} \Big) \\ &= \quad \Big(\frac{2m}{m+1} \frac{1}{G} - \frac{m+1}{2m} G - \frac{2m-1}{x} \Big) - \Big(\frac{1}{G} - G - \frac{1}{x} \Big) \\ &= \quad \frac{m-1}{m+1} \frac{1}{G} + \frac{m-1}{2m} G - \frac{2(m-1)}{x} \\ &= \quad \frac{m-1}{m+1} \frac{1}{G} \Big[1 + \frac{m+1}{2m} G^2 - \frac{2(m+1)}{x} G \Big] \qquad \text{at } x = x_m \end{aligned}$$

If we take $m = l$, then $x_m = x_l = R_0$ and this implies that

$$\Lambda(R_0) < 0. \qquad \square$$

Remark 4.2. The mode l solution is uniquely determined up to a multiple of a special solution of the homogeneous system (4.15)–(4.18). It follows that by imposing the condition $\tau_{nl} = 0$ (i.e., $\int_0^{2\pi} \lambda_{ln}(\theta) \cos l\theta d\theta = 0$) we get a unique solution to (4.15)–(4.18). We have thus proved:

Theorem 4.4 *There exists a unique formal power series solution (4.4), (4.5), (4.6), (1.8) of (1.1)–(1.6) subject to the conditions (1.9)–(1.11).*

5 A change of variables

There is a serious difficulty in proving directly that the power series of the formal solution asserted in Theorem 4.4 is convergent (cf. [4]). We therefore proceed indirectly, by first transforming the free boundary problem into a problem in a fixed domain. We perform the change of variables

$$r' = r/(R_0 + f(\theta, \varepsilon)), \tag{5.1}$$

under which the original problem is reduced to a problem in a disc $\{r' < 1\}$. For simplicity, we shall still use r for the new variable (instead of r') and set

$$\Delta = \frac{1}{(R_0 + f(\theta, \varepsilon))^2} \mathcal{L}(D),$$

where

$$f(\theta, \varepsilon)) = \sum_{j=1}^{\infty} \varepsilon^j \lambda_j(\theta), \tag{5.2}$$

and

$$\begin{aligned} \mathcal{L}(D) = & \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{f_{\theta\theta}}{R_0 + f}\frac{1}{r}\frac{\partial}{\partial r} \\ & - \frac{2 f_\theta}{R_0 + f}\frac{1}{r}\frac{\partial^2}{\partial \theta \partial r} + \frac{2 f_\theta^2}{(R_0 + f)^2}\frac{1}{r}\frac{\partial}{\partial r} + \frac{f_\theta^2}{(R_0 + f)^2}\frac{\partial^2}{\partial r^2}. \end{aligned} \tag{5.3}$$

Then, in the new coordinates, the system becomes:

$$\mathcal{L}(D)\mu = \chi_{\{r<1\}}(R_0 + f)^2 \mu \quad \text{in } \{r < \infty\}, \tag{5.4}$$

$$\mathcal{L}(D)u = (R_0 + f)^2 u \quad \text{in } \{r < 1\}, \tag{5.5}$$

with the boundary conditions

$$u = 0 \quad \text{on } r = 1, \tag{5.6}$$

$$[\mu] = \left[\frac{\partial \mu}{\partial r}\right] = 0 \quad \text{on } r = 1, \tag{5.7}$$

$$\frac{\partial u}{\partial r} + \beta \frac{(R_0 + f)^2}{[(R_0 + f)^2 + f_\theta^2]^{1/2}} = 0 \quad \text{on } r = 1. \tag{5.8}$$

As before, seek for solutions of the form:

$$u = u_0(r) + \sum_{j=1}^{\infty} \varepsilon^j u_j(r,\theta), \tag{5.9}$$

$$\mu = \mu_0(r) + \sum_{j=1}^{\infty} \varepsilon^j \mu_j(r,\theta), \tag{5.10}$$

$$f(\theta,\varepsilon) = R_0 + \varepsilon\cos\theta + \sum_{j=2}^{\infty} \varepsilon^j \lambda_j(\theta), \tag{5.11}$$

$$\beta = \beta_0 + \sum_{j=1}^{\infty} \varepsilon^j \beta_j. \tag{5.12}$$

The zeroth order solution in the new variables is given by

$$\mu_0 = \begin{cases} \dfrac{I_0(R_0 r)}{R_0 I_1(R_0)} & \text{for } r < 1 \\ \log r + \dfrac{I_0(R_0)}{R_0 I_1(R_0)} & \text{for } r > 1 \end{cases}, \tag{5.13}$$

$$u_0(r) = \frac{1}{I_1(R_0)} \int_r^1 I_1(R_0\xi)d\xi \quad \text{for } r < 1, \tag{5.14}$$

$$\beta_0 = \frac{1}{R_0}. \tag{5.15}$$

Clearly

$$\frac{\partial \mu_0}{\partial r} = \begin{cases} \dfrac{I_1(R_0 r)}{I_1(R_0)} & \text{for } r < 1 \\ \dfrac{1}{r} & \text{for } r > 1, \end{cases} \tag{5.16}$$

$$\frac{\partial u_0}{\partial r} = -\frac{I_1(R_0 r)}{I_1(R_0)} \quad \text{for } r < 1. \tag{5.17}$$

Substituting (5.2), (5.9)–(5.12) into the system (5.4), (5.5), (5.8), we find that $(\mu_n, u_n, \lambda_n, \beta_n)$ satisfy the following system

$$\Delta\mu_n - \chi_{\{r<1\}}(R^2\mu_n + 2R\lambda_n\mu_0) - \frac{1}{R}\lambda_{n,\theta\theta}\frac{1}{r}\frac{\partial\mu_0}{\partial r} = F^{1n}(r,\theta) \quad \text{in } \mathbb{R}^2, \tag{5.18}$$

$$\Delta u_n + (R^2\mu_n + 2R\lambda_n\mu_0) - \frac{1}{R}\lambda_{n,\theta\theta}\frac{1}{r}\frac{\partial u_0}{\partial r} = F^{2n}(r,\theta) \quad \text{in } \{r<1\}, \tag{5.19}$$

where $R = R_0 = R_{0l}$, with the boundary conditions

$$[\mu_n] = \left[\frac{\partial \mu_n}{\partial r}\right] = 0 \quad \text{on } r = 1, \tag{5.20}$$

$$u_n = 0 \quad \text{on } r = 1, \tag{5.21}$$

$$\frac{\partial u_n}{\partial r} + \beta_n R + \frac{1}{R}\lambda_n = F^{3n}(\theta) \quad \text{on } r = 1. \tag{5.22}$$

The F^{jn} depend only on the $\mu_m, u_m, \lambda_m, \beta_m$ for $m < n$.

By reversing the mapping

$$r \to \frac{r}{R_0 + f(\theta, \varepsilon)}$$

and using the results of §4 we can deduce that the system (5.18)–(5.22) has a solution which is even in θ.

In the next section we prove general lemmas which will enable us (in §7) to derive bounds for the system (5.18)–(5.22) that ensure convergence of the power series (5.9)–(5.12).

6 Fundamental lemmas

We introduce the norm

$$\|N\|_{H_r^2(B_1)} = \left\| |N| + \frac{1}{r}\left|\frac{\partial N}{\partial r}\right| + \left|\frac{\partial^2 N}{\partial r^2}\right| + \frac{1}{r}\left|\frac{\partial^2 N}{\partial r \partial \theta}\right| + \frac{1}{r^2}\left|\frac{\partial^2 N}{\partial \theta^2}\right| \right\|_{L^2(B_1)}$$

where B_1 is the unit ball in $\mathbb{R}^2$. This rather unusual norm is related to the form of the operators in (5.8), (5.9), and will be very useful. It is easily seen that

$$\|N\|_{H^2(B_1)} \leqslant C\|N\|_{H_r^2(B_1)}. \tag{6.1}$$

As proved in [4], if $N = 0$ on ∂B_1 and

$$\int_0^{2\pi} N \cos\theta d\theta = \int_0^{2\pi} N \sin\theta d\theta = 0,$$

then also

$$\|N\|_{H_r^2(B_1)} \leqslant C\|N\|_{H^2(B_1)}; \tag{6.2}$$

the condition $N = 0$ on ∂B_1 may actually be dropped by applying the last inequality to $N - N_0$ where $\Delta N_0 = 0$ in B_1, $N_0 = N$ on ∂B_1, and then estimating $\|N_0\|_{H_r^2(B_1)}$ using its expansion into a series $\sum_{m\neq\pm1} a_m r^m e^{im\theta}$.

Consider the system for $M = M(r,\theta), U = U(r,\theta), \Lambda = \Lambda(\theta)$:

$$\Delta M - \chi_{\{r<1\}}(R^2 M + 2R\Lambda\mu_0) - \frac{1}{R}\Lambda_{\theta\theta}\frac{1}{r}\frac{\partial \mu_0}{\partial r} = F^1(r,\theta) \quad \text{in } \mathbb{R}^2, \tag{6.3}$$

$$\Delta U + (R^2 M + 2R\Lambda\mu_0) - \frac{1}{R}\Lambda_{\theta\theta}\frac{1}{r}\frac{\partial u_0}{\partial r} = F^2(r,\theta) \quad \text{in } \{r<1\}, \tag{6.4}$$

with the boundary conditions

$$[M] = \left[\frac{\partial M}{\partial r}\right] = 0 \quad \text{on } r = 1, \tag{6.5}$$

$$U = 0 \quad \text{on } r = 1, \tag{6.6}$$

$$\frac{\partial U}{\partial r} + \beta R + \frac{1}{R}\Lambda = F^3(\theta) \quad \text{on } r = 1; \tag{6.7}$$

here β is a given real number.

Lemma 6.1 *Set $R = R_0 = R_{0l}$ and let F^j be even functions of θ such that $F^1 \in L^2(\mathbb{R}^2), F^2 \in L^2(B_1), F^3 \in H^{1/2}(\partial B_1)$,*

$$\int_0^{2\pi} F^1(r,\theta)\cos m\theta d\theta \equiv 0, \quad \int_0^{2\pi} F^2(r,\theta)\cos m\theta d\theta \equiv 0,$$
$$\int_0^{2\pi} F^3(\theta)\cos m\theta d\theta = 0 \quad \textit{for } m = 1, l, \tag{6.8}$$

and, for $r > 1$,

$$\begin{gathered} F^1(r,\theta) = \frac{1}{r^2}F^{10}(\theta) + \frac{1}{r^2}F^{11}(r,\theta), \\ \int_0^{2\pi} F^{10}(\theta)d\theta = 0, \quad \|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)} < \infty. \end{gathered} \tag{6.9}$$

Set

$$|||F||| = (\|F^1\|_{L^2(B_1)} + \|F^{10}\|_{L^2(\partial B_1)} + \|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|F^2\|_{L^2(B_1)} + \|F^3\|_{H^{1/2}(\partial B_1)}).$$

Then there exists a unique solution (M, U, Λ) of (6.3)–(6.7), even in θ, such that

$$\begin{gathered} \int_0^{2\pi} M(r,\theta)\cos m\theta d\theta \equiv 0, \quad \int_0^{2\pi} U(r,\theta)\cos m\theta d\theta \equiv 0, \\ \int_0^{2\pi} \Lambda(\theta)\cos m\theta d\theta = 0 \quad \text{for } m = 1, l, \end{gathered} \tag{6.10}$$

and the following estimates hold

$$\begin{aligned} &\Big\|\Big(|M - M_0| + \Big|\frac{1}{r}\frac{\partial(M - M_0)}{\partial r}\Big| + \Big|\frac{1}{r}\frac{\partial M}{\partial \theta}\Big| + \Big|\frac{\partial^2(M - M_0)}{\partial r^2}\Big| + \Big|\frac{1}{r}\frac{\partial^2 M}{\partial r\partial\theta}\Big|\Big)\Big\|_{L^2(B_1)} \\ &+\|M + U - C_{20}\|_{H^2(B_1)} + \|\Lambda + \beta R^2\|_{H^{3/2}(\partial B_1)} + \Big|C_{20} - \beta R\Big(\frac{I_0^2(R)}{I_1^2(R)} - 1\Big)\Big| \leqslant C|||F|||, \end{aligned} \tag{6.11}$$

and

$$\Big\|r^2\Big(\Big|\frac{1}{r}\frac{\partial M}{\partial r}\Big| + \Big|\frac{\partial^2 M}{\partial r^2}\Big| + \Big|\frac{1}{r}\frac{\partial^2 M}{\partial r\partial\theta}\Big|\Big)\Big\|_{L^2(\mathbb{R}^2\setminus B_1)} \leqslant C|||F|||, \tag{6.12}$$

where C_{20} is a constant,

$$M_0(r) = M_0^{(1)}(r) + M_0^{(3)}(r) \quad \text{for } r < 1, \tag{6.13}$$

and

$$M_0^{(3)}(r) = -\frac{\Lambda_0}{R}\Big[\frac{I_0(Rr)}{I_0(R)} - \frac{rI_1(Rr)}{I_1(R}\Big], \tag{6.14}$$

$$M_0^{(1)}(r) = C_{20}\frac{I_0(Rr)}{I_0(R)}, \tag{6.15}$$

$$\Lambda_0 = \frac{1}{2\pi}\int_0^{2\pi}\Lambda(\theta)d\theta;$$

C is a constant independent of β.

We recall that the norms H^k for k which is not as integer can be defined either by the Fourier transform, or by interpolation [1].

It is worth pointing out that the first term on left-hand side of (6.11) is not the H_r^2 norm of $M - M_0$, as the pure second order θ derivative is missing. Note also that in the left-hand side of (6.12) there is a weight r^2 inside the L^2-norm; this is enabled by the factor $1/r^2$ in (6.9).

Remark 6.1. It may look rather surprising that $\Lambda + \beta R^2$ has one more derivative than F^3, since this is not apparent from the boundary condition (6.7). The reason for this gain of one derivative is roughly the following: Equation (6.3) suggests that the regularity of M is the same as the regularity of Λ. Since $M+U$ satisfies a "nice" Poisson equation, $\partial U/\partial r$ is "likely" to have the same regularity as $\partial M/\partial r$ at $r=1$, i.e., as Λ_θ. Thus (6.7) is in some sense a pseudo-differential operator of the first order for Λ, and thus should have one more derivative than F^3.

Proof. By the assumptions on the F^j we can write

$$\begin{aligned} F^j(r,\theta) &= \sum_{m\neq 1, m\neq l} F^j_m(r)\cos m\theta, \qquad j=1,2,\\ F^3(\theta) &= \sum_{m\neq 1, m\neq l} F^3_m \cos m\theta. \end{aligned}$$

Furthermore, for $r>1$,

$$F^1_m(r) = \frac{1}{r^2}(F^{10}_m + F^{11}_m(r)),$$

where

$$\begin{aligned} F^{10}_m &= \frac{1}{\pi}\int_0^{2\pi} F^{10}(\theta)\cos m\theta d\theta \quad \text{ if } m\geqslant 2,\\ F^{10}_0 &= 0,\\ F^{11}_m(r) &= \frac{1}{\pi}\int_0^{2\pi} F^{11}(r,\theta)\cos m\theta d\theta \quad \text{ if } m\geqslant 2,\\ F^{11}_0(r) &= \frac{1}{2\pi}\int_0^{2\pi} F^{11}(r,\theta) d\theta. \end{aligned} \tag{6.16}$$

In view of (6.10) the solution Λ, M, U must have the form

$$\Lambda = \sum_{m\neq 1, m\neq l} \Lambda_m \cos m\theta, \tag{6.17}$$

$$M = \sum_{m\neq 1, m\neq l} M_m(r)\cos m\theta, \tag{6.18}$$

$$U = \sum_{m\neq 1, m\neq l} U_m(r)\cos m\theta. \tag{6.19}$$

Then (M_m, U_m, Λ_m) satisfy

$$M_m'' + \frac{1}{r}M_m' - \frac{m^2}{r^2}M_m - \chi_{\{r<1\}}(R^2 M_m + 2R\Lambda_m\mu_0) + \frac{1}{R}m^2\Lambda_m\frac{1}{r}\frac{\partial \mu_0}{\partial r} = F^1_m \quad \text{ in } \mathbb{R}^2, \tag{6.20}$$

$$U_m'' + \frac{1}{r}U_m' - \frac{m^2}{r^2}U_m + (R^2 M_m + 2R\Lambda_m\mu_0) + \frac{1}{R}m^2\Lambda_m\frac{1}{r}\frac{\partial u_0}{\partial r} = F^2_m \quad \text{ for } r<1, \tag{6.21}$$

with the boundary conditions

$$[M_m] = \left[\frac{\partial M_m}{\partial r}\right] = 0 \quad \text{ on } r=1, \tag{6.22}$$

$$U_m = 0 \quad \text{ on } r=1, \tag{6.23}$$

$$\frac{\partial U_m}{\partial r} + \frac{1}{R}\Lambda_m + \beta R\delta_{m0} = F^3_m \quad \text{ on } r=1. \tag{6.24}$$

The general solution of (6.20) can be written in the form

$$M_m = M_m^{(1)} + M_m^{(2)} + M_m^{(3)}, \tag{6.25}$$

where $M_m^{(1)}$ corresponds to the general homogeneous solution, namely,

$$M_m^{(1)} = \begin{cases} C_{1m} I_m(Rr) & \text{for } r < 1 \\ C_{2m} r^{-m} & \text{for } r > 1; \end{cases} \tag{6.26}$$

$M_m^{(2)}$ corresponds to an inhomogeneous solution involving $F_m^1(r)$, and $M_m^{(3)}$ corresponds to an inhomogeneous solution involving Λ_m. Writing $M_m^{(2)} = g_m(r) I_m(Rr)$ for $r < 1$ and $M_m^{(2)} = g_m(r) r^{-m}$ for $r > 1$, we derive a first order ODE for $g_m'(r)$ which can be readily solved by integration. This leads to

$$M_m^{(2)} = \begin{cases} -I_m(Rr) \displaystyle\int_r^1 \frac{ds}{s I_m^2(Rs)} \int_0^s \tau I_m(R\tau) F_m^1(\tau) d\tau & \text{for } r < 1 \\ -\dfrac{1}{r^m} \displaystyle\int_1^r s^{2m-1} ds \int_s^\infty \frac{F_m^1(\tau)}{\tau^{m-1}} d\tau & \text{for } r > 1. \end{cases} \tag{6.27}$$

Finally, recalling (5.13) and (5.16), a solution $M_m^{(3)}$ is obtained by replacing, in (6.27), $F_m^1(r)$ by

$$\begin{cases} \dfrac{\Lambda_m}{I_1(R)} \Big\{ 2I_0(Rr) - \dfrac{m^2}{Rr} I_1(Rr) \Big\} & \text{for } r < 1, \\ -\dfrac{m^2}{R} \dfrac{1}{r^2} \Lambda_m & \text{for } r > 1. \end{cases}$$

This yields, for $r > 1$,

$$\begin{aligned} M_m^{(3)}(r) &= \frac{\Lambda_m}{r^m} \int_1^r s^{2m-1} ds \int_s^\infty \frac{m^2}{R\tau^{m+1}} d\tau = \frac{\Lambda_m}{R} \Big(1 - \frac{1}{r^m} \Big) \quad \text{for } m \geqslant 2, \\ M_0^{(3)}(r) &= 0. \end{aligned} \tag{6.28}$$

To compute $M_m^{(3)}(r)$ for $r < 1$, we use Theorem A.6:

$$\begin{aligned} M_m^{(3)}(r) &= \frac{\Lambda_m}{R} \frac{I_m(Rr)}{I_1(R)} \int_r^1 \frac{ds}{s I_m^2(Rs)} \int_0^{Rs} \frac{1}{R} I_m(\tau) \Big\{ m^2 I_1(\tau) - 2\tau I_0(\tau) \Big\} d\tau \\ &= \frac{\Lambda_m}{R} \frac{I_m(Rr)}{I_1(R)} \int_r^1 \frac{Rs}{I_m^2(Rs)} \Big\{ \frac{m-2}{Rs} I_1(Rs) I_m(Rs) \\ &\quad - \Big[I_2(Rs) I_m(Rs) - I_1(Rs) I_{m+1}(Rs) \Big] \Big\} ds, \end{aligned}$$

so that

$$M_m^{(3)}(r) = \frac{\Lambda_m}{R} \frac{I_m(Rr)}{I_1(R)} B_m(r) \quad \text{for } m \geqslant 2, \tag{6.29}$$

where in view of the relations $\frac{I_2(r)}{I_1(r)} = \frac{I_0(r)}{I_1(r)} - \frac{2}{r}$,

$$B_m(r) = \int_r^1 \frac{I_1(Rs)}{I_m(Rs)} \Big\{ m - (Rs) \Big[\frac{I_0(Rs)}{I_1(Rs)} - \frac{I_{m+1}(Rs)}{I_m(Rs)} \Big] \Big\} ds \quad \text{if } m \geqslant 2. \tag{6.30}$$

The same formula is valid for $m = 0$ with

$$\begin{aligned} B_0(r) &= -\int_r^1 \frac{ds}{s I_0^2(Rs)} \int_0^{Rs} \frac{2}{R} \tau I_0^2(\tau) d\tau \\ &= -\int_r^1 \frac{ds}{Rs I_0^2(Rs)} (\tau)^2 [I_0^2(\tau) - I_1^2(\tau)] \Big|_0^{Rs} \\ &= -\int_r^1 (Rs) \Big[1 - \frac{I_1^2(Rs)}{I_0^2(Rs)} \Big] ds \;=\; -s \frac{I_1(Rs)}{I_0(Rs)} \Big|_r^1, \end{aligned}$$

or,

$$B_0(r) = -\frac{I_1(R)}{I_0(R)} + r\frac{I_1(Rr)}{I_0(Rr)}; \tag{6.31}$$

the second and third equalities above can be verified by differentiating the right-hand sides and using the relations $(xI_1(x))' = xI_0(x)$, $I_0'(x) = I_1(x)$. In particular we conclude that (6.14) is valid.

We next add the equations (6.20) and (6.21) to get

$$(U_m + M_m)'' + \frac{1}{r}(U_m + M_m)' - \frac{m^2}{r^2}(U_m + M_m) = F_m^1 + F_m^2,$$

which leads to

$$U_m = -M_m + C_{3m}r^m - r^m \int_r^1 \frac{ds}{s^{2m+1}} \int_0^s \tau^{m+1}\Big[F_m^1(\tau) + F_m^2(\tau)\Big]d\tau. \tag{6.32}$$

We divide the rest of the proof of the lemma into six steps.

Step 1. Solving the system for $(C_{1m}, C_{2m}, C_{3m}, \Lambda_m/R)$.

Substituting the expressions for M_m, U_m from (6.25), (6.32) into the boundary conditions (6.22)–(6.24), we find after using the relation (see (A.5)) $R\frac{I_{m+1}(R)}{I_m(R)} = R\frac{I_{m-1}(R)}{I_m(R)} - 2m$,

$$C_{1m}I_m(R) = C_{2m}, \tag{6.33}$$

$$\begin{aligned} &C_{1m}RI_m'(R) + \frac{1}{I_m(R)}\int_0^1 \tau I_m(R\tau)F_m^1(\tau)d\tau + \frac{\Lambda_m}{R}\Big[m + R\Big(\frac{I_0(R)}{I_1(R)} - \frac{I_{m-1}(R)}{I_m(R)}\Big)\Big] \\ &\quad = -mC_{2m} + \frac{\Lambda_m}{R}m - \int_1^\infty \frac{F_m^1(\tau)}{\tau^{m-1}}d\tau, \end{aligned} \tag{6.34}$$

$$C_{2m} = C_{3m}, \tag{6.35}$$

$$\begin{aligned} &mC_{3m} + \int_0^1 \tau^{m+1}\Big[F_m^1(\tau) + F_m^2(\tau)\Big]d\tau + \frac{\Lambda_m}{R} + \delta_{m0}\beta R \\ &\quad = -mC_{2m} + \frac{\Lambda_m}{R}m - \int_1^\infty \frac{F_m^1(\tau)}{\tau^{m-1}}d\tau + F_m^3. \end{aligned} \tag{6.36}$$

We eliminate C_{1m} and C_{3m} using (6.33) and (6.35), obtaining a system for C_{2m} and Λ_m/R which after noticing that $R\frac{I_m'(R)}{I_m(R)} + m = \frac{RI_{m-1}(R)}{I_m(R)}$, takes the form:

$$\begin{aligned} &C_{2m}\frac{RI_{m-1}(R)}{I_m(R)} + \frac{\Lambda_m}{R}R\Big(\frac{I_0(R)}{I_1(R)} - \frac{I_{m-1}(R)}{I_m(R)}\Big) \\ &\quad = -\int_1^\infty \frac{F_m^1(\tau)}{\tau^{m-1}}d\tau - \frac{1}{I_m(R)}\int_0^1 \tau I_m(R\tau)F_m^1(\tau)d\tau, \end{aligned} \tag{6.37}$$

$$\begin{aligned} &C_{2m}2m + \frac{\Lambda_m}{R}(1-m) = F_m^3 - \delta_{m0}\beta R \\ &\quad - \int_0^1 \tau^{m+1}\Big[F_m^1(\tau) + F_m^2(\tau)\Big]d\tau - \int_1^\infty \frac{F_m^1(\tau)}{\tau^{m-1}}d\tau. \end{aligned} \tag{6.38}$$

The coefficients matrix for C_{2m} and Λ_m/R is

$$\begin{pmatrix} R\frac{I_{m-1}(R)}{I_m(R)} & R\Big(\frac{I_0(R)}{I_1(R)} - \frac{I_{m-1}(R)}{I_m(R)}\Big) \\ 2m & -m+1 \end{pmatrix},$$

and its determinant is

$$T_m = R\Big[(m+1)\frac{I_{m-1}(R)}{I_m(R)} - 2m\frac{I_0(R)}{I_1(R)}\Big].$$

Since $R = R_{0l}$, Theorem 2.5 implies that $T_m \neq 0$ if $m \neq 1, m \neq l$, and, consequently, $(C_{1m}, C_{2m}, C_{3m}, \Lambda_m/R)$ is uniquely determined. We conclude that the system (6.20)–(6.24) has a unique solution.

Remark 6.2. If $m = 0$ then the quantity

$$\frac{\Lambda_m}{R}\Big[m + R\Big(\frac{I_0(R)}{I_1(R)} - \frac{I_{m-1}(R)}{I_m(R)}\Big)\Big]$$

in (6.34) (which was the expression for $\partial M_m^{(3)}/\partial r$ at $r = R$) should be replaced by the expression $B_0'(1)I_0(R)/I_1(R)$, which is equal to

$$\frac{\Lambda_0}{R}R\Big(\frac{I_0(R)}{I_1(R)} - \frac{I_1(R)}{I_0(R)}\Big).$$

This implies that, in (6.37), if $m = 0$ then $I_{m-1}(R)/I_m(R)$ should be replaced by $I_1(R)/I_0(R)$.

Step 2. Estimating $\|\Lambda + \beta R\|_{H^{3/2}(\partial B_1)}$.

By (A.2), $I_m(R)/I_{m-1}(R) \leqslant R/2m \to 0$ as $m \to \infty$. Hence by (A.5),

$$\frac{I_{m-1}(R)}{I_m(R)} = \frac{I_{m+1}(R)}{I_m(R)} + \frac{2m}{R} \sim \frac{2m}{R} \quad \text{for large } m,$$

so that $T_m \sim 2m^2$ for m large. It is also clear from (A.4) that $RI_m'(R)/I_m(R) \sim m$, and from (A.9) that

$$\frac{I_m(R\tau)}{I_m(R)} \sim \tau^m \quad \text{uniformly in } \tau, \quad 0 < \tau < 1. \tag{6.39}$$

Using these relations in (6.37), (6.38), we find that for $m \geqslant 2$, $m \neq l$,

$$\begin{aligned} m\Big(\Big|\frac{\Lambda_m}{R}\Big| + |C_{2m}|\Big) &\leqslant C\Big\{\Big|\int_1^\infty \frac{F_m^1(\tau)}{\tau^m}\tau d\tau\Big| + \Big|\int_0^1 \frac{I_m(R\tau)}{I_m(R)}F_m^1(\tau)\tau d\tau\Big| \\ &\quad + |F_m^3| + \Big|\int_0^1 \tau^m(F_m^1(\tau) + F_m^2(\tau))\tau d\tau\Big|\Big\} \\ &\leqslant C\Big(\int_0^\infty [F_m^1(\tau)]^2\tau d\tau\Big)^{1/2}\Big(\int_0^1 \tau^{2m+1}d\tau + \int_1^\infty \frac{d\tau}{\tau^{2m-1}}\Big)^{1/2} \\ &\quad + C|F_m^3| + C\Big(\int_0^1 [F_m^2(\tau)]^2\tau d\tau\Big)^{1/2}\Big(\int_0^1 \tau^{2m+1}d\tau\Big)^{1/2} \\ &\leqslant \frac{C}{m^{1/2}}\Big(\|F_m^1\|_{L^2(\mathbb{R}^2)} + \|F_m^2\|_{L^2(B_1)}\Big) + C|F_m^3|. \end{aligned} \tag{6.40}$$

Consider now the case $m = 0$. By (6.16),

$$\Big|\int_1^\infty \tau F_0^1(\tau)d\tau\Big| = \Big|\frac{1}{2\pi}\int_0^{2\pi}\int_1^\infty \frac{F^{11}(\tau,\theta)}{\tau}d\tau d\theta\Big| \leqslant C\|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)}.$$

Recall also that $M_0^{(3)}$ is computed from (6.17), which means that in (6.37) we replace the quantity $I_{m-1}(R)/I_m(R)$ by $I_1(R)/I_0(R)$ (see Remark 6.1). Solving (6.38) for $\frac{\Lambda_0}{R} + \beta$ and then substituting this into (6.37) and observing that

$$C_{20}\frac{RI_1}{I_0} - \beta R^2\Big(\frac{I_0}{I_1} - \frac{I_1}{I_0}\Big) = \frac{RI_1}{I_0}\Big[C_{20} - \beta R\Big(\frac{I_0^2}{I_1^2} - 1\Big)\Big],$$

we get

$$\Big|\frac{\Lambda_0}{R}+\beta R\Big|+\Big|C_{20}-\beta R\Big(\frac{I_0^2(R)}{I_1^2(R)}-1\Big)\Big|\leqslant C\Big(\|F_0^1\|_{L^2(B_1)}+\|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)}+\|F_0^2\|_{L^2(B_1)}+|F_0^3|\Big). \tag{6.41}$$

The estimates (6.40) imply that

$$\sum_{m\neq 1} m^3|\Lambda_m|^2\leqslant C\Big\{\sum_{m\geqslant 2}\Big(\|F_m^1\|^2_{L^2(\mathbb{R}^2)}+\|F_m^2\|^2_{L^2(B_1)}\Big)+\sum_{m\geqslant 2} m|F_m^3|^2\Big\}, \tag{6.42}$$

and, together with (6.41), we conclude that

$$\begin{aligned}
&\|\Lambda-\Lambda_0\|_{H^{3/2}(\partial B_1)}\leqslant C\Big(\|F^1\|_{L^2(\mathbb{R}^2)}+\|F^2\|_{L^2(B_1)}+\|F^3\|_{H^{1/2}(\partial B_1)}\Big),\\
&\Big|\frac{\Lambda_0}{R}+\beta R\Big|+\Big|C_{20}-\beta R\Big(\frac{I_0^2(R)}{I_1^2(R)}-1\Big)\Big|\\
&\qquad\leqslant C\Big(\|F^1\|_{L^2(B_1)}+\|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)}+\|F^2\|_{L^2(B_1)}+\|F^3\|_{H^{1/2}(\partial B_1)}\Big).
\end{aligned} \tag{6.43}$$

Step 3. Estimating $M^{(1)}$.

For the region $r>1$, if $m\geqslant 2$,

$$\begin{aligned}
&m^2\int_1^\infty\Big[r\frac{\partial}{\partial r}M_m^{(1)}(r)\Big]^2 rdr+\int_1^\infty\Big[r^2\frac{\partial^2}{\partial r^2}M_m^{(1)}(r)\Big]^2 rdr\\
&\leqslant\ Cm^4|C_{2m}|^2\int_1^\infty r^{-2m}rdr\\
&\leqslant\ Cm|F_m^3|^2+C\Big(\|F_m^1\|^2_{L^2(\mathbb{R}^2)}+\|F_m^2\|^2_{L^2(B_1)}\Big),
\end{aligned} \tag{6.44}$$

which implies the bound

$$\begin{aligned}
&\|rM_r^{(1)}\|_{L^2(\mathbb{R}^2\setminus B_1)}+\|rM_{r\theta}^{(1)}\|_{L^2(\mathbb{R}^2\setminus B_1)}+\|r^2M_{rr}^{(1)}\|_{L^2(\mathbb{R}^2\setminus B_1)}\\
&\leqslant C\Big(\|F^1\|_{L^2(\mathbb{R}^2)}+\|F^2\|_{L^2(B_1)}\Big)+C\|F^3\|_{H^{1/2}(\partial B_1)}.
\end{aligned} \tag{6.45}$$

Next we consider a bound on $M^{(1)}$ for $r<1$. Since $C_{1m}=C_{2m}/I_m(R)$, we have, by (6.26) and (6.39),

$$\begin{aligned}
m^4\int_0^1\frac{1}{r^4}[M_m^{(1)}(r)]^2rdr\ &\leqslant\ m^4|C_{2m}|^2\int_0^1\frac{1}{r^4}\left(\frac{I_m(Rr)}{I_m(R)}\right)^2rdr\\
&\leqslant\ Cm^4|C_{2m}|^2\int_0^1 r^{2m-4}rdr\\
&\leqslant\ Cm|F_m^3|^2+C\Big(\|F_m^1\|^2_{L^2(\mathbb{R}^2)}+\|F_m^2\|^2_{L^2(B_1)}\Big),
\end{aligned}$$

if $m\geqslant 2$, where the last inequality follows by (6.40). Also, by (A.3)–(A.5) we easily check that

$$\begin{aligned}
\Big|\frac{\partial}{\partial r}I_m(r)\Big|&\leqslant Cm\frac{I_m(r)}{r},\\
\Big|\frac{\partial^2}{\partial r^2}I_m(r)\Big|&\leqslant Cm^2\frac{I_m(r)}{r^2},
\end{aligned}$$

so that

$$
\begin{aligned}
&m^2\int_0^1 \frac{1}{r^2}\Big[\frac{\partial}{\partial r}M_m^{(1)}(r)\Big]^2 rdr + \int_0^1 \Big[\frac{\partial^2}{\partial r^2}M_m^{(1)}(r)\Big]^2 rdr\\
&\leqslant \; Cm^4|C_{2m}|^2\int_0^1\left(\frac{I_m(Rr)}{I_m(R)}\right)^2 rdr\\
&\leqslant \; Cm|F_m^3|^2 + C\Big(\|F_m^1\|^2_{L^2(\mathbb{R}^2)} + \|F_m^2\|^2_{L^2(B_1)}\Big),
\end{aligned}
$$

if $m \geqslant 2$. Thus, altogether,

$$
\begin{aligned}
\|M^{(1)} - M_0^{(1)}(r)\|_{H_r^2(B_1)} \leqslant C\Big(&\|F^1\|_{L^2(B_1)} + \|F^{10}\|_{L^2(\partial B_1)} + \|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)}\\
&+\|F^2\|_{L^2(B_1)} + \|F^3\|_{H^{1/2}(\partial B_1)}\Big).
\end{aligned} \tag{6.46}
$$

Step 4. Estimating $M^{(2)}$.

For $r<1$, $M^{(2)} = \sum M_m^{(2)}(r)\cos m\theta$ satisfies

$$
\begin{aligned}
&\Delta M^{(2)} - R^2 M^{(2)} = F^1 \quad \text{in } B_1,\\
&M^{(2)} = 0 \quad \text{on } \partial B_1.
\end{aligned}
$$

By elliptic L^2 estimates, $\|M^{(2)}\|_{H^2(B_1)} \leqslant C\|F^1\|_{L^2(B_1)}$ so that, by (6.1)

$$
\|M^{(2)}\|_{H_r^2(B_1)} \leqslant C\|F^1\|_{L^2(B_1)}. \tag{6.47}
$$

To estimate $M^{(2)}(r,\theta)$ in $r>1$, we use (6.27) and (6.16). We can then write $M^{(2)} = M^{(20)} + M^{(21)}$,

$$
\begin{aligned}
M^{(20)}(r,\theta) &= \sum M_m^{(20)}(r)\cos m\theta,\\
M^{(21)}(r,\theta) &= \sum M_m^{(21)}(r)\cos m\theta,
\end{aligned}
$$

where

$$
\begin{aligned}
&M_m^{(20)}(r) = -\frac{1}{r^m}\int_1^r s^{2m-1}\int_s^\infty \frac{F^{10}}{\tau^{m+1}}d\tau = -\,\frac{F^{10}}{m^2}\Big(1-\frac{1}{r^m}\Big) \quad \text{if } m \geqslant 2,\\
&M_0^{(20)} = 0,\\
&M_m^{(21)}(r) = -\frac{1}{r^m}\int_1^r s^{2m-1}\int_s^\infty \frac{F_m^{11}(\tau)}{\tau^{m+1}}d\tau.
\end{aligned}
$$

Hence

$$
m^2\int_1^\infty \Big(r\frac{\partial}{\partial r}M_m^{(20)}(r)\Big)^2 rdr + \int_1^\infty \Big(r^2\frac{\partial^2}{\partial r^2}M_m^{(20)}(r)\Big)^2 rdr \leqslant \frac{C}{m}|F_m^{10}|^2 \leqslant C|F_m^{10}|^2,
$$

and this allows us to deduce that

$$
\|rM_r^{(20)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|rM_{r\theta}^{(20)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^2 M_{rr}^{(20)}\|_{L^2(\mathbb{R}^2\setminus B_1)} \leqslant C\|F^{10}\|_{L^2(\partial B_1)}. \tag{6.48}
$$

Next, by direct computation ($m=0$ is included in the computation)

$$
\begin{aligned}
\frac{\partial}{\partial r}M_m^{(21)}(r) &= \frac{m}{r^{m+1}}\int_1^r s^{2m-1}\int_s^\infty \frac{F_m^{11}(\tau)}{\tau^{m+1}}d\tau, -r^{m-1}\int_r^\infty \frac{F_m^{11}(\tau)}{\tau^{m+1}}d\tau\\
\frac{\partial^2}{\partial r^2}M_m^{(21)}(r) &= -\;\frac{m(m+1)}{r^{m+2}}\int_1^r s^{2m-1}\int_s^\infty \frac{F_m^{11}(\tau)}{\tau^{m+1}}d\tau + r^{m-2}\int_r^\infty \frac{F_m^{11}(\tau)}{\tau^{m+1}}d\tau + \frac{F_m^{11}(r)}{r^2}.
\end{aligned}
$$

Therefore, for all $m \geqslant 0$,

$$\begin{aligned}
&\Big|(m+1)r\frac{\partial}{\partial r}M_m^{(21)}(r)\Big| + \Big|r^2\frac{\partial^2}{\partial r^2}M_m^{(21)}(r)\Big| \\
\leqslant\ & C\Big\{\frac{m(m+1)}{r^m}\int_1^r s^{2m-1}\int_s^\infty \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}d\tau + (m+1)r^m\int_r^\infty \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}d\tau + |F_m^{11}(r)|\Big\}.
\end{aligned}$$

We proceed to estimate the terms on the right-hand side by the method used in [4, Lemma 8.1]. We can write

$$\begin{aligned}
&\frac{m(m+1)}{r^m}\int_1^r s^{2m-1}\int_s^\infty \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}d\tau \\
=\ & \frac{m(m+1)}{r^m}\int_1^\infty \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}\int_1^{\min(r,\tau)} s^{2m-1}ds\,d\tau \\
\leqslant\ & \frac{m(m+1)}{r^m}\Big\{\int_1^r \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}\frac{\tau^{2m}}{m}d\tau + \int_r^\infty \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}\frac{r^{2m}}{m}d\tau\Big\} \\
\leqslant\ & \frac{(m+1)}{r^m}\int_1^r |F_m^{11}(\tau)|\tau^{m-1}d\tau + (m+1)r^m\int_r^\infty \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}d\tau,
\end{aligned} \tag{6.49}$$

(notice that the left-hand side of the last inequality is 0 when $m=0$) so that, for all $m \geqslant 0$,

$$\begin{aligned}
&\Big|(m+1)r\frac{\partial}{\partial r}M_m^{(21)}(r)\Big| + \Big|r^2\frac{\partial^2}{\partial r^2}M_m^{(21)}(r)\Big| \\
\leqslant\ & C\Big\{\frac{m}{r^m}\int_1^r |F_m^{11}(\tau)|\tau^{m-1}d\tau + (m+1)r^m\int_r^\infty \frac{|F_m^{11}(\tau)|}{\tau^{m+1}}d\tau + |F_m^{11}(r)|\Big\} \\
\equiv\ & C(\mathcal{Q}_{1m} + \mathcal{Q}_{2m} + |F_m^{11}(r)|).
\end{aligned} \tag{6.50}$$

It remains to estimate L^2 norms of $\mathcal{Q}_{1m}$ and $\mathcal{Q}_{2m}$. Substituting $\tau = r\xi$ into the integrand of $\mathcal{Q}_{2m}$, we get

$$\begin{aligned}
&\Big[\int_1^\infty |\mathcal{Q}_{2m}(r)|^2 r dr\Big]^{1/2} \\
=\ & (m+1)\Big[\int_1^\infty r^{2m}\Big(\int_1^\infty \frac{F_m^{11}(r\xi)}{r^{m+1}\xi^{m+1}}r d\xi\Big)^2 r dr\Big]^{1/2} \\
=\ & (m+1)\Big[\int_1^\infty \Big(\int_1^\infty \frac{F_m^{11}(r\xi)}{\xi^{m+1}}d\xi\Big)^2 r dr\Big]^{1/2} \\
\leqslant\ & (m+1)\int_1^\infty \Big(\int_1^\infty |F_m^{11}(r\xi)|^2 r dr\Big)^{1/2}\frac{d\xi}{\xi^{m+1}} \qquad \text{(by Minkowski's inequality)} \\
=\ & (m+1)\int_1^\infty \Big(\int_\xi^\infty |F_m^{11}(\tau)|^2 \tau d\tau\Big)^{1/2}\frac{d\xi}{\xi^{m+2}} \quad (r = \tau/\xi) \\
\leqslant\ & \Big(\int_1^\infty |F_m^{11}(\tau)|^2\tau d\tau\Big)^{1/2}.
\end{aligned}$$

Similarly, for $m \geqslant 2$ (notice that $\mathcal{Q}_{10} = 0$),

$$\begin{aligned}
\Big[\int_1^\infty |\mathcal{Q}_{1m}(r)|^2 r dr\Big]^{1/2} &= m\Big[\int_1^\infty \frac{1}{r^{2m}}\Big(\int_{1/r}^1 F_m^{11}(r\xi)r^{m-1}\xi^{m-1}r d\xi\Big)^2 r dr\Big]^{1/2} \\
&= m\Big[\int_1^\infty \Big(\int_{1/r}^1 F_m^{11}(r\xi)\xi^{m-1}d\xi\Big)^2 r dr\Big]^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leqslant m\int_0^1 \Big(\int_1^\infty |F_m^{11}(r\xi)|^2\chi_{\{r\xi>1\}}rdr\Big)^{1/2}\xi^{m-1}d\xi \\
&\qquad\text{(by Minkowski's inequality)}\\
&= m\int_0^1 \Big(\int_1^\infty |F_m^{11}(\tau)|^2\tau d\tau\Big)^{1/2}\xi^{m-2}d\xi \\
&\leqslant \frac{m}{m-1}\Big(\int_1^\infty |F_m^{11}(\tau)|^2\tau d\tau\Big)^{1/2}.
\end{aligned}$$

Substituting these estimates into (6.50), we obtain, for all $m \geqslant 0$,

$$\int_1^\infty \Big[(m+1)r\frac{\partial}{\partial r}M_m^{(21)}(r)\Big]^2 rdr + \int_1^\infty \Big[r^2\frac{\partial^2}{\partial r^2}M_m^{(21)}(r)\Big]^2 rdr \leqslant C\int_1^\infty |F_m^{11}(\tau)|^2\tau d\tau. \quad (6.51)$$

This implies that

$$\|rM_r^{(21)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|rM_{r\theta}^{(21)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^2M_{rr}^{(21)}\|_{L^2(\mathbb{R}^2\setminus B_1)} \leqslant C\|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)}. \quad (6.52)$$

Combining this inequality with (6.48), we get

$$\begin{aligned}
&\|rM_r^{(2)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|rM_{r\theta}^{(2)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^2M_{rr}^{(2)}\|_{L^2(\mathbb{R}^2\setminus B_1)}\\
&\qquad\leqslant C\{\|F^{10}\|_{L^2(\partial B_1)} + \|F^{11}\|_{L^2(\mathbb{R}^2\setminus B_1)}\}.
\end{aligned} \quad (6.53)$$

Step 5. Estimating $M^{(3)}$.

Using (6.28) we see that, for $r > 1$, the term with $m = 0$ disappears, whereas if $m \geqslant 2$,

$$m^2\int_1^\infty \Big[r\frac{\partial}{\partial r}M_m^{(3)}(r)\Big]^2 rdr + \int_1^\infty \Big[r^2\frac{\partial^2}{\partial r^2}M_m^{(3)}(r)\Big]^2 rdr \leqslant Cm^3\Big|\frac{\Lambda_m}{R}\Big|^2.$$

Therefore, by (6.42),

$$\begin{aligned}
&\|rM_r^{(3)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|rM_{r\theta}^{(3)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^2M_{rr}^{(3)}\|_{L^2(\mathbb{R}^2\setminus B_1)}\\
&\qquad\leqslant C\Big(\|F^1\|_{L^2(\mathbb{R}^2)} + \|F^2\|_{L^2(B_1)} + \|F^3\|_{H^{1/2}(\partial B_1)}\Big).
\end{aligned} \quad (6.54)$$

For $r < 1$,

$$M_m^{(3)}(r) = \frac{\Lambda_m}{R}\,\frac{I_m(Rr)}{I_1(R)}B_m(r).$$

Each $M_m^{(3)}$ is a power series in r which does not contain a first order term. Consequently,

$$\Big(|M_m^{(3)}| + \Big|\frac{1}{r}\frac{\partial M_m^{(3)}}{\partial r}\Big| + \Big|\frac{\partial^2 M_m^{(3)}}{\partial r^2}\Big|\Big) \leqslant C\frac{|\Lambda_m|}{R} \quad \text{for } r < 1,$$

for each $m \geqslant 0$ (C depends on m). For m large, since $Rs\frac{I_{m+1}(Rs)}{I_m(Rs)} = O(\frac{1}{m})$ (by (A.9)), we have

$$|M_m^{(3)}| \;\leqslant\; Cm\frac{|\Lambda_m|}{R}\int_r^1 \Big(\frac{r}{s}\Big)^m sds = \frac{C|\Lambda_m|}{R}(r^2 - r^m) \leqslant \frac{C|\Lambda_m|}{R}r^2.$$

Next,

$$\begin{aligned}
\frac{\partial}{\partial r}M_m^{(3)} &= \frac{\Lambda_m}{RI_1(R)}\Big(RI_m'(Rr)B_m(r) + I_m(Rr)B_m'(r)\Big)\\
&= \frac{\Lambda_m}{RI_1(R)}\Big\{RI_m'(Rr)B_m(r) + I_1(Rr)\Big(-m + Rr\Big[\frac{I_0(Rr)}{I_1(Rr)} - \frac{I_{m+1}(Rr)}{I_m(Rr)}\Big]\Big)\Big\}.
\end{aligned}$$

Using the relations

$$\frac{I_m'(Rr)}{I_m(Rr)} = -\frac{m}{Rr} + \frac{I_{m+1}(Rr)}{I_m(Rr)} = -\frac{m}{Rr} + O(1), \quad B_m(r)I_m(Rr) = O(1),$$

which follow by(A.9), we find that

$$\begin{aligned}\frac{\partial}{\partial r}M_m^{(3)} &= \frac{\Lambda_m}{RI_1(R)}\Big(-\frac{m}{r}I_m(Rr)B_m(r) - mI_1(Rr) + O(1)\Big) \\ &= \frac{\Lambda_m}{RI_1(R)}\Big(\frac{m^2}{r}\int_r^1 \frac{r^m}{s^m}I_1(Rs)ds - mI_1(Rr) + O(1)\Big) \qquad \text{(by (6.30),}\end{aligned}$$

so that, by integration by parts,

$$\frac{\partial}{\partial r}M_m^{(3)} = \frac{\Lambda_m}{RI_1(R)}\Big(\frac{m^2}{m-1}\int_r^1 \frac{r^{m-1}}{s^{m-1}}RI_1'(Rs)ds - \frac{m^2}{m-1}I_1(R)r^{m-1} + \frac{m}{m-1}I_1(Rr) + O(1)\Big).$$

Thus

$$\Big|\frac{\partial}{\partial r}M_m^{(3)}\Big| \leqslant C\frac{|\Lambda_m|}{R}(mr^{m-1}+1).$$

Finally, by a similar (but longer) computation, we get the bound

$$\Big|\frac{\partial^2}{\partial r^2}M_m^{(3)}\Big| \leqslant C\frac{|\Lambda_m|}{R}(m^2r^{m-2}+1).$$

It follows that, uniformly in m,

$$\begin{aligned}&(m+1)^3\int_0^1 \Big[M_m^{(3)}(r)\Big]^2 rdr + (m+1)^2\int_0^1 \Big[\frac{1}{r}\frac{\partial}{\partial r}M_m^{(3)}(r)\Big]^2 rdr + \int_0^1 \Big[\frac{\partial^2}{\partial r^2}M_m^{(3)}(r)\Big]^2 rdr \\ &\leqslant C(m+1)^3\Big|\frac{\Lambda_m}{R}\Big|^2.\end{aligned}$$

Hence

$$\begin{aligned}\Big\|\Big(|M^{(3)} - M_0^{(3)}| + \Big|\frac{1}{r}\frac{\partial(M^{(3)} - M_0^{(3)})}{\partial r}\Big| + \Big|\frac{1}{r}\frac{\partial M^{(3)}}{\partial\theta}\Big| \\ + \Big|\frac{\partial^2(M^{(3)} - M_0^{(3)})}{\partial r^2}\Big| + \Big|\frac{1}{r}\frac{\partial^2 M^{(3)}}{\partial r\partial\theta}\Big|\Big)\Big\|_{L^2(B_1)} \leqslant C\|\Lambda - \Lambda_0\|_{H^{3/2}(\partial B_1)};\end{aligned} \tag{6.55}$$

the right-hand side was already estimated in (6.43).

Step 6. Estimating $M+U$.

By (6.32),

$$M_m + U_m = C_{3m}r^m - r^m\mathcal{P}_m,$$

where $r^m\mathcal{P}_m$ can be estimated by elliptic estimates and (6.2) in the same way as $M_m^{(2)}$ in $r<1$. Using also the fact that $C_{2m} = C_{3m}$ and the bound (6.40), we conclude that

$$\|M + U - C_{20}\|_{H^2(B_1)}$$

is bounded by the right-hand side of (6.11).

Collecting (6.41), (6.43), (6.46), (6.47) and (6.55), the proof of (6.11) is complete, and collecting (6.44), (6.45), (6.53) and (6.54), the proof of (6.12) also follows.

Uniqueness is a by-product of the proof. Actually, for uniqueness we need only to require that the solution is even in θ, that is satisfies (6.10), and that the left-hand sides of (6.11) and (6.12) are finite. □

In estimating the derivatives of μ_n, u_n, λ_n inductively from the system (5.18)–(5.22), we need to have good enough estimates on the terms which appear in the F^{jn} so that we can apply Lemma 6.1. But some of the terms in F^{1n}, F^{2n} involve $\partial_\theta^2\lambda_m$ for $m < n$, whereas we only have $H^{3/2}$ estimates on the λ_m's. This means that the assumption made in Lemma 6.1 that F^1, F^2 belong to L^2 is too restrictive for estimating solutions corresponding to the right-hand sides terms of F^1, F^2 which involve the $\partial_\theta^2\lambda_m$. Fortunately these terms have a very special structure, to which the next lemma can be applied.

Lemma 6.2 *Suppose that F^1, F^2 are even functions of θ such that (6.8) holds and*

$$\begin{aligned}
&F^1(r,\theta) = \frac{\partial}{\partial\theta}G^1(r,\theta) \quad \text{for } r<1,\\
&F^1(r,\theta) = \frac{\partial}{\partial\theta}G^1(r,\theta) + \frac{\partial}{\partial\theta}\Big(\frac{g(\theta)}{r^2}\Big) \quad \text{for } r>1,\\
&F^2(r,\theta) = \frac{\partial}{\partial\theta}G^2(r,\theta) \quad \text{for } r<1,\\
&F^3(\theta) \equiv 0,\\
&\beta = 0,
\end{aligned} \tag{6.56}$$

where G^1, G^2, g are odd functions in θ which do not contain modes 1 and l terms, with

$$\begin{aligned}
J_1(G) \equiv\ &\|G_1\|_{L^2(B_1)} + \|rG_r^1\|_{L^2(B_1)} + \|r^2G^1\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^3G_r^1\|_{L^2(\mathbb{R}^2\setminus B_1)}\\
&+\|G^2\|_{L^2(B_1)} + \|rG_r^2\|_{L^2(B_1)} < \infty,\\
J_2(G) \equiv\ &\|G_1(1-,\cdot)\|_{H^{1/2}(\partial B_1)} + \|G_1(1+,\cdot)\|_{H^{1/2}(\partial B_1)} + \|g\|_{H^{1/2}(\partial B_1)} < \infty.
\end{aligned} \tag{6.57}$$

Then there exists a unique solution (M,Λ,U) of (6.3)–(6.7) which is even in θ and satisfies (6.10) and the following inequalities:

$$\begin{aligned}
&\|\Lambda\|_{H^{3/2}(\partial B_1)} + \|M\|_{L^2(B_1)} + \Big\|\Big(\Big|\frac{1}{r}\frac{\partial M}{\partial r}\Big| + \Big|\frac{\partial^2 M}{\partial r^2}\Big| + \Big|\frac{1}{r}\frac{\partial^2 M}{\partial r\partial\theta}\Big|\Big)\Big\|_{L^2(B_1)}\\
&\qquad + \|M+U\|_{L^2(B_1)} + \Big\|\Big(\Big|\frac{1}{r}\frac{\partial(M+U)}{\partial r}\Big| + \Big|\frac{\partial^2(M+U)}{\partial r^2}\Big| + \Big|\frac{1}{r}\frac{\partial^2(M+U)}{\partial r\partial\theta}\Big|\Big)\Big\|_{L^2(B_1)}\\
&\leqslant C(J_1(G)+J_2(G)),
\end{aligned} \tag{6.58}$$

$$\begin{aligned}
&\Big\|r^2\Big(\Big|\frac{1}{r}\frac{\partial M}{\partial r}\Big| + \Big|\frac{\partial^2 M}{\partial r^2}\Big| + \Big|\frac{1}{r}\frac{\partial^2 M}{\partial r\partial\theta}\Big|\Big)\Big\|_{L^2(\mathbb{R}^2\setminus B_1)}\\
&\leqslant C(J_1(G)+J_2(G)).
\end{aligned} \tag{6.59}$$

Remark 6.3. It would appear more consistent with the statement of Lemma 6.1 to replace $G^1(r,\theta)$ by $G^1(r,\theta)/r^2$ for $r>1$, and similarly change

$$\|r^2G^1\|_{L^2(\mathbb{R}^2\setminus B_1)} \quad \text{to } \|G^1\|_{L^2(\mathbb{R}^2\setminus B_1)} \quad \text{etc.;}$$

however the present notation is more convenient for the subsequent applications of the lemma.

Proof. Proceeding as in the proof from Lemma 6.1, we can solve $M^{(1)}, M^{(2)}, M^{(3)}, M+U, \Lambda$ as before. Since F^1, F^2 are no longer in L^2, we are forced to estimate the integrals in (6.40) in a different way. Since $F^3 \equiv 0$, $\beta = 0$, no mode 0 terms appear; by (6.8) also no modes 1 and l appear in the solution, and thus we need to consider only modes $m \geqslant 2$, $m \neq l$. In particular, $M_0(r) \equiv 0$, $M_0^{(3)}(r) \equiv 0$ and therefore $C_{20} = 0$.

Step 1. Estimate for $M^{(1)}$.

For $r > 1$,

$$\begin{aligned} F_m^1(r) &= \frac{1}{\pi}\int_0^{2\pi} F^1(r,\theta)\cos m\theta \\ &= \frac{-m}{\pi}\int_0^{2\pi}\Big(G^1(r,\theta) + \frac{g(\theta)}{r^2}\Big)\sin m\theta d\theta \;\equiv\; -m\Big[G_m^1(r) + \frac{1}{r^2}g_m\Big], \end{aligned} \tag{6.60}$$

so that, if $m \geqslant 3$,

$$\begin{aligned} &\Big|\int_1^\infty \frac{F_m^1(r)}{r^{m-1}}dr\Big| \\ &= \Big|\frac{1}{m-2}F_m^1(1+) + \frac{1}{m-2}\int_1^\infty r^{2-m}\frac{\partial}{\partial r}F_m^1(r)dr\Big| \\ &= \Big|\frac{-m}{m-2}[G_m^1(1+) + g_m] - \frac{m}{m-2}\int_1^\infty r^{2-m}\Big(\frac{\partial}{\partial r}G_m(r) - \frac{2}{r^3}g_m\Big)dr\Big| \\ &\leqslant 3|G_m^1(1+) + g_m| + 3\Big[\Big(\int_1^\infty \Big[r\frac{\partial}{\partial r}G_m(r)\Big]^2 rdr\Big)^{1/2} + 2|g_m|\Big]\Big(\int_1^\infty r^{-2m}rdr\Big)^{1/2} \\ &\leqslant 3|G_m^1(1+)| + 9|g_m| + \frac{6}{m^{1/2}}\Big\|r\frac{\partial}{\partial r}G_m(r)\Big\|_{L^2(\mathbb{R}^2\setminus B_1)}, \end{aligned} \tag{6.61}$$

whereas if $m = 2$,

$$\Big|\int_1^\infty \frac{F_m^1(r)}{r^{m-1}}dr\Big| \leqslant \|F_2^1\|_{L^2(\mathbb{R}^2\setminus B_1)} \leqslant 2(\|G_2^1(r)\|_{L^2(\mathbb{R}^2\setminus B_1)} + |g_2|). \tag{6.62}$$

Similarly, for all $m \geqslant 2$, $r < 1$,

$$\begin{aligned} \Big|\int_0^1 \frac{I_m(R\tau)}{I_m(R)}F_m^1(\tau)\tau d\tau\Big| &= \Big|F_m^1(1-)\int_0^1 \frac{I_m(R\tau)}{I_m(R)}\tau d\tau - \int_0^1\Big(\int_0^r \frac{I_m(R\tau)}{I_m(R)}\tau d\tau\Big)\frac{\partial}{\partial r}F_m^1(r)dr\Big| \\ &\leqslant \frac{C}{m}|F_m^1(1-)| + \frac{C}{m}\int_0^1 r^{m+2}\Big|\frac{\partial}{\partial r}F_m^1(r)\Big|dr \\ &\leqslant C|G_m^1(1-)| + \frac{C}{m^{1/2}}\Big\|r\frac{\partial}{\partial r}G_m^1(r)\Big\|_{L^2(B_1)}. \end{aligned} \tag{6.63}$$

We can estimate $\int_0^1 \tau^m(F_m^1(\tau)+F_m^2(\tau))\tau d\tau$ in the same manner. Substituting these estimates into the first inequality in (6.40) and recalling that $F_3 = 0$, we get

$$\begin{aligned} m\Big(\Big|\frac{\Lambda_m}{R}\Big| + |C_{2m}|\Big) &\leqslant C(|G_m^1(1\pm)| + |g_m|) \\ &\quad + \frac{C}{m^{1/2}}\Big\{\Big\|r\frac{\partial}{\partial r}G_m(r)\Big\|_{L^2(\mathbb{R}^2\setminus B_1)} + \Big\|r\frac{\partial}{\partial r}G_m(r)\Big\|_{L^2(B_1)}\Big\}. \end{aligned} \tag{6.64}$$

Using this estimate (instead of (6.40)), we can then follow the proof of (6.42)–(6.46) to derive the bounds

$$\begin{aligned} \|\Lambda\|_{H^{3/2}(\partial B_1)} \leqslant C\Big(&\|rG_r^1\|_{L^2(B_1)} + \|rG_r^1\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|G^1(1+,\cdot)\|_{H^{1/2}(\partial B_1)} \\ &+\|G^1(1-,\cdot)\|_{H^{1/2}(\partial B_1)} + \|g\|_{H^{1/2}(\partial B_1)}\Big), \end{aligned} \tag{6.65}$$

$$\begin{aligned} &\|rM_r^{(1)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|rM_{r\theta}^{(1)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^2M_{rr}^{(1)}\|_{L^2(\mathbb{R}^2\setminus B_1)} \\ &\leqslant \; C\Big(\|rG_r^1\|_{L^2(B_1)} + \|rG_r^1\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|G^1(1+,\cdot)\|_{H^{1/2}(\partial B_1)} \\ &\qquad +\|G^1(1-,\cdot)\|_{H^{1/2}(\partial B_1)} + \|g\|_{H^{1/2}(\partial B_1)}\Big), \end{aligned} \tag{6.66}$$

and

$$\begin{aligned} \|M^{(1)}\|_{H^2(B_1)} \leqslant C\Big(&\|rG_r^1\|_{L^2(B_1)} + \|rG_r^1\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|G^1(1,\cdot)\|_{H^{1/2}(\partial B_1)} \\ &+\|g\|_{H^{1/2}(\partial B_1)}\Big). \end{aligned} \tag{6.67}$$

Step 2. Estimate for $M^{(2)}$.

Let W be the solution of

$$\begin{aligned} &\Delta W - R^2 W = G^1(r,\theta) \quad \text{in } B_1, \\ &W = 0 \quad \text{on } \partial B_1. \end{aligned}$$

Then

$$M^{(2)} = \frac{\partial}{\partial\theta} W \quad \text{in } B_1. \tag{6.68}$$

By elliptic estimates and (6.2),

$$\|W\|_{H_r^2(B_1)} \leqslant C\|W\|_{H^2(B_1)} \leqslant C\|G^1\|_{L^2(B_1)}. \tag{6.69}$$

A direct computation shows that the function $\varphi(r,\theta) = rW_r$ satisfies

$$\Delta\varphi - R^2\varphi = rG_r^1 + 2W_{rr} + \frac{2}{r}W_r + \frac{2}{r^2}W_{\theta\theta} \equiv \mathcal{G} \quad \text{in } B_1, \tag{6.70}$$

$$\varphi_r = G^1(1,\theta) \quad \text{on } \partial B_1, \tag{6.71}$$

and, by (6.2) and (6.69),

$$\|\mathcal{G}\|_{L^2(B_1)} \leqslant C(\|G^1\|_{L^2(B_1)} + \|rG_r^1\|_{L^2(B_1)}).$$

Therefore, by L^2 elliptic estimates for the Neumann boundary value problem and by (6.2),

$$\begin{aligned} \|rW_r\|_{H_r^2(B_1)} &\leqslant C\|rW_r\|_{H^2(B_1)} = C\|\varphi\|_{H^2(B_1)} \\ &\leqslant C(\|G^1\|_{L^2(B_1)} + \|rG_r^1\|_{L^2(B_1)} + \|G^1(1-,\cdot)\|_{H^{1/2}(\partial B_1)}). \end{aligned} \tag{6.72}$$

One can readily check that

$$\begin{aligned} &\|M^{(2)}\|_{L^2(B_1)} + \|M_{rr}^{(2)}\|_{L^2(B_1)} + \Big\|\frac{1}{r}M_r^{(2)}\Big\|_{L^2(B_1)} + \Big\|\frac{1}{r}M_{r\theta}^{(2)}\Big\|_{L^2(B_1)} \\ &\leqslant \; C\Big(\Big\|r\Big(\frac{1}{r}W_\theta\Big)\Big\|_{L^2(B_1)} + \Big\|\frac{1}{r}(rW_r)_{r\theta}\Big\|_{L^2(B_1)} + \Big\|\frac{1}{r}W_{r\theta}\Big\|_{L^2(B_1)} + \Big\|\frac{1}{r^2}(rW_r)_{\theta\theta}\Big\|_{L^2(B_1)}\Big) \\ &\leqslant \; C(\|G^1\|_{L^2(B_1)} + \|rG_r^1\|_{L^2(B_1)} + \|G^1(1-,\cdot)\|_{H^{1/2}(\partial B_1)}), \end{aligned} \tag{6.73}$$

where the second inequality is a consequence of (6.69) and (6.72).

Next we estimate $M^{(2)}$ for $r > 1$. We use the formula (6.27), as before. Let

$$\begin{aligned} F^1_m(r) &= \frac{1}{r^2} F^{10}_m + F^{11}_m(r), \\ F^{10}_m &= \frac{1}{\pi}\int_0^{2\pi} g'(\theta)\cos m\theta d\theta = \frac{-m}{\pi}\int_0^{2\pi} g(\theta)\sin m\theta d\theta = -m g_m, \\ F^{11}_m(r) &= \frac{1}{\pi}\int_0^{2\pi} G^1_\theta(r,\theta)\cos m\theta d\theta = \frac{-m}{\pi}\int_0^{2\pi} G^1(r,\theta)\sin m\theta d\theta = -m G^1_m(r). \end{aligned} \tag{6.74}$$

We can then write $M^{(2)} = M^{(20)} + M^{(21)}$, where

$$\begin{aligned} M^{(20)}(r,\theta) &= \sum_{m\geqslant 2} M^{(20)}_m(r)\cos m\theta, \\ M^{(21)}(r,\theta) &= \sum_{m\geqslant 2} M^{(21)}_m(r)\cos m\theta, \end{aligned}$$

and

$$\begin{aligned} M^{(20)}_m(r) &= -\frac{1}{r^m}\int_1^r s^{2m-1}\int_s^\infty \frac{F^{10}}{\tau^{m+1}} d\tau = -\frac{F^{10}}{m^2}\Big(1 - \frac{1}{r^m}\Big), \\ M^{(21)}_m(r) &= -\frac{1}{r^m}\int_1^r s^{2m-1}\int_s^\infty \frac{F^{11}_m(\tau)}{\tau^{m-1}} d\tau. \end{aligned}$$

As before, we have

$$m^2\int_1^\infty \Big(r\frac{\partial}{\partial r} M^{(20)}_m(r)\Big)^2 r dr + \int_1^\infty \Big(r^2\frac{\partial^2}{\partial r^2} M^{(20)}_m(r)\Big)^2 r dr \leqslant \frac{C}{m}|F^{10}_m|^2 \leqslant C m |g_m|^2,$$

so that

$$\|r M^{(20)}_r\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r M^{(20)}_{r\theta}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^2 M^{(20)}_{rr}\|_{L^2(\mathbb{R}^2\setminus B_1)} \leqslant C\|g\|_{H^{1/2}(\partial B_1)}. \tag{6.75}$$

Next, by direct computation as in the previous lemma,

$$\frac{\partial}{\partial r} M^{(21)}_m(r) = \frac{m}{r^{m+1}}\int_1^r s^{2m-1}\int_s^\infty \frac{F^{11}_m(\tau)}{\tau^{m-1}} d\tau - r^{m-1}\int_r^\infty \frac{F^{11}_m(\tau)}{\tau^{m-1}} d\tau, \tag{6.76}$$

$$\begin{aligned} \frac{\partial^2}{\partial r^2} M^{(21)}_m(r) \;=\; & -\frac{m(m+1)}{r^{m+2}}\int_1^r s^{2m-1}\int_s^\infty \frac{F^{11}_m(\tau)}{\tau^{m-1}} d\tau \\ & + r^{m-2}\int_r^\infty \frac{F^{11}_m(\tau)}{\tau^{m-1}} d\tau + F^{11}_m(r). \end{aligned} \tag{6.77}$$

By integration by parts, we obtain, for $m \geqslant 3$,

$$\begin{aligned} \int_r^\infty \frac{F^{11}_m(\tau)}{\tau^{m-1}} d\tau &= \frac{r^{2-m}}{m-2} F^{11}_m(r) + \frac{1}{m-2}\int_r^\infty \tau^{2-m}\frac{\partial}{\partial\tau} F^{11}_m(\tau) d\tau \\ &= -\frac{m}{m-2} r^{2-m} G^1_m(r) - \frac{m}{m-2}\int_r^\infty \tau^{2-m}\frac{\partial}{\partial\tau} G^1_m(\tau) d\tau, \end{aligned}$$

$$
\begin{aligned}
&\int_1^r s^{2m-1}\int_s^\infty \frac{F_m^{11}(\tau)}{\tau^{m-1}}d\tau \\
&= -\frac{m}{m-2}\int_1^r s^{m+1}G_m^1(s)ds - \frac{m}{m-2}\int_1^r s^{2m-1}\int_s^\infty \tau^{2-m}\frac{\partial}{\partial\tau}G_m^1(\tau)d\tau \\
&= \frac{m}{m^2-4}\Big[G_m^1(1) - r^{m+2}G_m^1(r)\Big] + \frac{m}{m^2-4}\int_1^r \tau^{m+2}\frac{\partial}{\partial\tau}G_m^1(\tau)d\tau \\
&\qquad - \frac{m}{m-2}\int_1^r s^{2m-1}\int_s^\infty \tau^{2-m}\frac{\partial}{\partial\tau}G_m^1(\tau)d\tau.
\end{aligned}
$$

Substituting these equalities into (6.76)–(6.77), we find that the $O(m^0)$ order terms of $rG_m^1(r)$ in $\partial M_m^{(21)}(r)/\partial r$ and $O(m^1)$ order terms of $G_m^1(r)$ in $\partial^2 M_m^{(21)}(r)/\partial r^2$ cancel out, and we obtain

$$
\begin{aligned}
&\Big|mr\frac{\partial}{\partial r}M_m^{(21)}(r)\Big| + \Big|r^2\frac{\partial^2}{\partial r^2}M_m^{(21)}(r)\Big| \\
&\leqslant C\Big\{\frac{m^2}{r^m}\int_1^r s^{2m-1}\int_s^\infty \tau^{2-m}\Big|\frac{\partial}{\partial\tau}G_m^1(\tau)\Big|d\tau + mr^m\int_r^\infty \tau^{2-m}\Big|\frac{\partial}{\partial\tau}G_m^1(\tau)\Big|d\tau \\
&\qquad + \frac{m}{r^m}\int_1^r \tau^{m+2}\Big|\frac{\partial}{\partial\tau}G_m^1(\tau)\Big|d\tau + \frac{m}{r^m}G_m^1(1) + |r^2G_m^1(r)|\Big\} \\
&\equiv C(\mathcal{Q}_{1m} + \mathcal{Q}_{2m} + \mathcal{Q}_{3m} + \frac{m}{r^m}G_m^1(1) + |r^2G_m^1(r)|).
\end{aligned}
$$

The same computation as in (6.49) yields the estimate $\mathcal{Q}_{1m} \leqslant \mathcal{Q}_{2m} + \mathcal{Q}_{3m}$. Following the same procedure as in Lemma 6.1, we obtain

$$
\int_1^\infty [\mathcal{Q}_{2m}(r)]^2 rdr + \int_1^\infty [\mathcal{Q}_{3m}(r)]^2 rdr \leqslant C\int_1^\infty \Big|\tau^3\frac{\partial}{\partial\tau}G_m^1(\tau)\Big|^2\tau d\tau. \tag{6.78}
$$

Thus, for $m \geqslant 3$,

$$
\begin{aligned}
&m^2\int_1^\infty \Big[r\frac{\partial}{\partial r}M_m^{(21)}(r)\Big]^2 rdr + \int_1^\infty \Big[r^2\frac{\partial^2}{\partial r^2}M_m^{(21)}(r)\Big]^2 rdr \\
&\leqslant C\Big\{m[G_m^1(1)]^2 + \int_1^\infty \Big|\tau^2G_m^1(\tau)\Big|^2\tau d\tau + \int_1^\infty \Big|\tau^3\frac{\partial}{\partial\tau}G_m^1(\tau)\Big|^2\tau d\tau\Big\}.
\end{aligned} \tag{6.79}
$$

For $m = 2$, $|F_2^{11}(r)| = 2|G_2^1(r)|$, and we just have to slightly modify the above proof in order to derive the same inequality. Summing over $m \geqslant 2$, we deduce that

$$
\begin{aligned}
&\|rM_r^{(21)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|rM_{r\theta}^{(21)}\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^2M_{rr}^{(21)}\|_{L^2(\mathbb{R}^2\setminus B_1)} \\
&\leqslant C\Big[\|G^1(1+,\cdot)\|_{H^{1/2}(\partial B_1)} + \|r^2G^1\|_{L^2(\mathbb{R}^2\setminus B_1)} + \|r^3G_r^1\|_{L^2(\mathbb{R}^2\setminus B_1)}\Big].
\end{aligned} \tag{6.80}
$$

Step 3. Estimate for $M^{(3)}$.

The estimate for $M^{(3)}$ proceeds as in (6.55) with $\|\Lambda - \Lambda_0\|_{H^{3/2}}$ replaced by $\|\Lambda\|_{H^{3/2}}$.

Step 4. Estimate for $M + U$.

This is similar to the estimate in Lemma 6.1, but using the technique in Step 2 of this lemma for estimating $M^{(2)}$ in $r < 1$.

Combining the above estimates, the proof of the lemma is complete. □

We next consider the case of mode l only, namely,

$$
\begin{aligned}
&F^1(r,\theta) = F_l^1(r)\cos l\theta \quad \text{for } r < 1, \\
&F^1(r,\theta) = \frac{1}{r^2}F_l^{10}\cos l\theta + \frac{1}{r^2}F_l^{11}(r)\cos l\theta, \quad \text{for } r > 1, \\
&F^2(r,\theta) = F_l^2(r)\cos l\theta \quad \text{for } r < 1, \\
&F^3(\theta) = F_l^3\cos l\theta.
\end{aligned} \tag{6.81}
$$

As before, we obtain a linear system of equations for $(C_{1l}, C_{2l}, C_{3l}, \Lambda_l/R)$. Since in this case the determinant T_l is zero, the system (6.37), (6.38) (with $m = l$) can be solved if and only if the following compatibility condition is satisfied:

$$\det\begin{pmatrix} \dfrac{RI_{l-1}(R)}{I_l(R)}, & -\displaystyle\int_1^\infty \frac{F_l^1(\tau)}{\tau^{l-1}}d\tau - \frac{1}{I_l(R)}\int_0^1 \tau I_l(R\tau)F_l^1(\tau)d\tau \\ 2l, & F_l^3 - \displaystyle\int_0^1 \tau^{l+1}\{F_l^1(\tau)+F_l^2(\tau)\}d\tau - \int_1^\infty \frac{F_l^1(\tau)}{\tau^{l-1}}d\tau \end{pmatrix} = 0.$$

The solution is unique up to a multiple by a homogeneous solution. Hence if we impose the condition $\Lambda_l = 0$ then the solution of (6.37), (6.38) is unique. For later references we state:

Lemma 6.3 *For the data (6.81) (mode l) a solution which is even in θ exists if and only if*

$$\begin{aligned}\Big(2l - R\frac{I_{l-1}(R)}{I_l(R)}\Big)\int_1^\infty \frac{F_l^1(\tau)}{\tau^{l-1}}d\tau + \frac{2l}{I_l(R)}\int_0^1 \tau I_l(R\tau)F_l^1(\tau)d\tau \\ +R\frac{I_{l-1}(R)}{I_l(R)}\Big[F_l^3 - \int_0^1 \tau^{l+1}\{F_l^1(\tau)+F_l^2(\tau)\}d\tau\Big] = 0\end{aligned} \tag{6.82}$$

and the solution is uniquely determined by the condition that $\Lambda_l = 0$.

The next lemma summarizes Lemmas 6.1–6.3.

Lemma 6.4 *Assume that in (6.3)–(6.4), $F^j = \widehat{F}^j + \widetilde{F}^j + \overline{F}^j$, where $\widehat{F}^j$, $\widetilde{F}^j$ and $\overline{F}^j$ satisfy the assumptions of Lemmas 6.1, 6.2 and 6.3, respectively, i.e.,*

(a) $\widehat{F}^j$ and $\widetilde{F}^j$ do not contain modes 1 and l; $\overline{F}^j$ contains mode l only.

(b) $\widehat{F}^j$ satisfies the conditions of Lemma 6.1 and $\widetilde{F}^j$ satisfies the conditions of Lemma 6.2.

If the compatibility condition (6.82) is satisfied for the $\overline{F}^j$, then there exists a unique solution (M, U, Λ) which is even in θ, satisfies the condition $\Lambda_l = 0$, and can be decomposed into a sum of three parts, satisfying, respectively, the properties asserted in Lemmas 6.1–6.3.

Remark 6.4. If we differentiate the equations (6.3)–(6.7) in θ, we get an equation which is exactly the same except that the solutions will be odd in θ and we need to work with Fourier sine series. The conclusion in Lemmas 6.1–6.4 are still valid (with the estimates and the right-hand sides replaced by their θ-derivative). If we differentiate in θ a second time, the solutions will become even in θ and Lemmas 6.1–6.4 can again be directly applied.

7 Convergence

In this section we estimate inductively the $(\mu_n, u_n, \lambda_n, \beta_{n-1})$. We denote the mode k of F^{jn} in (5.18), (5.19), (5.22) by F_k^{jn}. By Lemma 6.4 the system (5.18)–(5.22) can be uniquely solved for any mode k, $k \neq l$. For mode l the system can be solved if and only if the compatibility condition (6.82) is satisfied. This condition will take the form $\widetilde{\Lambda}\beta_{n-1} = \widetilde{F_n}$ where $\widetilde{F_n}$ is determined by the $\mu_s, u_s, \lambda_s, \beta_{s-1}$ for $s < n$ and, as the next lemma shows, $\widetilde{\Lambda} \neq 0$ and $\widetilde{\Lambda}$ is independent of n.

Lemma 7.1 *The expression*

$$\widetilde{\Lambda} \equiv \frac{\partial}{\partial\beta_{n-1}}\Big\{\Big(2l - R\frac{I_{l-1}(R)}{I_l(R)}\Big)\int_1^\infty \frac{F_l^{1n}(\tau)}{\tau^{l-1}}d\tau + \frac{2l}{I_l(R)}\int_0^1 \tau I_l(R\tau)F_l^{1n}(\tau)d\tau \\ +R\frac{I_{l-1}(R)}{I_l(R)}\Big[F_l^{3n} - \int_0^1 \tau^{l+1}\{F_l^{1n}(\tau)+F_l^{2n}(\tau)\}d\tau\Big]\Big\} \neq 0 \tag{7.1}$$

is independent of n and is nonzero.

Proof. The compatibility condition is invariant under the change of variable (5.1). Therefore the assertion that $\widetilde{\Lambda} \neq 0$ (which is equivalent to the statement that the compatibility condition (6.82) holds for some choice of β_{n-1}) follows from Theorem 4.3. We next proceed to derive an expression for $\widetilde{\Lambda}$ which, although very complicated, nevertheless shows that it is independent of n. We begin by writing explicit formulas for F^{jn}. Substituting (5.2), (5.9)–(5.12) into (5.4), (5.5), (5.8), we find that $(\mu_n, u_n, \lambda_n, \beta_n) = (M, U, \Lambda, \beta)$ satisfy (6.3)–(6.7) with $F^j = F^{jn}$ defined as the ε^n order terms in the following expansion:

$$\sum_{n\geqslant 1}\varepsilon^n F^{1n} = \Big(-\frac{f_{\theta\theta}}{R+f}+\frac{f_{\theta\theta}}{R}\Big)\frac{1}{r}\frac{\partial\mu_0}{\partial r} + \frac{f_\theta^2}{(R+f)^2}\Big(\frac{\partial^2\mu_0}{\partial r^2}+\frac{2}{r}\frac{\partial\mu_0}{\partial r}\Big) \\ -\frac{f_{\theta\theta}}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial\mu_n}{\partial r} - \frac{2f_\theta}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial^2\mu_n}{\partial\theta\partial r} \\ +\frac{f_\theta^2}{(R+f)^2}\sum_{n\geqslant 1}\varepsilon^n\Big(\frac{\partial^2\mu_n}{\partial r^2}+\frac{2}{r}\frac{\partial\mu_n}{\partial r}\Big) - \chi_{\{r<1\}}\Big((2Rf+f^2)\sum_{n\geqslant 1}\varepsilon^n\mu_n + f^2\mu_0\Big), \tag{7.2}$$

$$\sum_{n\geqslant 1}\varepsilon^n(F^{1n}+F^{2n}) = -\frac{f_{\theta\theta}}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial(\mu_n+u_n)}{\partial r} - \frac{2f_\theta}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial^2(\mu_n+u_n)}{\partial\theta\partial r} \\ +\frac{f_\theta^2}{(R+f)^2}\sum_{n\geqslant 1}\varepsilon^n\Big(\frac{\partial^2(\mu_n+u_n)}{\partial r^2}+\frac{2}{r}\frac{\partial(\mu_n+u_n)}{\partial r}\Big), \tag{7.3}$$

$$-\sum_{n\geqslant 1}\varepsilon^n F^{3n} = \Big(\sum_{n\geqslant 0}\varepsilon^n\beta_n\Big)\Big[\frac{(R+f)^2}{[(R+f)^2+f_\theta^2]^{1/2}} - R\Big] - \frac{1}{R}f. \tag{7.4}$$

Consider the system for $(\mu_{n-1}, u_{n-1}, \lambda_{n-1})$, whose solution is given in §6 (Lemma 6.4) with $F^j = F^{j,n-1}$. The solution depends on β_{n-1} only through its zero mode (see (6.33)–(6.36) with $\beta = \beta_{n-1}$ and note that $\delta_{m0}\beta = 0$ if $m > 0$). We need to find the explicit dependence of the F_m^{jn} on β_{n-1}. We therefore first solve the system (6.33)–(6.36) with $F_m^j = F_m^{j,n-1}$, $m = 0$ and $\beta = \beta_{n-1}$. By Remark 6.2, if $m = 0$ then I_{m-1} in (6.34) is replaced by I_1. Solving (6.37), (6.38) and (6.33), (6.35), we find

$$\frac{1}{R}\Lambda_0^{n-1} = -\beta_{n-1}R + \cdots, \\ C_{20}^{n-1} = C_{30}^{n-1} = \beta_{n-1}R\Big(\frac{I_0^2(R)}{I_1^2(R)} - 1\Big) + \cdots, \\ C_{10}^{n-1} = \beta_{n-1}\frac{R}{I_0(R)}\Big(\frac{I_0^2(R)}{I_1^2(R)} - 1\Big) + \cdots,$$

where "$\cdots$" refers to terms independent of β_{n-1}. Then, from the formula for $M = \mu_{n-1}$ in Lemma 6.4 (or actually the special case stated as Lemma 6.1),

$$\frac{\partial}{\partial\beta_{n-1}}\mu_{n-1} = I_0(Rr)\frac{\partial C_{10}^{n-1}}{\partial\beta_{n-1}} + \frac{I_0(Rr)}{I_1(R)}B_0(r)\frac{\partial}{\partial\beta_{n-1}}\frac{\Lambda_0^{n-1}}{R}$$

$$= \frac{RI_0(Rr)}{I_0(R)}\Big\{\frac{I_0^2(R)}{I_1^2(R)} - 1 - \frac{I_0(R)}{I_1(R)}B_0(r)\Big\} \tag{7.5}$$
$$= \frac{RI_0(R)I_0(Rr)}{I_1^2(R)} - \frac{Rr}{I_1(R)}I_1(Rr) \quad \text{for } r < 1,$$

$$\frac{\partial}{\partial\beta_{n-1}}\mu_{n-1} = \frac{\partial}{\partial\beta_{n-1}}C_{20}^{n-1} = R\Big(\frac{I_0^2(R)}{I_1^2(R)} - 1\Big) \quad \text{for } r > 1 \quad (\text{since } M_0^{(3)} \equiv 0), \tag{7.6}$$

$$\frac{\partial}{\partial\beta_{n-1}}(\mu_{n-1} + u_{n-1}) = \frac{\partial}{\partial\beta_{n-1}}C_{30}^{n-1} = R\Big(\frac{I_0^2(R)}{I_1^2(R)} - 1\Big) \quad \text{for } r < 1, \tag{7.7}$$

$$\frac{1}{(n-1)!}\frac{\partial}{\partial\beta_{n-1}}\frac{\partial^{n-1}}{\partial\varepsilon^{n-1}}\Big|_{\varepsilon=0} f = \frac{\partial}{\partial\beta_{n-1}}\lambda_{n-1} = \frac{\partial}{\partial\beta_{n-1}}\Lambda_0^{n-1} = -R^2. \tag{7.8}$$

Clearly, the μ_k, u_k, λ_k do not depend on β_{n-1} if $k < n-1$. Since the right-hand sides of (7.6), (7.7) are constants independent of r, we find from

$$\frac{\partial}{\partial\beta_{n-1}}F^{1n}(r,\theta) = \frac{1}{n!}\frac{\partial}{\partial\beta_{n-1}}\frac{\partial^n}{\partial\varepsilon^n}\Big|_{\varepsilon=0}\Big(\text{the right-hand side of (7.2)}\Big)$$

that the only nonzero contribution comes from the first term

$$\Big(-\frac{f_{\theta\theta}}{R+f} + \frac{f_{\theta\theta}}{R}\Big)\frac{1}{r}\frac{\partial\mu_0}{\partial r} = \frac{f_{\theta\theta}f}{R(R+f)}\frac{1}{r}\frac{\partial\mu_0}{\partial r}.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial\beta_{n-1}}F^{1n}(r,\theta) &= \frac{\lambda_1''(\theta)}{R^2}\frac{1}{r}\frac{\partial\mu_0}{\partial r}\frac{\partial\lambda_{n-1}(\theta)}{\partial\beta_{n-1}} \\ &= -\lambda_1''(\theta)\frac{1}{r^2} = \lambda_1(\theta)\frac{l^2}{r^2} \quad \text{for } r > 1.\end{aligned} \tag{7.9}$$

Similarly

$$\begin{aligned}&\frac{\partial}{\partial\beta_{n-1}}\Big(F^{1n}(r,\theta) + F^{2n}(r,\theta)\Big) \\ &= \frac{1}{n!}\frac{\partial}{\partial\beta_{n-1}}\frac{\partial^n}{\partial\varepsilon^n}\Big|_{\varepsilon=0}\Big(\text{the right-hand side of (7.3)}\Big) \\ &= 0 \quad \text{for } r < 1,\end{aligned} \tag{7.10}$$

A direct computation shows that

$$\begin{aligned}-\frac{\partial}{\partial\beta_{n-1}}F^{3n}(\theta) &= \frac{1}{n!}\frac{\partial}{\partial\beta_{n-1}}\frac{\partial^n}{\partial\varepsilon^n}\Big|_{\varepsilon=0}\Big(\text{the right-hand side of (7.4)}\Big) \\ &= \lambda_1(\theta) + \beta_1\frac{\partial}{\partial\beta_{n-1}}\Lambda_0^{n-1} = \lambda_1(\theta) - R^2\beta_1.\end{aligned} \tag{7.11}$$

Finally, for $r < 1$,

$$\begin{aligned}\frac{\partial}{\partial\beta_{n-1}}F^{1n}(r,\theta) &= \frac{1}{n!}\frac{\partial}{\partial\beta_{n-1}}\frac{\partial^n}{\partial\varepsilon^n}\Big|_{\varepsilon=0}\Big(\text{the right-hand side of (7.2)}\Big) \\ &= \frac{\lambda_1''(\theta)}{R^2}\frac{1}{r}\frac{\partial\mu_0}{\partial r}\frac{\partial\lambda_{n-1}(\theta)}{\partial\beta_{n-1}} - \frac{\lambda_1''(\theta)}{R}\frac{1}{r}\frac{\partial}{\partial\beta_{n-1}}\frac{\partial\mu_{n-1}}{\partial r} \\ &\quad -\Big[2R\lambda_1(\theta)\frac{\partial\mu_{n-1}}{\partial\beta_{n-1}} + 2R\mu_1(r,\theta)\frac{\partial\lambda_{n-1}(\theta)}{\partial\beta_{n-1}} + 2\lambda_1(\theta)\mu_0\frac{\partial\lambda_{n-1}(\theta)}{\partial\beta_{n-1}}\Big]\end{aligned}$$

where the expression in brackets comes from the coefficient of $\chi_{\{r<1\}}$ in (7.2). Using (7.6), (7.8) we get

$$\begin{aligned}
&\frac{\partial}{\partial\beta_{n-1}}F^{1n}(r,\theta)\\
&= -\lambda_1''(\theta)\frac{1}{r}\frac{\partial\mu_0}{\partial r}-\lambda_1''(\theta)\frac{1}{r}\frac{\partial}{\partial r}\Big[\frac{I_0(R)I_0(Rr)}{I_1^2(R)}-\frac{r}{I_1(R)}I_1(Rr)\Big]\\
&\quad -2R\lambda_1(\theta)\Big[\frac{RI_0(R)I_0(Rr)}{I_1^2(R)}-\frac{Rr}{I_1(R)}I_1(Rr)\Big]+2R^3\mu_1(r,\theta)+2R^2\mu_0(r)\lambda_1(\theta). \qquad (7.12)\\
&= \lambda_1(\theta)\Big\{\frac{l^2I_1(Rr)}{rI_1(R)}+\frac{l^2}{r}\Big[\frac{RI_0(R)I_1(Rr)}{I_1^2(R)}-\frac{RrI_0(Rr)}{I_1(R)}\Big]\\
&\quad -2R\Big[\frac{RI_0(R)I_0(Rr)}{I_1^2(R)}-\frac{Rr}{I_1(R)}I_1(Rr)\Big]-2R^3\frac{l+1}{2l}\frac{I_l(Rr)}{RI_l(R)}+2R^2\mu_0(r)\Big\},
\end{aligned}$$

where in the last equality we have also used (3.15). Substituting (7.9)–(7.12) into (7.1), we conclude that

$$\begin{aligned}
\widetilde{\Lambda} &= \Big(2l-R\frac{I_{l-1}(R)}{I_l(R)}\Big)l+\frac{2l}{I_l(R)}\int_0^1 rI_l(Rr)\Big\{\frac{l^2I_1(Rr)}{rI_1(R)}+\frac{l^2}{r}\Big[\frac{RI_0(R)I_1(Rr)}{I_1^2(R)}-\frac{RrI_0(Rr)}{I_1(R)}\Big]\\
&\quad -2R\Big[\frac{RI_0(R)I_0(Rr)}{I_1^2(R)}-\frac{Rr}{I_1(R)}I_1(Rr)\Big]-2R^3\frac{l+1}{2l}\frac{I_l(Rr)}{RI_l(R)}+2R^2\mu_0(r)\Big\}dr-R\frac{I_{l-1}(R)}{I_l(R)}
\end{aligned} \qquad (7.13)$$

which is independent of n. □

Remark 7.1. As was already stated before, for the solution $(u_n,\mu_n,\lambda_n,\beta_{n-1})$ constructed by using Lemmas 6.4, 7.1, the mode l component is not unique, for we can add to it any multiple of (μ_1,u_1,λ_1). We shall henceforth fix this multiple uniquely by the condition

$$\int_0^{2\pi}\lambda_n(\theta)\cos l\theta=0 \quad \text{for } n\geqslant 2. \qquad (7.14)$$

This normalization is not necessarily the same as the normalization in Remark 4.2, because the methods of construction of the solution are different.

We shall say that a linear function space X with norm $\|\cdot\|_X$ has the *algebra property* if, whenever f,g belong to X, also fg belongs to X and

$$\|fg\|_X\leqslant C\|f\|_X\|g\|_X$$

where C is a constant independent of f,g. In the sequel we shall need the algebra property in a more general sense, whereby g belong to one space X_1, and f belongs to another space, X_2, and

$$\|fg\|_{X_2}\leqslant C\|g\|_{X_1}\|f\|_{X_2}.$$

In order to estimate the F^{jn} we shall need to have the algebra property for the various terms in the F^{jn}. The spaces $L^2(B_1)$ or $H^{1/2}(B_1)$ are not suitable for this purpose. Noticing that the functions defining F^j in (7.2)–(7.4) have the form $g(\theta)\varphi(r,\theta)$, it will be convenient to work with the norm

$$\|\psi(r,\theta)\|_{W_2^{m,n}}=\sum_{0\leqslant j\leqslant m,0\leqslant k\leqslant n}\|\partial_r^j\partial_\theta^k\psi\|_{L^2}$$

where m,n are integers; notice that this definition is a little different from the usual Sobolev norm since, for example, $1/r^2$ factor is not present in the ∂_θ^2 derivative. We shall need on one occasion to also use the norm

$$\|\psi(r,\theta)\|_{W_2^{1,3/2}}=\|\psi(r,\theta)\|_{W_2^{1,1}}+\|\partial_\theta^{1/2}\psi(r,\theta)\|_{W_2^{1,1}}.$$

The following estimates are well known (see, for example, [1])

Lemma 7.2

$$\|g(\theta)\varphi(r,\theta)\|_{W_2^{0,k}(B_1)} \leqslant \|g\|_{H^k(\partial B_1)}\|\varphi(r,\theta)\|_{W_2^{0,k}(B_1)} \quad \textit{for } k \geqslant 1, \tag{7.15}$$

$$\|g(\theta)h(\theta)\|_{H^s(\partial B_1)} \leqslant \|g\|_{H^s(\partial B_1)}\|h\|_{H^s(\partial B_1)} \quad \textit{for } s > 1/2, \tag{7.16}$$

$$\|\varphi(1,\theta)\|_{H^{m+1/2}(\partial B_1)} \leqslant C\|\varphi(r,\theta)\|_{W_2^{1,m+1}(B_1)} \quad \textit{for } m \geqslant 0. \tag{7.17}$$

The first two inequalities provide the algebra properties needed to estimate the F^{jn}. These inequalities are also valid if B_1 is replaced by $\mathbb{R}^2 \setminus B_1$.

Let H_0, H, B and Γ be positive constants $\geqslant 1$ and set

$$E(n,k) = k!\frac{H_0 H^{n-1}}{n^2}\frac{B^k}{k^2} \quad \text{for } n \geqslant 0,\ k \geqslant 0, \tag{7.18}$$

with the following convention:

$$H^{n-1} = 1 \quad \text{if } n = 0, \quad n^2 = 1 \quad \text{if } n = 0, \quad k^2 = 1 \quad \text{if } k = 0.$$

We inductively assume that, for $1 \leqslant s < n$,

$$\begin{aligned} \Big\| \Big(\Big|\partial_\theta^k(\mu_s - \mu_s^0(r))\Big| + \Big|\partial_\theta^k\Big(\frac{1}{r}\partial_r(\mu_s - \mu_s^0(r))\Big)\Big| \\ + \Big|\partial_\theta^k\Big(\frac{1}{r}\partial_r\partial_\theta\mu_s\Big)\Big| + \Big|\partial_\theta^k\partial_r^2(\mu_s - \mu_s^0(r))\Big|\Big)\Big\|_{W_2^{0,1}(B_1)} \leqslant E(s,k), \end{aligned} \tag{7.19}$$

$$\Big\| r^2\Big(\Big|\partial_\theta^k\mu_s\Big| + \Big|\partial_\theta^k\Big(\frac{1}{r}\partial_r\mu_s\Big)\Big| + \Big|\partial_\theta^k\Big(\frac{1}{r}\partial_r\partial_\theta\mu_s\Big)\Big| + \Big|\partial_\theta^k\partial_r^2\mu_s\Big|\Big)\Big\|_{W_2^{0,1}(\mathbb{R}^2\setminus B_1)} \leqslant E(s,k), \tag{7.20}$$

$$\begin{aligned} \Big\| \Big(\Big|\partial_\theta^k(\mu_s + u_s)\Big| + \Big|\partial_\theta^k\Big(\frac{1}{r}\partial_r(\mu_s + u_s)\Big)\Big| \\ + \Big|\partial_\theta^k\Big(\frac{1}{r}\partial_r\partial_\theta(\mu_s + u_s)\Big)\Big| + \Big|\partial_\theta^k\partial_r^2(\mu_s + u_s)\Big|\Big)\Big\|_{W_2^{0,1}(B_1)} \leqslant E(s,k), \end{aligned} \tag{7.21}$$

$$\Big\|\partial_\theta^k(\lambda_s + \beta_s R^2)\Big\|_{H^{5/2}(\partial B_1)} \leqslant E(s,k), \tag{7.22}$$

$$|\beta_s| \leqslant \Gamma E(s,0) \tag{7.23}$$

where $\mu_n^0(r)$ is defined in (6.13)–(6.15) with $\lambda_s^0 = \Lambda_0 = \frac{1}{2\pi}\int_0^{2\pi}\lambda_s(\theta)d\theta$ in (6.14), and $C_{20}^s = C_{20}$ in (6.15) is defined by (6.33)–(6.36) with $F^j \equiv F^{js}$:

$$\mu_s^0 = \frac{\lambda_s^0}{R}\Big[\frac{rI_1(Rr)}{I_1(R)} - \frac{I_0(Rr)}{I_0(R)}\Big] + C_{20}^s\frac{I_0(Rr)}{I_0(R)}. \tag{7.24}$$

Note that (7.23) implies that

$$\|\partial_\theta^k\partial_\theta^{3/2}(\lambda_s + \beta_s R^2)\|_{W_2^{0,1}(B_1)} \leqslant CE(s,k).$$

For $s = 1, 2$, the solutions (μ_s, u_s, λ_s) are given in terms of the explicit analytic functions and the estimates (7.19)–(7.23) are valid if we choose H_0 to be large enough and $H \geqslant 1$, $\Gamma \geqslant 1$.

We proceed to establish the estimates (7.19)–(7.23) for $s = n$ assuming $n \geqslant 3$ and H, Γ large enough (independent of n).

In proving these estimates we shall use several lemmas which deal with estimating derivatives of composite functions; these lemmas are stated and proved in Appendix B.

Step 1. Estimating $F^{j,n}$.

We need to estimate $F^{j,n}$ for $j = 1, 2, 3$. To estimate $F^{3,n}$, we rewrite (7.4) as

$$\begin{aligned}
&-\sum_{m\geqslant 1}\varepsilon^m F^{3m} \\
&= \Big(\sum_{m\geqslant 0}\varepsilon^m\beta_m\Big)\Big[\frac{(R+f)^2}{[(R+f)^2+f_\theta^2]^{1/2}} - R\Big] - \frac{1}{R}f \\
&= \Big(\sum_{m\geqslant 2}\varepsilon^m\beta_m\Big)\Big[\frac{(R+f)^2}{[(R+f)^2+f_\theta^2]^{1/2}} - R\Big] + \frac{1}{R}\Big[\frac{(R+f)^2}{[(R+f)^2+f_\theta^2]^{1/2}} - R - f\Big] \\
&\equiv \sum\varepsilon^m\mathcal{Q}_{1,m} + \sum\varepsilon^m\mathcal{Q}_{2,m}.
\end{aligned} \tag{7.25}$$

We can write

$$\frac{(R+\xi)^2}{[(R+\xi)^2+\zeta^2]^{1/2}} - R - \xi = -\,\frac{1}{2R}\zeta^2 + \zeta^2\sum_{1\leqslant i+j<\infty} a_{ij}\xi^i\zeta^{2j},$$

where a_{ij} are constants such that

$$|a_{ij}| \leqslant A_0 A^{i+j}.$$

Note that $\Big[\frac{(R+f)^2}{[(R+f)^2+f_\theta^2]^{1/2}} - R\Big]$ starts with ε order terms and $\Big[\frac{(R+f)^2}{[(R+f)^2+f_\theta^2]^{1/2}} - R - f\Big]$ starts with ε^2 order terms. Since the terms involving orders $> n$ on the right-hand sides do not appear among the terms of $F^{j,n}$, we may replace them by 0 when estimating $F^{j,n}$. By (7.23), $|\beta_{n-1}| \leqslant E(n-1,0) \leqslant \frac{\Gamma}{H}E(n,0)$. Using also (7.23) and applying (B.11) of Lemma B.4 and Lemma B.1 to the $\mathcal{Q}_{1,m}$ terms, we obtain

$$\|\partial_\theta^k\mathcal{Q}_{1,n}\|_{H^{3/2}(\partial B_1)} \leqslant \frac{C(\Gamma)}{H}E(n,k). \tag{7.26}$$

Using (7.22) and applying (B.12) of Lemma B.4 to the $\mathcal{Q}_{2m}$ terms, we also get

$$\|\partial_\theta^k\mathcal{Q}_{2,n}\|_{H^{3/2}(\partial B_1)} \leqslant \frac{C(\Gamma)}{H}E(n,k), \tag{7.27}$$

so that

$$\|\partial_\theta^k F^{3,n}\|_{H^{3/2}(\partial B_1)} \leqslant \frac{C(\Gamma)}{H}E(n,k). \tag{7.28}$$

We next estimate the terms in (7.2) and (7.3) contributing to $F^{1,n}$ and $F^{1,n}+F^{2,n}$. For $r>1$, we can write

$$\begin{aligned}
\Big(-\,\frac{f_{\theta\theta}}{R+f} + \frac{f_{\theta\theta}}{R}\Big)\frac{1}{r}\frac{\partial\mu_0}{\partial r} + \frac{f_\theta^2}{(R+f)^2}\Big(\frac{\partial^2\mu_0}{\partial r^2} + \frac{2}{r}\frac{\partial\mu_0}{\partial r}\Big) &= \Big[\Big(-\,\frac{f_\theta}{R+f} + \frac{f_\theta}{R}\Big)\frac{1}{r^2}\Big]_\theta \\
&\equiv \sum\varepsilon^m\frac{1}{r^2}\frac{\partial}{\partial\theta}g_m(\theta),
\end{aligned}$$

and $\Big(-\,\frac{f_\theta}{R+f} + \frac{f_\theta}{R}\Big)$ starts with ε^2 order terms. Applying (B.12) of Lemma B.4, we obtain

$$\|\partial_\theta^k g_n\|_{H^{3/2}(\partial B_1)} \leqslant \frac{C(\Gamma)}{H}E(n,k). \tag{7.29}$$

Similarly,

$$
\begin{aligned}
&-\frac{f_{\theta\theta}}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial\mu_n}{\partial r}-\frac{2f_\theta}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial^2\mu_n}{\partial\theta\partial r}+\frac{f_\theta^2}{(R+f)^2}\sum_{n\geqslant 1}\varepsilon^n\Big(\frac{\partial^2\mu_n}{\partial r^2}+\frac{2}{r}\frac{\partial\mu_n}{\partial r}\Big)\\
&= -\Big[\frac{f_\theta}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial\mu_n}{\partial r}\Big]_\theta+\Big\{\frac{f_\theta^2}{(R+f)^2}\sum_{n\geqslant 1}\varepsilon^n\Big(\frac{\partial^2\mu_n}{\partial r^2}+\frac{1}{r}\frac{\partial\mu_n}{\partial r}\Big)-\frac{f_\theta}{R+f}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial^2\mu_n}{\partial\theta\partial r}\Big\}\\
&\equiv \sum_m\varepsilon^m\frac{\partial}{\partial\theta}G_m^1(r,\theta)+\sum_m\varepsilon^m K_m(r,\theta).
\end{aligned}
$$

Using (7.20), (7.22) and Lemmas B.1, B.4, we obtain

$$
\|r^3\partial_\theta^k\partial_r G_n^1\|_{W_2^{0,1}(\mathbb{R}^2\setminus B_1)}+\|r^2\partial_\theta^k G_n^1\|_{W_2^{1,3/2}(\mathbb{R}^2\setminus B_1)}\leqslant\frac{C(\Gamma)}{H}E(n,k) \tag{7.30}
$$

$$
\|r^2\partial_\theta^k K_n\|_{W_2^{0,1}(\mathbb{R}^2\setminus B_1)}\leqslant\frac{C(\Gamma)}{H}E(n,k). \tag{7.31}
$$

This completes the estimate for $F^{1,n}$ in the region $r>1$. The estimate for $F^{1,n}$ and $F^{1,n}+F^{2,n}$ in the region $r<1$ can be carried out in a similar way. Using the equations of μ_n, u_n, λ_n and the equations obtained by differentiation $k+1$ times in θ, we can apply Lemmas 6.4 and Remark 6.4 to conclude that

$$
\text{[the left-hand sides of (7.19)–(7.22) for s=n]}\leqslant\frac{C(\Gamma)}{H}E(n,k)\quad\text{ for } k\geqslant 0. \tag{7.32}
$$

Note that the part of Lemma 6.4 which is based on Lemma 6.2 is used in handling $\frac{\partial}{\partial\theta}g_n$, $\frac{\partial}{\partial\theta}G_n^1$, and similar terms from $F^{1,n}$ in $\{r>1\}$ as well as from $F^{1,n}$ in $\{r<1\}$, whereas the part based on Lemma 6.1 is used in handling K_n and similar terms from F^{1n} in $\{r>1\}$.

Step 2. Estimating β_n.

Recall that β_n is determined by the equation (6.82) with $F_l^j=F_l^{j,n+1}$ for $j=1,2,3$. (Only mode l terms enter into the compatibility condition).

The coefficient for β_n in (6.82) is given by (7.13), which is independent of n and $\neq 0$. Consequently, β_n is estimated by those terms in $F_l^{j,n+1}$ which do not depend on β_n. We shall denote these terms by $\widetilde{F}_l^{j,n+1}$, i.e., $\widetilde{F}_l^{j,n+1}=F_l^{j,n+1}\Big|_{\beta_n=0}$. We need to estimate these terms and we begin with

a) $\widetilde{F}_l^{3,n+1}$.

By (7.25),

$$
\begin{aligned}
&-\sum_{m\geqslant 1}\varepsilon^m F^{3m}\Big|_{\beta_n=0}\\
&=\Big[\Big(\sum_{m\geqslant 2}\varepsilon^m\beta_m\Big)f-\Big(\sum_{m\geqslant 2}\varepsilon^m\beta_m\Big)\frac{f_\theta^2}{2R}+\Big(\sum_{m\geqslant 2}\varepsilon^m\beta_m\Big)f_\theta^2\sum_{1\leqslant i+j<\infty}a_{ij}\xi^i(f_\theta)^{2j}\\
&\qquad-\frac{1}{2R^2}f_\theta^2+\frac{f_\theta^2}{R}\sum_{1\leqslant i+j<\infty}a_{ij}\xi^i(f_\theta)^{2j}\Big]_{\beta_n=0} \qquad (7.33)\\
&\equiv\Big[\sum\varepsilon^m\mathcal{P}_{1,m}+\sum\varepsilon^m\mathcal{P}_{2,m}+\sum\varepsilon^m\mathcal{P}_{3,m}+\sum\varepsilon^m\mathcal{P}_{4,m}+\sum\varepsilon^m\mathcal{P}_{5,m}\Big]_{\beta_n=0}.
\end{aligned}
$$

Clearly, the second, third, and the fifth sum are products of at least 3 series. Thus, by Lemmas B.1 and B.4, using also the embedding inequality $\|g\|_{L^\infty}\leqslant C\|g\|_{H^s}$ for $s>1/2$,

$$
\Big[|\mathcal{P}_{2,n+1}|+|\mathcal{P}_{3,n+1}|+|\mathcal{P}_{5,n+1}|\Big]_{\beta_n=0}\leqslant\frac{C(\Gamma)}{H^2}E(n+1,0)\leqslant\frac{C(\Gamma)}{H}E(n,0). \tag{7.34}
$$

Next, by Remark 7.1, the $\lambda_m(\theta)$ for $m \geqslant 2$ do not contain mode l terms. Therefore the only term in $\sum \varepsilon^m \mathcal{P}_{1,m}$ of order ε^{n+1} and of mode l is given by $\beta_n \lambda_1(\theta)$; hence, $|\mathcal{P}_{1,n+1}|_{\beta_n=0} = 0$.

Finally,

$$\mathcal{P}_{4,n+1} = -\frac{1}{2R^2} \sum_{k=2}^{n-1} \partial_\theta \lambda_{n+1-k} \partial_\theta \lambda_k,$$

so that, by the inductive assumptions and Lemma B.1,

$$|P_{4,n+1}| \leqslant \frac{C}{H} E(n+1, 0) \leqslant C E(n, 0). \tag{7.35}$$

Combining these estimates, we conclude that

$$|\widetilde{F}_l^{3,n+1} + \beta_n| \leqslant \Big[C + \frac{C(\Gamma)}{H}\Big] E(n, 0). \tag{7.36}$$

(b) $\widetilde{F}_l^{1,n+1} + \widetilde{F}_l^{2,n+1}$.

Clearly, $\widetilde{F}_l^{1,n+1} + \widetilde{F}_l^{2,n+1} = \frac{1}{\pi} \int_0^{2\pi} (\widetilde{F}^{1,n+1} + \widetilde{F}^{2,n+1}) \cos l\theta d\theta$.

To estimate the contribution from the first term on the right-hand side of (7.3) we multiply by $\cos l\theta$ and integrate over θ, $0 \leqslant \theta \leqslant 2\pi$; in this way we obtain the coefficients of mode l terms. Next we integrate by parts to reduce the derivative $\partial_\theta^2 f$. We get

$$\begin{aligned}
& -\int_0^{2\pi} \frac{2f_\theta}{R+f} \sum_{m\geqslant 1} \varepsilon^m \frac{1}{r} \frac{\partial^2(\mu_m + u_m)}{\partial\theta\partial r} \cos l\theta d\theta \\
& + \int_0^{2\pi} \frac{2f_\theta^2}{(R+f)^2} \sum_{m\geqslant 1} \varepsilon^m \frac{1}{r} \frac{\partial(\mu_m + u_m)}{\partial r} \cos l\theta d\theta \\
& - \int_0^{2\pi} \frac{f_\theta}{R+f} \sum_{m\geqslant 1} \varepsilon^m \frac{1}{r} \frac{\partial(\mu_m + u_m)}{\partial r} \cos l\theta d\theta \\
\equiv \; & -J_1 + J_2 - J_3.
\end{aligned}$$

We also expand

$$\frac{1}{R+f} = \frac{1}{R} - \frac{1}{R} \sum_{m\geqslant 1} \frac{(-f)^m}{R^m}.$$

Now recall that f_θ is independent of Γ (but f depends on Γ). As we substitute the expansion of $1/(R+f)$ into the J_k, we observe that whenever a factor f^k appears in a particular term, it comes in a product of at least three series in ε, each with no zero order term. We can therefore apply Lemma B.1 and the inductive assumptions to get, at $\beta_n = 0$, the bound

$$C_0 \Big(\frac{C_0}{H}\Big)^{k-1} \frac{C(\Gamma)}{H} E(n, 0)$$

where C_0 depends only on H_0. There remain only the terms which do not depend on f and they come as product of two series. Using the inductive assumptions we can bound them, at $\beta_n = 0$, by $CE(n, 0)$. We thus conclude that the coefficient of mode l in ε^{n+1} of the first term on the right-hand side of (7.3) is bounded, at $\beta_n = 0$, by

$$\Big[C + \frac{C(\Gamma)}{H}\Big] E(n, 0). \tag{7.37}$$

The other terms on the right-hand side of (7.3) can be estimated in a similar manner. Thus

$$\int_0^1 \tau^{l+1} |\widetilde{F}_l^{1,n+1}(\tau) + \widetilde{F}_l^{2,n+1}(\tau)| d\tau \leqslant \Big[C + \frac{C(\Gamma)}{H}\Big] E(n,0). \tag{7.38}$$

(c) $\widetilde{F}_l^{1,n+1}(r)$, $r > 1$.

$\widetilde{F}_l^{1,n+1}$ in the region $r > 1$ can be handled as in (b), except for the first two terms in (7.2), which give

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi} \Big[\Big(-\frac{f_\theta}{R+f} + \frac{f_\theta}{R}\Big)\frac{1}{r^2}\Big]_\theta \cos l\theta d\theta &= \frac{l}{\pi}\int_0^{2\pi} \Big[\Big(-\frac{f_\theta}{R+f} + \frac{f_\theta}{R}\Big)\frac{1}{r^2}\Big] \sin l\theta d\theta \\ &= \frac{l}{\pi R r^2}\int_0^{2\pi} \frac{f_\theta f}{f+R} \sin l\theta d\theta.\end{aligned}$$

We write $f_\theta f/(f+R)$ in the form $f_\theta f/R + \cdots$, where the "$\cdots$" refers to terms which are products of at least three series for which we can apply Lemma B.4 to get an extra $1/H$ factor. The remaining term is:

$$\frac{l}{\pi R r^2}\int_0^{2\pi} \frac{f_\theta f}{R} \sin l\theta d\theta = \frac{l}{\pi R r^2}\int_0^{2\pi} \frac{f_\theta (f - f_0)}{R} \sin l\theta d\theta + \frac{l f_0}{\pi R r^2}\int_0^{2\pi} \frac{f_\theta}{R} \sin l\theta d\theta, \tag{7.39}$$

where $f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta$. Since f_θ is independent of Γ and $f - f_0$ can be estimated in the L^2 norm by the θ derivatives of f, the coefficient of $\varepsilon^{n+1}\cos l\theta$ in the first term on the right-hand side of (7.39), at $\beta_n = 0$, is bounded by

$$\frac{C}{H} E(n+1, 0) = C E(n,0).$$

To estimate the coefficient of $\varepsilon^{n+1}\cos l\theta$ in the second term, note that by (7.14)

$$\int_0^{2\pi} f_\theta \sin l\theta d\theta = \varepsilon \int_0^{2\pi} \lambda_1'(\theta) \sin l\theta d\theta = -\pi l \varepsilon.$$

It follows that this coefficient is equal to

$$-\frac{l^2}{R^2 r^2}\lambda_{n0}$$

where λ_{n0} is the zero mode of λ_n and, by (7.22), $|\lambda_{n0}| \leqslant E(n,0)$ if $\beta_n = 0$. Thus the second term on the right-hand side of (7.39) does not contribute anything to $\widetilde{F}_l^{1,n+1}$. In summary,

$$\int_1^\infty \tau^{-l+1} \Big|\widetilde{F}_l^{1,n+1}(\tau)\Big| d\tau \leqslant \Big[C + \frac{C(\Gamma)}{H}\Big] E(n,0). \tag{7.40}$$

(d) $\widetilde{F}_l^{1,n+1}(r)$, $r < 1$.

The estimate for $\widetilde{F}_l^{1,n+1}$ in $r < 1$ is the most difficult, since some of the derivatives of μ_n involve β_n. The terms that arise as the product of 3 series are treated in a same way as before. The remaining terms come from

$$\frac{f_{\theta\theta} f}{R^2}\frac{1}{r}\frac{\partial \mu_0}{\partial r} + \frac{f_\theta^2}{R^2}\Big(\frac{\partial^2 \mu_0}{\partial r^2} + \frac{2}{r}\frac{\partial \mu_0}{\partial r}\Big) - \frac{2 f_\theta}{R}\sum_{n\geqslant 1} \varepsilon^n \frac{1}{r}\frac{\partial^2 \mu_n}{\partial\theta\partial r}$$

$$
\begin{aligned}
&-\frac{f_{\theta\theta}}{R}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial\mu_n}{\partial r}-2Rf\sum_{n\geqslant 1}\varepsilon^n\mu_n-f^2\mu_0\\
=\ &\Big(\frac{f_\theta f}{R^2}\Big)_\theta\frac{1}{r}\frac{\partial\mu_0}{\partial r}+\frac{f_\theta^2}{R^2}\Big(\frac{\partial^2\mu_0}{\partial r^2}+\frac{1}{r}\frac{\partial\mu_0}{\partial r}\Big)-\frac{f_\theta}{R}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial^2\mu_n}{\partial\theta\partial r}\\
&-\Big(\frac{f_\theta}{R}\sum_{n\geqslant 1}\varepsilon^n\frac{1}{r}\frac{\partial\mu_n}{\partial r}\Big)_\theta-2Rf\sum_{n\geqslant 1}\varepsilon^n\mu_n-f^2\mu_0\\
\equiv\ &\sum\varepsilon^m\mathcal{R}_{1,m}+\sum\varepsilon^m\mathcal{R}_{2,m}+\sum\varepsilon^m\mathcal{R}_{3,m}+\sum\varepsilon^m\mathcal{R}_{4,m}+\sum\varepsilon^m\mathcal{R}_{5,m}+\sum\varepsilon^m\mathcal{R}_{6,m}.
\end{aligned}
$$

The ε^{n+1} terms in the series $\sum\varepsilon^m\mathcal{R}_{2,m}$ and $\sum\varepsilon^m\mathcal{R}_{3,m}$ are estimated by the inductive assumptions independently of Γ. Thus, as before,

$$
\Big[\|\mathcal{R}_{2,n+1}\|+\|\mathcal{R}_{3,n+1}\|\Big]_{\beta_n=0}\leqslant CE(n,0),
$$

where the norm $\|\cdot\|$ is defined in (7.37). The series $\sum\varepsilon^m\mathcal{R}_{1,m}$ can be treated in a same way as part (c):

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{2\pi}\Big(\frac{f_\theta f}{R^2}\Big)_\theta\frac{1}{r}\frac{\partial\mu_0}{\partial r}\cos l\theta d\theta\\
&\quad=\frac{l}{R^2\pi}\frac{1}{r}\frac{\partial\mu_0}{\partial r}\int_0^{2\pi}f_\theta(\theta)[f(\theta)-f_0]\sin l\theta d\theta+\frac{lf_0}{R^2\pi}\frac{1}{r}\frac{\partial\mu_0}{\partial r}\int_0^{2\pi}f_\theta\sin l\theta d\theta,
\end{aligned}
$$

where the estimate for the first term is independent of Γ whereas the coefficient of $\varepsilon^{n+1}\cos l\theta$ from the second term is zero when we set $\beta_n=0$.

To estimate the coefficient of mode l in $\mathcal{R}_{4,n+1}\Big|_{\beta_n=0}$ we write

$$
\mu_m=[\mu_m(r,\theta)-\mu_m^0(r)]+\mu_m^0(r). \tag{7.41}
$$

Then the series corresponding to $\mu_m(r,\theta)-\mu_m^0(r)$ (in $\sum\mathcal{R}_{4,m}$) can be treated as before, as product of two series, using Lemma B.1 and inductive assumption; since $\partial_r(\mu_m-\mu_m^0)$ is independent of Γ, we get the bound $CE(n,0)$. Thus it remains to estimate the coefficient of $\varepsilon^{n+1}\cos l\theta$ in

$$
-\Big(\frac{f_\theta}{R}\Big)_\theta\sum_{m\geqslant 1}\varepsilon^m\frac{1}{r}\frac{\partial\mu_m^0(r)}{\partial r}\quad\text{when }\beta_n=0,
$$

where $\mu_m^0(r)$ is defined as in (7.24) and the constant C_{20}^m is estimated by (6.11) with $\Lambda=\lambda_m, M=\mu_m, U=u_m, \beta=\beta_m$. Again, since $f-\varepsilon\lambda_1(\theta)$ has no mode l terms, the l mode in the coefficient of ε^n of the above series is

$$
\begin{aligned}
&-\frac{1}{\pi}\int_0^{2\pi}\Big(\frac{\lambda_1'(\theta)}{R}\Big)'\frac{1}{r}\frac{\partial\mu_n^0(r)}{\partial r}\cos l\theta d\theta\\
&=\ l^2\frac{1}{Rr}\frac{\partial\mu_n^0(r)}{\partial r}
\end{aligned}
$$

and, at $\beta_n=0$, this yields

$$
\frac{l^2}{Rr}\frac{\lambda_{n0}}{R}\frac{\partial}{\partial r}\Big[\frac{rI_1(Rr)}{I_1(R)}-\frac{I_0(Rr)}{I_0(R)}\Big]+C_{20}^n\frac{\partial}{\partial r}\frac{I_0(Rr)}{I_0(R)}
$$

where λ_{n0} (the zero mode of λ_n) and C_{20}^n are bounded, at $\beta_n = 0$, by $CE(n,0)$ (by Lemma 6.1). We conclude that

$$\Big| \int_0^{2\pi} \mathcal{R}_{4,n+1} \cos l\theta d\theta \Big|_{\beta_n=0} \leqslant CE(n,0).$$

Consider next $\sum \varepsilon^m \mathcal{R}_{5,m}$. If we decompose μ_m as in (7.41), then the part corresponding to $\mu_m - \mu_m^0$ can be treated similarly to (7.39) (with f_θ replaced by $\mu_m - \mu_m^0$). Thus it remains to estimate the coefficient of $\varepsilon^{n+1} \cos l\theta$ in

$$-2Rf \sum \varepsilon^m \mu_m^0 \quad \text{at } \beta_n = 0,$$

which is equal to $-2R\mu_n^0$ at $\beta_n = 0$. As before, if $\beta_n = 0$, then

$$|\mu_n^0| \leqslant C(|\lambda_{n0}| + |C_{20}^n|) \leqslant CE(n,0),$$

so that

$$\Big| \int_0^{2\pi} \mathcal{R}_{5,n+1} \cos l\theta d\theta \Big| \leqslant CE(n,0).$$

Consider finally $\sum \varepsilon^m \mathcal{R}_{6,m}$. Writing

$$f^2 = (f - f_0)^2 + 2(f - f_0) f_0 + f_0^2,$$

the terms $(f - f_0)^2$ and $(f - f_0) f_0$ can be treated as before and we obtain the bound $CE(n,0)$ on the coefficient of $\varepsilon^{n+1} \cos l\theta$. Since f_0^2 is a constant, it has no mode l terms, and thus altogether,

$$\Big| \int_0^{2\pi} \mathcal{R}_{6,n+1} \cos l\theta d\theta \Big|_{\beta_n=0} \leqslant CE(n,0).$$

Combining all these estimates, the compatibility condition for β_n yields

$$|\widetilde{\Lambda} \beta_n| \leqslant \Big[C + \frac{C(\Gamma)}{H} \Big] E(n,0), \tag{7.42}$$

where $\widetilde{\Lambda}$ was defined in Lemma 7.1, and $|\widetilde{\Lambda}| \geqslant c > 0$.

Choosing Γ and H such that

$$\frac{C(\Gamma)}{H} \leqslant 1, \quad \max\{1, \frac{1}{c}\}(C+1) \leqslant \Gamma, \tag{7.43}$$

(for the $C(\Gamma)$ that appear in both (7.42) and (7.32)), we conclude that (7.23) holds for $s = n$ and then, by (7.32), also (7.19)–(7.22) hold for $s = n$.

This completes the induction proof.

Remark 7.2. Having proved (7.19)–(7.22) for all s, we get (with Γ now fixed) $|\lambda_s^0| \leqslant CE(s,0)$ (from (7.22), (7.23)) and $|C_{20}^s| \leqslant CE(s,0)$ (by (6.44) applied with $F^j = F^{j,s}$, $\beta = \beta_s$, $C_{20} = C_{20}^s$). Then also $|\mu_s^0| \leqslant CE(s,0)$, by (7.25). Consequently (7.19)–(7.22) are valid after deleting the μ_n^0 and β_n terms from the left-hand sides, and changing H to a larger number on the right-hand sides, if necessary.

Using Remark 7.2, we conclude:

Theorem 7.3 *There exists a solution of (5.4)–(5.8) of the form (5.2), (5.9)–(5.12) where the series are convergent for $|\varepsilon| \leqslant \varepsilon_0$, for some $\varepsilon_0 > 0$, and define analytic functions in (θ, ε) for $|\varepsilon| \leqslant \varepsilon_0$, $0 \leqslant \theta \leqslant 2\pi$; the solution is unique under the assumption that it is even in θ, it contains no mode 1 terms, and*

$$\int_0^{2\pi} \lambda_n(\theta) \cos l\theta d\theta = 0 \ (n \geqslant 2).$$

8 Bifurcation branches of analytic solution for the original problem

The estimates (7.19)–(7.22) can be extended to include derivatives with respect to r to any order, namely:

$$\begin{aligned} \|r^m\partial_r^m\partial_\theta^k\mu_n\|_{W_2^{0,1}(B_1)} &\leqslant \frac{H_0H^{n-1}}{n^2}(k+m)!\frac{B^k}{k^2}D_0D^m \\ \|r^m\partial_r^m\partial_\theta^k\mu_n\|_{W_2^{0,1}(\mathbb{R}^n\setminus B_1)} &\leqslant \frac{H_0H^{n-1}}{n^2}(k+m)!\frac{B^k}{k^2}D_0D^m \\ \|r^m\partial_r^m\partial_\theta^k u_n\|_{W_2^{0,1}(B_1)} &\leqslant \frac{H_0H^{n-1}}{n^2}(k+m)!\frac{B^k}{k^2}D_0D^m \end{aligned} \tag{8.1}$$

The proof is by induction on m. For $m = 1, 2$ this is already proved in (7.19)–(7.20) (if we choose $D \gg B$). Suppose now that (8.1) is true for all $m < j$. To prove it for $m = j$ we apply $r^jD_r^{j-2}D_\theta^k$ to the equations (6.3)–(6.4) with $M = \mu_n$, $U = u_n$, $\Lambda = \lambda_n$ and then move all the terms to the right-hand side, except for $r^jD_r^jD_\theta^k\mu_n$, to get

$$\begin{aligned} r^jD_r^jD_\theta^k\mu_n &= r^jD_r^{j-2}D_\theta^kF^{1n} + \chi_{\{r<1\}}r^jD_r^{j-2}D_\theta^k[R^2\mu_n + 2R\lambda_n\mu_0] \\ &\quad + \frac{1}{R}r^jD_r^{j-2}\Big(\frac{1}{r}\frac{\partial\mu_0}{\partial r}\Big)D_\theta^{k+2}\lambda_n - r^jD_r^{j-2}D_\theta^k\Big(\frac{1}{r}D_r\mu_n\Big) - r^jD_r^{j-2}\Big(\frac{1}{r^2}D_\theta^{k+2}\mu_n\Big). \end{aligned}$$

From (7.2)–(7.4) we find that the terms in F^{1n} are of the form to which Lemma B.5 can be applied. Thus these terms can be estimated in the same way as in §7, namely, we consider $D_\theta^nF^{1n}$ as a function to which we apply successive r-derivatives and use Lemma B.5 and the induction assumption on j. Similarly other terms can be estimated using induction. We remark here that in the last term, which involves $D_\theta^{k+2}D_r^{j-2}\mu_n$, we "loose" two θ derivatives, but we "gain" two r-derivatives; the $(k+m)!$ factor together with the choice $D \gg B$ enable us to carry out the induction for μ_n, and similarly for u_n.

The estimates in (8.1) show that μ is analytic in (x,ε) if $0 < |x| \leqslant 1$, $|\varepsilon| \leqslant \varepsilon_0$ for some $\varepsilon_0 > 0$, and in $1 \leqslant |x|, |\varepsilon| \leqslant \varepsilon_0$ and u is analytic in (x,ε) if $0 < |x| \leqslant 1, |\varepsilon| \leqslant \varepsilon_0$, for some ε_0; recall that the free boundary is analytic in (θ,ε) if $0 \leqslant \theta \leqslant 2\pi, |\varepsilon| \leqslant \varepsilon_0$.

These facts allow us to reverse the mapping (5.1) and obtain well defined analytic functions

$$\mu(x,\varepsilon), u(x,\varepsilon)$$

and free boundary $r = R_{0l} + f(\theta,\varepsilon)$ in the original variables, which satisfy the system (1.1)–(1.6) with β as in (1.8); more precisely,

$$\begin{aligned} &\mu^+(x,\varepsilon) \text{ is analytic in } (x,\varepsilon) \text{ for } |x| \geqslant R_{0l} + f(\theta,\varepsilon), |\varepsilon| \leqslant \varepsilon_0 \\ &\mu^-(x,\varepsilon) \text{ is analytic in } (x,\varepsilon) \text{ for } 0 < |x| \leqslant R_{0l} + f(\theta,\varepsilon), |\varepsilon| \leqslant \varepsilon_0 \end{aligned}$$

and these functions can actually be extended analytically into some δ_0-neighborhood of the free boundary where δ_0 is a positive number independent of ε.

It is now easy to extend both μ^- and u^- as analytic functions in (x,ε) also in a neighborhood of $x = 0$, as in [4]. Indeed, since μ^- is bounded in $0 < |x| \leqslant 1/2$, we can represent it in the form

$$\mu^-(x,\varepsilon) = \int_{|x|=\delta}\frac{\partial G}{\partial n}(x-y)\mu^-(y,\varepsilon)dS_y - \int_{|x|=\delta}G(x-y)\frac{\partial}{\partial n}\mu^-(y,\varepsilon)dS_y$$

for $0 < |x| \leqslant \delta$ ($\delta < 1/2$) where G is a fundamental solution of $\Delta - 1$. The right-hand side provides the analytic extension of $\mu^-(x,\varepsilon)$ to $|x| \leqslant \delta$, $|\varepsilon| \leqslant \varepsilon_0$. Next, since $\mu^-(x,\varepsilon) + u(x,\varepsilon)$ is harmonic and bounded in $0 < |x| \leqslant 1/2$, we can similarly extend it to $|x| \leqslant \delta, |\varepsilon| \leqslant \varepsilon_0$, and thus conclude that the function $u(x,\varepsilon)$ also has analytic extension to $|x| \leqslant \delta, |\varepsilon| \leqslant \varepsilon_0$.

We have thus completed the proof of the following theorem, which is the main result of this paper:

Theorem 8.1 *For any integer $l \geqslant 2$ there exists a family of solutions of (1.1)–(1.6) with free boundary $r = R_{0l} + f(\theta,\varepsilon)$ where*

$$f(\theta,\varepsilon) = \varepsilon \cos l\theta + O(\varepsilon^2).$$

The solution $\mu, u, r = R_{0l} + f(\theta,\varepsilon), \beta$ has the form (1.13), (1.14), (1.7), (1.8) and the series converge and define analytic function in (x,ε) for μ, u and in (θ,ε) for f; more precisely

$$\begin{aligned}&\mu^+(x,\varepsilon) \text{ is analytic in } |x| \geqslant R_{0l} - \delta_1, |\varepsilon| \leqslant \varepsilon_0,\\ &\mu^-(x,\varepsilon) \text{ and } u(x,\varepsilon) \text{ are analytic in } |x| \leqslant R_{0l} + \delta_1, |\varepsilon| \leqslant \varepsilon_0,\end{aligned}$$

for some $\varepsilon_0 > 0, \delta_1 > 0$; furthermore, the solution is even in θ and satisfies the conditions (1.9)–(1.11). Finally, R_{0l} and the solution with all the above propertied are unique.

A Some facts about Bessel functions

A.1 Basic Properties of Bessel functions

In this appendix we establish various facts about the Bessel functions $I_m(x)$ for $m \geqslant 0$, $x \geqslant 0$. We recall that $I_m(x)$ satisfies the differential equation

$$I''_m(x) + \frac{1}{r} I'_m(x) - \Big(1 + \frac{m^2}{x^2}\Big) I_m(x) = 0 \tag{A.1}$$

and is given by

$$I_m(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{m+2k}}{k!\Gamma(m+k+1)}. \tag{A.2}$$

Furthermore,

$$I'_m(x) + \frac{m}{x} I_m(x) = I_{m-1}(x), \quad m \geqslant 1, \tag{A.3}$$

$$I'_m(x) - \frac{m}{x} I_m(x) = I_{m+1}(x), \tag{A.4}$$

so that

$$I_{m-1}(x) - I_{m+1}(x) = \frac{2m}{x} I_m(x), \quad m \geqslant 1, \tag{A.5}$$

and

$$r^{m+1} I_m(x) = \frac{d}{dx}\Big(x^{m+1} I_{m+1}(x)\Big), \tag{A.6}$$

$$I_m(x) = \Big(\frac{2}{\pi x}\Big)^{1/2} e^x [1 + O(\frac{1}{x})] \quad \text{if } x \to \infty, \tag{A.7}$$

$$I_m(x) I_n(x) = \sum_{k=0}^{\infty} \frac{\Gamma(m+n+2k+1)(x/2)^{m+n+2k}}{k!\Gamma(m+k+1)\Gamma(n+k+1)\Gamma(m+n+k+1)}. \tag{A.8}$$

In particular,

$$I_0'(x) = I_1(x), \qquad xI_0(x) = \frac{d}{dx}\Big(rI_1(x)\Big).$$

We also recall [9, p. 225] the relation

$$I_m(x) = \sqrt{\frac{1}{2m\pi}}\Big(\frac{ex}{2m}\Big)^m\Big(1+O(\frac{1}{m})\Big) \quad m\to\infty \tag{A.9}$$

uniformly in x in any bounded set.

Consider the function

$$f_m(x) = \frac{I_0(x)I_m(x)}{I_1(x)I_{m-1}(x)}, \quad m\geqslant 2. \tag{A.10}$$

Theorem A.1 *If $m\geqslant 2$, then*

$$\frac{d}{dx}f_m(x) > 0 \quad \textit{for all } x>0. \tag{A.11}$$

To prove the theorem we first study the functions

$$S_m(x) = \log I_m(x). \tag{A.12}$$

Lemma A.2 *For any $m\geqslant 1$,*

$$S_m'(x) - S_{m-1}'(x) > 0 \quad \textit{for all } x>0, \tag{A.13}$$

and, consequently,

$$S_m'(x) - S_0'(x) > 0 \quad \textit{for all } x>0. \tag{A.14}$$

Proof. From (A.1) we have

$$S_m'' + (S_m')^2 + \frac{1}{x}S_m' = 1 + \frac{m^2}{x^2}. \tag{A.15}$$

Similarly,

$$S_{m-1}'' + (S_{m-1}')^2 + \frac{1}{x}S_{m-1}' = 1 + \frac{(m-1)^2}{x^2}$$

so that

$$(S_m - S_{m-1})'' + (S_m' + S_{m-1}')(S_m' - S_{m-1}') + \frac{1}{x}(S_m' - S_{m-1}') = \frac{m^2-(m-1)^2}{x^2} > 0. \tag{A.16}$$

As $x\to 0$

$$S_m = m\log x + O(1), \qquad S_{m-1} = (m-1)\log x + O(1)$$

so that

$$S_m - S_{m-1} = \log x + O(1)$$

and

$$S_m' - S_{m-1}' = \frac{1}{x} + O(1).$$

Thus (A.13) holds if x is small.

If (A.13) does not hold for all $x > 0$ then there is a smallest value $x = x_0 > 0$ for which (A.13) is not satisfied. Clearly

$$S'_m(x_0) - S'_{m-1}(x_0) = 0 \tag{A.17}$$

and

$$S''_m(x_0) - S''_{m-1}(x_0) \leqslant 0.$$

On the other hand, from (A.16) and (A.17) we infer that

$$S''_m(x_0) - S''_{m-1}(x_0) = \frac{m^2 - (m-1)^2}{x_0^2} > 0,$$

which is a contradiction. □

Lemma A.3 *For any $m \geqslant 1$*

$$S'_m(x) - S'_0(x) < \frac{m}{x} \quad \textit{for all } x > 0. \tag{A.18}$$

Proof. In view of (A.12), (A.18) is equivalent to

$$\frac{I'_m(x)}{I_m(x)} - \frac{I'_0(x)}{I_0(x)} < \frac{m}{x}$$

or, by (A.4), to

$$\frac{m}{x} + \frac{I_{m+1}(x)}{I_m(x)} - \frac{I'_0(x)}{I_0(x)} < \frac{m}{x}$$

which is the same as

$$\frac{I_{m+1}}{I_m} < \frac{I_1}{I_0} \quad \text{for } x > 0. \tag{A.19}$$

Using the product formula (A.8) it was proved in [4] that each term in the power series of $I_{m+1}(x)I_{m-1}(x)$ is smaller that each term in the power series of $(I_m(x))^2$, so that

$$\frac{I_{m+1}}{I_m} < \frac{I_m}{I_{m-1}}$$

and (A.19) then follows by iterating this inequality. □

Proof of Theorem A.1. Introduce the function

$$\varphi(x) = \Big(S_m(x) - S_{m-1}(x)\Big) - \Big(S_1(x) - S_0(x)\Big), \tag{A.20}$$

the assertion (A.11) is equivalent to showing that

$$\varphi'(x) > 0 \quad \text{if } x > 0. \tag{A.21}$$

Writing (A.16) for $m = 1$ and subtracting from (A.16), we obtain

$$\varphi'' + \Big\{(S'_m + S'_{m-1})(S'_m - S'_{m-1}) - (S'_1 + S'_0)(S'_1 - S'_0)\Big\} + \frac{1}{x}\varphi' = \frac{2(m-1)}{x^2} > 0 \quad \text{if } m \geqslant 2. \tag{A.22}$$

For x small,

$$
\begin{aligned}
\frac{I_m}{I_{m-1}}\frac{I_0}{I_1} &= \frac{(\frac{x}{2})^m[\frac{1}{\Gamma(m+1)}+(\frac{x}{2})^2\frac{1}{\Gamma(m+2)}]}{(\frac{x}{2})^{m-1}[\frac{1}{\Gamma(m)}+(\frac{x}{2})^2\frac{1}{\Gamma(m+1)}]}\,\frac{1+(\frac{x}{2})^2}{\frac{x}{2}[1+\frac{1}{2}(\frac{x}{2})^2]}[1+O(x^2)] \\
&= \frac{1}{m}\frac{1+\frac{z}{m+1}}{1+\frac{z}{m}}\frac{1+z}{1+\frac{z}{2}}(1+O(z)) \qquad (z=(x/2)^2) \\
&= \frac{1}{m}\Big[1+\Big(\frac{1}{2}-\frac{1}{m(m+1)}\Big)z\Big]+O(z^2)
\end{aligned}
$$

and, since $m \geqslant 2$, (A.21) holds for x small.

If (A.21) does not hold for all $x>0$, then there is a smallest $x=x_0$ such that

$$\varphi'(x_0)=0 \tag{A.23}$$

and, clearly, $\varphi''(x_0) \leqslant 0$, so that, by (A.22),

$$\frac{2(m-1)}{x^2} \leqslant \Big\{(S'_m+S'_{m-1})(S'_m-S'_{m-1})-(S'_1+S'_0)(S'_1-S'_0)\Big\} \quad \text{at } x=x_0. \tag{A.24}$$

Writing (A.23) in the form

$$S'_m-S'_{m-1}=S'_1-S'_0 \quad \text{at } x=x_0, \tag{A.25}$$

we see that the right-hand side of (A.24) is equal to

$$
\begin{aligned}
&[(S'_m+S'_{m-1})-(S'_1+S'_0)](S'_1-S'_0) \\
=\ &[(S'_{m-1}+(S'_1-S'_0)+S'_{m-1})-(S'_1+S'_0)](S'_1-S'_0) \\
=\ &(2S'_{m-1}-2S'_0)(S'_1-S'_0)
\end{aligned}
$$

which, by Lemmas A.2 and A.3, is smaller than

$$2\frac{m-1}{x}\frac{1}{x};$$

this is a contradiction to (A.24). □.

From (A.2) and (A.7) we see that

$$
\begin{aligned}
f_m(x) &\to 1 \quad \text{if } x\to\infty, \\
f_m(x) &\to \frac{1}{m} \quad \text{if } x \to 0
\end{aligned}
$$

so that, if $m \geqslant 2$,

$$f_m(x)-\frac{m+1}{2m}$$

is negative for small x and positive for large x. Hence:

Corollary A.4 *For any $m \geqslant 2$ there exists a unique solution $x=R_{0m}$ of the equation*

$$\frac{I_0(R_{0m})}{I_1(R_{0m})}\frac{I_m(R_{0m})}{I_{m-1}(R_{0m})}=\frac{m+1}{2m}. \tag{A.26}$$

Theorem A.5 *For any $m \neq l$, $m \neq 1$, $l \geqslant 2$,*

$$\frac{I_0(R_{0l})}{I_1(R_{0l})}\frac{I_m(R_{0l})}{I_{m-1}(R_{0l})} \neq \frac{m+1}{2m}; \tag{A.27}$$

furthermore,

$$R_{0l} < R_{0m} \quad \text{if } 2 \leqslant l < m. \tag{A.28}$$

The proof is given in Sections A.2 and A.3 and uses, among other things the interesting fact (proved in Section A.4) that the function I_1/I_0 is a concave.

We recall the following result due to Friedman and Reitich [4]

Theorem A.6 *The following identity holds for any $m \geqslant 2$:*

$$\frac{1}{x^2}\int_0^x \Big[m^2 I_1(s) I_m(s) - 2sI_0(s)I_m(s)\Big]\, ds - \frac{m-2}{x}I_1(x)I_m(x) + \Big[I_2(x)I_m(x) - I_1(x)I_{m+1}(x)\Big] \equiv 0. \tag{A.29}$$

A.2 Proof of Theorem A.5

By (A.4)

$$\frac{I'_{m-1}(x)}{I_{m-1}(x)} - \frac{m-1}{x} = \frac{I_m(x)}{I_{m-1}(x)}, \tag{A.30}$$

so that the equation

$$\frac{I_0(x)}{I_1(x)} \cdot \frac{I_m(x)}{I_{m-1}(x)} = \frac{m+1}{2m} \tag{A.31}$$

can be written also in the form

$$\frac{I'_{m-1}(x)}{I_{m-1}(x)} = \frac{m-1}{x} + \frac{m+1}{2m}\frac{I_1(x)}{I_0(x)}. \tag{A.32}$$

By Corollary A.5 this equation has unique positive solution $x_m = R_{0m}$ and, setting

$$z_m(x) = \frac{I'_{m-1}(x)}{I_{m-1}(x)}, \tag{A.33}$$

$$G(x) = \frac{I_1(x)}{I_0(x)}, \tag{A.34}$$

we have

$$\begin{aligned} z_m(x) &< \frac{m-1}{x} + \frac{m+1}{2m}G(x) \quad \text{if } x < x_m, \\ z_m(x) &> \frac{m-1}{x} + \frac{m+1}{2m}G(x) \quad \text{if } x > x_m, \end{aligned} \tag{A.35}$$

for any $m \geqslant 1$.

Theorem A.5 can then be restated in the following form:

Theorem A.7 *For any $m \geqslant 1$,*

$$z_{m+1}(x) < \frac{m}{x} + \frac{m+2}{2(m+1)} G(x) \qquad \text{at } x = x_m, \tag{A.36}$$

so that $x_{m+1} > x_m$.

We shall need several facts about the function G.

Lemma A.8 *The function $G(x)$ satisfies:*

$$0 < G(x) < 1 \qquad \text{if } x > 0. \tag{A.37}$$

Proof. By (A.13) (with $m = 1$), we get

$$G'(x) > 0 \quad \text{for all } x > 0. \tag{A.38}$$

It is also easily seen that

$$G(x) = \frac{x}{2} + O(x^2) \qquad \text{as } x \to 0. \tag{A.39}$$

Since further, by (A.7),

$$G(x) \to 1 \qquad \text{if } x \to \infty, \tag{A.40}$$

the lemma follows. □

Lemma A.9 *There holds*

$$G''(x) \leqslant 0 \quad \text{for all } x > 0. \tag{A.41}$$

The proof, which is quite lengthy, is given in Section A.4.

Using the relation $I_1' = -I_0/x + I_0$ we find that G satisfies the differential equation

$$G' + G^2 + \frac{1}{x} G = 1. \tag{A.42}$$

This equation will be needed later on.

Proof of Theorem A.7. From (A.30) and (A.33) we have

$$z_m(x) - \frac{m-1}{m} = \frac{I_m(x)}{I_{m-1}(x)}. \tag{A.43}$$

It will be more convenient to work with the function

$$V_m(x) = z_m(x) - \frac{m-1}{x} \qquad \left(= \frac{I_m(x)}{I_{m-1}(x)} \right). \tag{A.44}$$

By (A.3), (A.4) and (A.2), $V_m(x)$ satisfies

$$V_m'(x) + V_m^2(x) + \frac{2m-1}{x} V_m(x) = 1, \tag{A.45}$$

$$V_m(x) = \frac{x}{2m} + O(x^2), \quad x \to 0. \tag{A.46}$$

Similarly,

$$V'_{m+1}(x) + V^2_{m+1}(x) + \frac{2m+1}{x} V_{m+1}(x) = 1, \tag{A.47}$$

$$V_{m+1}(x) = \frac{x}{2(m+1)} + O(x^2), \quad x \to 0. \tag{A.48}$$

Theorem A.7 can be restated as follows: If

$$V_m(x_m) = \frac{1}{2} \frac{m+1}{m} G(x_m), \tag{A.49}$$

then

$$V_{m+1}(x_m) < \frac{1}{2} \frac{m+2}{m+1} G(x_m), \tag{A.50}$$

or, in terms of the function

$$W(x) = \frac{(m+1)^2}{m(m+2)} V_{m+1}(x), \tag{A.51}$$

$$W(x_m) < \frac{1}{2} \frac{m+1}{m} G(x_m). \tag{A.52}$$

Here we have introduced the function W in order to have the right-hand side of (A.52) the same as for (A.49). Our intention is to compare $V_m(x)$ with $W(x)$ for $0 < x \leqslant x_m$.

By (A.47), W satisfies the differential equation

$$W' + \frac{m(m+2)}{(m+1)^2} W^2 + \frac{2m+1}{x} W = \frac{(m+1)^2}{m(m+2)}, \quad x > 0 \tag{A.53}$$

and, by (A.48),

$$W(x) = \frac{m+1}{2m(m+2)} x + O(x^2), \quad x \to 0.$$

In view of (A.46),

$$W(x) < V_m(x) \quad \text{for } x \text{ near } 0. \tag{A.54}$$

If we can show that

$$\frac{(m+1)^2}{m(m+2)} - \frac{m(m+2)}{(m+1)^2} W^2 - \frac{2m+1}{x} W < 1 - W^2 - \frac{2m-1}{x} W \quad \text{for } 0 < x \leqslant x_m, \tag{A.55}$$

then by comparison we deduce that $W(x) < V_m(x)$ for all $0 < x \leqslant x_m$, and (A.52) follows. Indeed otherwise there is a smallest $\overline{x}$ such that $0 < \overline{x} \leqslant x_m$ and $(W - V_m) = 0$, $(W - V_m)' \geqslant 0$ at $\overline{x}$; however, in view of (A.55) at $x = \overline{x}$ and (A.45), (A.53), we also have $W'(\overline{x}) < V'_m(\overline{x})$, which is a contradiction.

We have thus reduced the proof of Theorem A.7 to establishing the inequality (A.55), or

$$\frac{1}{m(m+2)} + \frac{1}{(m+1)^2} W^2 < \frac{2}{x} W \quad \text{for } 0 < x \leqslant x_m. \tag{A.56}$$

The above analysis shows that as long as (A.56) holds for $0 < x < \widetilde{x}$, $W(x) < V_m(x)$ for $0 < x < \widetilde{x}$. Hence, in proving (A.56) it suffices to consider functions $W(x)$ satisfying

$$W(x) < V_m(x). \tag{A.57}$$

Since

$$V_m(x) < \frac{1}{2}\,\frac{m+1}{m}G(x) \quad \text{if } x < x_m,$$

we may replace (A.57) by the simple inequality

$$W(x) < \frac{1}{2}\,\frac{m+1}{m}G(x), \quad 0 < x < x_m. \tag{A.58}$$

Lemma A.10 *There holds*

$$\frac{m+1}{m(m+2)}G(x) \leqslant W(x) \quad \textit{for all } x > 0. \tag{A.59}$$

Proof. We shall construct a subsolution

$$\widetilde{W}(x) = \lambda G(x), \quad \lambda > 0$$

to the function W. By (A.53), this means that λ has to be such that

$$\lambda G' + \lambda^2 \frac{m(m+2)}{(m+1)^2}G^2 + \frac{(2m+1)\lambda}{x}G < \frac{(m+1)^2}{m(m+2)}$$

or, in view of (A.42),

$$G^2\left[\lambda^2\frac{m(m+2)}{(m+1)^2} - \lambda\right] + \frac{2m\lambda}{x}G < \frac{(m+1)^2}{m(m+2)} - \lambda.$$

But for $\lambda \leqslant (m+1)/[m(m+2)]$, this inequality is a consequence of

$$\frac{2m\lambda}{x}G < \frac{(m+1)^2}{m(m+2)} - \lambda,$$

or

$$\frac{2}{x}G < 1, \quad \textit{i.e.}, \; I_1(x) < \frac{x}{2}I_0(x),$$

which is indeed true for all $x > 0$ (by comparing the two series term-wise).

As $x \to 0$

$$\lambda G \sim \frac{\lambda x}{2}, \quad W \sim \frac{m+1}{2m(m+2)}x$$

so that the inequality $\lambda G(x) < W(x)$ holds for x near zero if

$$\lambda < \frac{m+1}{m(m+2)}.$$

But then, by comparison, $\lambda G(x) < W(x)$ for all $x > 0$. This yields the assertion (A.59). □

We summarize: In order to complete the proof of Theorem A.5 it suffices to prove that (A.56) holds for $W(x)$ satisfying (A.58) and (A.59). We shall state this in a different way:

If we introduce the function

$$\Phi(z, x) = \frac{1}{m(m+2)} + \frac{z^2}{(m+1)^2}G^2(x) - \frac{2}{x}zG(x), \tag{A.60}$$

then, in view of (A.58) and (A.59), what we have to prove is that

$$\Phi(z,x) < 0 \quad \text{for } 0 < x < x_m, \;\; z \in \left(\frac{m+1}{m(m+2)}, \frac{m+1}{2m}\right). \tag{A.61}$$

Set

$$z_1 = \frac{m+1}{m(m+2)}, \qquad z_2 = \frac{m+1}{2m}.$$

Since the function $z \to \Phi(z,x)$ is a parabola, it is sufficient to prove (A.61) just at the extreme points z_1 and z_2. But

$$\Phi(z_1,x) < 0 \quad \text{reduces to } 1 + \frac{G^2}{m(m+2)} < \frac{2(m+1)}{x} G,$$
$$\Phi(z_2,x) < 0 \quad \text{reduces to } \frac{1}{m+2} + \frac{G^2}{4m} < \frac{m+1}{x} G,$$

and since (by Lemma A.8) $0 < G < 1$, it suffices to prove that

$$1 + \frac{1}{m(m+2)} < \frac{2(m+1)}{x} G,$$

and

$$\frac{1}{m+2} + \frac{1}{4m} < \frac{m+1}{x} G.$$

Noting that the second inequality is a consequence of the first one, it remains to prove that the function

$$F(x) = 2(m+1)G(x) - \theta_m x \tag{A.62}$$

is positive for $0 < x \leqslant x_m$, where

$$\theta_m = \frac{(m+1)^2}{m(m+2)}. \tag{A.63}$$

Observe that for x near 0

$$F(x) \sim (m+1)x - \frac{(m+1)^2}{m(m+2)} x = ax, \quad a > 0$$

so that

$$F(0) = 0, \quad F'(0) > 0.$$

By Lemma A.9, $F(x)$ is a concave function. Hence, in order to prove that $F(x) > 0$ for all $0 < x \leqslant x_m$ it suffices to show that $F(x_m) > 0$, i.e., that

$$2(m+1)G(x_m) - \theta_m x_m > 0. \tag{A.64}$$

A.3 Proof of (A.64)

Introduce the positive solution $\overline{V}_m$ of

$$\overline{V}_m^2(\xi) + \frac{2m-1}{\xi}\overline{V}_m(\xi) = 1, \tag{A.65}$$

i.e.,

$$\overline{V}_m(\xi) = -\frac{2m-1}{2\xi} + \Big[\Big(\frac{2m-1}{2\xi}\Big)^2 + 1\Big]^{1/2}. \tag{A.66}$$

Note that, by (A.65),

$$\overline{V}_m(\xi) < \frac{\xi}{2m-1}. \tag{A.67}$$

Differentiating (A.65) we obtain

$$2\overline{V}_m\overline{V}'_m + \frac{2m-1}{\xi}\overline{V}'_m = \frac{2m-1}{\xi^2}\overline{V}_m \tag{A.68}$$

and hence, upon using (A.67),

$$\overline{V}'_m = \frac{\frac{2m-1}{\xi^2}\overline{V}_m}{2\overline{V}_m + \frac{2m-1}{\xi}} < \frac{\frac{1}{\xi}}{\frac{2m-1}{\xi}} = \frac{1}{2m-1}. \tag{A.69}$$

Lemma A.11 *There holds*

$$V_m(x) \geqslant \Big(\frac{2m-1}{2m}\Big)^{1/2}\overline{V}_m\Big(\Big(\frac{2m-1}{2m}\Big)^{1/2}x\Big) \tag{A.70}$$

for all $x > 0$.

Proof. Consider the function

$$z(x) = \lambda\overline{V}_m(\lambda x), \quad \lambda > 0 \tag{A.71}$$

and set $\xi = \lambda x$. Then

$$\begin{aligned} z' + z^2 + \frac{2m-1}{x}z - 1 &= \lambda^2[\overline{V}'_m(\xi) + \overline{V}_m^2(\xi) + \frac{2m-1}{\xi}\overline{V}_m(\xi) - 1] + \lambda^2 - 1 \\ &= \lambda^2\overline{V}'_m(\xi) + \lambda^2 - 1 \qquad \text{(by (A.65))} \\ &< \frac{\lambda^2}{2m-1} + \lambda^2 - 1 \qquad \text{(by (A.69))} \\ &= \lambda^2\frac{2m}{2m-1} - 1 \; < \; 0 \end{aligned}$$

if

$$\lambda^2 < \frac{2m-1}{2m},$$

so that z is a subsolution of (A.45). Since also, for x near 0,

$$\begin{aligned} z(x) &\sim \lambda\frac{2m-1}{2\xi}\frac{1}{2}\Big(\frac{2\xi}{2m-1}\Big)^2 \qquad \text{(by (A.66))} \\ &= \lambda^2\frac{x}{2m-1} \; < \; \frac{x}{2m} \; \sim \; V_m(x), \end{aligned}$$

we conclude that, by comparison, that $z(x) < V_m(x)$ for all $x > 0$, and (A.70) follows. □

Lemma A.12 *The function* $\overline{V}_m(\xi)$ *satisfies:* $\overline{V}'_m(\xi) > 0$, $\overline{V}''_m(\xi) < 0$.

Proof. The first inequality follows from (A.68). To prove the second inequality we set $V = \overline{V}_m$ and differentiate (A.68). We get

$$2VV'' + 2(V')^2 + \frac{2m-1}{\xi}V'' - \frac{2(2m-1)}{\xi^2}V' + \frac{2(m-1)}{\xi^3}V = 0.$$

Since $V > 0$, we see that

$$\begin{aligned} \operatorname{sgn} V'' &= \operatorname{sgn}\Big\{ -2(V')^2 + \frac{2(2m-1)}{\xi^2}V' - \frac{2(2m-1)}{\xi^3}V \Big\} \\ &\leqslant \operatorname{sgn}\Big\{ \frac{2(2m-1)}{\xi^2}\Big(V' - \frac{V}{\xi}\Big)\Big\} = \operatorname{sgn}\Big\{\frac{2}{\xi}(-2VV')\Big\} \quad \text{by (A.68)} \end{aligned}$$

which is negative since $V' > 0$. $\square$

The lower bound for V_m derived in Lemma A.11 will henceforth be used to deduce that x_m in (A.35) is sufficiently large, which is an important step in the proof of (A.64).

Set

$$\begin{aligned} Q_m(x) &= \frac{2m}{m+1}\Big(\frac{2m-1}{2m}\Big)^{1/2}\overline{V}_m\Big(\Big(\frac{2m-1}{2m}\Big)^{1/2}x\Big), \\ P_m(x) &= \frac{2m}{m+1}V_m(x), \qquad (V_m \text{ defined in (A.44)}) \\ L_m(x) &= \frac{1}{2(m+1)}\theta_m x = \frac{m+1}{2m(m+2)}x. \end{aligned}$$

By (A.49)

$$P_m(x_m) = G(x_m), \tag{A.72}$$

and (A.64) is equivalent to

$$L_m(x_m) < G(x_m). \tag{A.73}$$

For x near 0

$$G(x) \sim \frac{x}{2},$$

so that

$$G(0) = L_m(0) = 0, \qquad G'(0) > L'_m(0),$$

whereas, for x large, $G(x) < 1 < L_m(x)$. Since G is concave it follows that there exists a unique point $\widetilde{x}_m$ such that

$$\begin{aligned} L_m(x) < G(x) \quad &\text{if } x < \widetilde{x}_m, \\ L_m(x) > G(x) \quad &\text{if } x > \widetilde{x}_m. \end{aligned} \tag{A.74}$$

To prove (A.73), let $\widetilde{x}$ be the point where

$$L_m(\widetilde{x}) = 1, \quad i.e., \ \widetilde{x} = \frac{2m(m+2)}{m+1}. \tag{A.75}$$

Since $G(x) < 1$, (A.74) implies that

$$\widetilde{x}_m < \widetilde{x}. \tag{A.76}$$

Suppose

$$Q_m(\widetilde{x}) > 1 = L_m(\widetilde{x}). \tag{A.77}$$

Then from the concavity of Q_m (Lemma A.12) and the fact that $Q_m(0) = L_m(0) = 0$ it follows that

$$Q_m(x) > L_m(x) \quad \text{if } x < \widetilde{x}. \tag{A.78}$$

By monotonicity of Q_m (Lemma A.11)

$$Q_m(x) > Q_m(\widetilde{x}) > 1 \quad \text{if } x > \widetilde{x}$$

whereas by (A.78), (A.74),

$$Q_m(x) > L_m(x) > G(x) \quad \text{if } \widetilde{x}_m < x < \widetilde{x}.$$

Thus altogether

$$Q_m(x) > G(x) \quad \text{if } \widetilde{x}_m < x < \infty$$

and since $P_m(x) > Q_m(x)$ for all $x > 0$ (Lemma A.11), we conclude that

$$P_m(x) > G(x) \quad \text{if } \widetilde{x}_m < x < \infty.$$

Hence, by (A.72), $x_m < \widetilde{x}_m$ and, recalling (A.74), the assertion (A.73) follows.

In order to complete the proof of Theorem A.7 it remains to prove that (A.77) holds.

Set

$$A_m = \frac{m+1}{2m}\Big(\frac{2m}{2m-1}\Big)^{1/2}, \quad \sigma_m = \Big(\frac{2m-1}{2m}\Big)^{1/2}\widetilde{x}.$$

Then (A.77) reduces to

$$\overline{V}_m(\sigma_m) > A_m. \tag{A.79}$$

Since

$$\overline{V}_m^2(\sigma_m) + \frac{2m-1}{\sigma_m}\overline{V}_m(\sigma_m) = 1,$$

if

$$A_m^2 + \frac{2m-1}{\sigma_m}A_m < 1 \tag{A.80}$$

then, (A.79) follows by monotonicity. Substituting $\widetilde{x}$ from (A.75) into σ_m, the inequality (A.80) reduces to

$$\frac{(m+1)^2(3m+1)}{2m(2m-1)(m+2)} < 1$$

which is valid if $m \geqslant 4$.

We have thus completed the proof of Theorem A.5 for $m \geqslant 4$. The proof for $m \leqslant 3$ can be obtained by explicit calculations. Indeed, the solution of

$$\frac{I_2(x)}{I_1(x)} = \frac{3}{4}\frac{I_1(x)}{I_0(x)}$$

is $x = a = 3.773474$ and

$$\frac{I_3(a)}{I_2(a)} - \frac{2}{3}\frac{I_1(a)}{I_0(a)} = -0.0686071 < 0,$$

whereas the solution of

$$\frac{I_3(x)}{I_2(x)} = \frac{2}{3}\frac{I_1(x)}{I_0(x)}$$

is $x = b = 5.119174$ and

$$\frac{I_4(b)}{I_3(b)} - \frac{5}{8}\frac{I_1(b)}{I_0(b)} = -0.058144 < 0.$$

A.4 Concavity of G

The function $\overline{G}(x) = \frac{1+\varepsilon}{2}x$ ($\varepsilon > 0$) is a supersolution of G, i.e.,

$$\overline{G}' > 1 - \overline{G}^2 - \frac{\overline{G}}{x}.$$

Since also $G(x) \sim \frac{x}{2} < \overline{G}(x)$ for x near 0, it follows that

$$G(x) \leqslant \frac{x}{2} \quad \text{for all } x > 0. \tag{A.81}$$

Differentiating (A.42) we get

$$\begin{aligned} G'' &= -2GG' - \frac{G'}{x} + \frac{1}{x^2}G \\ &= -2G(1 - G^2 - \frac{G}{x}) - \frac{1}{x}(1 - G^2 - \frac{G}{x}) + \frac{G}{x^2} \\ &= -2G + 2G^3 + \frac{3G^2}{x} + \frac{2G}{x^2} - \frac{1}{x} \equiv K(G,x). \end{aligned} \tag{A.82}$$

If $x \leqslant 1$, then by (A.81)

$$G'' < -2G + xG^2 + \frac{3}{2}G + \frac{1}{x} - \frac{1}{x} = G(-\frac{1}{2} + xG) < 0$$

and thus G is concave.

We next want to show that

$$K(G(x),x) < 0 \quad \text{if } 1 < x < 2. \tag{A.83}$$

To do that note that since $G(x) > G(1) = 0.4464$,

$$\frac{3G^2(x)}{\xi} + \frac{2G(x)}{\xi^2} - \frac{1}{\xi}$$

is monotone decreasing in ξ, $0 < \xi < 1$, and

$$\frac{\partial K}{\partial G} = -2 + 6G^2 + \frac{6G}{x} + \frac{2}{x^2} > 0 \quad \text{if } G = G(\xi),\ \xi > 1,\ 1 < x < 2.$$

Hence, if

$$H(x,y) \equiv -2G(x) + 2G^3(x) + \frac{3G^2(x)}{y} + \frac{2G(x)}{y^2} - \frac{1}{y} < 0 \tag{A.84}$$

for a pair (x, y) with $1 \leqslant y < x \leqslant 2$, then

$$K(G(\xi), \xi) < 0 \quad \text{for } y \leqslant \xi \leqslant x.$$

We shall use this remark with points

$$a_1 = 1.0,\ a_2 = 1.1,\ a_3 = 1.2,\ a_4 = 1.3,\ a_5 = 1.5,\ a_6 = 1.7,\ a_7 = 1.9,\ a_8 = 2.0.$$

By direct computation we find that $H(a_{j+1}, a_j) < 0$ for all j. Hence (A.83) holds and, consequently, $G(x)$ is concave for $1 \leqslant x \leqslant 2$.

It remains to prove the concavity of $G(x)$ for $x > 2$. To do that we shall first derive rather sharp upper and lower bounds on G:

Lemma A.13 *The function G satisfies*

$$G(x) < 1 - \frac{1}{2x} - \frac{1}{8x^2} - \frac{1}{8x^3} \quad \textit{if } x \geqslant 2, \tag{A.85}$$

Proof. Writing

$$G = 1 - \frac{1}{2x} + \varphi,$$

(A.42) becomes

$$\varphi' = -2\varphi - \varphi^2 - \frac{1}{4x^2}.$$

By direct calculation one shows that the function

$$\psi(x) = -\frac{1}{8x^2} - \frac{1}{8x^3}$$

satisfies

$$\psi' > -2\psi - \psi^2 - \frac{1}{4x^2},$$

and thus

$$\overline{G} \equiv 1 - \frac{1}{2x} - \frac{1}{8x^2} - \frac{1}{8x^3}$$

is a supersolution to (A.42). Since

$$\overline{G}(2) = 1 - \frac{1}{4} - \frac{1}{32} - \frac{1}{64} \approx 0.703 > 0.697 \approx G(2),$$

the assertion (A.85) follows. □

We have

$$6G^2(x) \geqslant 6G^2(2) \approx 2.91 > 2 \quad \text{if } x \geqslant 2$$

and, therefore,

$$\frac{\partial K}{\partial G} = -2 + 6G^2 + \frac{6G}{x} + \frac{2}{x^2} > 0 \quad \text{if } x \geqslant 2.$$

Consequently (recall (A.82)) in order to prove that $G''(x) < 0$, or that $K(G, x) < 0$, it suffices to prove that $K(\overline{G}, x) < 0$ where $\overline{G}(x)$ is the supersolution given by the right-hand side of (A.85). By direction computation,

$$K(\overline{G}, x) = \frac{\overline{G}}{x}\left\{1 + \frac{1}{2x} - \frac{5}{8x^2} - \frac{3}{32x^3} + \frac{1}{16x^4} + \frac{1}{32x^5}\right\} - \frac{1}{x},$$

which can be expanded to

$$K(\overline{G},x)=\frac{1}{x^3}\Big(-1+\frac{1}{32}\,\frac{1}{x}+\frac{1}{8}\,\frac{1}{x^2}+\frac{23}{256}\,\frac{1}{x^3}-\frac{3}{256}\,\frac{1}{x^4}-\frac{3}{256}\,\frac{1}{x^5}-\frac{1}{256}\,\frac{1}{x^6}\Big).$$

For $x\geqslant 1$, the sum of the terms with positive sign is less than $1/32+1/8+23/256$ which is less than 1. Therefore, $K(\overline{G},x)<0$ if $x\geqslant 1$. This completes the proof of the concavity of G. □

B Some lemmas on analyticity

In this section we establish norm estimates on the derivatives of composite functions of the same type as appeared in [1] and [4, Sec. 9].

Throughout this section we assume that the norm $\|\cdot\|$ satisfies the property

$$\|fg\|\leqslant\|f\|\,\|g\|. \tag{B.1}$$

Lemma B.1 *Suppose*

$$\Big\|\frac{1}{k!}\partial_\theta^k w_m^{(i)}(\theta)\Big\|\leqslant\frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2}\qquad (k^2=1\ \textit{if}\ k=0) \tag{B.2}$$

for all $m\geqslant 1, 0\leqslant k\leqslant K, 1\leqslant i\leqslant q$ *and set*

$$\prod_{i=1}^{q}\Big(\sum_{m=1}^{\infty}w_m^{(i)}(\theta)\varepsilon^m\Big)=\sum_{m=q}^{\infty}W_m^q(\theta)\varepsilon^m\quad (q=1,2,3,\cdots).$$

Then

$$\Big\|\frac{1}{k!}\partial_\theta^k W_m^q(\theta)\Big\|\leqslant\frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2}\Big(\frac{48H_0}{H}\Big)^{q-1}\qquad \textit{for}\ m\geqslant q,\ 0\leqslant k\leqslant K. \tag{B.3}$$

Proof. First, we have

$$\begin{aligned}\sum_{k=1}^{m-1}\frac{1}{k^2(m-k)^2}&=\sum_{k=1}^{m-1}\Big(\frac{1}{m^2k^2}+\frac{2}{m^3k}+\frac{1}{m^2(m-k)^2}+\frac{2}{m^3(m-k)^3}\Big)\\&\leqslant 2\sum_{k=1}^{m-1}\Big(\frac{1}{m^2k^2}+\frac{2}{m^3k}\Big)\\&<\frac{2}{m^2}\Big\{\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{2}{m^2}\Big(1+\int_1^{m-1}\frac{dx}{x}\Big)\Big\}\\&=\frac{2}{m^2}\Big\{\frac{\pi^2}{6}+\frac{2}{m}[1+\ln(m-1]\Big\}\leqslant\frac{2}{m^2}\Big\{\frac{\pi^2}{6}+\frac{2}{3}(1+\ln 2)\Big\}<\frac{6}{m^2}.\end{aligned} \tag{B.4}$$

The estimate (B.3) is valid for $q=1$, by assumption. We proceed by induction from $q-1$ to q. By (B.4), we also have (recall the convention $k^2=1$ if $k=0$)

$$\sum_{k=0}^{m}\frac{1}{k^2(m-k)^2}=\frac{2}{m^2}+\sum_{k=1}^{m-1}\frac{1}{k^2(m-k)^2}\leqslant\frac{8}{m^2}. \tag{B.5}$$

Note that

$$W_m^q(\theta) = \sum_{j=1}^{m-q+1} W_{m-j}^{q-1}(\theta) w_j^{(q)}(\theta),$$

and

$$\frac{1}{k!}\partial_\theta^k W_m^q(\theta) = \sum_{j=1}^{m-q+1}\sum_{l=0}^{k} \frac{1}{l!}\partial_\theta^l W_{m-j}^{q-1} \frac{1}{(k-l)!}\partial_\theta^{k-l} w_j^{(q)}.$$

Therefore

$$\begin{aligned}
\left\|\frac{1}{k!}\partial_\theta^k W_m^q(\theta)\right\| &\leqslant \sum_{j=1}^{m-q+1}\sum_{l=0}^{k} \frac{B^l}{l^2}\frac{H_0 H^{m-j-1}}{(m-j)^2}\Big(\frac{48H_0}{H}\Big)^{q-2}\frac{B^{k-l}}{(k-l)^2}\frac{H_0H^{j-1}}{j^2} \\
&\leqslant B^k H_0 H^{m-1}\Big(\frac{48H_0}{H}\Big)^{q-1}\frac{1}{48}\sum_{j=1}^{m-q+1}\frac{1}{(m-j)^2}\frac{1}{j^2}\sum_{l=0}^{k}\frac{1}{l^2}\frac{1}{(k-l)^2} \\
&\leqslant \frac{B^k}{k^2}\frac{H_0H^{m-1}}{m^2}\Big(\frac{48H_0}{H}\Big)^{q-1} \quad \text{(by (B.4) and (B.5)).} \qquad \square
\end{aligned}$$

Taking $w_m^{(1)}(\theta) = w_m^{(2)}(\theta) = \cdots = w_m^{(q)}(\theta) = w_m(\theta)$ in Lemma B.1, we get

Lemma B.2 *Suppose*

$$\left\|\frac{1}{k!}\partial_\theta^k w_m(\theta)\right\| \leqslant \frac{B^k}{k^2}\;\frac{H_0H^{m-1}}{m^2} \qquad (k^2 = 1 \text{ if } k=0) \tag{B.6}$$

for all $m \geqslant 1, 0 \leqslant k \leqslant K$ *and set*

$$\Big(\sum_{m=1}^{\infty} w_m(\theta)\varepsilon^m\Big)^q = \sum_{m=q}^{\infty} W_m^q(\theta)\varepsilon^m \quad (q = 1,2,3,\cdots).$$

Then

$$\left\|\frac{1}{k!}\partial_\theta^k W_m^q(\theta)\right\| \leqslant \frac{B^k}{k^2}\;\frac{H_0H^{m-1}}{m^2}\Big(\frac{48H_0}{H}\Big)^{q-1} \qquad \text{for } m \geqslant q,\; 0 \leqslant k \leqslant K. \tag{B.7}$$

Lemma B.3 *Consider the formal power series*

$$f(w,\theta) = \sum_{m=1}^{\infty} f_m(\theta) w^m,$$

$$w(\theta,\varepsilon) = \sum_{m=1}^{\infty} w_m(\theta)\varepsilon^m$$

and set

$$F(\theta,\varepsilon) = f[w(\theta,\varepsilon),\theta] = \sum_{m=1}^{\infty} F_m(\theta)\varepsilon^m.$$

Suppose that

$$\left\|\frac{1}{k!}\partial_\theta^k f_m\right\| \leqslant A_0 A^{m-1}\frac{B^k}{k^2}, \qquad \left\|\frac{1}{k!}\partial_\theta^k w_m\right\| \leqslant \frac{B^k}{k^2}\;\frac{H_0H^{m-1}}{m^2} \tag{B.8}$$

hold for all $m \geqslant 1$ *and all* $0 \leqslant k \leqslant K$ *and* $H/A \geqslant 96H_0$. *Then*

$$\left\|\frac{1}{k!}\partial_\theta^k F_m\right\| \leqslant 16A_0\frac{B^k}{k^2}\;\frac{H_0H^{m-1}}{m^2} \qquad \text{for } m \geqslant 1. \tag{B.9}$$

Proof. Using the notation from previous lemmas, we have

$$F_m = \sum_{q=1}^{m} f_q W_m^q.$$

Thus

$$\frac{1}{k!}\partial_\theta^k F_m = \sum_{q=1}^{m}\sum_{l=0}^{k} \frac{1}{l!}\partial_\theta^l f_q \frac{1}{(k-l)!}\partial_\theta^{k-l} W_m^q,$$

we now estimate

$$\begin{aligned}\left\|\frac{1}{k!}\partial_\theta^k F_m\right\| &\leqslant \sum_{q=1}^{m}\sum_{l=0}^{k} A_0 A^{q-1}\frac{B^l}{l^2}\cdot\frac{B^{k-l}}{(k-l)^2}\frac{H_0 H^{m-1}}{m^2}\left(\frac{48H_0}{H}\right)^{q-1} \\ &\leqslant \frac{8A_0B^k}{k^2}\cdot\frac{H_0H^{m-1}}{m^2}\sum_{q=1}^{\infty}\frac{1}{2^{q-1}} \quad \text{(by (B.5))} \\ &= 16\frac{A_0B^k}{k^2}\cdot\frac{H_0H^{m-1}}{m^2}. \qquad \square\end{aligned}$$

The above lemma extends to double power series:

Lemma B.4 *Consider the formal power series*

$$\begin{aligned} f(x,y,\theta) &= \sum_{1\leqslant i+j<\infty} f_{ij}(\theta)x^i y^j, \\ w_1(\theta,\varepsilon) &= \sum_{m=1}^{\infty} w_m^{(1)}(\theta)\varepsilon^m, \\ w_2(\theta,\varepsilon) &= \sum_{m=1}^{\infty} w_m^{(2)}(\theta)\varepsilon^m, \end{aligned}$$

and set

$$F(\theta,\varepsilon) = f[w_1(\theta,\varepsilon), w_2(\theta,\varepsilon),\theta] = \sum_{m=1}^{\infty} F_m(\theta)\varepsilon^m.$$

Suppose that

$$\left\|\frac{1}{k!}\partial_\theta^k f_{ij}\right\| \leqslant A_0 A^{i+j-1}\frac{B^k}{k^2}, \qquad \left\|\frac{1}{k!}\partial_\theta^k w_m^{(i)}\right\| \leqslant \frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2}, \quad i=1,2. \tag{B.10}$$

hold for all $m \geqslant 1, i+j \geqslant 1$ and all $k \geqslant 0$ and $H/A \geqslant 96H_0$. Then

$$\left\|\frac{1}{k!}\partial_\theta^k F_m\right\| \leqslant 32\cdot A_0\frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2}. \tag{B.11}$$

If, furthermore,

$$f_{10}(\theta) \equiv f_{01}(\theta) \equiv 0,$$

i.e., the series for f starts with second order terms, then

$$\left\|\frac{1}{k!}\partial_\theta^k F_m\right\| \leqslant 32\cdot 48A_0\frac{B^k}{k^2}\,\frac{H_0^2H^{m-2}}{m^2}. \tag{B.12}$$

Similarly if the series for f starts with m-th order terms, then there is an extra $(48H_0/H)^{m-1}$ factor in the resulting estimates (right-hand side of (B.11)).

Proof. We write

$$\Big(\sum_{m=1}^{\infty} w_m^{(1)}(\theta)\varepsilon^m\Big)^i\Big(\sum_{m=1}^{\infty} w_m^{(2)}(\theta)\varepsilon^m\Big)^j = \sum_{m=i+j}^{\infty} F_m^{(i,j)}(\theta)\varepsilon^m.$$

Then by Lemma B.1,

$$\Big\|\frac{1}{k!}\partial_\theta^k F_m^{(i,j)}\Big\| \leqslant \frac{B^k}{k^2}\frac{H_0 H^{m-1}}{m^2}\Big(\frac{48H_0}{H}\Big)^{i+j-1}. \tag{B.13}$$

Since

$$F_m(\theta) = \sum_{1\leqslant i+j\leqslant m} f_{ij}(\theta)F_m^{(i,j)}(\theta),$$

we then have

$$\begin{aligned}
\Big\|\frac{1}{k!}\partial_\theta^k F_m(\theta)\Big\| &\leqslant \sum_{1\leqslant i+j\leqslant m}\sum_{l=0}^{k}\Big\|\frac{1}{l!}\partial_\theta^l f_{ij}(\theta)\Big\|\Big\|\frac{1}{(k-l)!}\partial_\theta^{k-l}F_m^{(i,j)}\Big\| \\
&\leqslant \sum_{1\leqslant i+j\leqslant m}\sum_{l=0}^{k} A_0 A^{i+j-1}\frac{B^l}{l^2}\,\frac{B^{k-l}}{(k-l)^2}\,\frac{H_0H^{m-1}}{m^2}\Big(\frac{48H_0}{H}\Big)^{i+j-1} \\
&\leqslant 8A_0\frac{B^k}{k^2}\frac{H_0H^{m-1}}{m^2}\sum_{1\leqslant i+j\leqslant m}\Big(\frac{1}{2}\Big)^{i+j-1} \qquad \text{(by (B.5))} \\
&= 32\cdot A_0\frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2}.
\end{aligned}$$

In case $f_{10}(\theta)\equiv f_{01}(\theta)\equiv 0$, $i+j$ is at least 2 for the nonvanishing terms and there is an extra factor $48H_0/H$ factor. Similarly if $i+j$ starts with m, then we get the extra factor $(48H_0/H)^{m-1}$. □

In the next lemma, we use two norms, $\|\cdot\|_{X_1}$ for functions $g(\theta)$ and $\|\cdot\|_{X_2}$ for functions $f(r,\theta)$.

Lemma B.5 *Suppose that the norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$ satisfy the following algebra property*

$$\|g(\theta)f(r,\theta)\|_{X_2} \leqslant C^*\|g\|_{X_1}\|f\|_{X_2}.$$

Assume that

$$\Big\|\partial_\theta^k w_m(\theta)\Big\|_{X_1} \leqslant k!\frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2} \qquad (k^2 = 1 \text{ if } k=0) \tag{B.14}$$

for all $m\geqslant 1, 0\leqslant k\leqslant K$ and fixed i, and

$$\Big\|\partial_r^i\partial_\theta^k u_m(r,\theta)\Big\|_{X_2} \leqslant (k+i)!\frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2}. \tag{B.15}$$

Set

$$\Big(\sum_{m=1}^{\infty} w_m(\theta)\varepsilon^m\Big)\Big(\sum_{m=1}^{\infty} u_m(r,\theta)\varepsilon^m\Big) = \sum_{m=2}^{\infty} U_m(r,\theta)\varepsilon^m.$$

Then

$$\Big\|\partial_r^i\partial_\theta^k U_m(r,\theta)\Big\|_{X_2} \leqslant C^*(k+i)!\frac{B^k}{k^2}\,\frac{H_0H^{m-1}}{m^2}\Big(\frac{48H_0}{H}\Big) \qquad \text{for } m\geqslant 2, 0\leqslant k\leqslant K. \tag{B.16}$$

Proof. The proof is almost the same as that for Lemma B.1. Note that

$$U_m(r,\theta) = \sum_{j=1}^{m-1} w_{m-j}(\theta) u_j(r,\theta),$$

and

$$\partial_r^i \partial_\theta^k U_m(r,\theta) = \sum_{j=1}^{m-1} \sum_{l=0}^{k} \binom{k}{l} \partial_\theta^l w_{m-j}(\theta) \partial_r^i \partial_\theta^{k-l} u_j(r,\theta).$$

Therefore

$$\begin{aligned}
&\left\| \frac{1}{(k+i)!} \partial_r^i \partial_\theta^k U_m(r,\theta) \right\|_{X_2} \\
&\leqslant C^* \sum_{j=1}^{m-1} \sum_{l=0}^{k} \frac{\binom{k}{l}}{\binom{k+i}{l}} \frac{B^l}{l^2} \frac{H_0 H^{m-j-1}}{(m-j)^2} \frac{B^{k-l}}{(k-l)^2} \frac{H_0 H^{j-1}}{j^2} \\
&\leqslant C^* B^k H_0 H^{m-1} \Big(\frac{48 H_0}{H} \Big) \frac{1}{48} \sum_{j=1}^{m-1} \frac{1}{(m-j)^2} \frac{1}{j^2} \sum_{l=0}^{k} \frac{1}{l^2} \frac{1}{(k-l)^2} \\
&\leqslant C^* \frac{B^k}{k^2} \frac{H_0 H^{m-1}}{m^2} \Big(\frac{48 H_0}{H} \Big).
\end{aligned}$$

The proof is complete. □

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