A STEFAN PROBLEM FOR MULTI-DIMENSIONAL REACTION DIFFUSION SYSTEMS

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A STEFAN PROBLEM FOR MULTI-DIMENSIONAL REACTION DIFFUSION SYSTEMS

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Abstract. This paper deals with a Stefan problem for a system of three weakly coupled semilinear parabolic equations. This system describes the dissolution of a particle in a solution. The dissolved species $A$ reacts chemically with species $B$ already in the solution, thereby forming species $C$. Species $C$ diffuses in the solution and some of it adsorbs to the particle’s boundary and causes either (i) decrease in the dissolution rate, or (ii) increase in the dissolution rate. It is proved that for the model in case (i) the solution is unstable in any small time interval, whereas for the model in case (ii) the problem has a unique solution in a small time interval.

1. The model. Consider a solid particle composed of chemical $A$ with uniform concentration $A^*$. The particle is in a solution. In the solution there is also another chemical $B$. As the particle dissolves, the $A$ that enters the solution reacts with $B$ to form $C$. Then species $C$ diffuses in the solution and some of it reaches the solid particle and adsorbs to its surface. We shall denote the concentrations of species $A$, $B$ and $C$ in the solution simply by $A$, $B$ and $C$, respectively.

We consider here the two dimensional model in polar coordinates $(r, \theta)$. Assuming that the solid particle is enclosed by a surface $r = g(\theta, t)$, the reaction-diffusion equations are

\begin{align}
(1.1) \quad \frac{\partial A}{\partial t} &= D_A \Delta A - KAB, \\
(1.2) \quad \frac{\partial B}{\partial t} &= D_B \Delta B - KAB, \\
(1.3) \quad \frac{\partial C}{\partial t} &= D_C \Delta C + KAB
\end{align}

in $\{r > g(\theta, t)\}$, where $K$ is the reaction rate and $D_A, D_B, D_C$ are the diffusion coefficients.

Let $\vec{n}$ denote the inward normal to the surface

$$\Gamma_t = \{r = g(\theta, t)\}$$

and $V_n$ the velocity of $\Gamma_t$ in the normal direction. Then

\begin{align}
(1.4) \quad \vec{n} &= (n_r, n_\theta) = \left( \frac{-g}{\sqrt{g^2 + g_\theta^2}}, \frac{g_\theta}{g\sqrt{g^2 + g_\theta^2}} \right), \\
(1.5) \quad V_n &= \frac{-g}{\sqrt{g^2 + g_\theta^2}} g_t.
\end{align}

By conservation of mass, the rate at which the particle’s boundary moves is proportional to the flux of species $A$ away from the particle, that is,

\begin{align}
(1.6) \quad V_n &= \alpha D_A \frac{\partial A}{\partial n} \quad \text{on} \quad \Gamma_t \quad (\alpha > 0).
\end{align}

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Since there is no flux of $B$ through, or adsorption of $B$ to, the particle's surface,

\[ \frac{\partial B}{\partial n} = 0 \quad \text{on } \Gamma_t. \]  

(1.7)

The adsorption of $C$ to the surface is proportional to the local saturation and is given by the empirical law $D_C \frac{\partial C}{\partial n} = -\gamma C^n$ for some positive constants $\gamma, n$. As in [2, Chapter 18] we take $n = 4$, i.e.,

\[ D_C \frac{\partial C}{\partial n} = -\gamma C^4; \]  

(1.8)

all the results of this paper, however, remain valid for general $n$.

As in [2, Chapter 18] [4] we take

\[ A(\infty, t) = 0, \quad B(\infty, t) = B^*, \quad C(\infty, t) = 0 \]  

(1.9)

where "\( \infty \)" means the limit as $r$ goes to $\infty$, uniformly in $\theta$, and $B^*$ is a positive constant.

We next define the boundary condition for $A$. Denote by $\zeta(\theta, t)$ the fraction of the concentration of $C$ which covers $A$ at the point $r = g(\theta, t)$; $\zeta(\theta, t) = 1$ if the point $r = g(\theta, t)$ is fully covered. We take the boundary condition for $A$ on the free boundary, as in [2, Chapter 18], to be

\[ D_A \zeta(\theta, t) \frac{\partial A}{\partial n} + (1 - \zeta(\theta, t))^+ (A - A^+) = 0 \quad \text{on } \Gamma_t. \]  

(1.10)

Finally we impose initial conditions:

\[ g(\theta, 0) = g_0(\theta), \quad \zeta(\theta, 0) = \zeta_0(\theta), \]  

(1.11)

\[ A(r, \theta, 0) = A_0(r, \theta), \quad B(r, \theta, 0) = B_0(r, \theta), \quad C(r, \theta, 0) = C_0(r, \theta). \]  

(1.12)

For simplicity we shall henceforth take $\alpha = 1, \gamma = 1$.

Before we can analyze the problems (1.1)-(1.12) we need to derive an equation for the evolution of $\zeta$. There are two factors affecting the change of $\zeta(\theta, t)$: the flux $-D_C \frac{\partial C}{\partial n}$ and the change in the surface element of $\Gamma_t$ along the normal.

Denote by $\frac{D}{Dt}$ the total derivative along the normal direction. Then

\[ \frac{D\zeta(\theta, t)}{Dt} = \lim_{\Delta t \to 0} \frac{\zeta(\theta + V_n n_\theta \Delta t, t + \Delta t) - \zeta(\theta, t)}{\Delta t} = \frac{\partial \zeta}{\partial t} + n_\theta V_n \frac{\partial \zeta}{\partial \theta}. \]  

(1.13)

If we introduce the surface element along $\Gamma_t$

\[ S(\theta, t) = \Psi(\theta, t) d\theta \quad \text{where } \Psi(\theta, t) = \sqrt{g^2(\theta, t) + g^2_\theta(\theta, t)}, \]  

then the evolution of $\zeta$ can be described in the form:

\[ \zeta(\theta + V_n n_\theta \Delta t, t + \Delta t)S(\theta + V_n n_\theta \Delta t, t + \Delta t) - \zeta(\theta, t)S(\theta, t) \]

\[ = -D_C \frac{\partial C}{\partial n} \cdot S(\theta, t) \Delta t + O((\Delta t)^2). \]  

\[ 2 \]
From this equation we easily derive

\begin{equation}
\frac{D\zeta}{Dt} + Q\zeta = -D_C \frac{\partial C}{\partial n}
\end{equation}

where

\begin{align*}
Q &= \lim_{\Delta t \to 0} \frac{S(\theta + V_n n_\theta \Delta t, t + \Delta t) - S(\theta, t)}{\Delta t \cdot S(\theta, t)} \\
&= \lim_{\Delta t \to 0} \frac{\Psi(\theta + V_n n_\theta \Delta t, t + \Delta t) d(\theta + V_n n_\theta \Delta t) - \Psi(\theta, t) d\theta}{\Delta t \cdot \Psi(\theta, t) d\theta} \\
&= \lim_{\Delta t \to 0} \frac{\Psi(\theta + V_n n_\theta \Delta t, t + \Delta t) [1 + \Delta t \cdot (\partial(V_n n_\theta)/\partial \theta)] - \Psi(\theta, t)}{\Delta t \cdot \Psi(\theta, t)} \\
&= \frac{\Psi_t + V_n n_\theta \Psi_\theta + \Psi(V_n n_\theta)_\theta}{\Psi} \\
&= \frac{1}{\Psi} \left\{ \frac{gg_t + g_\theta g_\theta}{\sqrt{g^2 + g_\theta^2}} + (\Psi V_n n_\theta)_\theta \right\} \quad \text{(by (1.5))} \\
&= \frac{1}{\Psi} \left\{ -V_n - g_\theta \left( \frac{\Psi V_n}{g} \right)_\theta + \left( \frac{g_\theta}{g} V_n \right)_\theta \right\} \quad \text{(by (1.4), (1.5))} \\
&= \frac{1}{\Psi} \left\{ -V_n + \frac{\Psi V_n}{g} \left( \frac{g_\theta}{\Psi} \right)_\theta \right\} = -\frac{V_n}{\Psi} + \frac{V_n}{g} \left( \frac{g_\theta}{\Psi} \right)_\theta.
\end{align*}

Substituting this into (1.14) and using (1.13), we get

\begin{equation}
\frac{\partial \zeta}{\partial t} + n_\theta V_n \frac{\partial \zeta}{\partial \theta} + V_n \left\{ -\frac{1}{\sqrt{g^2 + g_\theta^2}} + \frac{1}{g} \left( \frac{g_\theta}{\sqrt{g^2 + g_\theta^2}} \right)_\theta \right\} \zeta = -D_C \frac{\partial C}{\partial n}.
\end{equation}

It will be convenient to work with the variable

\begin{equation}
\xi(\theta, t) = \frac{1}{\zeta(\theta, t)}.
\end{equation}

Then equations (1.6), (1.11), (1.15) become:

\begin{equation}
V_n = \frac{-gg_t}{\sqrt{g^2 + g_\theta^2}} = D_A \frac{\partial A}{\partial n} = (\xi - 1)^+(A^* - A) \quad \text{on } \Gamma_t,
\end{equation}

\begin{equation}
\frac{\partial \xi}{\partial t} + \frac{V_n}{g} \frac{g_\theta}{\sqrt{g^2 + g_\theta^2}} \frac{\partial \xi}{\partial \theta} - \frac{V_n}{g} \left( \frac{g_\theta}{\sqrt{g^2 + g_\theta^2}} \right)_\theta \xi + \frac{V_n}{\sqrt{g^2 + g_\theta^2}} \xi = -\xi^2 C^4 \quad \text{on } \Gamma_t.
\end{equation}

When the initial data are independent of \( \theta \) it is shown \cite{4} that the system has a unique classical solution. However, the situation becomes more complicated when we allow the initial data to depend on \( \theta \). In fact, it will be shown that the problem is thus not well posed for classical solutions. In Section 2 we linearize the problem about a radial solution. We then show that the solution to the linearized problem is unique (Section 4) but may blow-up in
arbitrarily small time no matter how smooth the initial data are (Section 3). Furthermore, the full nonlinear problem is not stable near the radial solution (Section 5).

There are some similarities between our problem and the Stefan problem with supercooled water. The Mullin-Sekerka instabilities for the latter problem are not as bad as in our case; the linearized problem (for the supercooled water model) is well posed for all time provided the data are smooth, and the instability is only in the sense that the absolute value of the solution goes to $\infty$ as $t \to \infty$. To explain the origin of the instabilities for our problem consider a non-radial particle as in Figure 1.1.

![Figure 1.1](image)

At a convex point $A$ on the free boundary, as the particle dissolves the local area shrinks. This increases the concentration $\zeta$ which will then slow down the dissolution near $A$. The reverse situation occurs at a point $B$: the local area increases, $\zeta$ decreases and the dissolution increases. This process tends to accentuate the ripples in the free boundary, and will generally result in blow-up of the $C^{1+\beta}$ norm of $\Gamma_t$, at a very short time. This situation does not preclude the existence of a "weak solution", but the construction of an appropriate weak solution remains an open problem, even for much simpler Stefan problems such as in [3].

In the final Section 6 we shall consider the model where the adsorbed $C$ increases the dissolution rather than inhibits it. (A situation like this arises, for instance, for an oil drop in water, when soap is added to the water.) We shall establish in this case the existence and uniqueness of solutions, for some small time.

**Remark 1.1.** It will be shown that $\zeta(r, \theta)$ satisfies a nonlinear second order partial differential equation which is elliptic in case (i) where the adsorption of $C$ slows the dissolution rate, and hyperbolic in case (ii) where the adsorption of $C$ increases the dissolution rate; this equation is coupled, of course, to the equations for $A$, $B$ and $C$. In both cases the data for $\zeta$ are the Cauchy data. Since the Cauchy problem is well posed for hyperbolic equations and ill-posed for elliptic equations, this would explain, mathematically, why we get instability in case (i) and stability in case (ii).

**Remark 1.2.** The results of the paper extend to 3-dimensional particles; the formulas are more complicated but the methods are the same.
2. The linearized problem. When the initial conditions in (1.11), (1.12) are independent of \( \theta \) and satisfy some regularity and compatibility assumptions, it is proved in [4] that the nonlinear system has a unique global radial solution \((A^0, B^0, C^0, \zeta^0, R^0)\) and that there is a finite shut-down time \( T^* \), that is, \( \zeta(T^*) = 1 \) (and \( \zeta(t) > 1 \) if \( t > T^* \)). The radial solution satisfies:

\[
\begin{align*}
R^0, \ zeta^0 & \in C^2[0, T^*], \\
R^0(0) = R_0 > 0, \ zeta^0(0) = \delta \in (0, 1), \\
\frac{dR^0}{dt} < 0, \ \frac{d\zeta^0}{dt} > 0 \text{ for } 0 \leq t < T^*,
\end{align*}
\]

and

\[
\begin{align*}
A^0, B^0, C^0 & \in C^{2+\nu,1+\frac{\nu}{2}}(r \geq R^0(t), 0 \leq t < T^*), \\
\frac{\partial A^0}{\partial r} & \leq 0, \quad \frac{\partial B^0}{\partial r} \geq 0, \\
A^0 & \geq 0, \quad B^0 \geq 0, \quad C^0 \geq 0
\end{align*}
\]

where \( 0 < \nu < 1; \) it is assumed that initially (2.2) holds, that

\[
A^0(r, 0) = 0, \quad C^0(r, 0) = 0, \quad B^0(r, 0) = B^* \quad \text{if} \quad r \geq R_0 + \delta_0
\]

for some \( \delta_0 > 0, \) and that the compatibility conditions

\[
\frac{\partial^2 A^0(R_0, 0)}{\partial r^2} = \frac{\partial^2 B^0(R_0, 0)}{\partial r^2} = \frac{\partial^2 C^0(R_0, 0)}{\partial r^2} = 0, \quad A^0(R_0, 0) < A^*,
\]

hold.

We linearize the system (1.1)–(1.3), (1.17), (1.18) with boundary conditions (1.7), (1.8) about a radially symmetric solution by setting

\[
A = A^0 + \varepsilon A^1, \quad B = B^0 + \varepsilon B^1, \quad C = C^0 + \varepsilon C^1, \\
g = R^0 + \varepsilon h, \quad \xi = \xi^0 + \varepsilon \eta
\]

where \( \xi^0 = 1/\zeta^0. \) Substituting (2.5) into (1.1)–(1.3) and dropping \( O(\varepsilon^2) \) terms, we obtain

\[
\begin{align*}
\left( \frac{\partial}{\partial t} - D_A \Delta \right) A^1 &= -K(A^1 B^0 + A^0 B^1), \quad r > R^0(t), \\
\left( \frac{\partial}{\partial t} - D_B \Delta \right) B^1 &= -K(A^1 B^0 + A^0 B^1), \quad r > R^0(t), \\
\left( \frac{\partial}{\partial t} - D_C \Delta \right) C^1 &= K(A^1 B^0 + A^0 B^1), \quad r > R^0(t).
\end{align*}
\]

Clearly

\[
g^2 = \varepsilon^2 h^2
\]

Therefore, by (1.17),

\[
D_A \frac{\partial A}{\partial n} = -g_t + O(\varepsilon^2).
\]
Using (2.5) we get

\[
DA \left\{ \frac{\partial A^0}{\partial n} \bigg|_{r=R^0} - \frac{\partial A^0}{\partial n} \bigg|_{r=g} \right\} + \varepsilon DA \frac{\partial A^1}{\partial n} \bigg|_{r=g} + D_A \frac{\partial A^0}{\partial n} \bigg|_{r=R^0} \\
= - \frac{dR^0}{dt} - \varepsilon \frac{\partial h}{\partial t} + O(\varepsilon^2)
\]

so that, after dropping \(O(\varepsilon^2)\) terms,

\[
(2.7) \quad DA \frac{\partial A^1}{\partial n} - D_A \frac{\partial^2 A^0}{\partial r^2} h = -\frac{\partial h}{\partial t} \quad \text{on} \quad r = R^0(t).
\]

Similarly

\[
(2.8) \quad \frac{\partial B^1}{\partial n} - \frac{\partial^2 B^0}{\partial r^2} h = 0 \quad \text{on} \quad r = R^0(t),
\]

\[
(2.9) \quad \frac{\partial C}{\partial n} - \frac{\partial^2 C^0}{\partial r^2} h = 4(C^0)^3 C^1 + 4(C^0)^3 \frac{\partial C^0}{\partial r} h \quad \text{on} \quad r = R^0(t).
\]

We next substitute \(g\) and \(\xi\) from (2.5) into (1.17), (1.18) to obtain linearized equations for \(h\) and \(\eta\). From (1.17),

\[
(2.10) \quad V_n = -g_t + O(\varepsilon^2)
\]

\[
(2.11) \quad -g_t = (\xi - 1)(A^* - A) + O(\varepsilon^2).
\]

Substituting \(g\) and \(\xi\) from (2.5) into (2.11), we get

\[
- \frac{dR^0}{dt} - \varepsilon \frac{\partial h}{\partial t} = \varepsilon \eta(A^* - A^0) - \varepsilon A^1(\xi^0 - 1)
+ (\xi^0 - 1) \left\{ \frac{A^0 - A^0}{r=g} - (A^0 - A^0) \right\} + (\xi^0 - 1)(A^* - A^0) \bigg|_{r=R^0} + O(\varepsilon^2).
\]

Dropping \(O(\varepsilon^2)\) terms, we obtain

\[
(2.12) \quad - \frac{\partial h}{\partial t} = \eta(A^* - A^0) - (\xi^0 - 1)A^1 + \frac{1}{D_A}(\xi^0 - 1)^2(A^* - A^0)h.
\]

Next, substituting (2.10) into (1.18) and recalling that \(g_\theta = O(\varepsilon)\), we get

\[
\xi_t - \frac{g_t g_\theta}{g^2} \xi_\theta + \frac{g_\theta}{g} \left( \frac{g_\theta}{g} \right)_t \xi - \frac{g_t}{g} \xi = -\xi^2 C^4 + O(\varepsilon^2).
\]

Substituting \(g, \xi\) and \(C\) from (2.5) and dropping \(O(\varepsilon^2)\), we find that

\[
(2.13) \quad \eta_t + \frac{1}{(R^0)^2} \frac{dR^0}{dt} \xi^0 \eta_\theta - \frac{1}{R^0} \frac{dR^0}{dt} \eta - \frac{1}{R^0} \xi^0 h_t + \frac{1}{(R^0)^2} \frac{dR^0}{dt} \xi^0 h
= -2\xi^0(C^0)^4 \eta - 4(C^0)^3(\xi^0)^2 C^1 - 4(C^0)^3(\xi^0)^2 \frac{\partial C^0}{\partial r} h \quad \text{on} \quad r = R^0(t).
\]
We next express \( \eta \) in terms of \( h, h_t \) from (2.12) and substitute this into (2.13). This results in the elliptic equation

\[
\mathcal{L} h = \frac{\partial}{\partial t} \left( a_1(t) \frac{\partial h}{\partial t} \right) + a_2(t) \frac{\partial^2 h}{\partial \theta^2} + b(t) \frac{\partial h}{\partial t} + c(t) h \\
= f_1(t) C^1(R^0(t), \theta, t) + f_2(t) A^1(R^0(t), \theta, t) \\
+ \frac{\partial}{\partial t} \left[ f_3(t) A^1(R^0(t), \theta, t) \right]
\]

(2.14)

where

\[
a_1(t) = -\frac{1}{A^* - A^0(R^0(t), t)}, \quad (a_1(t) < 0),
\]

(2.15)

\[
a_2(t) = \frac{1}{(R^0(t))^2} \frac{dR^0(t)}{dt} \xi^0(t), \quad (a_2(t) < 0),
\]

(2.16)

\[
b(t) = -\frac{(\xi^0(t) - 1)^2}{D_A} - \frac{\xi^0(t)}{R^0(t)} - \left\{ 2\xi^0(t) \left( C^0(R^0(t), t) \right) \right\}^4 - \frac{1}{R^0(t)} \frac{dR^0(t)}{dt} \xi^0(t) \right\} \frac{1}{A^* - A^0(R^0(t), t)},
\]

(2.17)

\[
c(t) = -\frac{2}{D_A} \left( \xi^0(t) - 1 \right) \frac{d\xi^0(t)}{dt} + \frac{1}{(R^0(t))^2} \frac{dR^0(t)}{dt} \xi^0(t)
\]

\[
+ 4 \left( C^0(R^0(t), t) \right)^3 \left( \xi^0(t) \right)^2 \frac{\partial C^0}{\partial r} \left( R^0(t), t \right)
\]

\[
- \left\{ 2\xi^0(t) \left( C^0(R^0(t), t) \right) \right\}^4 - \frac{1}{R^0(t)} \frac{dR^0(t)}{dt} \left( \xi^0(t) - 1 \right)^2 \right\} \frac{1}{D_A},
\]

(2.18)

\[
f_1(t) = -4 \left( C^0(R^0(t), t) \right)^3 \left( \xi^0(t) \right)^2,
\]

(2.19)

\[
f_2(t) = \left\{ 2\xi^0(t) \left( C^0(R^0(t), t) \right) \right\}^4 - \frac{1}{R^0(t)} \frac{dR^0(t)}{dt} \xi^0(t) - 1 \right\} \frac{1}{A^* - A^0(R^0(t), t)},
\]

(2.20)

\[
f_3(t) = -\frac{\xi^0(t) - 1}{A^* - A^0(R^0(t), t)}.
\]

(2.21)

Notice that

\[
\frac{\partial C^0}{\partial r} = \frac{1}{D_C} (C^0)^4 \quad \text{at} \quad (R^0(t), t).
\]

Therefore all the coefficients \( a_1, a_2, b, c, f_1, f_2 \) and \( f_3 \) are Lipschitz continuous, under the assumptions (2.2), (2.3).

3. The linearized problem is unstable. In this section we prove that the linearized problem is unstable. More specifically we show that for any given small \( T > 0 \) and \( \beta > 0 \) and for any large positive integer \( m \) there exist initial data for \( h \) and \( \eta \) whose first \( m \) derivatives are bounded by 1 such that a smooth solution exist for \( 0 < t < T \) but the \( C^{1, \beta} \) norm of \( h \) become infinite at \( t = T \):

**Theorem 3.1.** Consider the linearized problem (2.6)–(2.9), (2.12) and (2.13), under the assumptions (2.1)–(2.4). For any small \( T > 0 \), \( 0 < \beta < \frac{1}{2} \) and positive integer \( m \) there exist a
$C^m$ solution $(A^1, B^1, C^1, h, \eta)$ for $0 < t < T$, such that

$$\left| D_{\theta}^j h(\theta, 0) \right| \leq 1, \quad \left| D_{\theta}^j h(\theta, 0) \right| \leq 1 \quad \text{for} \quad 0 \leq j \leq m,$$

(3.1)

$$A^1(r, \theta, 0) = B^1(r, \theta, 0) = C^1(r, \theta, 0) \equiv 0 \quad \text{for} \quad r \geq R^0(0),$$

(3.2)

but

$$\| h(\cdot, T - 0) \|_{C^{1+\beta}([0, 2\pi])} = +\infty.$$

(3.3)

**Proof.** Let

$$G_T = \{0 \leq \theta \leq 2\pi, 0 \leq t \leq T\} \cap \{h(\theta, t); h \text{ and } h_t \text{ belong to } C(G_T) \text{ and are } 2\pi\text{-periodic in } \theta\}.$$

Introduce the norm

$$\| h \|_X = \| h \|_{C(G_T)} + \| h_t \|_{C(G_T)}.$$

For each $h \in X$ we solve (2.6) with the boundary conditions (2.7)–(2.9), zero initial conditions, and zero boundary conditions at $r = \infty$. Clearly,

$$\| A^1 \|_{L^\infty} \leq C \| h \|_X + CT \| B^1 \|_{L^\infty},$$

$$\| B^1 \|_{L^\infty} \leq C \| h \|_X + CT \| A^1 \|_{L^\infty},$$

$$\| C^1 \|_{L^\infty} \leq C \| h \|_X + CT (\| A^1 \|_{L^\infty} + \| B^1 \|_{L^\infty}),$$

so that

$$\| A^1 \|_{L^\infty} + \| B^1 \|_{L^\infty} + \| C^1 \|_{L^\infty} \leq C \| h \|_X,$$

(3.4)

if $T$ is small enough.

Next we use (3.4) and Hölder estimates for parabolic equations [7] to obtain

$$\| A^1 \|_{C^{\alpha, \alpha/2}(\Omega_T)} + \| B^1 \|_{C^{\alpha, \alpha/2}(\Omega_T)} + \| C^1 \|_{C^{\alpha, \alpha/2}(\Omega_T)} \leq C \| h \|_X,$$

(3.5)

for some $0 < \alpha < 1$, where

$$\Omega_T = \left\{ (x_1, x_2, t); \sqrt{x_1^2 + x_2^2} \geq R^0(t), 0 \leq t \leq T\right\}.$$

The constant $C$ is independent of $T$.

In the sequel we assume without loss of generality that $\frac{1}{2} \beta < \alpha < \beta$.

Since $A^1, B^1, C^1$ have zero initial values, (3.5) implies that

$$\| A^1 \|_{L^\infty} + \| B^1 \|_{L^\infty} + \| C^1 \|_{L^\infty} \leq CT^{\alpha/2} \| h \|_X.$$

(3.6)
We now pick up any periodic function \( k(\theta) \) such that
\[
\left\| k \right\|_{C^{1+\beta}} = \infty, \quad \left\| k \right\|_{C^{1+\beta/2}} < \infty.
\]
and denote by \( \tilde{h} \) the solution to the elliptic problem
\[
\mathcal{L}(\tilde{h}) = f_1(t)C^1(R^0(t), \theta, t) + f_2(t)A^1(R^0(t), \theta, t) + \frac{\partial}{\partial t} \left[ f_3(t)A^1(R^0(t), \theta, t) \right],
\]
(3.8)
\[
\tilde{h} \text{ is } 2\pi \text{-periodic in } \theta,
\]
(3.9)
\[
\tilde{h}_t(\theta, 0) = 0, \quad \tilde{h}(\theta, T) = k(\theta).
\]
(3.10)
We define the map \( W \) by
\[
(W\tilde{h})(\theta, t) = \tilde{h}(\theta, t).
\]
We shall show that \( W \) has a fixed point, and this will give us the solution asserted in Theorem 3.1, (with \( \eta \) defined by (2.12)).

We first need to derive some estimates on \( h \) and its derivatives which will depend on the small parameter \( T \) in an appropriate way. It is convenient to scale variables by introducing
\[
s = \frac{t}{T}, \quad \varphi = \frac{\theta}{T}, \quad \tilde{h}(s, \varphi) = \tilde{h}(\theta, t).
\]
Setting
\[
\tilde{L}h \equiv \frac{\partial}{\partial s} \left( a_1(Ts) \frac{\partial \tilde{h}}{\partial s} \right) + a_2(Ts) \frac{\partial^2 \tilde{h}}{\partial \varphi^2} + Tb(Ts) \frac{\partial \tilde{h}}{\partial s} + T^2c(Ts) \tilde{h},
\]
we have
\[
\tilde{L}h = T^2f_1(Ts)C^1(R^0(Ts), T\varphi, Ts) + T^2f_2(Ts)A^1(R^0(Ts), T\varphi, Ts)
\]
(3.11)
\[
+ T \frac{\partial}{\partial s} \left[ f_3(Ts)A^1(R^0(Ts), T\varphi, Ts) \right]
\]
and
\[
\left. \frac{\partial \tilde{h}}{\partial s} \right|_{s=0} = 0, \quad \left. \tilde{h} \right|_{s=1} = k(T\varphi),
\]
(3.12)
\[
\tilde{h}(\varphi, s) \text{ is } \frac{2\pi}{T} \text{-periodic in } \varphi.
\]

To analyze the solution \( \tilde{h} \) we introduce auxiliary problem:
\[
a_1(Ts) \frac{\partial^2 v}{\partial s^2} + a_2(Ts) \frac{\partial^2 v}{\partial \varphi^2} = Tf_3(Ts)A^1(R^0(Ts), T\varphi, Ts),
\]
(3.13)
\[
v \Big|_{s=0} = 0, \quad \frac{\partial v}{\partial s} \Big|_{s=1} = 0,
\]
\[
v \text{ is } \frac{2\pi}{T} \text{-periodic in } \varphi.
\]
Then
\[ \|v\|_{L^\infty(\tilde{G})} \leq CT\|A^1\|_{L^\infty} \]
where
\[ \tilde{G} = \left[0, \frac{2\pi}{T}\right] \times [0, 1] \]
and \(\| \cdot \|\) denotes norms in the variable \((s, \varphi)\).

By Schauder’s interior-boundary estimates [5],
\begin{align*}
\|v\|_{C^{2+\alpha/2}(\tilde{G})} &\leq C \left( T\|A^1\|_{C^{\alpha/2}} + \|v\|_{L^\infty(\tilde{G})} \right) \\
&\leq CT\|A^1\|_{C^{\alpha/2}},
\end{align*}
(3.14)
where \(C^{\alpha/2}\) denotes Hölder space with exponent \(\alpha/2\) in both space and time variables. Differentiating (3.13) in \(s\) and subtracting from equation (3.11), we find that
\[
\tilde{L}_0(\tilde{h} - v_s) = T^2 f_1(Ts) C^1 (R^0(Ts), T\varphi, Ts) + T^2 f_2(Ts) A^1 (R^0(Ts), T\varphi, Ts) \\
-Tb(Ts) \frac{\partial^2 v}{\partial s^2} + T \frac{\partial a_2(Ts)}{\partial t} \frac{\partial^2 v}{\partial \varphi^2} - T^2 c(Ts)(\tilde{h} - v_s) - T^2 c(Ts)v_s,
\]
(3.15)
where
\[
\frac{\partial}{\partial s}(\tilde{h} - v_s) \bigg|_{s=0} = 0, \quad (\tilde{h} - v_s) \bigg|_{s=1} = k(Ts)
\]
By the maximum principle
\[ \|\tilde{h} - v_s\|_{L^\infty(\tilde{G})} \leq C \left( \|k\|_{L^\infty} + \text{the right-hand-side of (3.15)} \right). \]

Therefore, if we apply elliptic \(C^{1+\alpha}\) estimates [9] to (3.15), we get
\[ \|\tilde{h} - v_s\|_{C^{1+\alpha/2}(\tilde{G})} \leq C \left( \|k\|_{C^{1+\alpha/2}} + T^2 \|A^1\|_{C^{\alpha/2}} + \|C^1\|_{C^{\alpha/2}} + T^2 \|\tilde{h} - v_s\|_{L^\infty} \right). \]
Recalling (3.14) we conclude that
\[ \|\tilde{h}\|_{C^{1+\alpha/2}(\tilde{G})} \leq C \left( \|k\|_{C^{1+\alpha/2}} + T \left( \|A^1\|_{C^{\alpha/2}} + \|C^1\|_{C^{\alpha/2}} \right) \right) \]
provided \(T\) is small enough. The above estimate written in terms of the variable \((t, \theta)\) reads:
\begin{align*}
\|\tilde{h}\|_{L^\infty} + T \left( \|\tilde{h}_t\|_{L^\infty} + \|\tilde{h}_\theta\|_{L^\infty} \right) + T^{1+\alpha/2} \left( [\tilde{h}]_{C^{\alpha/2}} + [\tilde{h}_\theta]_{C^{\alpha/2}} \right) \\
&\leq C \left( \|k\|_{L^\infty} + T \|k_\theta\|_{L^\infty} + T^{1+\alpha/2} [k_\theta]_{C^{\alpha/2}} \right) \\
&+ CT \left( \|A^1\|_{L^\infty} + \|C^1\|_{L^\infty} \right) + CT^{1+\alpha/2} \left( [A^1]_{C^{\alpha/2}} + [C^1]_{C^{\alpha/2}} \right).
\end{align*}
(3.16)
Using (3.5), (3.6) we conclude that
\[
\|\bar{h}\|_{L^\infty} + T \left( \|\bar{h}_t\|_{L^\infty} + \|\bar{h}_\theta\|_{L^\infty} \right) + T^{1+\alpha/2} (\|\bar{h}_{\bar{t}}\|_{C^{\alpha/2}} + [\bar{h}_\theta]_{C^{\alpha/2}})
\]
\[
\leq C^* \left( \|k\|_{L^\infty} + T \|k_\theta\|_{L^\infty} + T^{1+\alpha/2} [k_\theta]_{C^{\alpha/2}} \right)
\]
\[
+ CT^{1+\alpha/2} \left( \|h\|_{L^\infty} + \|h_t\|_{L^\infty} \right).
\]

(3.17)

This estimate shows that if
\[
X_0 = \{ h \in X; \|h\|_{L^\infty} + T \|h_t\|_{L^\infty} \leq C^* \|k\|_{C^{1+\alpha/2}} + 1 \}
\]

then \(W\) maps \(X_0\) into itself provided \(T\) is small enough. It is also clear that \(WX_0\) is a precompact subset of \(X_0\) and that \(W\) is continuous. The Schauder fixed point theorem can then be applied to conclude that \(W\) has a fixed point \(h\), and this gives a solution to the linearized problem with the same \(h\).

Next we show that \(h\) and \(\eta\) are \(C^m\) smooth.

The estimate (3.17) implies that
\[
\|h\|_{C^{1+\alpha/2}(\Omega_T)} \leq C.
\]

(3.18)

Recall, by (3.10), that \(h_t|_{t=0} = 0\). Since we have assumed (2.4), the compatibility condition (for parabolic equations) is satisfied for \(A^1, B^1, C^1\). We can therefore apply to equations (2.6) with the boundary condition (2.7)–(2.9) the \(C^{1+\alpha}\) parabolic estimates [8, Theorem 1.2] and the Schauder estimates [7] to get
\[
\|A^1\|_{C^{1+\alpha/2,1/2+\alpha/4}(\Omega_T)} \leq C,
\]
\[
\|B^1\|_{C^{2+\alpha/2,1+\alpha/4}(\Omega_T)} + \|C^1\|_{C^{2+\alpha/2,1+\alpha/4}(\Omega_T)} \leq C.
\]

(3.19)

We now differentiate (2.14) in \(\theta\) and obtain the same equation for \(h_\theta\) but with \(A^1, C^1\) replaced by \(A^1_\theta, C^1_\theta\). If we use interior-boundary Schauder estimates (away from \(t = T\)), the procedure that led to (3.16)–(3.18) gives
\[
\|h_\theta\|_{C^{1+\alpha/4}(\Omega_{T-\epsilon})} \leq C_\varepsilon
\]

(3.20)

for any \(\varepsilon > 0\). By iterating this process \(m\) times we arrive at the estimate
\[
\left\| \frac{\partial^j h}{\partial \theta^j} \right\|_{C^{1+\alpha/2-j^{-1}\alpha}(\Omega_{T-\epsilon})} \leq C_{\varepsilon,j} \quad (0 \leq j \leq m).
\]

Since \(\eta\) is given by (2.13) we get similar estimates for \(\eta\). Noting that \(\varepsilon\) is arbitrary, it follows that the solution has \(m\) derivatives (which are actually continuous) in \(\theta\). Next, using (3.19) and interior-boundary Schauder estimates for \(h\) (away from \(t = T\)), we obtain
\[
\|h\|_{C^{1+(1/2+\alpha/4)}(\Omega_{T-\epsilon})} \leq C.
\]

(3.21)

The estimates (3.20) and (3.21) imply that
\[
\|h_t\|_{C^{1+\alpha/2,1/2+\alpha/4}(\Omega_{T-\epsilon})} \leq C,
\]
where \( C^{1+\alpha/2, 1/2+\alpha/4} \) means \( C^{1+\alpha/2} \) in \( \theta \) and \( C^{1/2+\alpha/4} \) in \( t \). Thus, by parabolic Schauder estimates,

\[
\| A^1 \|_{C^{2+\alpha/2, 1+\alpha/4}(-T < t < 0)} \leq C.
\]

Notice that the coefficients in (2.15)–(2.21) are \( C^\infty \) for \( t > 0 \) (see [4]). Therefore we can iterate the above process to conclude that \( h \) is \( C^m \) in \((\theta, t)\) away from \( t = 0 \) and \( t = T \).

Finally, by multiplying the solution by a small enough constant \( \delta \) we find that \( \delta h \) and \( \delta \eta \) have all their first \( m \) derivatives (in \( \theta \)) bounded by 1 at \( t = 0 \), the solution of the linearized problem has (continuous) \( m \) derivatives for all \( 0 \leq t < T \) since (by (3.10)) \( \delta h(\theta, T - 0) = \delta k(\theta) \), and \( \| \delta k \|_{C^{1+\beta}} = \infty \); this completes the proof of the theorem. \( \square \)

### 4. Uniqueness for the linearized problem.

**Theorem 4.1.** Suppose \((A^1, B^1, C^1, h, \eta)\) is a solution of the linearized problem for \( 0 \leq t < T_0 \) such that

\[
\begin{align*}
A^1 &= B^1 = C^1 = 0 \text{ at } t = 0, \\
h &= \eta = 0 \text{ at } t = 0,
\end{align*}
\]

and

\[
\begin{align*}
A^1, B^1, C^1 &\text{ belong to } C^{1+\alpha/(1+\alpha)/2}(\Omega_{T_0}), \\
h &\in C^{1+\alpha, 1+\alpha}(G_{T_0}), \quad \eta \in C^{\alpha, \alpha}(G_{T_0}).
\end{align*}
\]

Then

\[
A^1 \equiv B^1 \equiv C^1 \equiv 0 \text{ in } \Omega_{T_0}
\]

and

\[
h \equiv \eta \equiv 0 \text{ in } G_{T_0}.
\]

**Proof.** From (4.1), (4.2) and (2.12) it follows that

\[
h_t(\theta, 0) \equiv 0.
\]

Let \( \varphi(\theta) \) be any \( 2\pi \)-periodic function such that

\[
\varphi''(\theta) = -k^2 \varphi(\theta)
\]

where \( k \) is a nonnegative integer. Let

\[
\tilde{A}(r, t) = \int_0^{2\pi} \varphi(\theta)A^1(r, \theta, t)d\theta
\]

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and define $\tilde{B}(r, t), \tilde{C}(r, t), \tilde{h}(t)$ in a similar way. Then $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{h}$ satisfy the same boundary conditions (2.7)–(2.9) as $A, B, C, h$. Multiplying the equations for $A^1, B^1, C^1$ by $\varphi(\theta)$ and integrating with respect to $\theta$, we find that

$$
\tilde{A}_t = D_A \left( \tilde{A}_{rr} + \frac{1}{r} \tilde{A}_r - \frac{k^2}{r^2} \tilde{A} \right) - K(\tilde{A}B^0 + A^0\tilde{B}),
$$

$$
\tilde{B}_t = D_B \left( \tilde{B}_{rr} + \frac{1}{r} \tilde{B}_r - \frac{k^2}{r^2} \tilde{B} \right) - K(\tilde{A}B^0 - A^0\tilde{B}),
$$

$$
\tilde{C}_t = D_C \left( \tilde{C}_{rr} + \frac{1}{r} \tilde{C}_r - \frac{k^2}{r^2} \tilde{C} \right) + K(\tilde{A}B^0 + A^0\tilde{B})
$$

if $r > R^0(t)$. The same argument as in (3.4)–(3.6) show that

$$
\|\tilde{A}\|_{L^\infty} + \|\tilde{B}\|_{L^\infty} + \|\tilde{C}\|_{L^\infty} \leq CT^{\alpha/2} \left( \|\tilde{h}\|_{L^\infty} + \|\tilde{h}_t\|_{L^\infty} \right), \quad L^\infty = L^\infty(\Omega_T).
$$

Next, multiplying (2.14) by $\varphi(\theta)$ and integrating with respect to $\theta$, we obtain the differential equation:

$$
\frac{d}{dt} \left( a_1 \frac{d\tilde{h}}{dt} \right) - k^2 a_2 \tilde{h} + b\tilde{h}_t + c\tilde{h} = f_2(t)\tilde{C} \left( R^0(t), t \right)
$$

$$
+ f_3(t)\tilde{A} \left( R^0(t), t \right) + \frac{\partial}{\partial t} \left( f_3(t) \tilde{A} \left( R^0(t), t \right) \right).
$$

Since,

$$
\tilde{h} = \tilde{h}_t = 0 \quad \text{and} \quad \tilde{A} = 0 \quad \text{at} \quad t = 0,
$$

we get, by integration,

$$
\|\tilde{h}_t\|_{L^\infty} \leq C \left( \|\tilde{A}\|_{L^\infty} + \|\tilde{C}\|_{L^\infty} \right) + CT \left( (k^2 + 1)\|\tilde{h}\|_{L^\infty} + \|\tilde{h}_t\|_{L^\infty} \right).
$$

Clearly,

$$
\|\tilde{h}\|_{L^\infty} \leq T\|\tilde{h}_t\|_{L^\infty}.
$$

Substituting (4.7), (4.5) into (4.6) we find that if $T$ is small enough (depending on $k$) then

$$
\tilde{h} \equiv 0.
$$

It then also follows that $\tilde{A} \equiv \tilde{B} \equiv \tilde{C} \equiv 0$ and $\eta \equiv 0$ if $t < T$. We can continue the process step-by-step to deduce that $\tilde{h} \equiv 0$ if $0 \leq t < T_0$ ($T_0$ is independent of $k$).

Taking in particular $\varphi(\theta) = \sin kx$ or $\cos kx$ for any integers $k$, we see that the Fourier series of the continuously differentiable function $\theta \mapsto h(\theta, t)$ vanishes identically. Therefore $h \equiv 0$ and then also $A \equiv B \equiv C \equiv 0$ and $\eta \equiv 0$. 

\[\square\]
5. The nonlinear problem is unstable. Theorem 3.1 can be extended to the full nonlinear problem. To be more precise, we say that a radial solution \(( A^0, B^0, C^0, \xi^0, R^0)\) satisfying (2.1)–(2.4) is locally Lipschitz stable if there exist \( T > 0 \) and positive integer \( m \geq 3 \) such that for any solution \(( A, B, C, \xi, g)\) of the nonlinear problem, if
\[
(5.1) \quad \|(A - A^0)_{t=0}\|_{C^m} + \|(B - B^0)_{t=0}\|_{C^m} + \|(C - C^0)_{t=0}\|_{C^m} \leq \varepsilon \\
(\|(\xi - \xi^0)_{t=0}\|_{C^m} + \|(g - R^0)_{t=0}\|_{C^m} \leq \varepsilon \quad \forall \varepsilon > 0
\]
and if the compatibility conditions hold (at \( r = g(\theta, 0) \)), then
\[
(5.2) \quad \|A - A^0\|_{C^{2,1}} + \|B - B^0\|_{C^{2,1}} + \|C - C^0\|_{C^{2,1}} \leq C^* \varepsilon , \\
(5.3) \quad \|\xi - \xi^0\|_{C^{1,1}} \leq C^* \varepsilon , \quad \|g - R^0\|_{C^{2,1}} \leq C^* \varepsilon
\]
for some constant \( C^* \) independent of \( \varepsilon \).

In (5.2) \( A^0, B^0, C^0 \) have been extended as \( C^{2+\nu, \nu+\nu/2} \) functions in the domain
\[
\Omega_{T,0} = \left\{ \sqrt{x_1^2 + x_2^2} \geq R_0(t) - \delta_0, \quad 0 \leq t \leq T \right\}
\]
for some \( \delta_0 > 0 \), and \( C^{2,1} \) means \( C^{2,1}(\Omega_T) \) where
\[
\Omega_T = \{ r \geq g(\theta, t), \quad 0 \leq t \leq T \}.
\]
For \( T \) small enough \( \Omega_T \) is contained in \( \Omega_{T,0} \). In (5.3), \( C^{j,1} \) means \( C^{j,1}(G_T) \).

**Theorem 5.1.** Each radial solution is not locally Lipschitz stable.

**Proof.** Suppose the assertion is not true for a particular radial solution \(( A^0, B^0, C^0, \xi^0, R^0)\), that is, this solution is Lipschitz stable up to some fixed time \( T > 0 \).

Take any solution of the nonlinear problem with
\[
(5.4) \quad A = A^0 + \varepsilon A^1_\varepsilon, \quad B = B^0 + \varepsilon B^1_\varepsilon, \quad C = C^0 + \varepsilon C^1_\varepsilon, \\
g = R^0 + \varepsilon h_\varepsilon, \quad \xi = \xi^0 + \varepsilon \eta_\varepsilon
\]
where
\[
(5.5) \quad \|A^1_\varepsilon\|_{t=0}\|_{C^m} + \|B^1_\varepsilon\|_{t=0}\|_{C^m} + \|C^1_\varepsilon\|_{t=0}\|_{C^m} \leq 1 , \\
\|h_\varepsilon\|_{t=0}\|_{C^m} + \|\eta_\varepsilon\|_{t=0}\|_{C^m} \leq 1 .
\]
The Lipschitz stability implies that
\[
(5.6) \quad \|A^1_\varepsilon\|_{C^{2,1}(\Omega_T)} + \|B^1_\varepsilon\|_{C^{2,1}(\Omega_T)} + \|C^1_\varepsilon\|_{C^{2,1}(\Omega_T)} \leq C ,
\]
\[
(5.7) \quad \|\eta_\varepsilon\|_{C^{1,1}(G_T)} \leq C ,
\]
\[
(5.8) \quad \|h_\varepsilon\|_{C^{2,1}(G_T)} \leq C
\]
where \( C \) is a constant independent of \( \varepsilon \).
Differentiating the equation

\begin{equation}
\tag{5.9}
g_t = -\frac{1}{g^3} \sqrt{g^2 + g^3} (\xi - 1)(A^* - A)
\end{equation}

in \( \theta \) and applying the estimates (5.6)-(5.8), we derive the bound

\begin{equation}
\tag{5.10}
\|(h_\varepsilon)_{t\theta}\|_{L^\infty} \leq C.
\end{equation}

Next we differentiate (5.9) in \( t \) and use (5.6)-(5.8) and (5.10). We obtain,

\begin{equation}
\tag{5.11}
\|(h_\varepsilon)_{tt}\|_{L^\infty} \leq C.
\end{equation}

The estimates (5.8), (5.10), (5.11) combined mean that

\begin{equation}
\tag{5.12}
\|h_\varepsilon\|_{C^{2,\alpha}(\bar{G}_T)} \leq C.
\end{equation}

We now take a sequence of initial values as above with \( \varepsilon \rightarrow 0 \). For a subsequence \( \varepsilon = \varepsilon_j \rightarrow 0 \),

\[ A^1_\varepsilon \to A^1, \quad B^1_\varepsilon \to B^1, \quad C^1_\varepsilon \to C^1 \quad \text{in} \quad C^{1+\alpha, (1+\alpha)/2}, \]

\[ \eta_\varepsilon \to \eta \quad \text{in} \quad C^{\alpha, \alpha}, \]

\[ h_\varepsilon \to h \quad \text{in} \quad C^{1+\alpha, 1+\alpha} \]

for any \( 0 < \alpha < 1 \), and

\begin{equation}
\tag{5.13}
\|\eta_t\|_{L^\infty} + \|\eta_\theta\|_{L^\infty} + \|h_{t\theta}\|_{L^\infty} + \|h_{tt}\|_{L^\infty} + \|h_{t\theta}\|_{L^\infty} < \infty.
\end{equation}

We can then proceed as in Section 2 (but rigorously!) to prove that \( (A^1, B^1, C^1, \eta, h) \) satisfies the linearized equations and boundary conditions (on \( r = R_0 + \varepsilon h_\varepsilon \)) with an error term of order \( O(\varepsilon) \). Letting \( \varepsilon = \varepsilon_j \rightarrow 0 \) we deduce that \( (A^1, B^1, C^1, \eta, h) \) is a solution to the linearized problem. Choose the initial values for the linearized problem to be as in Theorem 3.1. By uniqueness (Theorem 4.1) \( (A^1, B^1, C^1, \eta, h) \) must coincide with the linearized solution established in Theorem 3.1 and, consequently,

\[ \|h(\cdot, T)\|_{C^{1+\alpha}} = \infty, \]

which is a contradiction to (5.13).

\[
\square
\]

6. A model with accelerated dissolution. In this section we consider the case where the adsorbed \( C \) increases the dissolution of the grain:

\begin{equation}
\tag{6.1}
D_A \frac{\partial A}{\partial n} + P(\zeta)(A - A^*) = 0 \quad \text{on} \quad \Gamma_t
\end{equation}

where \( P(s) \) is a smooth function and

\begin{equation}
\tag{6.2}
P(s) > 0, \quad P'(s) > 0 \quad \text{for} \quad s \geq 0.
\end{equation}

Setting

\begin{equation}
\tag{6.3}
k = g_\theta, \quad F = A^* - A(g(\theta, t), \theta, t)
\end{equation}

k = g_\theta, \quad F = A^* - A(g(\theta, t), \theta, t)

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we then have, by (1.6),

\[ V_n = P(\zeta) F \]

and, by (1.13)–(1.15),

\[ \zeta_t = -\frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F \zeta_\theta - \frac{P(\zeta) F}{g} \left( \frac{k}{\sqrt{g^2 + k^2}} \right)_g + \frac{P(\zeta) F}{\sqrt{g^2 + k^2}} + C^4, \]

or

\[ \zeta_t = -\frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F \zeta_\theta - \frac{P(\zeta) F g}{(g^2 + k^2)^{3/2}} k_\theta + P(\zeta) F \left[ \frac{k^2}{(g^2 + k^2)^{3/2}} + \frac{1}{\sqrt{g^2 + k^2}} \right] + C^4. \] (6.4)

Differentiating the equation (1.5), written in the form

\[ g_t = -\frac{\sqrt{g^2 + k^2}}{g} P(\zeta) F, \]

in \( \theta \), we obtain

\[
\begin{align*}
\kappa_t &= -\frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F \kappa_\theta - \frac{\sqrt{g^2 + k^2}}{g} P'(\zeta) F \zeta_\theta \\
&= -\frac{\sqrt{g^2 + k^2}}{g} P(\zeta) F_\theta + P(\zeta) F \frac{k^3}{g^2 \sqrt{g^2 + k^2}}.
\end{align*}
\] (6.5)

The system (6.4)–(6.6) is a nonlinear hyperbolic system. To transform it into a diagonalized form, introduce

\[
\begin{align*}
\ell(\theta, t) &= \int_0^{k(\theta, t)} \frac{g(\theta, t)}{g^2(\theta, t) + s^2} ds + \int_{c_0}^{\zeta(\theta, t)} \left( \frac{P'(s)}{s P(s)} \right)^{1/2} ds, \\
m(\theta, t) &= \int_0^{k(\theta, t)} \frac{g(\theta, t)}{g^2(\theta, t) + s^2} ds - \int_{c_0}^{\zeta(\theta, t)} \left( \frac{P'(s)}{s P(s)} \right)^{1/2} ds, \quad c_0 > 0.
\end{align*}
\]

Then

\[
\begin{align*}
\ell_t &= \frac{g}{g^2 + k^2} \kappa_t + \left( \frac{P'(\zeta)}{\zeta P(\zeta)} \right)^{1/2} \zeta_t + g_t(\theta, t) \int_0^{k(\theta, t)} \frac{s^2 - g^2(\theta, t)}{(g^2(\theta, t) + s^2)^2} ds, \\
\ell_\theta &= \frac{g}{g^2 + k^2} \kappa_\theta + \left( \frac{P'(\zeta)}{\zeta P(\zeta)} \right)^{1/2} \zeta_\theta + k(\theta, t) \int_0^{k(\theta, t)} \frac{s^2 - g^2(\theta, t)}{(g^2(\theta, t) + s^2)^2} ds,
\end{align*}
\]

and similar formulas hold for \( m_t, m_\theta \).

A direct computation shows that

\[ \ell_t = -\left( \frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F + \frac{P'(\zeta) \zeta P(\zeta)}{g^2 + k^2} F \right) \ell_\theta \]

(6.8)
\[ + \left( \frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F + \sqrt{\frac{P'(\zeta) \zeta P(\zeta)}{g^2 + k^2}} F \right) k \int_0^k \frac{s^2 - g^2}{(s^2 + g^2)^2} ds \\
+ g_1 \int_0^k \frac{s^2 - g^2}{(s^2 + g^2)^2} ds - \frac{P(\zeta)}{\sqrt{g^2 + k^2}} F_0 + \frac{P(\zeta) F k^3}{g(g^2 + k^2)^{3/2}} \\
+ \sqrt{P'(\zeta) \zeta P(\zeta)} F \left[ \frac{k^2}{(g^2 + k^2)^{3/2}} + \frac{1}{\sqrt{g^2 + k^2}} \right] + C^4 \sqrt{\frac{P'(\zeta)}{\zeta P(\zeta)}}. \]

(6.9) \[ m_t = - \left( \frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F - \sqrt{\frac{P'(\zeta) \zeta P(\zeta)}{g^2 + k^2}} F \right) m_\theta \]

+ \left( \frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F - \sqrt{\frac{P'(\zeta) \zeta P(\zeta)}{g^2 + k^2}} F \right) k \int_0^k \frac{s^2 - g^2}{(s^2 + g^2)^2} ds \\
+ g_1 \int_0^k \frac{s^2 - g^2}{(s^2 + g^2)^2} ds - \frac{P(\zeta)}{\sqrt{g^2 + k^2}} F_0 + \frac{P(\zeta) F k^3}{g(g^2 + k^2)^{3/2}} \\
- \sqrt{P'(\zeta) \zeta P(\zeta)} F \left[ \frac{k^2}{(g^2 + k^2)^{3/2}} + \frac{1}{\sqrt{g^2 + k^2}} \right] - C^4 \sqrt{\frac{P'(\zeta)}{\zeta P(\zeta)}}.

Substituting (6.5) into (6.8), (6.9), we get

(6.10) \[ \ell_t + a \ell_\theta = \varphi(g, h, \zeta, F, F_\theta, E), \]

(6.11) \[ m_t + b m_\theta = \psi(g, h, \zeta, F, F_\theta, E) \]

where \[ E = C^4 (g(\theta, t), \theta, t), \]

\( \varphi \) and \( \psi \) are smooth functions as long as \( g \geq c_0 > 0, \zeta \geq c_0 \), and

(6.12) \[ a = \frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F + \left( \frac{P'(\zeta) \zeta P(\zeta)}{g^2 + k^2} \right)^{1/2} F, \]

(6.13) \[ b = \frac{k}{g \sqrt{g^2 + k^2}} P(\zeta) F - \left( \frac{P'(\zeta) \zeta P(\zeta)}{g^2 + k^2} \right)^{1/2} F. \]

We impose initial conditions

(6.14) \[ g|_{t=0} = g_0(\theta), \quad \zeta|_{t=0} = \zeta_0(\theta) \]

where \( g_0(\theta), \zeta_0(\theta) \) are periodic functions such that

(6.15) \[ g_0(\theta) \geq 2c_0, \quad \zeta_0(\theta) \geq 2c_0 \quad (c_0 > 0), \]

\[ \|g_0\|_{C^{2+\alpha}} \leq M, \quad \|\zeta_0\|_{C^{1+\alpha}} \leq M \]

where \( M < \infty. \)
The system (6.5), (6.10), (6.11) is hyperbolic with given initial data

\[(6.16) \quad g = g_0(\theta), \quad \ell|_{t=0} = \ell_0(\theta), \quad m|_{t=0} = m_0(\theta)\]

and

\[g_0(\theta) \geq 2c_0 > 0,\]
\[\ell_0(\theta) - m_0(\theta) \geq 2 \int_{c_0}^{2c_0} \left( \frac{P'(s)}{sP(s)} \right)^{1/2} ds \equiv 4c_1 > 0,\]
\[(6.17) \quad \ell_0(\theta) + m_0(\theta) \leq \pi - 2c_2 \quad \text{for some} \quad c_2 > 0,\]
\[\|g_0\|_{C^{2+\alpha}} \leq M, \quad \|\ell_0\|_{C^{1+\alpha}} \leq C^*(M, c_0),\]
\[\|m_0\|_{C^{1+\alpha}} \leq C^*(M, c_0).\]

In the sequel we denote by

\[\| \cdot \|_{C^{m+\alpha}_\theta(G_T)}\]

the $C^{m+\alpha}$ Hölder norm in $\theta$, taken uniformly in $t$ as $(\theta, t)$ varies in $G_T$, and by

\[\| \cdot \|_{C^{\alpha}_t(G_T)}\]

the $C^\alpha$ norm in $t$, uniformly in $\theta$ as $(\theta, t)$ varies in $G_T$. Finally, by

\[\|u\|_{C^{k+\alpha, k+\alpha}(G_T)}\]

we denote the sum of the $C^\alpha$ Hölder norms (in $(\theta, t) \in G_T$) of all the derivatives $\partial^I_\theta \partial^J_t u$ with $0 \leq i + j \leq k$.

**Lemma 6.1.** Given $F$, $E$ continuous in $G_T$ and $2\pi$-periodic in $\theta$, such that

\[(6.18) \quad \|F\|_{C^{2+\alpha}_\theta(G_T)} \leq M_1, \quad \|E\|_{C^{1+\alpha}_\theta(G_T)} \leq M_1,\]

there exists a unique solution $(\zeta, g, k)$ of the system (6.4)–(6.6) with initial conditions (6.14) and with $g_\theta = k$ for $0 \leq t \leq T$ provided $T$ is small enough, and

\[(6.19) \quad \|\zeta\|_{C^{1+\alpha}_\theta(G_T)} \leq C(M, c_0),\]
\[(6.20) \quad \|\zeta\|_{C^{1+\alpha, 1+\alpha}_t(G_T)} \leq C(M, M_1, c_0),\]
\[(6.21) \quad g(\theta, t) \geq c_0, \quad \zeta(\theta, t) \geq c_0 \quad \text{in} \quad \Omega_T;\]

here $T$ depends on $M$, $M_1$ and $c_0$.

**Lemma 6.2.** If $(\hat{F}, \hat{E})$ is another pair satisfying (6.18) then the corresponding solution $\hat{\zeta}, \hat{g}$ satisfies:

\[(6.22) \quad \|\zeta - \hat{\zeta}\|_{C^{\alpha}_\theta(G_t)} + \|g - \hat{g}\|_{C^{1+\alpha}_\theta(G_t)} \leq C(M, M_1, c_0) t \left[ \|F - \hat{F}\|_{C^{1+\alpha}_\theta(G_t)} + \|E - \hat{E}\|_{C^{\alpha}_\theta(G_t)} \right]\]

for $0 \leq t \leq T$. 

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Proof of Lemma 6.1. From (6.7) we see that

\[ k = g \tan \left( \frac{\ell + m}{2} \right) \quad \text{if} \quad |\ell + m| < \pi \]

and

\[ \int_{c_0}^{\zeta} \left( \frac{P'(s)}{sP(s)} \right)^{1/2} ds = \frac{\ell - m}{2} \quad \text{if} \quad \ell - m > 0. \]

Since we are interested only in solutions for small time, without loss of generality we may take \( P(s) = s \) for \( s \) large. Then (6.24) defines uniquely \( \zeta \) as a function of \((\ell - m)/2, \)

\[ \zeta = Y \left( \frac{\ell - m}{2} \right), \]

and \( Y(s) \) is smooth for \( s \geq c_1, c_1 \) as in (6.17).

Substituting (6.23), (6.25) into the right-hand sides of (6.10)–(6.13) and (6.5), we find that (6.5) and (6.10), (6.11) form a quasilinear hyperbolic system for \((g, \ell, m)\). By standard results \[1\] [6] this system with the initial conditions (6.16) has a unique solution such that

\[ \|g\|_{C^{1+\alpha, 1+\alpha}(\varSigma_T)} + \|\ell\|_{C^{1+\alpha, 1+\alpha}(\varSigma_T)} + \|m\|_{C^{1+\alpha, 1+\alpha}(\varSigma_T)} \leq C(M, M_1, c_0), \]

provided \( T \) is small enough; the smallness of \( T \) is required also to ensure that

\[ g(\theta, t) \geq c_0, \quad |\ell + m| \leq \pi - c_2, \quad \ell - m \geq c_1 \]

in \( \Omega_T \) (so that (6.23) and (6.25) determine \( k \) and \( \zeta \) as smooth functions of \( \ell, m \)); (6.21) is then also satisfied.

We next show that \( k = g_\theta \). Differentiating (6.5) formally in \( \theta \) and comparing with (6.6), we easily get

\[ (g_\theta - k)_t = P(\zeta) F \frac{k^2}{g^2} \frac{k^2}{g^2} (g_\theta - k) \]

and, since \((g_\theta - k) = 0 \at t = 0\), we conclude that \( g_\theta \equiv k \). The differentiation in \( \theta \) in (6.5) can be justified by first integrating in \( t \) and then differentiating in \( \theta \).

Since \( g_\theta = k \), the estimates (6.26) imply (6.20). To prove (6.19) we consider the characteristics for \( \ell \), given by

\[ \frac{d\xi(\theta, t)}{dt} = a(\xi, t), \quad \xi(\theta, t) = \theta. \]

It follows that

\[ \frac{d\xi_\theta}{dt} = a(\xi_\theta, t) \xi_\theta, \quad \xi_\theta(\theta, 0) = 1 \]

and therefore

\[ \exp \left[ -C(M, M_1, c_0)t \right] \leq \xi_\theta \leq \exp \left[ C(M, M_1, c_0)t \right], \]
and so

\[
\frac{1}{2} \leq \xi_\theta(\theta, t) \leq 2
\]

if \( T \) is small enough. This allows us to establish the \( C^\alpha \) estimate of \( \ell_\theta \):

\[
\| \ell \|_{C^{1+\alpha}(G_T)} \leq C(M, c_0);
\]

the constant \( C(M, c_0) \) is independent of \( M_1 \). A similar estimate can be established for \( m \), and together they yield the assertion (6.19).

\[ \square \]

**Remark 6.1.** The fact that the constant \( \hat{C}(M, c_0) \) is independent of \( M_1 \) is crucial for establishing existence for the full nonlinear problem.

**Proof of Lemma 6.2.** The proof follows by standard arguments (cf. [1]), integrating along characteristics and applying the Gronwall inequality.

\[ \square \]

**Remark 6.2.** From (6.20) we have:

\[
[g_t]_{C^{2+\alpha/2}(G_T)} + \| D^2 g \|_{C^{2+\alpha/2}(G_T)} + \| \xi_t \|_{C^{2+\alpha/2}(G_T)} + \| D_\theta \xi \|_{C^{2+\alpha/2}(G_T)} \leq C(M, M_1, c_0) T^{\alpha/2} \leq 1
\]

if \( T \) is small enough.

Since \( 0 \leq F \leq A^* \), (6.4), (6.5) and (6.19) imply that

\[
\| g_t \|_{C(G_T)} + \| \xi_t \|_{C(G_T)} \leq C(M, c_0, A^*),
\]

\[
\| g_t \|_{C^2(G_T)} \leq C(M, c_0) \left( 1 + \| F \|_{C^2(G_T)} \right).
\]

Combining the estimates (6.29)–(6.31), we then have:

\[
\| \xi \|_{C^{1+\alpha/(1+\alpha)}(G_T)} \leq \hat{C}(M, c_0, A^*),
\]

\[
\| g \|_{C^{2+\alpha, 1+\alpha/2}(G_T)} \leq \hat{C}(M, c_0) \left[ 1 + \| F \|_{C^2(G_T)} \right].
\]

We now consider the full nonlinear problem with initial conditions (6.14) satisfying (6.15), and

\[
A|_{t=0} = A_0(r, \theta), \quad B|_{t=0} = B_0(r, \theta), \quad C|_{t=0} = C_0(r, \theta)
\]

for \( r \geq g_0(\theta) \), where

\[
A_0 \equiv 0, \quad B \equiv B^*, \quad C_0 \equiv 0 \quad \text{for} \quad r > R^*
\]

for some large \( R^* > R_0 \). We further assume that

\[
0 \leq A_0 < A^*, \quad 0 \leq B_0 \leq B^*, \quad C_0 \geq 0,
\]

\[
\| A_0 \|_{C^{2+\alpha}} + \| B_0 \|_{C^{2+\alpha}} + \| C_0 \|_{C^{2+\alpha}} \leq M
\]

in the region \( \sqrt{x_1^2 + x_2^2} \geq g_0(\theta) \), and

\[
D_A \frac{\partial A_0}{\partial n} + P(\zeta_0)(A_0 - A^*) = 0,
\]

\[
\frac{\partial B_0}{\partial n} = 0, \quad D_C \frac{\partial C_0}{\partial n} = -C_0^4 \quad \text{on} \quad r = g_0(\theta).
\]
For simplicity we have taken the $M$ in (6.37) to be the same as (6.15).

**Theorem 6.3.** The full nonlinear problem with initial data satisfying (6.15), (6.17) and (6.35)–(6.38) has a unique classical solution for $0 \leq t \leq T$ for some small enough $T > 0$; $T$ depends only on the constants appearing in the conditions (6.15), (6.17).

The solution is such that $g$ and $\zeta$ belong to $C^{1+\alpha}_{\theta}(G_T)$.

**Proof.** We shall prove existence by a fixed point argument for a mapping $W$ defined in the class

$$K = \left\{(g, \zeta) ; \left\|g\right\|_{C^{2+\alpha}_{\theta}(G_T)} \leq \tilde{C}(M, c_0), \quad \left\|\zeta\right\|_{C^{1+\alpha, (1+\alpha)/2}_{\theta}(G_T)} \leq \tilde{C}(M, c_0, A^\ast), \right.$$

$$\left. \left\|g\right\|_{C^{2+\alpha, 1+\alpha/2}_{\theta}(G_T)} \leq \tilde{C}(M, c_0)2\pi \left[1 + A^\ast + (1 + M)(1 + \tilde{C}(M, c_0))\right], \right.$$

$$g|_{t=0} = g_0(\theta), \quad \zeta|_{t=0} = \zeta_0(\theta), \quad g(\theta, t) \geq c_0, \quad \zeta(\theta, t) \geq c_0\right\}$$

where the constants $\tilde{C}, \tilde{C}$ are taken from (6.32), (6.33) and (6.19).

Given $(g, \zeta) \in K$, we substitute it into the boundary condition for $A$ and then solve for $A, B, C$. By the maximum principle,

$$0 \leq A(r, \theta, t) \leq A^\ast, \quad 0 \leq B(r, \theta, t) \leq B^\ast$$

and hence

$$0 \leq C(r, \theta, t) \leq C(M, c_0)$$

where the constant is independent of $T$.

By the Schauder estimates [7]

$$\|A\|_{C^{2+\alpha, 1+\alpha/2}(Q_T)} + \|B\|_{C^{2+\alpha, 1+\alpha/2}(Q_T)} + \|C\|_{C^{2+\alpha, 1+\alpha/2}(Q_T)} \leq C(M, c_0)$$

with another constant $C(M, c_0)$ independent of $T$.

Consider the functions

$$F(\theta, t) = A^\ast - A(g(\theta, t), \theta, t), \quad E(\theta, t) = C^4(g(\theta, t), \theta, t).$$

Then $0 \leq F \leq A^\ast$ and

$$[F]_{C^2(\Gamma_T)} \leq 2\pi \|F_\theta\|_{C(\Omega_T)}$$

$$\leq 2\pi \left[\|A_r\|_{C(\Gamma_T)}\|g_\theta\|_{C(\Omega_T)} + \|A_\theta\|_{C(\Omega_T)}\right]$$

and, by (6.39), (6.37) and the definition of $K$, we easily get

$$\|F\|_{C^{2+\alpha}_{\theta}(G_T)} \leq 2\pi(M + 1)(1 + \tilde{C}(M, c_0))$$

(6.40)

It is also clear that

$$\|F\|_{C^{2+\alpha}_{\theta}(G_T)} + \|E\|_{C^{1+\alpha}_{\theta}(G_T)} \leq C_1(M, c_0) \equiv M_1.$$

(6.41)

We now solve (6.4), (6.5) with $(F, E)$ as above and devote the solution by $(\tilde{g}, \tilde{\zeta})$, and then define a mapping $W$ by

$$W(g, \zeta) = (\tilde{g}, \tilde{\zeta}).$$

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By Lemma 6.1 and the estimates (6.32), (6.33), (6.40), $W$ maps $K$ into itself.

Introduce a topology on $K$ by the norm
\[
\|(g, \zeta)\| = \|g\|_{C(G_T)} + \|g_\theta\|_{C(G_T)} + \|g_\theta\|_{C(G_T)} + \|\zeta\|_{C(G_T)} + \|\zeta_\theta\|_{C(G_T)}.
\]

Then $K$ is a closed convex set and, by (6.20), $W : K \to K$ is compact. Since the mapping $W$ is uniquely defined, it is then also continuous, and so, by Schauder’s fixed point theorem, $W$ has a fixed point. This gives a solution to the full nonlinear problem.

To prove uniqueness suppose $(\hat{A}, \hat{B}, \hat{C}, \hat{g}, \hat{\zeta})$ is another solution. Then
\[
(6.42) \quad \|\hat{\ell}_\theta\|_{C^1_\theta(G_T)} \leq C, \quad \|\hat{m}_\theta\|_{C^1_\theta(G_T)} \leq C
\]

Using the differential equations for $g, \ell, m$ and (6.42), (6.28), we easily find that
\[
(6.43) \quad \|\zeta - \hat{\zeta}\|_{C^1(G_T)} + \|g_\theta - \hat{g}_\theta\|_{C^1_\theta(G_T)} + \|g_\ell - \hat{g}_\ell\|_{C(G_T)}
\]
\[
\leq C \left[\|\hat{F} - \hat{F}\|_{C^{1+\alpha}(G_T)} + \|E - \hat{E}\|_{C^1_\theta(G_T)}\right] = J.
\]

Hence
\[
\|\zeta - \hat{\zeta}\|_{C^{1/2}(G_T)} + \|g_\theta - \hat{g}_\theta\|_{C^{1/2}(G_T)} \leq CT^{1/2}J.
\]

Using also Lemma 6.2 we get
\[
(6.44) \quad \|\zeta - \hat{\zeta}\|_{C^{0, \alpha/2}(G_T)} + \|g - \hat{g}\|_{C^{1+\alpha, (1+\alpha)/2}(G_T)} \leq CT^{1/2}J.
\]

We now proceed as in [3, Section 8]. Let
\[
V(T) = \|g - \hat{g}\|_{L^\infty(G_T)}
\]

and consider $A, \hat{A}$ in the common domain
\[
\tilde{\Omega}_T = \{r > \hat{g}(r, \theta) + V(T)\}.
\]

As in [3]
\[
\left[\begin{array}{c}
DA \frac{\partial(A - \hat{A})}{\partial n} + P(\zeta)(A - \hat{A}) \\
\end{array}\right]_{r = \hat{g}(\theta, t) + V(T)} \leq C \left\{\|g - \hat{g}\|_{C^{1+\alpha, (1+\alpha)/2}(G_T)} + \|\zeta - \hat{\zeta}\|_{C^{0, \alpha/2}(G_T)}\right\}.
\]

Similar estimates hold for $B$ and $C$. By parabolic $C^{1+\alpha, (1+\alpha)/2}$ estimates [8] we then have:
\[
\|A - \hat{A}\|_{C^{1+\alpha}(\hat{\Omega}_T)} + \|B - \hat{B}\|_{C^{1+\alpha}(\hat{\Omega}_T)} + \|C - \hat{C}\|_{C^{1+\alpha}(\hat{\Omega}_T)}
\]
\[
\leq C \left[\|g - \hat{g}\|_{C^{1+\alpha, (1+\alpha)/2}(G_T)} + \|\zeta - \hat{\zeta}\|_{C^{0, \alpha/2}(G_T)}\right] = L(T).
\]

Using this and the $C^{2+\alpha}$ regularity of $A$ and $\hat{A}$, we find that
\[
\|F - \hat{F}\|_{C^{1+\alpha}(\tilde{\Omega}_T)} \leq CL(T)
\]
A similar estimate holds for $E - \hat{E}$. Substituting these two estimates into (6.44) (recall the definition of $J$ in (6.43)) we see that

$$L(T) \leq C T^{\alpha/2} L(T)$$

and, therefore, $L(T) = 0$ if $T$ is small enough. Similarly $L(t) \equiv 0$ if $0 \leq t \leq T$ and then also $A = \hat{A}, B = \hat{B}, C = \hat{C}$ if $0 \leq t \leq T$.

The uniqueness proof can be extended to any time interval for which the solutions exist. □

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