GLOBAL ASYMPTOTIC LIMIT OF SOLUTIONS OF THE CAHN–HILLIARD EQUATION

Xinfu Chen[1]
Department of Mathematics & Statistics
University of Pittsburgh
Pittsburgh, PA 15260
e-mail: xinfu+@pitt.edu

ABSTRACT. We study the asymptotic limit, as $\varepsilon \searrow 0$, of solutions of the Cahn–Hilliard equation

$$u_t^\varepsilon = \Delta(-\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon))$$

under the assumption that the initial energy

$$\int_\Omega \Big[\frac{\varepsilon}{2}|\nabla u^\varepsilon(\cdot,0)|^2 + \frac{1}{\varepsilon}F(u^\varepsilon(\cdot,0))\Big]$$

is bounded independent of ε. Here $f = F'$ and F is a smooth function taking its global minimum 0 only at $u = \pm 1$. We show that there is a subsequence of $\{u^\varepsilon\}_{0<\varepsilon\le 1}$ converging to a weak solution of an appropriately defined limit Cahn–Hilliard problem. We also show that, in case of radial symmetry, all the interfaces of the limit has multiplicity one for almost all time $t > 0$, regardless of initial energy distributions.

KEYWORDS. Cahn–Hilliard equation, Hele–Shaw problem, Mullins–Sekerka problem, asymptotic limit, functions of bounded variation, Radon measure, varifold, first variation of varifolds, mean curvature.

AMS subject classifications (1991). 35K22, 35D05, 35R35, 49Q20.

1 Introduction.

In this paper, we shall study the asymptotic limit, as $\varepsilon \searrow 0$, of the solutions of the Cahn–Hilliard equation

$$\begin{cases} u_t^\varepsilon(x,t) = \Delta v^\varepsilon(x,t), & (x,t) \in \Omega\times(0,\infty), \\ v^\varepsilon = -\varepsilon\Delta u^\varepsilon + \varepsilon^{-1}f(u^\varepsilon), & (x,t)\in\Omega\times[0,\infty), \\ \frac{\partial}{\partial n}u^\varepsilon = \frac{\partial}{\partial n}v^\varepsilon = 0, & (x,t)\in\partial\Omega\times[0,\infty), \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \ge 2)$ and $f(u)$ is the derivative of a potential F satisfying

$$\begin{cases} \text{(a) } F\in C^3(\mathbb{R}),\ F(\pm1) = 0, \text{ and } F(u) > 0 \text{ for all } u \ne \pm 1; \\ \text{(b) } F' = f \text{ and for some } p > 2 \text{ and } c_0 > 0,\ f'(u) \ge c_0|u|^{p-2} \text{ if } |u| \ge 1 - c_0. \end{cases} \tag{1.2}$$

For the initial data u_0^ε, we assume

$$\begin{cases} \sup\limits_{0<\varepsilon\le1}\int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u_0^\varepsilon(x)|^2 + \frac{1}{\varepsilon}F(u_0^\varepsilon(x))\Big)dx \le \mathcal{E}_0 < \infty, \\ \frac{1}{|\Omega|}\int_\Omega u_0^\varepsilon(x) = m_0 \in (-1,1) \qquad \forall\,\varepsilon\in(0,1]. \end{cases} \tag{1.3}$$

[1] Partially supported by the Alfred P. Sloan Research Fellowship and the National Science Foundation Grant DMS–9404773. The author thanks Professor H. Mete Soner for many helpful discussions.