LIQUID CRYSTALS AND ENERGY ESTIMATES FOR \( S^2 \)-VALUED MAPS

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This report summarizes results obtained in collaboration with J.M. Coron and
E. Lieb (see [3] and [4]); it answers some questions raised by J. Ericksen and
D. Kinderlehrer. The original motivation comes from the theory of liquid crystals
(see [7], [8], [10]), and is well explained in other contributions to this volume.

We deal with maps \( \phi \) from a domain \( \Omega \subset \mathbb{R}^3 \) with values into \( S^2 \) which
admit a finite number of singularities. We consider two different kinds of
problems. In the first type of problem the location and the degree of the
singularities is prescribed; the main result is an explicit formula, when \( \Omega = \mathbb{R}^3 \),
for the minimum value of the deformation energy. In the second type of problem
the number, the location and the degree of the singularities are "free"; our main
result asserts that if \( \phi \) is a minimizer then all its singularities have degree
\( \pm 1 \), moreover, the first order expansion shows that \( \phi \) (or \(-\phi\)) acts like a
rotation near every singularity - a fact which agrees with experimental and
numerical evidence (see [5] and [6]).

1. Prescribed Singularities

Fix \( N \) points \( a_1, a_2, \ldots, a_N \) in \( \mathbb{R}^3 \) (the desired location of the
singularities). Consider maps \( \phi \) which are smooth on \( \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \{a_i\} \), with values
in \( S^2 \), and with finite energy, i.e.

\[
E(\phi) = \int_{\mathbb{R}^3} |\nabla \phi|^2 < \infty
\]

[The most general energy of interest in the theory of liquid crystals is

\[
E(\phi) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \phi|^2 + \frac{1}{4} |\nabla \phi|_2^2 + \frac{1}{2} \langle \phi, \nabla \phi \rangle^2
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where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( S^2 \).]

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