

**ENERGY MINIMIZING CONFIGURATIONS FOR MIXTURES
OF TWO IMPERFECTLY BONDED CONDUCTORS**

By

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ENERGY MINIMIZING CONFIGURATIONS FOR MIXTURES OF TWO IMPERFECTLY BONDED CONDUCTORS

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Abstract

We consider a domain filled with a polydisperse suspension of electrically conducting spheres of conductivity σ_p embedded in a matrix of lesser conductivity σ_m . It is assumed that there exists an electric contact resistance at the sphere - matrix interface. The contact resistance is characterized by a scalar β , which has dimensions of conductivity per unit length. A current flux is prescribed on the domain boundary and we seek the energy minimizing configuration among all suspensions satisfying a resource constraint on the total volume of spheres. We establish the existence of an energy minimizing configuration within the class of polydisperse suspensions of spheres. The optimal suspension is shown to contain only spheres of radii greater than or equal to $R_{cr} = \beta^{-1}(\sigma_m^{-1} - \sigma_p^{-1})^{-1}$. Here R_{cr} is the ratio between the interfacial resistance and the mismatch between the resistivity of each phase.

1 Introduction

We consider suspensions of electrically conducting spheres embedded in a matrix of lesser electrical conductivity. Here we allow the suspensions to contain spheres of different radii. This class of suspensions is referred to as the class of *polydisperse* suspensions of spheres. The suspension is contained inside a convex domain $\Omega \subset R^3$ with Lipschitz continuous boundary $\partial\Omega$. The conductivities of the spheres and matrix are assumed isotropic and are specified by σ_p and σ_m respectively, with $\sigma_p > \sigma_m$. We consider the technologically important case when there is an interfacial contact resistance between the two phases. The contact resistance is characterized by a scalar β with dimensions of conductivity per unit length.

The region occupied by the i^{th} sphere in the suspension is denoted by B_i , and the configuration of spheres given by their union $\cup B_i$ is denoted by A . The two phase interface

is denoted by $\Gamma = \cup \partial B_i$. We assume that the spheres are strictly contained inside Ω , ie., $A \subset \Omega$ and $\Gamma \cap \partial\Omega = \emptyset$. The local resistivity tensor inside the composite is described by $\sigma^{-1}(x) = \sigma_p^{-1}\chi_A + \sigma_m^{-1}(1 - \chi_A)$, where χ_A equals one in A and zero otherwise. For a prescribed heat flux $g \in H^{-1/2}(\partial\Omega)$, such that $\int_{\partial\Omega} g ds = 0$, the thermal energy dissipated inside the composite is given by $E(A, g)$, where

$$E(A, g) = \min\{C(A, j) : j \in L^2(\Omega)^3, \operatorname{div} j = 0, j \cdot n = g \text{ on } \partial\Omega\} \quad (1.1)$$

and

$$C(A, j) = \int_{\Omega} \sigma^{-1}(x) j \cdot j dx + \beta^{-1} \int_{\Gamma} (j \cdot n)^2 ds. \quad (1.2)$$

Here $\operatorname{div} j = 0$ holds in the sense of distributions, ds is the element of surface area, and the vector n is the unit normal pointing into the matrix phase. The first term of the functional $C(A, j)$ is associated with bulk energy dissipation, while the second term gives the energy dissipation at the two-phase interface. The minimizer j_A is precisely the heat flux in the composite.

We consider the problem of minimizing the thermal energy dissipation among polydisperse suspensions, subject to a resource constraint on the total volume occupied by the spheres. We introduce the class C_{θ_p} of all polydisperse suspensions containing a finite number of spheres, satisfying the resource constraint $\operatorname{meas}(A) \leq \theta_p \operatorname{meas}(\Omega)$. Here θ_p is an upper bound on the volume fraction occupied by the suspension. Note that there are no constraints on the size or number of spheres for configurations in C_{θ_p} . We suppose that the spheres do not touch each other. To make this requirement precise we consider a suspension in C_{θ_p} consisting of N spheres and denote the center and radius of the i^{th} sphere by x_i and r_i respectively. We surround the i^{th} sphere by an open ball S_i with center x_i and slightly larger radius $(1 + \lambda)r_i$, where λ is a fixed positive constant. We require that the open balls do not overlap, ie.,

$$S_i \cap S_j = \emptyset \quad i \neq j, \quad (1.3)$$

and

$$S_i \cap \partial\Omega = \emptyset \quad i = 1, 2, \dots, N. \quad (1.4)$$

The class of suspensions in C_{θ_p} satisfying (1.3) and (1.4) is denoted by $C_{\theta_p, \lambda}$. This class is sufficiently large to allow for the possible appearance of fine structure in minimizing sequences of configurations. As an example, we introduce the composite sphere assemblage of Hashin [6]. We refer to the i^{th} spherical particle and its surrounding ball as a composite sphere. It is evident that the ratio between the radii of the sphere and its surrounding ball is independent of the size of the spherical particle and is given by $(1 + \lambda)^{-1}$. The volume fraction occupied by the spherical particle in the composite sphere is given by $(1 + \lambda)^{-3}$. We consider a volume filling configuration of composite spheres. Such a configuration has a

gradation of particle size down to the infinitesimal. It is clear that the total volume occupied by the particle suspension is $(1 + \lambda)^{-3} \text{meas}(\Omega)$. This type of configuration is referred to as the composite sphere assemblage. For parameter values of λ consistent with the resource constraint $(1 + \lambda)^{-3} \leq \theta_p$, one easily constructs sequences of configurations in $C_{\theta_p, \lambda}$ that converge to the composite sphere assemblage.

For a prescribed current flux $g \in H^{-1/2}$, such that $\int_{\partial\Omega} g ds = 0$ we consider the problem,

$$\min\{E(A, g) : A \in C_{\theta_p, \lambda}\} \quad (1.5)$$

In this paper it is shown that an energy minimizing configuration exists in the class $C_{\theta_p, \lambda}$. Moreover the optimal suspension depends upon the size of the domain Ω and consists of spheres with radii greater than or equal to $R_{cr} = \beta^{-1}(\sigma_m^{-1} - \sigma_p^{-1})^{-1}$, or no spheres at all: see Theorem 1.3. Here R_{cr} has the dimensions of length and is the ratio between the interfacial resistance and the mismatch between the resistivity of each phase. Thus we see that the presence of a contact resistance provides a small scale cutoff for the minimum sphere radius appearing in the optimal suspension.

The functional without the interfacial energy term in (1.2) has been widely studied. Indeed, in the absence of surface energy, it is well known from the fundamental work of Murat and Tartar [11] and Lurie and Cherkaev [10] that problems of the type (1.5) are most often illposed and exhibit minimizing sequences composed of arbitrarily fine mixtures of the two conductors.

Recently Ambrosio and Buttazzo [1] have considered functionals with bulk energies similar to the first term in (1.2) augmented by a penalization proportional to the perimeter of the two phase interface. They allow for arbitrary configurations of the two phases, placing a resource constraint on the better conductor. The perimeter penalization used in Ambrosio and Buttazzo's work rules out the appearance of arbitrarily fine mixtures in minimizing sequences by assigning an infinite value to them. Their penalization gives the necessary compactness and forces the optimal configuration to lie within the class of sets of finite perimeter that are (up to subsets of measure zero) open.

The approach taken here does not use an explicit perimeter penalization, but instead the penalization opposing the formation of fine scale mixtures follows from the electrical contact resistance at the two phase interface. The explicit mechanism by which fine scale minimizing sequences are eliminated is seen in the following inequality established in Lipton [9].

Theorem 1.1 Energy dissipation inequality.

Let B denote a sphere of radius a such that $A \cup B$ is a suspension in $C_{\theta_p, \lambda}$. then,

$$E(A \cup B, g) \geq E(A, g). \quad (1.6)$$

if

$$a \leq R_{cr} \tag{1.7}$$

for all $g \in H^{-1/2}(\partial\Omega)$ such that $\int_{\partial\Omega} g ds = 0$.

We remark that similar inequalities hold for suspensions of particles with Lipschitz boundaries, see Lipton [9].

It is evident from Theorem 1.1 that R_{cr} gives the critical sphere radius for which the benefits of a highly conducting inclusion are spoiled by the contact resistance at the particle surface. It follows that if a suspension contains spheres of radii less than R_{cr} then there is no advantage to keeping them in the suspension. We introduce the subclass $SC_{\theta_p, \lambda}$ of suspensions in $C_{\theta_p, \lambda}$ consisting only of spheres with radii greater than or equal to R_{cr} .

In view of Theorem 1.1, we see that the size of the domain Ω effects the optimal configuration. Indeed, the domain may have physical dimensions for which the class of configurations $SC_{\theta_p, \lambda}$ is empty. Thus, the class $C_{\theta_p, \lambda}$ consists of spheres of radius less than R_{cr} . For this case, it is evident that the optimal configuration is made from pure matrix material with no particles at all. We summarize these observations in the following,

Theorem 1.2 Necessary conditions of optimality.

If the class $SC_{\theta_p, \lambda}$ is not empty and if the configuration \check{A} is a minimizer for problem (1.5), then $\check{A} \in SC_{\theta_p, \lambda}$.

We now assert the existence of an energy minimizing configuration in the class $C_{\theta_p, \lambda}$.

Theorem 1.3 Existence of an energy minimizing suspension.

If the class $SC_{\theta_p, \lambda}$ is not empty, then there exists a minimizing configuration \check{A} for problem (1.5) and $\check{A} \in SC_{\theta_p, \lambda}$. If $SC_{\theta_p, \lambda}$ is empty then the minimizing configuration is $\check{A} = \emptyset$.

It is interesting to compare these results with the minimization problem treated in Ambrosio and Buttazzo [1], when the phases are perfectly bonded but with a penalization proportional to the surface area of the two phase interface. For a configuration A consisting of a polydisperse suspension of N spheres of radius a_1, a_2, \dots, a_N , we introduce the volume average of the reciprocal radii given by:

$$\langle a^{-1} \rangle = \sum_{i=1}^N a_i^{-1} \frac{|Y_i|}{|A|}, \tag{1.8}$$

where, $|Y_i|$ is the volume of the i^{th} sphere and $|A|$ is the volume of the suspension. We easily obtain the following necessary condition of optimality for an energy minimizing configuration in the class $C_{\theta_p, \lambda}$ given by:

Theorem 1.4 Necessary condition of optimality for perfectly bonded composites with surface area penalization.

If the energy minimizing configuration lies in the class $C_{\theta_p, \lambda}$ then the size distribution of sphere radii must satisfy:

$$\langle a^{-1} \rangle^{-1} > 0. \quad (1.9)$$

Unlike Theorems 1.2 and 1.3, it is evident from (1.9) that the perimeter penalization provides no length scale associated with the minimum sphere radius appearing in an optimal suspension.

Sections 2,3 and 4 of this paper are devoted to proving Theorem 1.3 in the event that $SC_{\theta_p, \lambda}$ is not empty. Clearly, Theorem 1.2 implies that all energy minimizing configurations must lie in the class $SC_{\theta_p, \lambda}$. Thus our analysis focuses on proving the existence of minimizers within the class $SC_{\theta_p, \lambda}$. Existence is proved through the direct method of the calculus of variations. It is shown in Section 2 that the set of characteristic functions associated with all configurations in $SC_{\theta_p, \lambda}$ is closed and compact with respect to strong L^1 convergence. In Section 3, the functionals $E(A, g)$ are shown to be continuous with respect to L^1 convergence and existence follows. Section 4 provides a derivation of the energy estimates used in section 3. In this Section, a higher regularity result for the trace of the potential on either side of the two phase boundary is obtained. This is used to establish a Poincare like inequality from which uniform bounds on the energy dissipation follow.

We remark that there exists a uniform constraint on the Lipschitz constant of ∂A for all configurations $A \in SC_{\theta_p, \lambda}$. Much work has focused on the problem of optimal shape design over such classes of sets: see Pironneau [12]. The approach taken in Sections 2 and 3 makes use of the fundamental theorems derived in the work of D. Chenais as given in [2] and [3]. We conclude the paper, proving Theorem 1.4 in Section 5.

2 Compactness of the design space

In light of Theorem 1.2 we need only to consider minimizing sequences of configurations in the class $SC_{\theta_p, \lambda}$. Thus we consider a sequence $\{A^\nu\}_{\nu=1}^\infty$ of configurations in $SC_{\theta_p, \lambda}$ and state the following:

Theorem 2.1 Given $\{A^\nu\}_{\nu=1}^\infty$ such that $A^\nu \in SC_{\theta_p, \lambda}$, $\nu = 1, 2, \dots$, there exists a subsequence also denoted by $\{A^\nu\}_{\nu=1}^\infty$ and a configuration $\check{A} \in SC_{\theta_p, \lambda}$ for which $meas(A^\nu \Delta \check{A}) \rightarrow 0$. Equivalently we have $\chi_{A^\nu} \rightarrow \chi_{\check{A}}$, strongly in $L^1(\Omega)$.

Proof:

We first note that there exists an upper bound on surface area for all configurations belonging to $SC_{\theta_p, \lambda}$. Indeed, a crude upper bound is given by the surface area associated with the configuration obtained by packing Ω with spheres of radius R_{cr} . This upper bound guarantees the existence of a subsequence of configurations also denoted by $\{A^\nu\}_{\nu=1}^\infty$ with characteristic functions χ_{A^ν} converging strongly in L^1 to the characteristic function $\chi_{\check{A}}$

where \check{A} is a set of finite perimeter. To finish the proof we show that \check{A} lies in $SC_{\theta_p, \lambda}$. We observe that all configurations in $SC_{\theta_p, \lambda}$ correspond to a class of domains that uniformly satisfy a constraint on the Lipschitz constant associated with the boundary of each domain. Thus we are in a position to apply a Theorem of D. Chenaïs [3]. Indeed, from [3] it follows that there exists a positive number ϵ such that given two elements A^ν and A^μ in $SC_{\theta_p, \lambda}$, if $meas(A^\nu \Delta A^\mu) \leq \epsilon$, then A^ν and A^μ are homeomorphic. Since the sequence $\{A^\nu\}_{\nu=1}^\infty$ is Cauchy, it follows that all configurations are eventually homeomorphic. Noting that the topology of a set is preserved under homeomorphism, it follows that eventually all configurations consist of the same number of spheres. We denote the common number of spheres by " p " and observe that the configurations A^ν are specified by a vector \underline{v}^ν of length $4p$ consisting of the radius and center of each sphere in the suspension. Since the sequence $\{A^\nu\}_{\nu=1}^\infty$ is Cauchy it follows that the sequence of configuration vectors \underline{v}^ν is Cauchy and therefore must converge to a limit configuration vector $\check{\underline{v}}$. It is clear that the vector $\check{\underline{v}}$ describes a configuration in $SC_{\theta_p, \lambda}$. Lastly, since, $meas(A^\nu \Delta \check{A}) \rightarrow 0$ and $\underline{v}^\nu \rightarrow \check{\underline{v}}$ it follows that $\check{\underline{v}}$ gives the configuration of \check{A} , and the Theorem is proved.

3 Continuity of the energy dissipation functional

In this section we establish the continuity of the functional $E(A, g)$. We state the following theorem:

Theorem 3.1 Continuity of the energy dissipation.

Given a sequence $\{A^\nu\}_{\nu=1}^\infty$ in $SC_{\theta_p, \lambda}$ and a set \check{A} in $SC_{\theta_p, \lambda}$, such that $meas(A^\nu \Delta \check{A}) \rightarrow 0$, then

$$\lim_{\nu \rightarrow \infty} E(A^\nu, g) = E(\check{A}, g) \tag{3.1}$$

Proof:

With out loss of generality, may suppose that the limit configuration \check{A} consists of " p " spheres. We observe from the proof of Theorem 2.1 that eventually all elements in the sequence $\{A^\nu\}_{\nu=1}^\infty$ consist of p spheres. Thus we assume that we are far enough out in the sequence, so that each configuration A^ν consists of p spheres. For the configuration A^ν we introduce the characteristic function χ_i^ν associated with the i^{th} sphere. The characteristic functions of the spheres are related to the characteristic function of the configuration A^ν by $\chi_{A^\nu} = \sum_{i=1}^p \chi_i^\nu$. Moreover, the convergence of $\{A^\nu\}_{\nu=1}^\infty$ to \check{A} imply $\chi_i^\nu \rightarrow \check{\chi}_i$ strongly in L^1 where $\check{\chi}_i$ is the characteristic function of the i^{th} sphere in the configuration \check{A} . The proof is facilitated by introducing the equations of state solved by the electric potential u_{A^ν} for the configuration A^ν . The current is related to the gradient of the potential by the constitutive law: $j_{A^\nu} = \sigma^\nu(x) \nabla u_{A^\nu}$ and

$$div(\sigma^\nu(x) \nabla u_{A^\nu}) = 0 \text{ in } \Omega \setminus \Gamma^\nu. \tag{3.2}$$

Across the interface one has

$$[j_{A^\nu} \cdot n] = 0 \text{ on } \Gamma^\nu, \quad (3.3)$$

and

$$j_{A^\nu} \cdot n|_p = -\beta[u_{A^\nu}] \text{ on } \Gamma^\nu, \quad \sigma_m \nabla u_{A^\nu} \cdot n = g \text{ on } \partial\Omega. \quad (3.4)$$

Here Γ^ν is the two phase interface $\sigma^\nu(x) = \sigma_p \chi_{A^\nu} + \sigma_m(1 - \chi_{A^\nu})$, and $[u_{A^\nu}] = u_{A^\nu}|_p - u_{A^\nu}|_m$. where the subscripts indicate the side of the interface where the trace is taken. The requirement, $\int_{\partial\Omega} g ds = 0$ is the solvability condition for the equation of state, and the potential $u_{A^\nu} \in H^1(\Omega \setminus \Gamma^\nu)$ is determined uniquely up to a constant. We provide a useful weak formulation of the boundary value problem (3.2) – (3.4). Introducing the space of vector fields $\underline{w} = (w_0, w_1, \dots, w_p)$ belonging to $H^1(\Omega)^{p+1}$, the weak formulation is given by:

$$\begin{aligned} & \sum_{i=1}^p \int_{\Omega} (\chi_i^\nu \sigma_p \nabla u_{A^\nu} \cdot \nabla w_i) dx + \int_{\Omega} (1 - \chi_{A^\nu}) \sigma_m \nabla u_{A^\nu} \cdot \nabla w_0 dx + \\ & + \beta \sum_{i=1}^p \int_{\partial B_i^\nu} [u_{A^\nu}] (w_i - w_0) ds - \int_{\partial\Omega} w_0 g ds = 0, \end{aligned} \quad (3.5)$$

for all $\underline{w} \in H^1(\Omega)^{p+1}$. Here ∂B_i^ν denotes the boundary of the i^{th} sphere in the configuration A^ν . One can establish the existence of a sequence of constants $\{c^\nu\}_{\nu=1}^\infty$ such that the normalized sequence of potentials $\{u_{A^\nu} - c^\nu\}_{\nu=1}^\infty$ is uniformly bounded, ie.,

$$\sup_{\nu} \|u_{A^\nu} - c^\nu\|_{H^1(\Omega \setminus \Gamma^\nu)} < \infty \quad (3.6)$$

This estimate is derived in Section 4: see Theorem 4.2. The normalized potential is a solution of the equation of state (3.2)–(3.4) and for the remainder of this Section we continue to denote it by u_{A^ν} . Next we observe that there is a uniform bound on the Lipschitz constant associated with the boundary of any sphere B_i^ν , that holds independently of the indices i and ν . Thus we may apply the Theorem of D. Chenais [2] to assert the existence of a positive number K and $p+1$ linear and continuous extension operators, $M_0^\nu, M_1^\nu, \dots, M_p^\nu$ such that for all $A^\nu = \cup_{i=1}^p B_i^\nu \in \text{SC}_{\theta_p, \lambda}$,

$$M_0^\nu : H^1(\Omega \setminus (A^\nu \cup \Gamma^\nu)) \rightarrow H^1(R^3), \quad (3.7)$$

$$M_i^\nu : H^1(B_i^\nu) \rightarrow H^1(R^3), \quad (3.8)$$

for $i = 1, 2, \dots, p$. where

$$\|M_0^\nu\| \leq K \text{ and } \|M_i^\nu\| \leq K \quad (3.9)$$

for $i = 1, 2, \dots, p$.

It is evident from (3.6) and (3.9) that

$$\sup_{\nu} \|M_0^{\nu} u_{A^{\nu}}\|_{H^1(\Omega)} < \infty \quad (3.10)$$

and

$$\sup_{\nu} \|M_i^{\nu} u_{A^{\nu}}\|_{H^1(\Omega)} < \infty, \quad (3.11)$$

for $i = 1, 2, \dots, p$.

From (3.10) and (3.11) we may pass to a subsequence if necessary to find that there exist functions $u_0^{\infty}, u_1^{\infty}, \dots, u_p^{\infty}$ all in $H^1(\Omega)$ such that:

$$M_0^{\nu} u_{A^{\nu}} \rightarrow u_0^{\infty}, \text{ weakly in } H^1(\Omega) \quad (3.12)$$

and

$$M_i^{\nu} u_{A^{\nu}} \rightarrow u_i^{\infty}, \text{ weakly in } H^1(\Omega), \quad (3.13)$$

for $i = 1, 2, \dots, p$.

It follows from the weak formulation (3.5) that the choice $w_0 = M_0^{\nu} u_{A^{\nu}}, w_i = M_i^{\nu} u_{A^{\nu}}, i = 1, 2, \dots, p$, gives:

$$E(A^{\nu}, g) = \int_{\partial\Omega} M_0^{\nu} u_{A^{\nu}} g ds = \int_{\partial\Omega} u_{A^{\nu}} g ds. \quad (3.14)$$

From the weak convergence (3.12), it follows that:

$$\lim_{\nu \rightarrow \infty} E(A^{\nu}, g) = \int_{\partial\Omega} u_0^{\infty} g ds. \quad (3.15)$$

Next we let $\chi_{\check{A}}$ denote the characteristic function of \check{A} and $\chi_{\check{A}} = \sum_{i=1}^{i=p} \check{\chi}_i$, where $\check{\chi}_i$ is the characteristic function of the i^{th} sphere in the limit configuration \check{A} . It is evident from (3.15) that the theorem follows once we show that, $u^{\infty} \equiv u_0^{\infty}(1 - \chi_{\check{A}}) + \sum_{i=1}^{i=p} \check{\chi}_i u_i^{\infty}$ is the solution of:

$$\begin{aligned} & \sum_{i=1}^p \int_{\Omega} (\check{\chi}_i \sigma_p \nabla u_i^{\infty} \cdot \nabla w_i) dx + \int_{\Omega} (1 - \chi_{\check{A}}) \sigma_m \nabla u_0^{\infty} \cdot \nabla w_0 dx + \\ & + \beta \sum_{i=1}^p \int_{\partial\check{B}_i} (u_i^{\infty} - u_0^{\infty})(w_i - w_0) ds - \int_{\partial\Omega} w_0 g ds = 0, \end{aligned} \quad (3.16)$$

for all w in $H^1(\Omega)^{p+1}$, where $\partial\check{B}_i$ is the boundary of the i^{th} sphere in the limit configuration. To show this we pass to the limit in the weak formulation (3.5) to find that it agrees with (3.16). We observe first that the weak convergence $\nabla M_i^{\nu} u_{A^{\nu}} \rightarrow \nabla u_i^{\infty}, i = 1, 2, \dots, p$ and $\nabla M_0^{\nu} u_{A^{\nu}} \rightarrow \nabla u_0^{\infty}$, together with the almost everywhere convergence of χ_i^{ν} to $\check{\chi}_i$, implies that the first two terms of (3.5) converge to the first two terms of (3.16). To expedite the presentation we denote the differences $u_i^{\infty} - u_0^{\infty}$ and $w_i - w_0$, defined everywhere on Ω , by $[u^{\infty}]_i$ and δ_i respectively. We consider the difference between the third terms of (3.5) and (3.16), given by:

$$\begin{aligned}
& \beta \left\{ \sum_{i=1}^p \int_{\partial B_i^\nu} [u_{A^\nu}] \delta_i ds - \sum_{i=1}^p \int_{\partial \tilde{B}_i} [u^\infty]_i \delta_i ds \right\} \\
= & \beta \left\{ \sum_{i=1}^p \left\{ \int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i) \delta_i ds + \left(\int_{\partial B_i^\nu} [u^\infty]_i \delta_i ds - \int_{\partial \tilde{B}_i} [u^\infty]_i \delta_i ds \right) \right\} \right\}. \quad (3.17)
\end{aligned}$$

We show for each term in the sum (3.17) that the difference

$$\int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i) \delta_i ds \rightarrow 0, \text{ for all } \delta_i \in H^1(\Omega). \quad (3.18)$$

To do this we observe,

$$\int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i) \delta_i ds = \int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i) \delta_i n^\nu \cdot n^\nu ds, \quad (3.19)$$

where n^ν is the unit normal pointing out of ∂B_i^ν . Extending the normal inside B_i^ν , we apply the divergence theorem to find:

$$\begin{aligned}
& \int_{\partial B_i^\nu} ([u_{A^\nu}] - [u^\infty]_i) \delta_i ds = \int_{\partial B_i^\nu} ((M_i^\nu u_{A^\nu} - M_0^\nu u_{A^\nu}) - [u^\infty]_i) \delta_i n^\nu \cdot n^\nu ds \\
= & \int_{\Omega} \chi_i^\nu \{ (\nabla M_i^\nu - \nabla u_i^\infty) + (\nabla u_0^\infty - \nabla M_0^\nu u_{A^\nu}) \} \cdot n^\nu \delta_i dx \\
+ & \int_{\Omega} \chi_i^\nu \{ (M_i^\nu - u_i^\infty) + (u_0^\infty - M_0^\nu u_{A^\nu}) \} (div n^\nu \delta_i + n^\nu \cdot \nabla \delta_i) dx \quad (3.20)
\end{aligned}$$

For the i^{th} sphere, the extension of the unit normal vector is simply $(x - x_i^\nu)(a_i^\nu)^{-1}$ and $div n^\nu = 3(a_i^\nu)^{-1}$, where a_i^ν and x_i^ν are the radii and center of the i^{th} sphere in the configuration A^ν . Clearly the products $\chi_i^\nu n^\nu \equiv \chi_i^\nu (x - x_i^\nu)(a_i^\nu)^{-1}$ and $\chi_i^\nu div n^\nu$ converge almost everywhere to $\chi_{\tilde{A}} \check{n}$ and $\chi_{\tilde{A}} div \check{n}$, where \check{n} is the extended normal associated with the limit configuration. Thus from the weak convergence (3.12) and (3.13) of the extended fields, it follows from (3.20) that the difference (3.18) vanishes in the limit. Last, we observe that for each term in the sum (3.17) the difference:

$$\int_{\partial B_i^\nu} [u^\infty]_i \delta_i ds - \int_{\partial \tilde{B}_i} [u^\infty]_i \delta_i ds \quad (3.21)$$

vanishes in the limit; this follows from the continuity of the trace, see Lions and Magenes [8]. Thus passing to the limit in (3.5) we recover (3.16) and the Theorem follows.

4 The Poincare inequality and the energy estimate.

In this Section we provide the energy estimates for the electric potential satisfying the equations of state given by (3.2) – (3.4). In view of the discussion in Sections 2 and 3 we

will only consider potentials associated with configurations in the class $SC_{\theta_p, \lambda}$ consisting of at most p spheres. For these configurations there exists a boundary layer L_λ of thickness $\frac{\lambda}{2}R_{cr}$ in which no sphere of conductivity σ_p is present. We let $\tilde{\Omega}$ denote the subset of Ω obtained by removing the boundary layer L_λ from Ω , ie., $\tilde{\Omega} = \Omega \setminus L_\lambda$.

We start with an elementary observation on the regularity of the potential on the two phase interface. The trace of the electric potential on the two phase interface is denoted by $u_{A^\nu}|_p$ and $u_{A^\nu}|_m$ where the subscripts m and p denote the side of the interface where the trace is taken. We give the following :

Lemma 4.1 The traces $u_{A^\nu}|_p$ and $u_{A^\nu}|_m$ lie in $C^0(\Gamma^\nu)$ for all $\nu = 1, 2, \dots$

Proof:

We observe from (3.2) – (3.4) that in each phase the solution u_{A^ν} satisfies the following set of Neumann problems:

$$\int_{B_i^\nu} \sigma_p \nabla u_{A^\nu} \cdot \nabla \varphi dx = -\beta_{H^{-1/2}(\partial B_i^\nu)} \langle [u_{A^\nu}], \varphi \rangle_{H^{1/2}(\partial B_i^\nu)}, \quad (4.1)$$

for all $\varphi \in H^1(B_i^\nu)$, $i = 1, 2, \dots, p$, and

$$\int_{\Omega \setminus A^\nu} \sigma_m \nabla u_{A^\nu} \cdot \nabla \varphi dx = -\beta_{H^{-1/2}(\Gamma^\nu)} \langle [u_{A^\nu}], \varphi \rangle_{H^{1/2}(\Gamma^\nu)} + \beta_{H^{-1/2}(\Omega)} \langle g, \varphi \rangle_{H^{1/2}(\Omega)}, \quad (4.2)$$

for all $\varphi \in H^1(\Omega \setminus A^\nu)$.

Observing that $u_{A^\nu} \in H^1(\Omega \setminus \Gamma^\nu)$ we see that the jump $[u_{A^\nu}]$ lies in $H^{1/2}(\Gamma^\nu)$ and we appeal to the regularity theory for the Neumann problem, (see Grisvard [5]) to conclude that $u_{A^\nu} \in H^2(\tilde{\Omega} \setminus \Gamma^\nu)$. The Lemma now follows from the Sobolev imbedding theorem.

Since Γ^ν is of class C^∞ , we can iterate the procedure used to prove Lemma 4.1 to find that $u_{A^\nu}|_p$ and $u_{A^\nu}|_m$ lie in $C^\infty(\Gamma^\nu)$.

Letting d be the diameter of the domain Ω and setting ω_3 equal to the volume of the unit sphere in three dimensions we state the following Poincare like inequality:

Theorem 4.1 For any sequence of configurations in the class $SC_{\theta_p, \lambda}$ consisting of p spheres, there exists a constant $M = 2\left(\frac{d^3}{3|L_\lambda|}\right)^2(3\omega_3^{2/3})^2|\Omega|^{2/3}$, such that for all potentials u_{A^ν} satisfying the equations of state:

$$\|u_{A^\nu} - c^\nu\|_{L^2(\Omega)}^2 \leq pM \left(\int_{\Omega} |\nabla u_{A^\nu}|^2 dx + \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds \right), \quad (4.3)$$

where $c^\nu = \frac{1}{|L_\lambda|} \int_{L_\lambda} u_{A^\nu} dx$.

The proof of Theorem 4.2 proceeds in two steps, first we introduce the Riesz and double layer potentials given by:

$$V_{1/3}(|\nabla u_{A^\nu}|)(x) = \int_{\Omega} |y - x|^{-2} |\nabla u_{A^\nu}(y)| dy \quad (4.4)$$

and

$$P^\nu(|[u_{A^\nu}]|)(x) = \int_{\Gamma^\nu} |[u_{A^\nu}(y)]| |\partial_{n_y} E(x, y)| ds_y, \quad (4.5)$$

where n is the outward pointing normal and $E(x, y)$ is the Newtonian potential $|y - x|^{-1}$.

We state the following:

Lemma 4.2

$$|u_{A^\nu}(x) - c^\nu| \leq \frac{d^3}{3|L_\lambda|} \{V_{1/3}(|\nabla u_{A^\nu}|)(x) + P^\nu(|[u_{A^\nu}]|)(x)\}. \quad (4.6)$$

Proof:

For x and y in Ω we write:

$$u_{A^\nu}(x) - u_{A^\nu}(y) = - \int_0^{|x-y|} D_r u_{A^\nu}(x + r\omega) dr - \sum_{j=1}^{\ell} [u_{A^\nu}(x + r_j\omega)], \quad (4.7)$$

where $0 \leq \ell \leq p$, $\omega = (y - x)/|y - x|$. Here $x + r_j\omega$ lies on the intersection of the j^{th} interface and the line segment connecting the points x and y , and $[]$ indicates the jump of u_{A^ν} across the surface of a spherical particle. Next we integrate (4.7) with respect to the y variable over the boundary layer L_λ to obtain:

$$|L_\lambda|(u_{A^\nu}(x) - c^\nu) = - \int_{L_\lambda} dy \int_0^{|x-y|} D_r u_{A^\nu}(x + r\omega) dr - \int_{L_\lambda} dy \left(\sum_{j=1}^{\ell} [u_{A^\nu}(x + r_j\omega)] \right). \quad (4.8)$$

We write

$$|L_\lambda|(u_{A^\nu}(x) - c^\nu)| = \left| \int_{L_\lambda} dy \int_0^{|x-y|} D_r u_{A^\nu}(x + r\omega) dr \right| + \left| \int_{L_\lambda} dy \left(\sum_{j=1}^{\ell} [u_{A^\nu}(x + r_j\omega)] \right) \right|. \quad (4.9)$$

Proceeding as in Gilbarg and Trudinger [4], the first term in (4.9) is estimated above by:

$$\left| \int_{L_\lambda} dy \int_0^{|x-y|} D_r u_{A^\nu}(x + r\omega) dr \right| \leq \frac{d^3}{3} \{V_{1/3}(|\nabla u_{A^\nu}|)(x)\}. \quad (4.10)$$

From the convexity of the domain Ω we may integrate the second term in (4.9) using the polar coordinates $dy = \rho^2 d\rho d\omega$, to obtain

$$\left| \int_{L_\lambda} dy \left(\sum_{j=1}^{\ell} [u_{A^\nu}(x + r_j\omega)] \right) \right| \leq \frac{d^3}{3} \sum_{i=1}^p \int_{\Omega_i} |[u_{A^\nu}(x + r_j(\omega)\omega)]| d\omega, \quad (4.11)$$

where Ω_i is the solid angle subtended by the i^{th} sphere and $x + r_j(\omega)\omega$ ranges over all points on its surface. We apply a standard change of variables, (cf., Jackson [7]), to obtain:

$$\sum_{i=1}^p \int_{\Omega_i} |[u_{A^\nu}(x + r_j(\omega)\omega)]| d\omega = \sum_{i=1}^p \int_{B_i^\nu} |[u_{A^\nu}]| |\partial_{n_y} E(x, y)| ds_y, \quad (4.12)$$

and the Lemma follows.

Application of Lemma 4.2 and Cauchy's inequality gives:

$$\|u_{A^\nu} - c^\nu\|_{L^2(\Omega)} \leq \frac{\sqrt{2}}{3} \frac{d^3}{|L_\lambda|} \left(\int_{\Omega} |V_{1/3}(|\nabla u_{A^\nu}|)|^2 dx + \int_{\Omega} |P^\nu(|[u_{A^\nu}]|)|^2 dx \right)^{1/2}. \quad (4.13)$$

From Gilbarg and Trudinger [4] we have:

$$\int_{\Omega} |V_{1/3}(|\nabla u_{A^\nu}|)|^2 dx \leq (3\omega_3^{2/3})^2 |\Omega|^{2/3} \|\nabla u_{A^\nu}\|_{L^2(\Omega)}^2. \quad (4.14)$$

Next we estimate the surface layer term on the righthand side of (4.13). We write,

$$P^\nu(|[u_{A^\nu}]|) = \sum_{i=1}^p P_i^\nu(|[u_{A^\nu}]|), \quad (4.15)$$

where $P_i^\nu(|[u_{A^\nu}]|) = \int_{\partial B_i^\nu} |[u_{A^\nu}(y)]| |\partial_{n_y} E(x, y)| ds_y$. Application of Cauchy's inequality gives:

$$\int_{\Omega} |P^\nu(|[u_{A^\nu}]|)|^2 dx \leq p \sum_{i=1}^p \int_{\Omega} |P_i^\nu(|[u_{A^\nu}]|)|^2 dx. \quad (4.16)$$

The Poincare inequality now follows from (4.13), (4.14), (4.16) and the following estimate:

Lemma 4.3

$$\int_{\Omega} |P_i^\nu(|[u_{A^\nu}]|)|^2 dx \leq (3\omega_3^{2/3})^2 |\Omega|^{2/3} \int_{\partial B_i^\nu} ([u_{A^\nu}(y)])^2 ds_y. \quad (4.17)$$

Proof:

Let $K(x, y) = |\partial_{n_y} E(x, y)|$, write $|[u_{A^\nu}]|K = ([u_{A^\nu}]^2 K)^{1/2} K^{1/2}$, and apply Cauchy's inequality to obtain:

$$|P_i^\nu(|[u_{A^\nu}]|)|^2 \leq \left(\int_{\partial B_i^\nu} ([u_{A^\nu}(y)])^2 K ds_y \right) \left(\int_{\partial B_i^\nu} K ds_y \right). \quad (4.18)$$

Elementary estimates, (cf., Jackson [7]), show that:

$$\left\| \int_{\partial B_i^\nu} K(x, y) ds_y \right\|_{L^\infty(\Omega)} \leq 4\pi, \quad (4.19)$$

thus,

$$\int_{\Omega} |P_i^\nu(|[u_{A^\nu}]|)|^2 dx \leq 4\pi \int_{\Omega} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}(y)])^2 K ds_y \right) dx. \quad (4.20)$$

Let $W \subset \Omega$, be a thin shell containing the boundary ∂B_i^ν then,

$$\int_{\Omega} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx = \int_{\Omega \setminus W} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx + \int_W \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx. \quad (4.21)$$

For $x \in \Omega \setminus W$ we have $K(x, y) \leq |x - y|^{-2}$ and application of Fubini's theorem gives:

$$\begin{aligned} \int_{\Omega \setminus W} \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx &\leq \int_{\partial B_i^\nu} \left(\int_{\Omega \setminus W} |x - y|^{-2} dx \right) ([u_{A^\nu}])^2 ds_y \\ &\leq \sup_{y \in \Omega} \{V_{1/3}(1)(x)\} \int_{\partial B_i^\nu} ([u_{A^\nu}])^2 ds_y \\ &\leq (3\omega_3^{2/3})^2 |\Omega|^{2/3} \int_{\partial B_i^\nu} ([u_{A^\nu}])^2 ds_y. \end{aligned} \quad (4.22)$$

To estimate the second term on the right hand side of (4.21) we recall from Lemma 4.1, that $[u_{A^\nu}]$ lies in $C^0(\Gamma^\nu)$ thus $\|[u_{A^\nu}]\|_{L^\infty(\partial B_i^\nu)} \leq \infty$ and

$$\int_W \left(\int_{\partial B_i^\nu} ([u_{A^\nu}])^2 K ds_y \right) dx \leq 4\pi |W| \|[u_{A^\nu}]\|_{L^\infty(\partial B_i^\nu)}^2. \quad (4.23)$$

Lemma 4.3 follows upon choosing W such that its volume tends to zero.

We conclude this Section by providing a uniform estimate on the normalized potential fields given by:

Theorem 4.2 Given a sequence of suspensions consisting of p spheres in the class $SC_{\theta_p, \lambda}$, the associated potential u_{A^ν} satisfies:

$$\sup_{\nu} \|u_{A^\nu} - c^\nu\|_{H^1(\Omega \setminus \Gamma^\nu)} < \infty, \quad (4.24)$$

where

$$c^\nu = |L_\lambda|^{-1} \int_{L_\lambda} u_{A^\nu}(y) dy. \quad (4.25)$$

Proof:

We observe that $u_{A^\nu} - c^\nu$ is a solution to the equations of state (3.2)–(3.4) and:

$$\begin{aligned} E(A^\nu, g) &= \int_{\partial \Omega} g u_{A^\nu} ds = \int_{\partial \Omega} g (u_{A^\nu} - c^\nu) ds \\ &\leq \|g\|_{H^{-1/2}(\partial \Omega)} \|u_{A^\nu} - c^\nu\|_{H^{1/2}(\partial \Omega)} \\ &\leq \|g\|_{H^{-1/2}(\partial \Omega)} \|u_{A^\nu} - c^\nu\|_{H^1(\Omega \setminus \Gamma^\nu)}. \end{aligned} \quad (4.26)$$

From the Poincare inequality (4.3), it follows that

$$\|u_{A^\nu} - c^\nu\|_{H^1(\Omega \setminus \Gamma^\nu)}^2 \leq (1 + pM) \left(\int_{\Omega} |\nabla u_{A^\nu}|^2 dx + \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds \right). \quad (4.27)$$

Next, set $\alpha = \min\{\sigma_p, \sigma_m, \beta\}$ and make the substitution $\underline{w} = (u_{A^\nu}, u_{A^\nu}, \dots, u_{A^\nu})$ in (3.5) to obtain:

$$\begin{aligned} E(A^\nu, g) &= \int_{\Omega} \sigma^\nu(x) |\nabla u_{A^\nu}|^2 dx + \beta \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds \\ &\geq \alpha \left(\int_{\Omega} |\nabla u_{A^\nu}|^2 dx + \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds \right). \end{aligned} \quad (4.28)$$

It is evident from (4.26), (4.27), and (4.28) that

$$\sup_{\nu} \int_{\Omega} |\nabla u_{A^\nu}|^2 dx \leq \infty, \quad \sup_{\nu} \int_{\Gamma^\nu} ([u_{A^\nu}])^2 ds \leq \infty, \quad (4.29)$$

and the theorem follows in view of (4.27) and (4.29).

5 Energy minimizing suspensions of perfectly bonded conductors with surface area penalization.

Here we consider an energy minimization problem when the two conductors are perfectly bonded. For this case the thermal energy dissipated inside the composite is given by $\tilde{E}(A, g)$ where,

$$\tilde{E}(A, g) = \min\{\tilde{C}(A, j) : j \in L^2(\Omega)^3, \operatorname{div} j = 0, j \cdot n = g \text{ on } \partial\Omega\} \quad (5.1)$$

and

$$\tilde{C}(A, j) = \int_{\Omega} \sigma^{-1}(x) j \cdot j dx. \quad (5.2)$$

Here $\operatorname{div} j = 0$ holds in the sense of distributions. The minimizer j_A is precisely the heat flux in the composite and on the boundary the prescribed heat flux g is an element of $H^{-1/2}(\partial\Omega)$, and satisfies the solvability condition $\int_{\partial\Omega} g ds = 0$.

We denote the surface area of the two phase interface by $P(A)$ and consider the minimization problem

$$\min\{\tilde{E}(A, g) + P(A) : A \in C_{\theta_p, \lambda}\}. \quad (5.3)$$

The cost functional in (5.3) is similar to the one treated in the work of Ambrosio and Buttazzo [1].

We now prove the necessary condition of optimality for this problem as given in Theorem 1.4. It is evident from the cost functional that the minimizing configuration in $C_{\theta_p, \lambda}$ must have finite interfacial surface area, ie., $P(A) < \infty$. Theorem 1.4 follows immediately upon noting that for a suspension of N spheres of radii a_1, a_2, \dots, a_N that:

$$P(A) = 3|A| < a^{-1} >. \quad (5.4)$$

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