A Mathematical Model for Ore Stack Drying

Group 1

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1 Introduction

The procedure for extracting iron ore from a mine follows the next steps: First the extraction which is basically digging and blasting, then the processing which includes crushing and washing the rocks and finally the storage of the rocks in stockpiles. The stockpiles are subsequently transported by rail to the place of final destination. We concentrate our attention on the storage within the stockpiles. See figures (4) and (1).

During the mining of iron ore it is common to use water to facilitate the extraction and crushing. But for transportation it is preferable that the ore have as low a water content as possible. However, the water content must not be too low or too much dust will be created. For iron ore the storage period may last from a few days up to two weeks.

![Figure 1](Image)

Figure 1: *Pictures of how the rock are moved by the stacker to form the stockpiles*

A typical stockpile is 20 m high, 1000 m long and 40 m wide and contains particles from 10 cm to 1 mm in size. Iron ore leaving the washery contains about 40% by volume of water. The desirable percentage of water in the stockpiles before the transportation is 16% by volume.

The aim is to develop a model of moisture movement in stockpiles to ore management to accelerate the drying process.

We consider three mechanisms that would contribute to model the moisture loss:

- Evaporation
- Inhomogeneous stockpile formation
1 INTRODUCTION

Figure 2: Top view of how the rock are moved by the stacker to form the stockpiles

- Drainage

and try to answer the questions:

- How quickly does a stockpile dry?
- What is the water distribution in a stockpile?
- How the stockpiles should be made so they dry more quickly?

We will model the different aspects of the problem to decide how much some of them will contribute to the problem.

In section 2 a simple model was designed to determinate that the effect of the rain is not important to our problem. In section 3 a model using fluid dynamics was implemented to calculate the amount of water that evaporates for the worse case scenario. The next section studies the model of formations of the stockpiles. It is important to include and model the distribution of the stockpiles since it has been observed that the lower edges of the stockpile are significantly drier than the inner and the top part. This section gives information of the porosity and diameter of void space within the pile. This data are needed for studying how the water runs inside the stockpile. Section 4 models the distribution of the water in the stockpiles. This model is
based upon Poiseuille's Law. Section 6 uses Darcy's Law to analyze a more complicated model of the flow of the water within the stockpiles. Finally a steady state solution is given in the last section.

2 Rain Analysis

It is reasonable to believe that rainfall will affect the overall volume of water in the iron ore pile. Recall that after iron ore leaves the washery it is stacked into piles with the following characteristics:

- height = 20m
- length = 1000m
- width = 40m
- water = 40% (by volume)

Based on these parameters, the volume of the iron ore is approximately 400,000m³. As the water volume is 40% (160,000m³) of the pile and our goal for water volume is 16% (64,000m³) of the pile, we wish to remove approximately 100,000m³ of water during the storage period. In order to determine if rain is a primary contributor to the amount of water in the pile, we shall analyze the effect of exceptionally heavy rains during the two week storage period.

Assuming it rains one inch (.025m) we will accrue 1000m³ of water assuming no water runs off the sides of the pile. In order for the rainfall to become significant (10% = 10,000m³), it would need 10 inches of rain over the two week period. As it would require extreme (and rare) conditions for mining areas to receive this amount of rain, we infer that rainfall is not a primary contributor to the overall water volume. Thus, we will not include rain as a factor in our model.

3 Evaporation Study

In this section we consider the effect of evaporation on loss of water in our ore pile. We wish to show that evaporation of water from the pile has a negligible effect on total water loss.

Evaporation can occur in two ways, by diffusion and by advection. Diffusion occurs in the absence of wind as water vapor is evaporated from a
surface and diffuses into the surrounding air. This process is very slow. The rate of evaporation increases dramatically when one considers advection. In this process, water vapor evaporates from a surface and is carried away by moving air, leaving dry air into which water evaporates much more quickly.

We wish to present a worst-case analysis of evaporation, thus we assume that evaporation occurs via advection as wind-forced air passes into one face of the ore pile, is humidified, and passes out the other face. To maximize this effect we assume that the wind is blowing perpendicular to the ridgeline as shown in Figure 3 below.

We model our ore pile as being permeated by many thin long pipes through which air may pass. We assume that all of the air that passes through the pile has become saturated with water (100% relative humidity).

For the purposes of modeling, we assume some worst-case conditions on temperature, wind speed and relative humidity shown in Table 1.

In order to determine the amount of water removed from the pile via evaporation, we need to calculate the amount of water in 1 cubic meter of
air at 100% relative humidity. The ideal gas model tells us that

\[ n = \frac{PV}{RT} \]  

(1)

The meaning of the variables \( P, V, T, R \) and \( n \) are given in Table 2.

We found that at 40\(^\circ\)C, the vapor pressure of water is \( P = 7.3814 \times 10^3 \) J/m\(^3\) thus the number of moles of water in 1 cubic meter of water vapor is obtained from (1) to be \( n = 2.836 \) mol. Using the atomic weight 18 of water and it’s density as a liquid 1000Kg/m\(^3\) we find that the volume of water (as a liquid) which will evaporate to give one cubic meter of water vapor at the above temperature is

\[ 51 \times 10^{-6} \text{m}^3 \text{ H}_2\text{O liquid}/\text{m}^3 \text{ vapor} \]

Actually, the amount of water in one cubic meter of saturated air will be smaller than this because of the presence of the air molecules, however this number gives us a good worst-case estimate.

We will now use fluid dynamics to study the flow of air through a single capillary. This will be modeled by the basic equations of conservation of mass and momentum.

\[ \frac{d\rho}{dt} + \nabla \cdot (\rho V) = 0 \]  

(2)

\[ \rho \left( \frac{\partial V}{\partial t} + V \cdot \nabla V \right) = -\nabla p + \nabla (\mu \nabla \cdot V) - \rho g \gamma \]  

(3)

where \( p \) is the air pressure, \( \mu \) is the viscosity of the air (considered constant), \( \rho \) is the density of the air (also constant in the incompresible case), and \( V \) is the vector velocity of the wind.

For the steady state solution, we have that \( \frac{\partial V}{\partial t} = 0 \). Therefore the equations (2) and (3) can be written as:
\[ \nabla \cdot (\rho V) = 0 \]  
(4)

\[ \rho \left( V \cdot \nabla V \right) = -\nabla p + (\mu \nabla^2 V) \]  
(5)

Now, we proceed to make dimensionless equations by the following change of variables:

\[
\begin{align*}
x &= L \tilde{x} \\
y &= A \tilde{y} \\
p &= p_l + (p_0 - p_l) \tilde{p}
\end{align*}
\]

where \( A \) is the typical radius of the pipe, \( p_l \) is the pressure at the entrance of the pipe and \( p_0 \) is the pressure at the end and \( L \) is the length of the pipe.

\[ \tilde{u}_x + \tilde{v}_y = 0 \]  
(6)

\[ \frac{\rho (p_0 - p_l) A^4}{L^2 \mu^2} \left( \tilde{u} \tilde{u}_x + \tilde{v} \tilde{u}_y \right) = -\tilde{p}_x + \tilde{u}_{yy} + \left( \frac{A}{L} \right)^2 \tilde{u}_{xx} \]  
(7)

\[ \frac{\rho (p_0 - p_l) A^6}{L^4 \mu^2} \left( \tilde{u} \tilde{u}_x + \tilde{v} \tilde{u}_y \right) = -\tilde{p}_y + \left( \frac{A}{L} \right)^2 \left[ \tilde{v}_{yy} + \left( \frac{A}{L} \right)^2 \tilde{v}_{xx} \right] \]  
(8)

We now consider \( A \ll L \) and viscosity \( \mu \) as a large value compared to inertia, thus we have the following:

\[ \tilde{u}_x + \tilde{v}_y = 0 \]  
(9)

\[ 0 = -\tilde{p}_x + \tilde{u}_{yy} \]  
(10)

\[ 0 = -\tilde{p}_y \]  
(11)

Now let \( r = f(z) \) the radius of our pipe at position \( z \). We impose the “no-slip” boundary conditions \( \tilde{u}(f(z)) = \tilde{v}(f(z)) = 0 \) for all \( z \). Changing the above equations to polar coordinates and eliminating the bars we have:

\[ u_x + v_r = 0 \]  
(12)

\[ -p_z + \frac{1}{r} \frac{d}{dr} \left( ru_r \right) = 0 \]  
(13)
\[ -p_r = 0 \]  
(14)

The following equation can then be found for pressure and velocity:

\[ p(z) = p_1 + \left(p_0 - p_1\right) \frac{\int_0^z f^{-3}(s)ds}{\int_0^L f^{-3}(s)ds} \]  
(15)

\[ u(z) = \frac{1}{4\mu} p'(z) \left(r^2 - f^2(z)\right) \]  
(16)

Using Bernoulli's equation where \( u_0 \) is the initial velocity, \( p_0 \) is the initial pressure,

\[ u_0^2 = \frac{2(p_0 - p)}{\rho} + u^2 \]  
(17)

we can consider \( u \) small and solve for the volume flux using the assumption that we have a constant radius \( R \). Thus \( f(z) = R \)

\[ V_f = \int_0^{2\pi} \int_0^{f(z)} ru(r,z)drd\theta \]  
(18)

\[ V_f = \frac{\pi \rho u^2 R^4}{16\mu L} \]  
(19)

Since this is the volume flux for one pipe passing through the stockpile, we now need to estimate the number of pipes. Since we have 40% water, we take the largest cross sectional area to find a worst case scenario for the number of pipes, \( N \).

\[ N = \frac{8 \cdot 10^3}{\pi R^2} \]  
(20)

The total volume of air \( (V_a) \) passing through the stockpile is now computed

\[ V_a = \frac{8 \cdot 10^3 \rho u^2 R^2}{16L\mu} \]  
(21)

Using the previous calculations, we have \( 51 \times 10^{-6}m^3 \) of water in \( 1m^3 \) of completely saturated air. We will now examine the worst case scenario with the following conditions:

- Wind speed \( u = 30mph = 13.41 \frac{m}{s} \)
• Temperature $T = 40^\circ$ C

• Constant viscosity of the air $\mu = 1.9 \cdot 10^{-5}$ $\frac{kg}{m \cdot s}$

• Average length of the pipe $L = 20m$

• Constant density of the air $\rho = \frac{1}{m^3}$

• Average radius of each pipe $R = 1mm = 10^{-3}m$

• Time interval $t = 2$ weeks $= 1209600s$

With all of these considerations the volume of water removed from the stockpile by evaporation is $14,620m^3$ which represents 9% of the initial amount of total water in the stock piles. This appears to be a sizable amount of water removed. However, we must point out the following unrealistic assumptions used in our calculations. First of all, the actual channels will have an irregular twisting shape which will greatly reduce the velocity of the wind. Thus, the wind will not be able to pass straight through all of the capillaries. Furthermore, the air is not likely to become fully saturated with water vapor after "passing" through the stockpile. In addition, wind speeds of $30mph$ lasting for two weeks is highly unlikely, and due to the wind flow, pressure inside the channels will be smaller. When all of the realistic observations are taken into account, it appears that evaporation will decrease the water amount by much less than 9%. Thus, we assert that evaporation is not the dominate contributor to water removal and will only play a factor in extreme conditions.

4 Initial Investigation of Water Flow

In this section we will begin to analyze the flow of water through the stockpile. For this initial investigation, we use the simple model of straight pipes with uniform radius passing vertically through the stock pile. This model is based upon Poiseuille's Law which describes the steady flow of a fluid in a straight pipe.

4.1 Model Description and Assumptions

We are assuming we have a steady, incompressible fluid with no slip conditions on the boundaries. Furthermore, the atmospheric pressure will remain
constant throughout the capillaries, thus the pressure gradient will be zero in this model. (See figure)

Using Poiseuille's Law (reference Currie) for steady flow in straight pipes, we have the following:

$$\rho \left( u \cdot \nabla \right) u = -\nabla p + \mu \nabla^2 u - g \rho \quad (22)$$

where $\rho$ is the water density, $u(y)$ is the velocity, $\mu$ is the dynamic viscosity of water, $g$ is the acceleration due to gravity and $p(y)$ is the pressure. Equation (22) has the following components:

- $\rho \left( u \cdot \nabla \right)$ represents the inertia
- $-\nabla p$ is the pressure gradient
\( \mu \nabla^2 \) accounts for the viscosity in the fluid

\( g \rho \) is the gravitational term

Since we have an incompressible fluid, we have \( \nabla \cdot u = 0 \). We will now compare the remaining components inside equation (22) to the gravitational term, which must be included in this model. When comparing the inertial and gravitational components, we find

\[
    u \sim \sqrt{Lg} \sim 14 \frac{m}{\sec}
\]

This implies that the velocity will be unrealistically large if inertia plays a dominant role in the equation (22). However, when we compare the viscous component with the gravitational term, we have for a given diameter, \( d \), of 1 mm:

\[
    u \sim \frac{gd^2}{\mu} \sim 10^{-4} \frac{m}{\sec}
\]

which is a fairly reasonable approximation for the velocity of the fluid. Thus we know for a small velocity, the inertial component is negligible (i.e. \( \rho (u \cdot \nabla) = 0 \)).

We can now rewrite equation (22) as

\[
    0 = \mu \nabla^2 u - g \rho \\
    0 = \mu (u_{zz} + u_{yy}) - g \rho \\
    u_{zz} + u_{yy} = \frac{g \rho}{\mu}
\]  

After converting the above equation into polar coordinates (assuming an axisymmetric pipe), we will have

\[
    \frac{1}{R} \frac{d}{dR} (R \frac{du}{dR}) = \frac{g \rho}{\mu}
\]  

\[
    u(R) = \frac{g \rho}{4 \mu} \left( a^2 - R^2 \right)
\]

Where \( a \) is the constant radius of the capillary and \( r \) is the radial coordinate where velocity is being analyzed.
Averaging the velocity function over a cross sectional area, we have

$$\bar{u} = \frac{g\rho}{\pi 4\mu a^2} \int_0^{2\pi} \int_0^a r \left( a^2 - r^2 \right) dr d\theta$$

(28)

$$\bar{u} = \frac{g\rho a^2}{8\mu}$$

(29)

We know the stockpile is usually ready to ship in 10 days. Thus we assume that the worst case scenario for the average velocity of the water is approximately:

$$\bar{u} = 1.6 \cdot 10^{-5} \frac{m}{sec}$$

(30)

By equating (29) and (30) we can find the approximate maximum radius for each capillary as

$$a = 10^{-6} m = 1 \mu m$$

(31)

This appears to be unrealistically small for the radius of the capillaries. Thus we can conclude that pressure inside the capillaries needs to be accounted for in our model. At this point we attempt to construct a more realistic model for fluid flow by analyzing the pressure acting on the water inside the capillaries.
5 Stockpile Formation

The motivation behind our study of stockpile formation developed from the group's hypothesis that composition of the pile influenced the flow of water within the pile. Our hypothesis, based on observations that the lower edges of the stockpile were significantly drier than the core, assumes the distribution of rock size and void space are not homogeneous throughout the cross-section of the stockpile. To determine whether our hypothesis is true a model which simulates pile formation was developed. From our study we hope to make conclusions about the distribution of porosity and diameter of void space within the pile.

5.1 Background

Our approach was based on the work of several authors. The principle theory in pile formation was developed by Bak, Tang and Weisenfeld[1], henceforth referred to as (BTW). Studies done by (BTW) on avalanches, simulated using sandpiles, were performed to understand the self-critical organizational nature of piles. (BTW) asserted that when the slope of a pile exceeded a critical angle an avalanche would result. To simulate such a situation a method known as cellular automata was employed. A cellular automata model involves dropping "cells" over random positions of a pile. A set of rules are established to define when these cells are unstable, normally expressed in the form of slope criteria, then rules for reorganization are imposed to establish a stable pile configuration. Specifically the rules to our model were based on the work of L. P. Kadanoff, et al[5].

An example of a one-dimensional cellular automaton model, similiar to one provided by H. M. Jaeger and S.R. Nagel[7], is given in (fig.1). In this model the cells are of equal size. Here the local rules which govern a "simple avalanche" are based on the height of a column relative to neighboring columns. Height is determined by the number of cells stacked in a column. If the height difference is greater than one, then a shift of the cell will occur. Initially the pile is stable (fig. 1.a). The shaded block is dropped in a random position. Since the critical height has been exceeded, the cell moves down and to the right until it has settled to a stable position (fig. 1.b-d). This model takes into account only local instability behavior, since it does not allow for global shifting in the pile. Hence once a cell becomes stable in the pile, it remains stable, a gross simplification.

In the following sections we introduce a cellular automata model developed to simulate "simple avalanche" dynamics for the iron-ore stockpile. We refer to this as a "simple avalanche" model since dynamics are controlled locally and discretely, as in the above example, rather than in a global framework, with the intent of defining the global structure of the stockpile. We look to our results to determine whether the model employed displays basic features of self-organized criticality.
5.2 Assumptions of Model

- The size of rocks were specified in the range of 10 cm to 1 mm. However, in our model we consider rocks, which are approximated as square blocks of two sizes, the sides of which are of ratio 2:1.

- We will then approximate the rocks as circles contained within the square block in order to represent void space between rocks in specific regions of the stockpile.

- For simplicity we consider any side elevation cross-section of the stockpile to be similar in composition. For this assumption to be valid we require the stacker to sweep uniformly back and forth depositing iron-ore, rather than dumping rock in one position for an extended period of time before moving to a new position. This method will produce a ridge of constant height as viewed from the front elevation.

- Since we view the composition of the pile as a cross-sectional area, we define percentage of rock in terms of area. In case(A) we assume that by cross-sectional area 50% of the stockpile will be composed of large blocks (2 × 2) and 50% by small blocks (1 × 1). In case(B) we let 80% of the cross-sectional area be composed of large blocks and 20% of small blocks.

- In the model one block drops at a time. The stockpile settles in a stable configuration before a new block is dropped.

5.3 The model

A matrix is defined to represent the cross-sectional area of the stockyard. The dimensions of the actual stockyard are preserved by the choice of rows and columns of the matrix. A coordinate \((m, n)\) in the matrix represents the column and row of the cellular model, respectively. A \(1 \times 1\) block is represented as a 1 in the matrix with position \((m, n)\). A \(2 \times 2\) block is represented as a quadrant of 2's with the lower left corner of the block's position denoted by \((m, n)\). The height of a stack of blocks are represented by the function \(H(m)\), where \(m\) specifies the column in which the blocks reside.
In simplified form the model has four key steps. Once the matrix and heights are initialized the size of a block to be dropped is determined. A random number generator is employed to insure the correct distribution of large to small blocks are dropped for case(A) and case(B). For instance in case(A), there is a 50% distribution of large blocks to small blocks in terms of area. This requires 20% of the rocks dropped to be large, 80% small. Next the drop column of the block is determined. To simulate the stacker, an array of 5 positions is established from which a block may be dropped. To assure randomness in the model we again use a random number generator to determine from which of the five positions the block starts. Once dropped, the stability of the block is checked. If the block is unstable, it will be shifted to a new position, at which point stability is again checked. After the block reaches a stable position, the process repeats for a new block.

5.3.1 Stability Requirements

The stability requirements are specific to each block size.

1 × 1 case: In the case of the small block height instability occurs only when a maximum slope is exceeded, similar to the model of L.P. Kadanoff, et al[5]. When a small block lands in its initial position the height function is incremented by

\[ H(m) = H(m) + 1 \]

If the slope to the left

\[ \sigma_L = H(m) - H(m - 1) > 1 \]

or the slope to the right

\[ \sigma_R = H(m) - H(m + 1) > 1 \]

then the block is shifted in the appropriate direction, one unit over and one unit down. (see fig 1) The new position is reevaluated until the block is stable, at which point a 1 is placed into the matrix.

2 × 2 case: In the case of a large block, instability can occur in one of two ways. The first instability evaluated is called a ledge instability. In this case a large block lands on a small block such that one half of the block is unsupported (see fig. 2) and is thus on a ledge. The block will reposition itself if there is room in the respective direction that it is unsupported to move. Since the large block has sides of length two, the block will move over two blocks and down one, if these positions in the matrix are void. If not, a large block will remain on the ledge, introducing void space into the pile. (see fig. 3)

Once a block is ledge stable the model checks for height stability. When a large block lands in a column the height function is incremented by

\[ H(m) = H(m) + 2 \]
If the slope to the left or right, similar to the $1 \times 1$ case, exceeds 3 units and the appropriate space exists, then the block will shift two units over in the direction of height instability and one unit down. The model continues the process of checking for instability, in the order specified, until the block is stable. Once stable a quadrant of 2's is placed in the matrix and a new block is dropped.

### 5.4 Results

To model the stockpile our code drops two different sizes of blocks within a given region. These blocks then settle onto the pile following the rules described above. We ran 15 trials of the pile formation under four sets of conditions. These 15 trials were averaged together, point by point, to get a temporal average of the cross section of the pile. The temporal averaging enabled us to understand
general trends in the evolution of the pile structure. Tendencies that recurred through all runs of the model emerged more strongly.

![Figure 5: Pile block distributions in terms of area.](image)

We examined two different ratios by area of large blocks to small blocks. In case(A) the ratio of the area of large to small blocks is 1:1, hence there will be four times as many small blocks as large blocks. (fig. 5.A) In case(B) we considered a 4:1 ratio by area of large blocks to small blocks, hence the number of each is equal. (fig. 5.B)

In fig.6 are temporal averages of two piles each with 5,000 blocks, representing case(A) and case(B). NOTE: The heavy lines along the slopes of the pile are artifacts of the temporal averaging, created by the irregular shape of the slopes on a fine level. These heavy lines do not affect the global behavior which has been analyzed. In case(B) where there are greater numbers of large blocks, we see sorting within the pile. The large white regions in figure (6) at the two lower edges of the pile correspond to areas where the temporal average is greater than 1.5. This corresponds to an area composed mainly of large blocks. As specified, there are more stability criteria that must be met for a large block to come to rest on the pile. When we have a substantial fraction of the pile comprised of these large blocks, they fall to the edges of the pile where they mix less with the smaller blocks.

![Figure 6: A pile of 5,000 blocks with ratio expressed in terms of area of large to small blocks.](image)

When the large blocks make up a smaller fraction of the pile by numbers, as in case(a), the distribution of block sizes is homogeneous.
The next figure (7) shows temporal averages of piles of 10,000 blocks. While the overall pile size increases, the structural patterns of large and small block sorting are qualitatively the same, as above.

![10000 particles at 1:1 by area](image)

![10000 particles at 4:1 by area](image)

Figure 7: A pile of 10,000 blocks with ratio expressed in terms of area of large to small blocks.

Hence these results support our hypothesis that sorting by self-organized criticality occurs within the pile only if the number of large blocks exceeds the number of small blocks in the pile.
6 Saturated & Unsaturated Flow Model

In this section we analyze a more sophisticated model for fluid flow within the ore pile. This model is based on the law of conservation of mass as well as a crucial relationship called Darcy's Law which relates the velocity of fluid through a porous media to the pressure of the liquid.

6.1 Model Description and Assumptions

Our pile of iron ore will consist of many different blocks with varying shapes and diameters piled up in such a way that there is a complicated system of water pathways formed from the spaces around the ore particles. Rather than try to model this network of flows around the rocks, we treat the entire pile as a continuous porous media through which water can flow. The ability of the porous material to conduct the flow of water is called the hydraulic conductivity and denoted by \( K \).

In this model we do not assume that the hydraulic conductivity is constant throughout the pile (so it is a heterogeneous media), but we do assume that at a specific point in the media, the resistance to water flow does not depend on the direction of the flow (so the media is isotropic).

One critical assumption that we make concerns the way in which the ore pile is built. Since the pile is being formed as the ore stacker moves along the ridge and deposits ore atop the pile, it is reasonable to assume that every cross section of the hill (perpendicular to the ridge-line) is essentially the same, thus fluid flow, water pressure, hydraulic conductivity, etc... do not depend on the position along the ridge. This assumption allows us to reduce our problem to a two dimensional model and consider fluid flow in a cross section of the ore pile. This is illustrated in Figure 6.1.

Our model assumes both the iron ore and the water are incompressible substances, thus the density \( \rho \) of water is constant. We also assume that no settling of ore particles occurs as fluid flow takes place (thus the hydraulic conductivity is constant with respect to time).

6.2 Conservation of Mass

Consider a small square of porous media as shown in Figure 6.1. Since the media is porous, it will not be completely filled by iron ore. The area in the square which is not occupied by ore is called the void space. The voids may be completely filled with water (in which case we say they are saturated)
or may be partially filled with water and partially filled with air (and are called unsaturated). The porosity \( n \) is defined as the ratio of the volume of the void space to the total volume. It is a percentage of total volume. The moisture content \( \theta \) is defined to be the ratio of the volume of the void space filled with water to the total volume of the square. We see that for saturated flow we have \( \theta = n \) but for unsaturated flow we have \( \theta < n \). It will be useful to define the degree of saturation \( S \) to be \( S = \theta / n \). Since we assume our ore media to be incompressible, \( n \) will not change with time.

We denote by \( v = (v_x, v_y) \) the velocity of the water at any point in the porous media. Note that \( \rho v \) represents the mass flow of water (per unit length) in the direction of the fluid flow. Further, the product \( \rho \theta \) is the total mass per unit area. The law of conservation of mass requires that the net rate of fluid mass flow into square of porous media be equal to the time rate of change of fluid mass within the square. Thus we have the equation:

\[
- \frac{\partial}{\partial x} \rho v_x - \frac{\partial}{\partial y} \rho v_y = \rho \frac{\partial \theta}{\partial t}
\]

(32)
Figure 6: Diagram of fluid flow in small square of porous media
6.3 Darcy's Law

We next wish to determine a relationship between velocity and pressure. To begin with, we make a distinction between real and gauge pressures. The real pressure of a liquid is the pressure including the atmospheric pressure \( p_a \). The gauge pressure refers to the pressure of the liquid above atmospheric pressure. While we cannot have a negative real pressure, it is common to encounter negative gauge pressures. In this paper when we speak of pressure \( p \) we are referring to gauge pressure, not real pressure.

There are three forces acting upon the water in the ore pile. First, pressure differences can cause water to move from an area where pressure is higher to an area of lower pressure. Second, gravity tends to cause the water to move downward. Third, the viscous friction of the water in the porous media exerts a force opposite to the direction of the fluid flow. These three effects are incorporated in an empirical law called *Darcy's law*. Darcy's law comes from much experimental evidence and has shown to be a very accurate tool for modeling fluid flow through a porous media. Darcy's law in our case is given by:

\[
\begin{align*}
    v_x &= -K \frac{\partial p}{\rho g \partial x} \\
    v_y &= -K \left( \frac{1}{\rho g} \frac{\partial p}{\partial y} + 1 \right)
\end{align*}
\]  

(33)

Where \( K \) is our hydraulic conductivity, and \( g \) is the gravitational acceleration constant. These equations relate the velocity of the water in the media to the pressure gradient.

In our model we will use Darcy's law to develop our boundary value problem. By substituting Darcy's law into the equation for conservation of mass, multiplying through by \( g \), moving some terms around and applying the chain rule we get:

\[
\frac{\partial}{\partial x} \left( K \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial p}{\partial y} \right) = \rho g \left( \frac{\partial \theta}{\partial p} \frac{\partial p}{\partial t} - \frac{\partial K}{\partial y} \right)
\]

(34)

6.4 Functional Relationship Modeling

In the equation above, the hydraulic conductivity \( K \) and the moisture content \( \theta \) will depend heavily on the water pressure. Before any solution to the above PDE can be obtained we need to model the functional relationships between \( \theta \) and \( p \) as well as \( K \) and \( p \).
6.4.1 Moisture Content Dependency on Pressure

When the gauge pressure of the water in our porous media is positive, no air can enter the media and the moisture content will simply be the porosity \((\theta = \eta)\). In this saturated condition, \(\theta\) will be constant with respect to \(p\). When the size of the ore particles is relatively small, there will actually be a small negative water pressure \(p_s\) for which our media still remains saturated. This is called the tension saturated pressure, and is illustrated in Figure 6.4.1. For pressures below \(p_s\) we need some experimental results to determine the relationship between \(\theta\) and \(p\).

Recall that \(S = \theta/\eta\) is the degree of saturation of our media. Because of hydrostatic attraction between the water and the media, even if we allow our media to completely drain, a small amount of water will remain in the ore stuck to the surfaces within. This final saturation value is called the irreducible wetting fluid saturation and is denoted \(S_0\). From this quantity we can define another type of saturation \(S_e\) called the effective saturation which is given by:

\[
S_e = \frac{S - S_0}{1 - S_0}
\]

Experiments documented in ?? have shown that if pressure is graphed as a function of effective saturation on log-log paper, a straight line is obtained except for \(S_e\) close to unity. The slope of this line \(\lambda\) is called the pore-size distribution. Mathematically we have

\[
\log_{10}\left(\frac{-p}{\rho g}\right) = -\lambda \log_{10}(S_e) + b
\]

Thus

\[
S_e = \left(\frac{-p}{B\rho g}\right)^{-1/\lambda}, \quad B = 10^b
\]

When \(p = p_s\) we have \(\theta = \eta\), thus \(S_e = 1\). Solving for \(B\) in the above equation we get:

\[
B = \frac{-p_s}{\rho g}
\]

This gives us an elegant model for the effective saturation as a function of pressure:
Figure 7: Functional Dependency of $\theta$ on $p_s$

\[
S_e = \left\{ \begin{array}{ll}
1 & \text{if } p \geq p_s \\
\left( \frac{p}{p_s} \right)^{-1/\lambda} & \text{if } p < p_s \\
\end{array} \right.
\]  

(35)

Converting this to an expression for $\theta$ as a function of $p$ we get

\[
\theta = \left\{ \begin{array}{ll}
n & \text{if } p \geq p_s \\
n \left( \frac{p}{p_s} \right)^{-1/\lambda} (1 - S_0) + S_0 & \text{if } p < p_s \\
\end{array} \right.
\]  

(36)

The graph of this function is shown in Figure 6.4.1.

6.4.2 Hydraulic Conductivity Dependency on Pressure

The hydraulic conductivity exhibits two different behaviors depending on the pressure in the porous media. When our media is saturated with water (i.e. for pressures $p > p_s$), the hydraulic conductivity $K$ does not depend on pressure. We call this the saturated hydraulic conductivity and denote it by $K_{SAT}$. Using the fluid dynamical results and assumptions from the previous section we may obtain a formula for $K_{SAT}$. 

\[ K_{\text{SAT}} = \frac{C a^2 \rho g}{8 \mu} \]

Here \( a^2 \) is the mean square radius of the pores at a particular place in the porous media and \( \mu \) is the viscosity of water.

Our current model, unlike the model of the previous section, takes into account that the path the water follows varies in diameter and does not go straight down. Here \( C \) is a constant which depends on various properties of the soil such as porosity, roundness and size of particles, etc.

When our porous media becomes unsaturated (when \( p < p_s \)), the hydraulic conductivity becomes dependent on pressure. Experimental evidence suggests that we have:

\[ K(S) = K_{\text{SAT}} S_e^3 \]

Where \( S_e \) is the effective saturation defined above. From equation 35 we obtain an expression for \( K \) as a function of \( p \).

\[ K = \begin{cases} 
K_{\text{SAT}} & \text{if } p \geq p_s \\
K_{\text{SAT}} \left( \frac{p}{p_s} \right)^{3/\lambda} & \text{if } p < p_s 
\end{cases} \]  \hspace{1cm} (37)

### 6.5 Boundary and Initial Conditions

At this point we have a complete partial differential equation describing the water pressure within the ore pile. We now need to determine the boundary conditions on the problem.

Our ore pile will have three bounding lines, namely the bottom and the two sides (refer to Figure 6.1). Consider first the boundary condition on the bottom of the pile. We assume that the ore pile sits on a non-porous rock bed, and thus no water can leave the pile from the bottom. Mathematically this says that

\[ u_y = 0 \text{ for } y = 0 \]

Applying Darcy’s law we get a boundary condition on the pressure at the bottom of the pile.

\[ \frac{\partial p}{\partial y} = -\rho g \text{ for } y = 0 \]  \hspace{1cm} (38)
The boundary conditions on the sides of the ore pile are a bit more complicated. The water table is the pressure level curve where \( p = p_s \). This is the dividing line between saturated and unsaturated flow. This is illustrated in Figure 6.5.

In an unsaturated region, no water can leave the ore pile from the sides, thus along the sides, the component of velocity in the direction of the normal vector \( n = (\pm H, L/2) \) must be zero. Applying Darcy's law to the expression \( v \cdot n = 0 \) we get the following boundary conditions:

\[
-H \frac{\partial p}{\partial x} + L \frac{\partial p}{\partial y} + \frac{Lp_s}{2} = 0 \quad \text{Left Unsaturated Boundary}
\]
\[
H \frac{\partial p}{\partial x} + L \frac{\partial p}{\partial y} + \frac{Lp_s}{2} = 0 \quad \text{Right Unsaturated Boundary}
\]

In the saturated region, we see that while water will flow out of the boundary, the pressure at the boundary will be 0 (atmospheric pressure), thus we get the boundary equations:

\[ p = 0 \quad \text{Left and Right Saturated Boundaries} \]
Notice at all points along the left and right boundaries, one of these boundary conditions will always hold, i.e. one of the equations will always be zero. Because of this we can multiply these together and obtain a boundary condition which holds over the entire side. This is a much better condition, because it does not require that we know where the unsaturated flow becomes saturated. The boundary conditions are given below.

Left Side:

\[ p \left( -H \frac{\partial p}{\partial x} + \frac{L \partial p}{2 \partial y} + \frac{L \rho g}{2} \right) = 0 \]  \hspace{1cm} (39)

Right Side:

\[ p \left( H \frac{\partial p}{\partial x} + \frac{L \partial p}{2 \partial y} + \frac{L \rho g}{2} \right) = 0 \]  \hspace{1cm} (40)

As for initial conditions, we assume that to start with, our ore pile is completely saturated with water, thus the water pressure within the pile must be greater than \( p_s \) everywhere. That is

\[ p > p_s \quad \text{for } t = 0 \]  \hspace{1cm} (41)

### 6.6 Non-dimensional Analysis

In this section we perform a non-dimensional analysis of the partial differential equation developed to model the pressure within the ore pile. This analysis allows us to determine which terms in the equation contribute significantly to the solution.

#### 6.6.1 Non-dimensional Equations

We make the following substitutions in order to obtain a set of non-dimensional variables:

\[ x = L \bar{x} \]
\[ y = L \bar{y} \]
\[ t = T \bar{t} \]
\[ p = -p_s \bar{p} \]
\[ K = K_{SAT} \bar{K} \]
\[ \theta = nS_0 + n(1 - S_0) \bar{\theta} \]  \hspace{1cm} (42)
With these substitutions our non-dimensional version of equation 34 becomes:

\[
\frac{\partial}{\partial x} \left( \bar{K} \frac{\partial \bar{p}}{\partial x} \right) + \frac{\partial}{\partial y} \left( \bar{K} \frac{\partial \bar{p}}{\partial y} \right) = Q_1 \left( \frac{\partial \bar{\theta}}{\partial t} \right) - Q_2 \frac{\partial \bar{K}}{\partial y} \tag{43}
\]

We should point out that the terms on the right hand side of the above equation represent the time-dependency and gravitational effects for the model. The relative size of the coefficients \(Q_1\) and \(Q_2\) will tell us the importance of each of these terms.

The coefficients \(Q_1\) and \(Q_2\) are given by:

\[
Q_1 = -\frac{\rho g n(1 - S_0) L^2}{TK_{SAT} p_s} \tag{44}
\]

\[
Q_2 = -\frac{\rho g L}{p_s} \tag{45}
\]

The equations for the non-dimensionalized hydraulic conductivity and moisture content become:

\[
\bar{K}(\bar{p}) = \begin{cases} 
1 & \text{if } \bar{p} \geq -1 \\
(-\bar{p})^{-3/\lambda} & \text{if } \bar{p} < -1 
\end{cases} \tag{46}
\]

\[
\bar{\theta}(\bar{p}) = \begin{cases} 
1 & \text{if } \bar{p} \geq -1 \\
(-\bar{p})^{-1/\lambda} & \text{if } \bar{p} < -1 
\end{cases} \tag{47}
\]

The bottom boundary condition becomes:

\[
\frac{\partial \bar{p}}{\partial \bar{y}} = \frac{\rho g L}{p_s} \text{ for } \bar{y} = 0 \tag{48}
\]

The left and right boundary conditions are

Left Side:

\[
\bar{p} \left( -\frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{p}}{\partial \bar{y}} - \frac{L \rho g}{p_s} \right) = 0 \tag{49}
\]

Right Side:

\[
\bar{p} \left( +\frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{p}}{\partial \bar{y}} - \frac{L \rho g}{p_s} \right) = 0 \tag{50}
\]

And our initial condition becomes

\[
\bar{p} > -1 \text{ for } t = 0 \tag{51}
\]
Table 3: Typical Parameter Values.

6.6.2 Typical Parameter Values

The following table gives a list some typical values for the parameters used in this non-dimensional analysis. These parameter values will be used in the next section to determine the relative importance of the terms in our non-dimensionalized equations above.

Using the values in the table above we can obtain values for the non-dimensional coefficients $Q_1$ and $Q_2$.

\[
Q_1 = \frac{4 \times 10^6}{T},
\]

$Q_2 = 67$

6.6.3 Non-dimensional Analysis

Consider now the coefficients $Q_1$ and $Q_2$. We see that over a small time scale (say an hour, $T = 3600s$), the time-dependent term will dominate the gravity term. Physically this means that initially the drainage from the pile controls the pressure within the pile. On a much larger time scale (say one week, $T = 604800s$) $Q_1$ will be two orders of magnitude smaller than $Q_2$. Physically this says that drainage no longer significantly affects the water pressure in the pile and that we have essentially achieved steady state.

In general our ore pile will be held in storage for over a week. What is crucial to know is how much water remains in the pile once it is time to ship it off for further processing. Our next step, then, is to determine a steady state solution for the ore pile (which we will essentially achieve in a weeks time) and determine the amount of water still present in the pile.
6.7 Steady State Solution

It was stated in the last section, that after approximately one week, we reached a steady state condition where no more water drains from the stockpile. The amount of water left in the stockpile is completely determined at this point by the steady state solution of the problem. Neglecting the time dependent term we get the following steady state equation:

\[
\frac{\partial}{\partial x} \left( K \frac{\partial \bar{p}}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial \bar{p}}{\partial y} \right) = -Q_2 \frac{\partial K}{\partial y}
\]  
(52)

It is straightforward to verify that the equation

\[
\bar{p}(\bar{x}, \bar{y}) = -Q_2 \bar{y} - 1
\]
(53)

satisfies both the steady state differential equation above as well as the boundary conditions given in equations (48), (49) and (50).

The dimensional solution is thus given by:

\[
p = -\rho g y + p_s
\]

We now use equation (36) to obtain an expression for water content as a function of height.

\[
\theta(y) = n \left( \left( \frac{-\rho g y}{p_s} + 1 \right)^{-1/\lambda} \left( 1 - S_0 \right) + S_0 \right)
\]
(54)

A graph of this function (for \( \lambda = 7.3 \)) is shown in Figure 6.7.

As one can see, the moisture content left in steady state is still very high (around 30%). This is a pessimistic result from the standpoint of shipping because an optimal water content for shipping is 16%.

The physical reason for our high water content is that most of the water, held in place by static pressure, never drains from the stack. The value of \( p_s \) and \( \lambda \) we used were for fine sand. Although we were not able to get true values for \( p_s \) or \( \lambda \) for our ore pile, we can use the values for silt which are \( p_s = 6860 Kg/m \cdot s \) and \( \lambda = 1.82 \). The steady state moisture content graph for these values is shown in Figure 6.7.

As can be seen from this graph, our ore pile had drained out to about 18% water content, close to the value we want for shipping.
Figure 9: Moisture Content vs. Height for $\lambda = 7.3$
Figure 10: Moisture Content vs. Height for $\lambda = 1.82$
7 Conclusions

We have found several interesting results about water flow through a pile of iron ore. We have shown that under normal conditions evaporation and rain do not significantly affect the moisture content of the pile. Furthermore, an accurate $\lambda$, pore-size distribution, is crucial to obtaining the correct moisture content of the pile. Physically, it is difficult to determine the pore-size distribution. One method is to analyze grain-size distribution using statistical analysis from cross-sections of the pile and drawing tentative conclusions about $\lambda$. Another approach to finding $\lambda$ is to use the computer model of the stockpile formation.

If one uses the computer model with equal numbers of large and small blocks, then the lower outer regions are composed almost exclusively of larger blocks. We hypothesize that $\lambda$ will be slightly smaller in these regions. Furthermore, $p_s$ will be closer to atmospheric pressure. Both of these factors imply the steady state solution for water content will be less in these regions. This explains the experimentally seen phenomena in piles.

When the computer model has four small blocks for each large block, then we have a well mixed pile. This produces a small $\lambda$ over the entire pile, which makes the water content relatively small overall. Similarly, porosity could also be found from the computerized model, which would also help in determining the accurate water content.

Further investigations into the structure of the pile formation would also improve our model. In particular, we would like to examine the role of different particle size ratios on pile formation. When the large particles are substantially larger than the small ones, we expect to find more void space. There will be more complex stability criteria that must be met for the larger block to come to rest. These new configurations will allow more opportunities for void space to be created under the large blocks. It would also be interesting to consider modelling stockpile formation with three different sizes of blocks. It would further be useful to model the pile structure that occurs when the stacker is moved to create two peaks in each cross-section. Modeling this interaction may help us to suggest improvements to pile creation methods that would optimize heterogeneity within the pile.
8 Appendix A - Definitions

**bulk volume** - Volume of entire pile of iron ore, including void space.

**hydraulic conductivity** - Denoted by \( K \). Indicates the ability of the material to conduct a fluid through it. Hydraulic conductivity depends upon the porosity of the material and the fluid which is draining.

**irreducible wetting water saturation** - Denoted by \( S_{w0} \). The percentage of water present in a substance when drainage is the only factor involved. For water in iron ore, \( S_{w0} = .16 \)

**gauge pressure** - Pressure above atmospheric pressure.

**pore-size distribution** - Denoted by \( \lambda \). The fraction of the total pore volume which contains pore diameters between \( \delta \) and \( \delta + d\delta \) where \( \delta \) is the diameter of the largest pore in a given region. The larger the disparity in pore sizes, the smaller the pore-size distribution. Often grain-size distribution is used to determine pore-size distribution.

**porosity** - Denoted by \( n \). The ratio of the volume of void space to bulk volume:

\[
 n = \frac{\text{volume of void space}}{\text{volume of bulk}}
\]

**saturated zone** - Where the capillaries are completely filled with water.

**saturation** - Denoted by \( S \). The ratio of the volume of fluid to the volume of voids, taken over a small volume of bulk:

\[
 S = \frac{\text{volume of fluids}}{\text{volume of voids}}
\]

**static pressure** - Denoted by \( p_e \). Pressure level required to force the fluid through the pore - negative quantity.

**steady flow** - Fluid velocity at any given point is constant in time.

**unsaturated zone** - Where the capillaries contain both water and gas.
**void space** - The portion of the pile not occupied by solid matter. This space includes water and/or air.

**water content** - (also moisture content) Denoted by $\theta$ ratio of weight of water to weight of solids within a sample.

\[
\theta = \frac{\text{weight of water}}{\text{weight of solids}}
\]

**water table** - The zone inside the pile where the pressure is static.
References


