NONLINEAR WAVE ANALYSIS
AND CONVERGENCE OF MUSCL SCHEMES

By

Huanan Yang

IMA Preprint Series # 697
September 1990
Nonlinear Wave Analysis and Convergence of MUSCL Schemes

Huanan Yang*

25 December 1989

ABSTRACT

The convergence of the general MUSCL schemes for strictly convex conservation laws is proved. The schemes are the outcome of the MUSCL TVD reconstructions and the E scheme building blocks. The proof is based on an observation of the theory on BV solutions due to Vol’pert, a series of Rankine–Hugoniot type wave analyses and a global entropy estimates.

Subject Classification: 65M05, 35L65

Key words: Conservation laws, Convergence of MUSCL schemes, Nonlinear wave analysis, Global entropy inequality
Nonlinear Wave Analysis and Convergence of MUSCL Schemes

Huanan Yang

School of Mathematics, University of Minnesota, Twin Cities
Minneapolis, Minnesota 55455

§1 Introduction

In this paper, we consider numerical solutions of Cauchy problems for non-linear hyperbolic scalar conservation laws in one space variable

\[
\begin{aligned}
& w_t + f(w)_x = 0 \\
& w(x, 0) = w_0(x).
\end{aligned}
\]  

(1.1)

It is well known that the solution of (1.1) may develop discontinuities in finite time even though the initial value \(w_0(x)\) is very smooth, say, a \(C^\infty\) function. Therefore, weak solutions in the sense of distributions must be considered. These weak solutions may not be unique. One has to introduce some entropy conditions which distinguish the physical solution from others[11].

During the last two decades, numerical methods for non-linear conservation laws have had rapid developments. Various schemes and techniques have been introduced. Nowadays, highly sophisticated numerical schemes meet following requirements numerically:

i) achieving high accuracy in smooth regions of the solutions,

ii) producing sharp profiles for the shocks and the contact discontinuities,

iii) getting correct positions and speed of the discontinuities, and

iv) avoiding superfluous oscillations and non-physical discontinuities (e.g., expansion shocks).

In this paper, we only consider semi-discrete conservative schemes of the form

\[
\frac{d}{dt} u_j(t) = -\frac{1}{h}(g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}).
\]
The Lax-Wendroff theorem[12] guarantees that the limit function of a sequence of the numerical solutions generated by such a scheme is a weak solution of the conservation law (1.1) provided that the numerical flux $g$ is consistent with the exact flux $f$. Among the numerous numerical methods, a class of schemes, Godunov-type schemes have been particularly efficient. These include the Godunov scheme[5], the MUSCL schemes[24],[25], the PPM schemes[28] and ENO schemes[8],[9],[10] [18], etc. These schemes share two common techniques: First, a piecewise polynomial is constructed based on the node values or the cell averages of the approximate solution. An essential requirement is that the piecewise polynomial should contain no untolerable oscillations. This technique is usually called projection or reconstruction. Second, a local Riemann solver, exact or approximate, is used to compute the numerical fluxes at the interfaces of the adjacent pieces of the polynomials. The Riemann solver is said to be the building block of the numerical scheme.

During the same period, the theory of numerical solutions of hyperbolic conservation laws have also developed rapidly. A central task of the theory is to investigate convergency of numerical schemes.

So far, there are mainly two approaches through which convergency of numerical solutions could be proved. The first approach is based on classical compactness arguments, e.g., the Helly’s theorem on functions with bounded variations. This approach requires a uniform TV(total variation) estimates. TVD schemes and TVB schemes in one space dimension satisfy this requirement. The following convergency proofs were achieved through this approach: Harten[7], Osher[16], Osher-Chakravathy[17], Tedmor[22], Vila[26], Shu[19], etc..

Since the TV estimates are not available for many schemes, another approach has been recently developed. In this approach, one first needs uniform boundedness. Convergency may be proved either by compensated compactness(see Murat[13], Tartar[21], DiPerna[3] and Tedmor[23]) or by the uniqueness result of the measure valued solutions of DiPerna[4](see Szepessy[20] and Coquel–Le Floch[2], etc.).

While the second approach is potentially powerful, the first one is still valuable since it
establishes non-oscillatory property of the numerical solutions which resemble the property of the exact solutions although it is generally difficult to get TV estimates for solutions of high order schemes and for solutions of systems of conservation laws, and there is no high order TVD schemes in the usual sense for multi-dimensional problems (see Goodman and LeVeque[6]). In this paper we will follow the first approach.

Despite the rapid development of the theory, it still lags far behind the development of practical methods. Convergency of many highly efficient methods has not been theoretically confirmed.

There are two essential difficulties in any convergency proof (consistency, conservativity and compatibility with initial conditions are relatively easy): one is getting stability estimates for certain compactness; another is establishing adequate entropy conditions for uniqueness. It is a common practice to design numerical schemes by combining essentially non-oscillatory reconstructions and entropy satisfying building blocks (Riemann solvers). However, to prove or disprove convergency of the resulted schemes can be a highly non-trivial task.

In this paper, we prove convergency of general MUSCL schemes for strictly convex fluxes under the conditions that the MUSCL reconstructions are bounded by minmod slope limiter at rarefaction and the building blocks are the E schemes—the most general class of 3-point schemes known to be convergent for general fluxes[15]. No further restriction except a natural Lipschitz condition is insert into the schemes. The convergency result is achieved by a combination of the entropy estimates of Osher[16] which control the entropy growth at rarefactions and some global wave analysis which controls it at shocks. The technical part of the proof involves some careful analyses on the propagations of the extrema of the numerical solutions which should be useful in more general cases.

In the next section, we first review the generalized MUSCL schemes and the convergency results of them. Constructed and analyzed by Osher[16], these schemes generalize Van Leer’s pioneering work[24], [25]. Our main theorem is also stated in this section. In §3, we establish some regularity results of the numerical solutions of the generalized
MUSCL schemes. These results enable us to trace the local space extrema of the numerical solutions. Previous results available in the literature appear far from adequate for our purpose. In §4, we review some relevant results of Vol’pert[27] and prove that if a sequence of numerical solutions generated by a TVD scheme converges to a weak solution which violates the entropy conditions, then the scheme can generate a sequence of solution which converges to a traveling expansion shock. This result and the regularity results of §3 constitute the basis of the non-linear wave analyses which are carried out in §5 and §6. In §5, we estimate the asymptotic propagation speed of the space extrema of the strong and sharp oscillations of a sequence of solutions. These oscillations are the most responsible for the entropy growth. In §6, we show that if a sequence of numerical solutions converges in $L_{^1}^{loc}$ to a traveling expansion shock, then, similarity transformations and displacements can be used to “push away” inconvenient oscillations, so that the path of the remaining strong oscillations converge to the path of the expansion shock. In §7, we complete the proof of our main theorem by a global entropy estimate which is made possible by the results of the two previous sections. In §8, the concluding remarks, we will briefly discuss some possibilities of further exploitations of the techniques developed in this paper.

We point out here that in an independent work[private communication], with drastically different methodology, P. L. Lions and P. E. Souganidis have proved results of the same nature in the framework of Hamilton–Jacobi equations. It is well known that in one space dimension, entropy solutions of a hyperbolic conservation law correspond to viscosity solutions of the Hamilton–Jacobi equation of the same flux function.

§2 Generalized MUSCL schemes.

As in Osher [16], we will consider conservative semi–discrete schemes only. The case of fully discrete schemes will be considered in a future work.

Let us partition the real line for the space variable into cells. The $j$-th cell is centered at $x_j = jh, j = 0, \pm 1, \pm 2, \ldots, h$ is the space step size. We denote the numerical approximation to the exact solution $w(x_j, t)$ or the cell average on the $j$-th cell $\bar{w}(x_j, t)$ by $u_j^t$, $0 \leq t \leq \infty$. 
A semi-discrete conservative scheme is of the form

\[
\frac{d}{dt} u_j(t) = -\frac{1}{h} (g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}})
\]

where

\[
g_{j+\frac{1}{2}} = g(u_{j-p+1}, u_{j-p+2}, \ldots, u_j, \ldots, u_{j+p}).
\]

g is Lipschitz continuous with respect to its every argument and is consistent with the conservation law in the sense that

\[
g(u, u, \ldots, u) \equiv f(u).
\]

The scheme is said to be similarity invariant if \( g \) is independent of \( h \).

The MUSCL schemes are based on a piecewise linear reconstruction:

\[
z(x, t) = u_j(t) + (x - x_j)s_j(t) \quad \text{for} \quad |x - x_j| < \frac{1}{2}h
\]

(2.1)

here \( s_j(t) \) are the slopes of the reconstruction in the \( j \)-th cell which satisfy

\[
s_j(t) = w_2(x_j, t) + O(h)
\]

(2.2)

if \( w(x, t) \) is smooth with respect to \( x \), and either \( u_j(t) = w(x_j, t) \) for all \( x_j \), or

\[
u_j(t) = \frac{1}{h} \int_{x_j-h/2}^{x_j+h/2} w(x_j + \xi, t) d\xi
\]

for all \( x_j \).

Let \( g(a, b) \) be an arbitrary first order accurate numerical flux function. The generalized MUSCL scheme is

\[
\frac{d}{dt} u_j(t) = -\frac{1}{h} \Delta^- g(u_{j+1} - \frac{h}{2}s_{j+1}, u_j + \frac{h}{2}s_j).
\]

(2.3)

The following lemmas were proved by Osher[16]:

About the order of accuracy of the scheme:

**Lemma 2.1.** At points \( w(x_j, t) = u_j(t) \) in a neighborhood of which \( g(u_{j+1}, u_j) \) is \( C^2 \) with Lipschitz continuous partial derivatives, and \( hs_{j+1}/\Delta t u_j = 1 + O(h) = hs_{j+1}/\Delta t u_j \), the algorithm (2.3) is at least second order accurate for smooth function \( w \).
About the TVD property:

**Lemma 2.2.** If \( g \) is a flux corresponding to an E scheme, then the scheme is TVD if

\[
0 \leq \frac{h s_i}{\Delta u_j} \frac{h s_{j+1}}{\Delta u_j} \leq 1
\]

for all \( j \).

**Lemma 2.3.** If \( g \) is a flux corresponding to a monotone scheme, then the scheme is TVD if

\[
1 \geq \begin{cases} 
\frac{h(s_{j+1} - s_j)}{2\Delta u_j}, & \text{near points where } g_1 \neq 0 \\
\frac{h(-s_{j+1} + s_j)}{2\Delta u_j}, & \text{near points where } g_1 \neq 0
\end{cases}
\]

for each \( j \). Here

\[
g_i = \frac{\partial}{\partial u_{j+i}} g(u_{j+1}, u_j), \quad i = 0, 1.
\]

Let \( U(w) \) be any convex function, \( F(w) \) be the corresponding entropy flux, i.e., \( F' = U'f' \). The convergence proof of [16] is based on some estimates of the quantity

\[
h\left(\frac{d}{dt} U(u_j) + D_F A(u_j)\right) = \int_{u_j}^{u_{j+1}} dw U''(w)[g_{j+\frac{1}{2}} - f(w)]
\]

where the consistent numerical entropy flux \( F_A \) is defined through

\[
F_A(u_j) = F(u_j) + U'(u_j)[g_{j-\frac{1}{2}} - f(u_j)].
\]

Thus a sufficient condition for the scheme (2.3) which satisfies (2.4) to converge to the unique entropy solution of (1.1) is that for all \( j \) and \( t \), the inequality

\[
\int_{u_j}^{u_{j+1}} dw U''(w)[g_{j+\frac{1}{2}} - f(w)] \leq 0
\]

holds for all convex \( U(w) \), a condition that characterize the E schemes which are only first order accurate. For high order schemes, one can not expect that (2.8) holds for everywhere for all convex \( U \). To prove the convergence of such a scheme, some global analysis is necessary. For non-convex flux functions, the problem becomes extremely difficult, since then we need entropy inequalities for a whole set of entropy functions. However, according
to DiPerna[3], if $f(w)$ is convex, the inequality (2.8) with $U(w) = \frac{1}{2}w^2$ is enough to guarantee convergency.

In the case $u_j > u_{j+1}$, let $\bar{u}_{j+\frac{1}{2}}$ be chosen so that

$$\int_{u_j}^{u_{j+1}} f'(w)(w - \bar{u}_{j+\frac{1}{2}})dw = 0.$$ 

Then, the following theorem holds.

**Theorem 2.4 (Osher[16]).** The sequence of approximate solutions satisfying (2.3) with an $E$ scheme building block converges a.e. to the unique solution of the scalar convex conservation law (1.1) provided that the initial data is in $BV$ and that for each $j$,

$$0 \leq \frac{hs_j}{\Delta^+ u_j}, \frac{hs_j}{\Delta^- u_j} \leq 1$$

(2.9)

and, in addition if $u_j > u_{j+1}$,

$$-hs_j \leq 2\max(\min((u_j - \bar{u}_{j+\frac{1}{2}}), (\bar{u}_{j-\frac{1}{2}} - u_j)), 0).$$

(2.10)

In order for the numerical flux to meet the Lipschitz continuous requirement, we make the following

**Assumption 2.5.** $s_j h$ is a Lipschitz continuous function of $u_{j-r}, \cdots, u_j, \cdots, u_{j+r}$ where $r$ is a fixed integer.

The main theorem of this paper is as follows.

**Theorem 2.6.** For strictly convex $f(w)$, the sequence of approximate solutions satisfying (2.3) and assumption 2.5 with an $E$ scheme building block converges a.e. to the unique solution of the scalar convex conservation law (1.1) provided that the initial data is in $BV$ and that for each $j$, (2.9) holds.

Hence, (2.10) is no longer required to ensure convergency. Actually, we can further release restriction at shocks. See remark 7.1 following the proof of theorem 2.6. At this point, it would be interesting to consider another entropy condition, Oleinik's $E$ condition[14]. Under this condition, for the solution $w(x,t)$ of a Cauchy problem of a strictly convex
scalar conservation law, there is a constant $E > 0$ such that for any $a > 0, t > 0$ and $x \in \mathbb{R}$,

$$\frac{u(x + a, t) - u(x, t)}{a} \leq \frac{E}{t}.$$ 

It is well known that both the conditions admit the unique physical solution and exclude non-physical weak solutions. In an interesting paper by Osher and Brenier[1], a MUSCL type scheme which allows overshoots at rarefactions is proven to converge to the physical solution. Form the construction of the generalised MUSCL schemes, we see that, at rarefactions where E condition might be violated, (2.8) is satisfied; at shocks where (2.8) might be violated, E condition is satisfied. One might think that convergency of the schemes could be proved by combining the two entropy conditions. Unfortunately, this is not a realistic strategy. In order to prove convergency without the restriction (2.10), we have to carry out some global analysis, i.e., we have to study dynamic properties of the numerical solution.

§3 The paths of the extrema

In this section we will give a detailed description of the propagations of the space extrema of numerical solutions generated by semi-discrete MUSCL schemes. Some conclusions in this section may be cliché. However, our task of rigorously proving the convergency of the schemes requires such a detailed description which seems unavailable in the literature.

I. The paths of the local extrema. Let

$$L = \{x_j\}_{j=-\infty}^{\infty} \times \mathbb{R}_+$$

be the set of the grid lines and $u_j(t), j = 0, \pm 1, \pm 2, \cdots$ be a numerical solution generated by a semi-discrete MUSCL scheme with an E-scheme building block.

Definition 3.1 A subset of $L$ denoted by $M_{t_1, t_2}$ is called the path of a local maximum from $t_1$ to $t_2$ if for any $t \in [t_1, t_2]$ the $t$-intersection

$$S_M(t) = \{(x, t) \in M_{t_1, t_2}\} \neq \emptyset$$
and the x-projection $P_M(t)$ of $S_M(t)$ is of the form

$$P_M(t) = \{x_{i(t)}, x_{i(t)+1}, \ldots, x_{j(t)}\}$$

which satisfies the following conditions:

1) There is a number $V_M(t)$, the value of the path, such that

$$u_k(t) \equiv V_M(t) \quad \text{for} \quad i(t) \leq k \leq j(t),$$

$$u_{i(t)-1}(t) < V_M(t)$$

and

$$u_{j(t)+1}(t) < V_M(t).$$

2) $M_{t_1,t_2}$ is “connected” on $[t_1, t_2]$ in the sense that for each $t$, there exists a neighborhood $U(t)$ such that whenever $t' \in U(t) \cap [t_1, t_2]$,

$$P_M(t') \subseteq P_M(t)$$

holds.

Throughout this section and in the relevant parts of other sections, we use the notations $M_{t_1,t_2}$, $S_M(t)$, $P_M(t)$ and $V_M(t)$ exclusively for the above notions. The results for minima follows the same argument obviously and we use $N_{t_1,t_2}$, $S_N(t)$, $P_N(t)$ and $V_N(t)$ as the corresponding notations for minima. We use the notations $E_{t_1,t_2}$, $S_E(t)$, $P_E(t)$ and $V_E(t)$ for extrema when there is no need to distinguish one from another. We will add superscripts for these notions associated with a sequence of solutions, a sequence of paths, or both. These conventions holds for the approximate paths which will be introduced later.

The main result of this section is the following theorem which constitutes the basis of our extremum tracing technique.

**Theorem 3.2** For any $T > 0$, if

$$u_{i-1}(T) < u_i(T) = \cdots = u_j(T) > u_{j+1}(T),$$

then there exists an $M_{0,T}$ with $P_M(T) = \{x_i, x_{i+1}, \ldots, x_j\}$. 

10
To prove this theorem, we need some lemmas.

**Lemma 3.3** Suppose at some time \( t_0 \),
\[
 u_{i-1}(t_0) < u_i(t_0) = \cdots = u_j(t_0) > u_{j+1}(t_0),
\] (3.1)
then there exist a \( \delta > 0 \), such that there is no strict minimum of \( u_k(t) \) in the interval \([x_i, x_j]\) for any \( t \in [t_0, t_0 + \delta) \).

**Proof** If \( j \leq i + 1 \), the lemma holds trivially. Otherwise, for any \( l \) with \( i < l < j \), define
\[
 D_l(t) = \min_{i \leq k < l} (\max_{i \leq k \leq l} (u_k(t) - u_l(t))_+), \max_{i < r \leq j} (u_r(t) - u_l(t))_+
\]
and
\[
 D(t) = \max_{i < l \leq i} D_l(t).
\]
Here \((a)_+ = a\), for \( a > 0 \) and \((a)_+ = 0\) otherwise. Clearly, \( D(t) \geq 0 \) and \( D(t) = 0 \) if and only if there is no strict minimum of \( u \) in the interval \([x_i, x_j]\). We prove the lemma by showing that \( D(t) \) is a non-increasing function of \( t \) in the interval \([t_0, t_0 + \delta)\) for some \( \delta > 0 \).

Since (3.1) holds at \( t = t_0 \), there exists a \( \delta > 0 \) such that for \( t \in [t_0, t_0 + \delta) \),
\[
 \max_{i \leq k \leq j} (u_i(t), u_{j+1}(t)) < \min_{i \leq k \leq j} u_k(t).
\]
Obviously, \( D(t) \) is continuous since \( u_k(t) \) for each \( k \), \((\cdot)_+\), \( \min(\cdot, \cdot) \) and \( \max(\cdot, \cdot) \) are all continuous and \( D(t) \) is a composition of them. Therefore \( \{ t : D(t) > 0 \} \) is a open set which is an union of open intervals: \( \cup_k I_k \). It suffices for us to show that \( D(t) \) is diminishing in each of \( I_k \subset [t_0, t_0 + \delta) \). Let \( t \in I_k \). There must be an \( l \) with \( i < l < j \), an \( l_1 \) with \( i \leq l_1 < l \) and an \( l_2 \) with \( l < l_2 \leq j \) such that \( D(t) = \min(u_{l_1}(t) - u_l(t), u_{l_2}(t) - u_l(t)) \). \( x_\nu, \nu = 1,2 \) must be strict maxima and \( x_i \) must be a strict minimum. Hence
\[
 \frac{du_\nu(t)}{dt} \leq 0 \quad \text{for} \quad \nu = 1,2 \quad \text{and} \quad \frac{du_l(t)}{dt} \geq 0.
\]
Since \(|du_j/dt| < C_1/h\) and the numerical flux is Lipschitz continuous, \( du_j/dt \) is Lipschitz continuous in \( t \) for fixed \( h \):
\[
 \left| \frac{du_j(t'')}{dt} - \frac{du_j(t')}{dt} \right| \leq (C_2/h^2)|t'' - t'|
\] (3.2)
where $C_2$ is independent of $j, h, t'$ and $t''$.

For any $\varepsilon > 0$, if $t' \in I_i, 0 < t - t' < h^2\varepsilon/2C_2$, then

$$\frac{du_i(s)}{dt} \geq -\frac{\varepsilon}{2}, \quad \frac{du_{i\nu}(s)}{dt} \leq \frac{\varepsilon}{2}, \quad \nu = 1, 2$$

for all $s \in [t', t]$. Therefore

$$u_i(t') \leq u_i(t) + \frac{\varepsilon}{2}(t - t')$$

and

$$u_{i\nu}(t') \geq u_{i\nu}(t) - \frac{\varepsilon}{2}(t - t'), \quad i = 1, 2.$$

Hence

$$D(t') \geq D_i(t') \geq \min_{i=1,2}(u_{i\nu}(t') - u_i(t'))$$

$$\geq \min_{i=1,2}(u_{i\nu}(t) - u_i(t)) - \varepsilon(t - t')$$

$$= D_i(t) - \varepsilon(t - t')$$

$$= D(t) - \varepsilon(t - t')$$

This clearly implies

$$D(s'') - D(s') \geq -\varepsilon(s' - s'')$$

for any $s', s'' \in I_i$ with $s' > s''$. The arbitrariness of $\varepsilon$ then implies

$$D(s'') - D(s') \geq 0. \quad \Box$$

Lemma 3.4 If at $t = t_0$,

$$u_{i-1}(t_0) > u_i(t_0) \geq \cdots \geq u_j(t_0) > u_{j+1}(t_0),$$

then there exists a $\delta > 0$ such that for $t \in [t_0, t_0 + \delta)$

$$u_{i-1}(t) > u_i(t) \geq \cdots \geq u_j(t) > u_{j+1}(t), \quad (3.3)$$

Proof Clearly, there exists a $\delta > 0$ such that

$$u_{i-1}(t) > \max_{i \leq i \leq j} u_i(t) \geq \min_{i \leq i \leq j} u_i(t) > u_{j+1}(t)$$
for $t \in [t_0, t_0 + \delta)$. Define

$$D(t) = \max_{i \leq l < i \leq 3 j} (u_i(t) - u_l(t))^+.)$$

We always have $D(t) \geq 0$. $D(t) = 0$ if and only if (3.3) holds. It therefore suffices to show that $D(t)$ diminishing in $t \in [t_0, t_0 + \delta)$. The proof is so similar to that for the lemma 3.3 that we omit it here. □

Lemma 3.5 Let $M_{t_1, t_2}$ be a path of a local maximum, then the corresponding $V_M(t)$ is a decreasing function on $[t_1, t_2]$.

Proof From the proof of lemma 3.3, there exists a $C_2 > 0$ such that (3.2) holds for any $j$, $h$, $t'$ and $t''$. Let $t_1 \leq t' < t'' \leq t_2$. For any $\varepsilon > 0$, any $t \in [t', t'']$ there is a $\delta(t) > 0$ such that the interval $O_t = (t - \delta(t), t + \delta(t))$ satisfies

$$O_t \subseteq (t', t''), \quad \text{if } \; t \in (t', t'') \quad (3.4)$$

$$P_M(s) \subseteq P_M(t), \quad \text{for } \; s \in O_t \quad (3.5)$$

$$\frac{du_j(s)}{ds} \leq \varepsilon, \quad \text{for } \; s \in O_t \text{ and } x_j \in P_M(t). \quad (3.6)$$

Since

$$\bigcup_{t \in [t', t'']} (t - \frac{1}{2} \delta(t), t + \frac{1}{2} \delta(t)) \supseteq [t', t''],$$

there exists a finite partition of $[t', t'']$:

$$t' = \tau_0 < \tau_1 < \cdots < \tau_n = t''$$

such that

$$\bigcup_{k=0}^{n} (\tau_k - \frac{1}{2} \delta(\tau_k), \tau_k + \frac{1}{2} \delta(\tau_k)) \supseteq [t', t''] \supseteq [t', t''].$$

We may obviously assume that

$$\frac{\delta(\tau_k) + \delta(\tau_{k+1})}{2} > \tau_{k+1} - \tau_k \quad \text{for each } \; k = 0, 1, \ldots, n - 1.$$

Hence, either $\delta(\tau_{k+1}) > \tau_{k+1} - \tau_k$ or $\delta(\tau_k) > \tau_{k+1} - \tau_k$. In either case, (3.5) and (3.6) imply

$$V_M(\tau_{k+1}) - V_M(\tau_k) \leq \varepsilon (\tau_{k+1} - \tau_k)$$
which in turn implies
\[ V_M(t'') - V_M(t') \leq \varepsilon(t'' - t'). \]

Hence
\[ V_M(t'') \leq V_M(t') \]
since \( \varepsilon \) is arbitrary. \( \square \)

**Lemma 3.6** Let \( M_{t_1, t_2} \) be a path of a local maximum with
\[ P_M(t) = \{ x_{i(t)}, \ldots, x_{j(t)} \} \quad \text{for each} \quad t \in [t_1, t_2]. \]

then
\[ R(t) = \inf_{j(t) < k < \infty} u_k(t) \]
is an increasing function on \([t_1, t_2]\).

**Proof** For any \( \varepsilon > 0 \), let \( t_1 \leq t' < t'' \leq t_2, t'' - t' < \frac{h^2 \varepsilon}{2C_2} \). Suppose \( \{ j_\nu \}_{\nu=1}^\infty \) satisfy \( j_\nu > j(t'') \) and
\[ \lim_{\nu \to \infty} u_{j_\nu}(t'') = R(t''), \]
then
\[ \lim_{\nu \to \infty} \inf \frac{du_{j_\nu}(t'')}{dt} \geq 0 \]
which implies that
\[ \lim_{\nu \to \infty} \inf \frac{du_{j_\nu}(t)}{dt} \geq -\frac{\varepsilon}{2} \]
uniformly for \( t \in [t', t''] \). Therefore
\[
R(t') = \inf_{j(t') < k < \infty} u_k(t') \\
\leq \lim_{\nu \to \infty} \inf u_{j_\nu}(t') \\
\leq \lim_{\nu \to \infty} u_{j_\nu}(t'') + \varepsilon(t'' - t') \\
= R(t'') + \varepsilon(t'' - t')
\]

Now, for any \( t', t'' \in [t_1, t_2] \), partitioning \([t', t'']\) into finite number of disjoint subintervals of length no more than \( h^2 \varepsilon / 2C_2 \), applying the above inequality to each of the subintervals
and summing the results up, one get

\[ R(t') \leq R(t'') + \varepsilon(t'' - t'). \]

Now we can conclude the proof of this lemma by noticing the arbitrariness of the \( \varepsilon \) again.

\( \square \)

The following lemma is useful both in our proof of the theorem 3.2 and in estimating the speed of a propagating oscillation.

**Lemma 3.7** Suppose \( M_{t_1,t_2} \) is a path of a local maximum such that

\[ x_i = \min[P_M(t_1) \cup P_M(t_2)] \]  \hspace{1cm} (3.7)

and

\[ x_j = \max[P_M(t_1) \cup P_M(t_2)], \]  \hspace{1cm} (3.8)

then

\[ \bigcup_{t_1 \leq t \leq t_2} P_M(t) \supseteq \{x_k : i \leq k \leq j\}. \]  \hspace{1cm} (3.9)

**Proof** Assume \( k \) is an integer between \( i \) and \( j \) such that

\[ x_k \notin \bigcup_{t_1 \leq t \leq t_2} P_M(t). \]

Then (3.7) and (3.8) imply that both

\[ Q_l = \{t \in [t_1, t_2] : \max P_M(t) < x_k\} \]

and

\[ Q_r = \{t \in [t_1, t_2] : \min P_M(t) > x_k\} \]

are not empty. Clearly, \( Q_l \cup Q_r = [t_1, t_2] \) and \( O_l \cap Q_r = \emptyset \). These contradict the connectivity of the interval \([t_1, t_2]\) since \( Q_l \) and \( Q_r \) are both open by the “connectivity” of \( M_{t_1,t_2} \) (see the definition 3.1). Therefore (3.9) holds.

\( \square \)

**Proof** of Theorem 3.2: Suppose \( T \) is the set of those \( t \) such that \( M_{t,T} \) exists and \( M_{t_2,T} \subseteq M_{t_1,T} \), for \( t_1 < t_2 \) when both \( t_1 \) and \( t_2 \) are in \( T \). We will prove the theorem by showing that \( T \) is both open and closed in \([0,T]\).
“Open”: Let $t' \in T, P_M(t') = \{x_p, \cdots, x_q\}$. There exists a $\delta_1 > 0$ such that for $t \in [t', t' - \delta_1)$,

$$\max(u_{p-1}(t), u_{q+1}(t)) < \min_{p \leq k \leq q} u_k(t). \quad (3.10)$$

Let $N(t)$ be the number of strict maximum sections (a strict maximum section is a finite sequence of consecutive grid points $\{x_i, \cdots, x_j\}$ such that $u_{i-1}(t) < u_i(t) = \cdots = u_j(t) > u_{j+1}(t)$) within the set $\{x_p, \cdots, x_q\}$ for $t \in [t' - \delta_1, t')$. The range of $N(t)$ is a finite set of integers $\{n_i\}_{i=1}^m$. Let $n_1 > n_2 > \cdots > n_m$. We claim that there exists a corresponding partition of $[t' - \delta_1, t']$:

$$t' - \delta_1 = \tau_0 < \tau_1 < \cdots < \tau_m = t',$$

such that $N(t) = n_k$ for $t \in [\tau_{k-1}, \tau_k)$ for each $k = 1, \cdots, m$. Actually, the claim is trivially true for $m = 1$. Assume it is true for $m = p - 1$. In the case $m = p$, consider the set $\{t \in [t' - \delta_1, t'] : N(t) = n_1\}$. Obviously, it is an open set in $[t' - \delta_1, t']$. If it is not the interval $[t' - \delta_1, \tau_1)$ for some $\tau_1 \in (t' - \delta_1, t']$, then there exists an open interval $(\bar{t}, \bar{t}) \subset (t' - \delta_1, t']$ such that $N(t) = n_1$ for $t \in (\bar{t}, \bar{t})$, but $N(\bar{t}) < n_1$. This contradicts the lemma 3.3 and 3.4. Therefore

$$\{t \in [t' - \delta_1, t'] : N(t) = n_1\} = [t' - \delta_1, \tau_1)$$

for some $\tau_1 \in (t' - \delta_1, t']$. By this and the induction hypothesis, the claim holds for $m = p$. This confirms the claim.

Now let $\delta = \tau_m - \tau_{m-1}$, then $\delta > 0$ and $N(t) \equiv n_m$ in $[t' - \delta, t')$. For any fixed $t \in [t' - \delta, t')$, We put the strict maximum sections within $P_M(t')$ in an order of increasing space variables: $P^{(1)}(t), \cdots, P^{(n_1)}(t)$. Fix any $i$ between 1 and $n_1$, for any $t \in (t' - \delta, t')$, define $M_{i,T}$ by letting $M_{i,T} \subset M_{T}$ and $P_M(s) = P^{(i)}(s)$ for $s \in [t, t')$. It is easy to verify that $M_{i,T}$ satisfies all the conditions in the definition 3.1.

“Close”: Let $t', \tau_i \in [0, T]$ for $i = 1, 2, \cdots$ and $\tau_i \downarrow t'$ as $i \to \infty$; suppose $M_{\tau_i,T}$ exists for each $i$, and $M_{\tau_i-1,T} \subset M_{\tau_i,T}$ for each $i = 2, 3, \cdots$. Lemma 3.5 and lemma 3.6 imply that $\{x_i(\tau_i)\}_{i=1}^\infty$ is a sequence with a upper bound. Similarly, $\{x_i(\tau_i)\}_{i=1}^\infty$ is a sequence with a lower bound. Here $i(\tau_i)$ and $i(\tau_i)$ are determined by $P_M(\tau_i) = \{x_i(\tau_i), \cdots, x_j(\tau_i)\}$. 

16
Let
\[ i' = \lim_{\nu \to \infty} \inf i(\tau_\nu), \quad \text{and} \quad j' = \lim_{\nu \to \infty} \sup j(\tau_\nu). \]

It is easy to see by the "connectedness" of \( M_{t_1, t_2} \) that \( i' \leq j' \) and
\[ u_{i'}(t') = \cdots = u_{j'}(t') = \lim_{\nu \to \infty} V(\tau_\nu). \]

Let \( i(t') \) and \( j(t') \) be the integers such that
\[ i(t') \leq i' \leq j' \leq j(t') \]

and
\[ u_{i(t')-1}(t') \neq u_{i(t')}(t'), \quad u_{j(t')+1}(t') \neq u_{j(t')}(t'). \]

\( i(t') \) and \( j(t') \) exist because of lemma 3.5 and lemma 3.6.

Claim:
\[ u_{i(t')-1}(t') < u_{i(t')}(t') \]

and
\[ u_{j(t')+1}(t') < u_{j(t')}(t') \]

hold. Assume it is not the case. Then by lemma 3.3 and lemma 3.4, there will be no strict maximum in \( \{ x_{i(t')-1}, \ldots, x_{j(t')+1} \} \) for \( t = \tau_\nu \) with sufficiently large \( \nu \). This contradicts the conditions for \( \tau_\nu \). This proves the claim and we can therefore let \( P_M(t') = \{ x_{i(t')}, \ldots, x_{j(t')} \} \) and \( M_{t', T} \) is now well defined. \( \square \)

The theorem 3.2 means the existence of the backward path of a local maximum. The next theorem means the uniqueness of the forward path, if any, of a local maximum.

**Theorem 3.8** Let \( M_{t_1, t_2}^{(1)} \) and \( M_{t_1, t_2}^{(2)} \) are the paths of two local maxima. If \( P_{M^{(1)}}(t_1) = P_{M^{(2)}}(t_1) \), then \( M_{t_1, t_2}^{(1)} = M_{t_1, t_2}^{(2)} \).

**Proof** It suffices to show that \( P_{M^{(1)}}(t) = P_{M^{(2)}}(t) \) for each \( t \in [t_1, t_2] \). Obviously,
\[ \{ t : P_{M^{(1)}}(t) \neq P_{M^{(2)}}(t) \} \]

is an open set. Hence if
\[ t' = \sup \{ t : P_{M^{(1)}}(s) = P_{M^{(2)}}(s) \quad \text{for} \quad t_1 \leq s \leq t \} < t_2, \]

17
$P_{M(1)}(t') = P_{M(2)}(t')$. Lemma 3.3 then implies that $P_{M(1)}(t) = P_{M(2)}(t)$ whenever $t \in [t', t' + \delta)$ for some $\delta > 0$. This contradicts the definition of $t'$. \qed

II. The approximate paths of the local extrema. $M_{t_1,t_2}$ is inconvenient for the purpose of our analysis. Ideally, we would like to have a function $x_M(t) : [t_1, t_2] \rightarrow \mathbb{R}$ which satisfies the following conditions:

1. There is a finite partition of $[t_1, t_2]$:

   $$t_1 = \tau_0 < \tau_1 < \cdots < \tau_n = t_2$$

such that $x_M(t)$ is constant on each subinterval $(\tau_{j-1}, \tau_j)$ for $1 \leq j \leq n$.

2. $x_M(t) \in P_M(t)$ for $t \in [t_1, t_2]$.

It is not known if every path $M_{t_1,t_2}$ has such a function $x_M(t)$. We therefore consider some approximate ones.

Lemma 3.9 For the path $M_{t_1,t_2}$ of each local maximum and for any $\varepsilon > 0$, there exists a function $x^*_{M}(t) : [t_1, t_2] \rightarrow \mathbb{R}$ which is piecewise constant: There is a finite partition of $[t_1, t_2]$:

$$t_1 = \tau_0 < \tau_1 < \cdots < \tau_n = t_2$$

such that $x_M(t) \equiv c_j$ and

$$c_j \in P_M(\tau_{j-1}) \cap P_M(\tau_j)$$

on each subinterval $(\tau_{j-1}, \tau_j)$ for $1 \leq j \leq n$. Furthermore, at any $t \in [t_1, t_2]$, for any $x_i$ between (and including) $x^*_{M}(t)$ and $P_M(t)$, we have

$$|V_M(t) - u_i(t)| < \varepsilon.$$ 

Finally, if $V^*_{M}(t) = u(x^*_{M}(t),t)$, then

$$TV_{t_1 \leq t \leq t_2}(V^*_{M}(t)) < V_M(t_1) - V_M(t_2) + \varepsilon.$$ 

Proof We repeat the construction of the finite covering as we have done in the proof of the lemma 3.5 with the following modifications: We replace $[t', t'']$ by $[t_1, t_2]$. For each
\( t \in [t_1, t_2] \) we choose the positive \( \delta(t) \) so small that in addition to (3.4) and (3.5), the followings also hold for \( s \in O_t \) and \( x_j \in P_M(t) \):

\[
|u_j(s) - u_j(t)| < \frac{\varepsilon}{2}
\]

(3.14)

and

\[
\frac{du_j(s)}{ds} < \frac{\varepsilon}{2(t_2 - t_1)}.
\]

(3.15)

Having constructed the finite covering, for each \( k = 0, 1, \ldots, n - 1 \), if \( \delta(\tau_k) > \tau_{k+1} - \tau_k \), we choose \( c_k \) to be any number in \( P_M(\tau_{k+1}) \), otherwise we choose \( c_k \) to be any number in \( P_M(\tau_k) \). Let

\[
x^\varepsilon_M(t) \equiv c_k, \text{ for } \tau_k < t < \tau_{k+1}.
\]

(3.16)

Now (3.11) is a easy consequence of the "connectivity" condition of \( M_{t_1, t_2} \). (3.12) follows (3.14). To show that (3.13) holds, notice that

\[
I = TV_{t_1 \leq t \leq t_2}(V_M^\varepsilon(t)) = \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} \left| \frac{dV_M^\varepsilon(t)}{dy} \right| dt.
\]

Suppose \( x_j(t) = c_k \), then

\[
I = \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} \frac{du_j(t)}{dt} \left| \frac{du_j(t)}{dt} \right| dt
\]

\[
= \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} \frac{du_j(t)}{dt} - \frac{du_j(t)}{dt} + 2\left( \frac{du_j(t)}{dt} \right)_+ dt
\]

\[
\leq V_M(t_1) - V_M(t_2) + \frac{2\varepsilon}{2(t_2 - t_1)} \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} dt
\]

\[
= V_M(t_1) - V_M(t_2) + \varepsilon.
\]

\( \square \)

We call the function \( x^\varepsilon_M(t) \) an \( \varepsilon \) - path of the local maximum.

§4 BV solutions for the conservation laws

For the complete theory of the BV functions and the BV solutions of the conservation laws, we refer the readers to [27]. In the first part of this section, we merely state some results contained in [27] which are relevant to our purpose. In the second part, we will make an important observation which facilitates our global analysis.
For any set $E \subset \mathbb{R}^n$, $\mu(E)$ is its Lebesque measure; $B_r(x_0)$ is the ball centered at $x_0$ with the radius $r$. Let $a, b$ be two vectors in $\mathbb{R}^n$, $(a, b)$ be their scalar product. Let $R_a(x_0)$ be the half space $(x - x_0, a) > 0$ in $\mathbb{R}^n$. A point of density (rarefaction) for the set $E$ is a point $x$ for which

$$\lim_{r \to 0} \frac{\mu(E \cap B_r(x))}{\mu(B_r(x))} = 1(0).$$

If $w(x)$ is a function defined on a set $E \subset \mathbb{R}^n$ and $x_0$ is not a point of rarefaction for $E$, then $L_E w(x_0)$ will denote the approximate limit of the function $w(x)$ at the point $x_0$ with respect to the set $E$: $\forall \varepsilon > 0, x_0$ is a point of rarefaction of the set

$$\{x : |w(x) - L_E w(x_0)| > \varepsilon, x \in E\}.$$

**Definition 4.1** Let $w(x)$ be a function defined on $\mathbb{R}^n$.

(α) A point $x_0 \in \mathbb{R}^n$ is said to be regular if there exists an unit vector $a$ such that $l_a w(x_0)$ and $l_{-a} w(x_0)$ exist and finite. Here, $l_a w(x_0) = L_{R_a(x_0)} w(x_0)$.

(β) The point $x_0$ is said to be a point of jump for $w(x)$ if it is regular and $l_a w(x_0) \neq l_{-a} w(x_0)$. The set of the jump points for $w(x)$ is denoted by $\Gamma(w)$.

(γ) If $x_0 \in \Gamma(w)$, then the value $a$ appearing in the definition (α) is called the normal to $\Gamma(w)$ at the point $x_0$. $a = (a_t, a_x)$ where $a_t$ is the time component of $a$, and $a_x$ the space component.

For completeness of the theory, here, we assume $w_t + f(w)_x = 0$ is a conservation law whose space variable $x \in \mathbb{R}^n$. Namely, $f$ is a vector of $n$ components: $f = (f_1, f_2, \ldots, f_n)'$, and $f(w)_x = \sum \frac{\partial}{\partial x_i} f_i(w)$. Suppose $w(x, t)$ is a BV solution of the conservation law. For any $x_0 \in \Gamma(w)$, let $a$ be the normal to $\Gamma(w)$ at the point $x_0$. Let $w_+ = l_a w(x_0), w_- = l_{-a} w(x_0)$.

Define

$$S(u, v) = [f(u) - f(v)]\text{sgn}(u - v).$$

the jump at $x_0$ is said to satisfy entropy condition if

$$|w_+ - c|, a_t + (S(w_+, c), a_x) \leq (|w_- - c|, a_t) + (S(w_-, c), a_x)$$

(4.2)
where $c$ is an arbitrary constant. Let $H_n$ be the $n$ dimensional Hausdorff measure. We need the following basic theorem:

**Theorem 4.2 (Vol'pert[27])** A necessary and sufficient condition for the bounded function $w \in BV$ to be a weak solution of $w_t + f(w)_x = 0$ and to satisfy the entropy conditions is that (4.2) holds for $H_n$--almost all points in $\Gamma(w)$.

Let $w(x, t)$ be a BV function and $M$ be a constant such that $|w(x, t)| < M$. Suppose also that $(x_0, t_0) \in \Gamma(w)$, that $a$ is the normal at $(x_0, t_0)$ to $\Gamma(w)$, and that $w_+$ and $w_-$ are defined as above.

Throughout this paper, similarity transforms and the similarity invariant property of the schemes play an important role. Let $S_{a, b}^{\varepsilon}$ be the similarity transform:

$$S_{a, b}^{\varepsilon}((x, t)) = (a + \varepsilon x, b + \varepsilon t).$$

This induces a transform in the set of the numerical solutions $u^h$ generated by a similarity invariant scheme:

$$T_{a, b}^{\varepsilon} u^h = u^h \circ S_{a, b}^{\varepsilon}.$$  

Some interesting properties of the transforms and their applications will be discussed in the next section. Define $w_\varepsilon(x, t)$ by

$$w_\varepsilon(x, t) = w \circ S_{x_0, t_0}^{\varepsilon}((x, t))$$

$$= w(x_0 + \varepsilon x, t_0 + \varepsilon t).$$

We then have

**Lemma 4.3** If $\{\varepsilon_j\}_{j=1}^{\infty}$ be a sequence of positive numbers, and $\lim_{j \to \infty} \varepsilon_j = 0$, then the sequence $\{w_\varepsilon\}$ converges in $L^1_{loc}$ to the function

$$W(x, t) = \begin{cases} 
    w_+ & \text{if } ((x, t), a) > 0 \\
    w_- & \text{if } ((x, t), a) < 0.
\end{cases} \quad (4.3)$$

**Proof.** Fix any positive number $R$. By the definition of the jump point, for any $\delta > 0$, there exists a number $\varepsilon(\delta)$ such that if $r < \varepsilon(\delta)R$, 

21
\[
\frac{\mu\left(\{(x, t) : |w(x, t) - W\left(\frac{x-x_0}{\varepsilon(t)}, \frac{t-t_0}{\varepsilon(t)}\right)| > \frac{\delta}{2V_R}\} \cap B_r(x_0, t_0)\right)}{\mu(B_r(x_0, t_0))} < \frac{\delta}{4MV_R}
\]

where \(V_r = \mu(B_r(0, 0))\) for any \(\gamma > 0\). Therefore, \(\forall j\) with \(\varepsilon_j < \varepsilon(\delta)\) and \(r_j = \varepsilon_j R\),

\[
\int_{B_r} |w_{\varepsilon_j}(x, t) - W(x, t)| \, dx \, dt
\]

\[
= \left(\int_{B_{r_j}(x_0, t_0)} |w(x, t) - W\left(\frac{x-x_0}{\varepsilon_j}, \frac{t-t_0}{\varepsilon_j}\right)| \, dx \, dt\right) \frac{\mu(B_R(0, 0))}{\mu(B_{r_j}(x_0, t_0))}
\]

\[
< 2M V_j \frac{V_R}{V_{r_j}} \frac{\delta}{4MV_R} + \frac{\delta}{2V_R} \frac{V_R}{V_{r_j}}
\]

\[
= \delta.
\]

This proves the lemma. \(\square\)

For the solutions of a conservation law, we have following simple lemma concerning a \(L^1_{loc}\) convergent sequences:

**Lemma 4.4** Suppose the sequence \(\{w_j\}_{j=1}^{\infty}\) converges strongly in \(L^1_{loc}\) to \(w\). If each \(w_j\) is a (entropy) weak solution of a conservation law, so is \(w\).

**Proof.** By the definition of (entropy) weak solutions of the conservation laws, fix any test function and pass to the limit. The lemma follows immediately. \(\square\)

As a consequence of the Vol'pert Theorem and the last two lemmas, the following Corollary holds.

**Corollary 4.5** A necessary and sufficient condition for the bounded function \(w \in BV\) to be a weak solution of \(w_t + f(w)_x = 0\) and to satisfy the entropy conditions is that (4.2) holds for all points in \(\Gamma(w)\).

The main result of this section is

**Theorem 4.6** Let \(\{u_{h_k}\}_{k=1}^{\infty}\) be a sequence of numerical solutions which converges in \(L^1_{loc}\) to a weak solution \(w\) of (1.1). Let \((x_0, t_0)\) be a jump point of the weak solution. If the numerical solution is similarity invariant, then we can construct a new sequence of numerical solutions of the same scheme such that it converges in \(L^1_{loc}\) to the traveling discontinuity \(W\) defined by (4.3).

**Proof.** Since the scheme is similarity invariant,
\[ \forall \varepsilon > 0, \]
\[ u^{h_j}(x,t) = u^{h_j}(x_0 + \varepsilon x, t_0 + \varepsilon t) \]
converges to \( w_\varepsilon \) in \( L^1_{loc} \) as \( j \to \infty \). Fixing an \( R_1 > 0 \), one can choose \( \varepsilon \) so that
\[ \int_{B_{R_1}} |w_\varepsilon(x,t) - W(x,t)| dx dt < \frac{1}{4} \]
and then choose \( j \) and let \( H_1 = h_j / \varepsilon \) so that
\[ \int_{B_{R_1}} |u^{H_1}(x,t) - w_\varepsilon(x,t)| < \frac{1}{4}. \]
Then
\[ \int_{B_{R_1}} |u^{H_1}(x,t) - W(x,t)| < \frac{1}{2}. \]
Let \( \{R_j\}_{j=1}^\infty \) be an increasing sequence of positive integers with \( \lim_{n \to \infty} R_n = \infty \). Repeating the same process as the above for each \( R_j \), we get a sequence \( \{u^{H_j}\}_{j=1}^\infty \) with \( H_j < \frac{1}{2j} \) such that
\[ \int_{B_{R_j}} |u^{H_j}(x,t) - W(x,t)| < \frac{1}{2j}. \]
Now clearly, \( \{u^{H_j}\}_{j=1}^\infty \) converges to \( W \) in \( L^1_{loc} \). \( \Box \)

**Remark 4.7** According to corollary 4.5 and theorem 4.6, the numerical approximations to the Cauchy problem (1.1) with any initial data \( w^0 \in BV \) produced by a TVB invariant scheme converges in \( L^1_{loc} \) to the unique entropy solution if there exists no sequence of numerical solutions of the scheme which have uniformly bounded total variations and which converge in \( L^1_{loc} \) to an entropy violating solution:

\[ W(x,t) = \begin{cases} w_- & \text{if } x < st, \\ w_+ & \text{if } x > st. \end{cases} \]  

(4.4)

We now return to the case of one space dimension and assume that \( f \) is strictly convex: \( f''(x) \geq c > 0 \). In particular, this implies that \( w_+ > w_- \). We use \( B \) to denote the set of all the solutions which can be generalized by the MUSCL scheme (2.4) and which satisfy inequality

\[ \max(\sup_{x,t}(|u^k(x,t)|), \sup_{t}(TV(u_k(\cdot,t)))) < B, \]  

(4.5)
where $B$ is a constant. We use $\Psi$ to denote the set of the sequences of the solutions in $B$ each of which converges in $L^1_{loc}$ to $W(x,t)$. Our goal is to prove that $\Psi$ is an empty set. We have the following simple yet useful lemma.

**Lemma 4.8**

(i) If $u \in B$, then $T_{a,b}^\varepsilon u \in B$.

(ii) If $\{u^k\}_{k=1}^\infty \in \Psi$, and $\{b_k\}_{k=1}^\infty$ is a bounded sequence of positive numbers then $\{T_{a_k,b_k}^\varepsilon u^k\}_{k=1}^\infty \in \Psi$ for $a_k = s b_k$.

(iii) If $\{u^k\}_{k=1}^\infty \in \Psi$, then any subsequence of $\{u^k\}_{k=1}^\infty$ also belongs to $\Psi$.

§5 Wave speed estimates

In this section we estimate asymptotically the propagation of a sequence of extrema associated with a sequence $\{u^k(x,t)\}_{k=1}^\infty \subset B$. Constants depending only on $B$, the flux function $f$, and the numerical flux $g$ are considered to be universal constants and are denoted by a single notation $C$ unless otherwise specified. Since the scheme is similarity invariant, we only consider the solutions in the domain $-\infty < x < +\infty, 0 \leq t \leq 1$.

Let $\{E^k_{0,1}\}_{k=1}^\infty$ be a sequence of local extrema corresponding to the sequence $\{u^k(t)\}_{k=1}^\infty$, and $\{x_{E^k_{0,1}}(x,t)\}_{k=1}^\infty$ a corresponding sequence of $\varepsilon$-paths. For each $k$, $x_{E^k_{0,1}}(t) = x_{I^k_{0,1}}(t) = I^k_{0,1}(t)h_k$ where $I^k_{0,1}(t)$ is an integer valued function. We make the convention that $I^k_{0,1}(t) = I^k_{0,1}(t) + 1$ if $I^k_{0,1}(t)$ is discontinuous at $t$. According to the definition in §3

$$V^\varepsilon_{E^k_{0,1}}(t) = u^k_{I^k_{0,1}}(t).$$

The following two technical lemmas will be used to estimate the wave speed.

**Lemma 5.1** Let $E_{0,1}$ be the path of a local extremum of the numerical solution $\{u_j(t)\}_{j=-\infty}^\infty$. Suppose $0 \leq t' < t'' \leq 1$. If the total variation of $V^\varepsilon_{E}(t)$ on $[t', t'']$ is $\sigma$, then

$$\int_{t'}^{t''} |g_{I^k_{0,1}}(t) + \frac{1}{2}(t) - f(V^\varepsilon_{E}(t))|dt \leq C\varepsilon|t'' - t'| + \sigma h$$

(5.1)

and

$$\int_{t'}^{t''} |g_{I^k_{0,1}}(t) + \frac{1}{2}(t) - f(V^\varepsilon_{E}(t))|dt \leq C\varepsilon|t'' - t'| + \sigma h.$$  (5.2)

**Proof** Without loss of generality, let the extremum be a maximum: $M_{0,1}$. First notice
that (5.2) is a direct consequence of (5.1) since \(|V_E^\varepsilon(t) - V_E(t)| < \varepsilon\). Next, we have
\[
\int_{t'}^{t''} |g_{I_{E}(t)} + \frac{1}{2}(t) - g_{I_{E}(t)} - \frac{1}{2}(t)|dt = h \int_{t'}^{t''} |\frac{d}{dt}V_E^\varepsilon(t)|dt \leq \sigma h.
\]

Let
\[
[t', t''] = D_1 \cup D_2 \cup D_3
\]
where
\[
D_1 = \{t \in [t', t'']: x_{I_{E}(t)} < \max_{x_i \in P_M(t)} x_i \}
\]
\[
D_2 = \{t \in [t', t'']: x_{I_{E}(t)} = \max_{x_i \in P_M(t)} x_i \}
\]
and
\[
D_3 = \{t \in [t', t'']: x_{I_{E}(t)} > \max_{x_i \in P_M(t)} x_i \}.
\]
We then have
\[
|g_{I_{E}(t)} + \frac{1}{2}(t) - f(u_{I_{E}(t)}(t))| < C\varepsilon, \quad \text{for} \quad t \in D_1,
\]
\[
|g_{I_{E}(t)} + \frac{1}{2}(t) - f(u_{I_{E}(t)}(t))| < |g_{I_{E}(t)} + \frac{1}{2}(t) - g_{I_{E}(t)} - \frac{1}{2}(t)|, \quad \text{for} \quad t \in D_2,
\]
and
\[
|g_{I_{E}(t)} - \frac{1}{2}(t) - f(u_{I_{E}(t)}(t))| < C\varepsilon \quad \text{for} \quad t \in D_3.
\]
Therefore
\[
\int_{t'}^{t''} \left| g_{I_{E}(t)} + \frac{1}{2}(t) - f(u_{I_{E}(t)}(t)) \right|dt \\
\leq \int_{D_1} \left| g_{I_{E}(t)} + \frac{1}{2}(t) - f(u_{I_{E}(t)}(t)) \right|dt + \int_{D_2 \cup D_3} \left| g_{I_{E}(t)} + \frac{1}{2}(t) - g_{I_{E}(t)} - \frac{1}{2}(t) \right|dt \\
+ \int_{D_3} \left| g_{I_{E}(t)} - \frac{1}{2}(t) - f(u_{I_{E}(t)}(t)) \right|dt \\
\leq C\varepsilon(t'' - t') + \sigma h.
\]

**Lemma 5.2** Suppose \(0 \leq t' < t'' \leq 1\) and
\[
t' = \tau_0 < \tau_1 < \cdots < \tau_m = t''
\]
is a partition of \([t', t'']\) such that \(I_{E}(t) = i_{\nu-1}\) for \(\tau_{\nu-1} < t < \tau_\nu\) and for each \(\nu = 1, 2, \cdots, m\), let \(\sigma\) be the total variation of \(V_E^\varepsilon(t)\) on \([t', t'']\). Then
\[
\sum_{\nu=1}^m (x_{i_{\nu}} - x_{i_{\nu-1}}) u_{i_{\nu}}(\tau_\nu) = (x_{I_{E}(t'')} - x_{I_{E}(t')}) u_{I_{E}(t'')} + R
\]

25
where

\[ |R| \leq \sigma \max_{1 \leq \nu \leq m-1} |x_{i_\nu} - x_{i_0}|. \]

**Proof** Let \( s_\nu = x_{i_\nu} - x_{i_0} \). Then

\[
\begin{align*}
\sum_{\nu=1}^{m} (x_{i_\nu} - x_{i_{\nu-1}})u_{i_\nu}(\tau_\nu) \\
= \sum_{\nu=1}^{m} (s_\nu - s_{\nu-1})u_{i_\nu}(\tau_\nu) \\
= - \sum_{\nu=1}^{m} s_{\nu-1}(u_{i_\nu}(\tau_\nu) - u_{i_{\nu-1}}(\tau_{\nu-1})) + (x_{I^*_k(t'')} - x_{I^*_k(t')})u_{L(t'')}(t'').
\end{align*}
\]

Thus

\[ R = - \sum_{\nu=1}^{m} s_{\nu-1}(u_{i_\nu}(\tau_\nu) - u_{i_{\nu-1}}(\tau_{\nu-1})) \]

and

\[
|R| \leq \sigma \max_{1 \leq \mu \leq m-1} |s_\mu| \sum_{\nu=1}^{m} |u_{i_\nu}(\tau_\nu) - u_{i_{\nu-1}}(\tau_{\nu-1})| \]

\[ \leq \sigma \max_{1 \leq \mu \leq m-1} |s_\mu|. \]

Now for each \( k \), let \( M^k_{0,1} \) be the path of a local maximum of \( u^k(t) \) and \( N^k_{0,1} \) be the path of a local minimum of \( u^k(t) \). Suppose \( x_{M^k(t)} = x_{I^*_k(t)} \) and \( x_{N^k(t)} = x_{J^*_k(t)} \). Denote

\[
\overline{df}(M, N, k, t) = \frac{f(V^k_M(t)) - f(V^k_N(t))}{V^k_M(t) - V^k_N(t)}.
\]

The following theorem is our first main result on wave speed estimate.

**Theorem 5.3** Suppose that a positive constant \( \rho \) exists so that \( V^k_M(t) - V^k_N(t) > \rho \)

for all \( k \) and \( t \in [0,1] \). If \( x_{J^*_k(t)} - x_{I^*_k(t)} \to 0 \) uniformly in \( t \) and in sufficiently small \( \varepsilon \) as \( k \to \infty \), then for any \( \delta > 0 \), an \( \varepsilon_0(\delta) > 0 \) and a \( K(\delta) \) exist so that

\[
|x_{I^*_k(t'')} - x_{I^*_k(t')} - \int_{t'}^{t''} \overline{df}(M, N, k, t)dt| < \delta
\]

for \( \varepsilon < \varepsilon_0(\delta) \), \( k > K(\delta) \) and \( t', t'' \in [0,1] \).

Let us first show the following preliminary estimate:

**Lemma 5.4** Under the conditions of theorem 5.3, \( \forall \eta > 0, \exists K_1(\eta) > 0 \) such that for \( \varepsilon < \rho/4, k > K_1(\eta) \)

\[
|x_{I^*_k(t'')} - x_{I^*_k(t')}| \leq C\frac{t'' - t'}{\rho} + \eta.
\]
**Proof**  Without loss of generality, suppose \( x_{I_k(t')} > x_{I_k(t')} \). The conditions of theorem 5.3 and the definition of \( \varepsilon \)-paths imply that \( \forall \varepsilon < \rho/4, V_{M_k}^\varepsilon(t) - V_{N_k}^\varepsilon(t) > \rho/2 \) and that \( \exists \varepsilon_0 > 0 \) such that
\[
x_{J^k(t)} - x_{I_k(t)} \to 0 \quad \text{as} \quad k \to \infty
\]
uniformly in \( \varepsilon \in (0, \varepsilon_0) \) and \( t \in [0,1] \). Therefore \( \forall \eta > 0, \exists K_1(\eta) > 0 \) such that for \( \varepsilon < \varepsilon_0, k > K_1(\eta) \) and \( t \in [0,1] \)
\[
|x_{J^k(t)} - x_{I_k(t)}| < \eta
\] (5.4)
holds.

If \( x_{I_k(t')} - x_{I_k(t')} \leq \eta \), the lemma holds trivially. Otherwise Lemma A.7 and the definition of \( \varepsilon \)-path imply that if \( J_k^\varepsilon(t) > I_k^\varepsilon(t) \) and
\[
x_{I_k(t')} + \eta \leq x_i \leq x_{I_k(t')},
\]
then \( \exists s_1 \) and \( s_2 \in [t', t'' \] with
\[
u_i^k(s_1) = V_{N_k}^\varepsilon(s_1) \quad \text{and} \quad u_i^k(s_2) = V_{M_k}^\varepsilon(s_2)
\] (5.5)
the same is true if \( J_k^\varepsilon(t) < I_k^\varepsilon(t) \) and
\[
x_{I_k(t')} \leq x_i \leq x_{I_k(t') } - \eta.
\]
(5.5) implies
\[
\int_{t'}^{t''} \left| \frac{du_i^k(t)}{dt} \right| dt \\
\geq \left| \int_{s_1}^{s_2} \left| \frac{du_i^k(t)}{dt} \right| dt \right| \\
\geq |V_{N_k}^\varepsilon(s_2) - V_{N_k}^\varepsilon(s_1)| \\
\geq \min(V_{M_k}^\varepsilon(s_1) - V_{N_k}^\varepsilon(s_1), V_{M_k}^\varepsilon(s_2) - V_{N_k}^\varepsilon(s_2)) \\
\geq \frac{\rho}{2}.
\]
Therefore

\[
C(t'' - t') \geq \int_{t'}^{t''} \sum_{i=L_k(t') + 1}^{L_k(t'')} |g_{i+\frac{1}{2}}(t) - g_{i-\frac{1}{2}}(t)|dt
\]

\[
= h_k \sum_{i=L_k(t') + 1}^{L_k(t'')} \int_{t'}^{t''} \left| \frac{du_i^k(t)}{dy} \right| dt
\]

\[
\geq \left( x_{L_k(t'')} - x_{L_k(t')} - \eta \right) \frac{\rho}{2}.
\]

I.e.

\[
x_{L_k(t'')} - x_{L_k(t')} \leq \frac{C}{\rho} (t'' - t') + \eta. \quad \square
\]

**Proof of theorem 5.3:** Without loss of generality, suppose \( J^k_\epsilon(t) < J^k_\epsilon(t) \) and consider the quantity

\[
S^k_\epsilon(t) = \sum_{i=L^k_\epsilon(t) + 1}^{J^k_\epsilon(t)} u_i^k(t) h_k.
\]

Assume \( \epsilon < \rho/4 \). Divide \([0, 1] \) into \( n \) subintervals of equal length:

\[
[0, 1] = \bigcup_{j=1}^{n} \Gamma_j \quad \text{where} \quad \Gamma_j = \left[ \frac{j-1}{n}, \frac{j}{n} \right].
\]

Suppose \( t' \in [t_{i-1}, t_i) \) and \( t'' \in [t_j, t_{j+1}) \). For convenience, let \( \hat{t}_{i-1} = t', \hat{t}_{j+1} = t'' \) and \( \hat{t}_j = t_j \) for each \( j = j_0, j_0 + 1, \ldots, j_1 \). Now fix a \( j, j_0 \leq j \leq j_1 + 1 \) and let

\[
\hat{t}_{j-1} = \tau_0 < \tau_1 < \cdots < \tau_{\nu_j} = \hat{t}_j
\]

be such a partition of \([\hat{t}_{j-1}, \hat{t}_j] \) that for \( t \in (\tau_{\nu-1}, \tau_\nu) \), both \( x_{L^k_\epsilon(t)} = x_{i_\nu} \) and \( x_{J^k_\epsilon(t)} = x_{j_\nu} \) are constants for each \( \nu = 1, \ldots, \nu_j \). The conservativity of the scheme implies that

\[
A = S^k_\epsilon(\hat{t}_j + 0) - S^k_\epsilon(\hat{t}_{j-1} + 0)
\]

\[
= \int_{\hat{t}_{j-1}}^{\hat{t}_j} (g_{L^k_\epsilon(t) + \frac{1}{2}}(t) - g_{J^k_\epsilon(t) + \frac{1}{2}}(t))dt
\]

\[
+ \sum_{\nu=1}^{\nu_j} (x_{j_\nu} - x_{j_{\nu-1}}) u^k_{j_\nu}(\tau_\nu)
\]

\[
- \sum_{\nu=1}^{\nu_j} (x_{i_\nu} - x_{i_{\nu-1}}) u^k_{i_\nu}(\tau_\nu)
\]

\[
= A_1 + A_2 - A_3
\]

28
Lemma 5.2 implies
\[|A_2 - (x_{J^k(t_i)} - x_{J^k(t_{i-1})})u^k_{J^k(t_i)}(\hat{t}_j)| \leq \gamma''_j \max_{1 \leq \nu \leq \nu_j - 1} |x_{j_\nu} - x_{j_0}|,\]
and
\[|A_3 - (x_{I^k(t_i)} - x_{I^k(t_{i-1})})u^k_{I^k(t_i)}(\hat{t}_j)| \leq \gamma'_j \max_{1 \leq \nu \leq \nu_j - 1} |x_{i_\nu} - x_{i_0}|\]
where \(\gamma'_j\) and \(\gamma''_j\) stand for the variations of \(V_{M^k}(t)\) and \(V_{N^k}(t)\) in \([\hat{t}_{j-1}, \hat{t}_j]\) respectively.

Hence
\[
\begin{align*}
| & (x_{I^k(t_i)} - x_{I^k(t_{i-1})})(u^k_{I^k(t_i)}(\hat{t}_j) - u^k_{J^k(t_i)}(\hat{t}_j)) \\
& - \int_{\hat{t}_{j-1}}^{\hat{t}_j} (g_{I^k(t) + \frac{1}{2}}(t) - g_{J^k(t) + \frac{1}{2}}(t)) dt | \\
\leq & \ |(x_{J^k(t_i)} - x_{J^k(t_{i-1})} - x_{I^k(t_i)} + x_{I^k(t_{i-1})})u^k_{J^k(t_i)}(\hat{t}_j)| \\
& + \gamma''_j \max_{1 \leq \nu \leq \nu_j - 1} |x_{j_\nu} - x_{j_0}| + \gamma'_j \max_{1 \leq \nu \leq \nu_j - 1} |x_{i_\nu} - x_{i_0}| \\
& + |A|.
\end{align*}
\]

Clearly, \(|A| < C_\eta\) and
\[
|[(x_{J^k(t_i)} - x_{I^k(t_i)}) - (x_{J^k(t_{i-1})} - x_{I^k(t_{i-1})})]u^k_{J^k(t_i)}(\hat{t}_j)| < C_\eta
\]
when \(k > K_1(\eta)\). Using this fact and applying lemma 5.4, we get
\[
\begin{align*}
\begin{align*}
| & (x_{I^k(t_i)} - x_{I^k(t_{i-1})})(u^k_{I^k(t_i)}(\hat{t}_j) - u^k_{J^k(t_i)}(\hat{t}_j)) \\
& - \int_{\hat{t}_{j-1}}^{\hat{t}_j} (g_{I^k(t) + \frac{1}{2}}(t) - g_{J^k(t) + \frac{1}{2}}(t)) dt | \\
\leq & \ C_\eta + \frac{\gamma'_j + \gamma''_j}{n \rho}
\end{align*}
\end{align*}
\]
for \(\varepsilon < \rho/4\) and \(k > K_1(\eta)\). Therefore
\[
|x_{I^k(t_i)} - x_{I^k(t_{i-1})} - \int_{\hat{t}_{j-1}}^{\hat{t}_j} \frac{g^k_{I^k(t) + \frac{1}{2}}(t) - g^k_{J^k(t) + \frac{1}{2}}(t)}{u^k_{I^k(t_i)}(\hat{t}_j) - u^k_{J^k(t_i)}(\hat{t}_j)} dt | \leq \frac{C_\eta + \gamma'_j + \gamma''_j}{\rho} (\eta + \frac{\gamma'_j + \gamma''_j}{n \rho}),
\]
for \(\varepsilon < \rho/4\) and \(k > K_1(\eta)\). Since \(u^k_{I^k(t_i)}(t) - u^k_{J^k(t_i)}(t) > \rho/2\) for \(\varepsilon < \rho/4\), lemma 5.1 can be applied to yield
\[
\left| \int_{\hat{t}_{j-1}}^{\hat{t}_j} \frac{g^k_{I^k(t) + \frac{1}{2}}(t) - g^k_{J^k(t) + \frac{1}{2}}(t)}{u^k_{I^k(t_i)}(\hat{t}_j) - u^k_{J^k(t_i)}(\hat{t}_j)} dt - \int_{\hat{t}_{j-1}}^{\hat{t}_j} \frac{\bar{df}(M, N, k, t)}{dt} dt \right|
\]
29
\[
\begin{align*}
\leq & \left| \int_{i_{j-1}}^{i_j} \frac{g_{k}^{\frac{k}{2}}(t) + \frac{1}{2}}{u_{k}^{\frac{k}{2}}(t)} - \frac{g_{k}^{\frac{k}{2}}(t) + \frac{1}{2}}{V_{M}(t) - V_{N}(t)} \right| dt \right.
\left. + \int_{i_{j-1}}^{i_j} \frac{g_{k}^{\frac{k}{2}}(t) + \frac{1}{2}}{V_{M}(t) - V_{N}(t)} \right| dt \right.
\left. - \int_{i_{j-1}}^{i_j} f(V_{M}(t)) - f(V_{N}(t)) \right| dt
\right.
\leq \frac{C}{\rho^2 n}(\varepsilon + \gamma_j' + \gamma_j'') + C\varepsilon + \frac{\gamma_j' + \gamma_j''}{\rho} h_k
\end{align*}
\]

Combining this and (5.6), we have

\[
\left| x_{I_i(t_j)} - x_{I_i(t_{j-1})} - \int_{i_{j-1}}^{i_j} \overline{df}(M, N, k, t) \right| dt \right|
\leq \frac{C}{\rho} \eta + \frac{C}{n\rho^2} (\gamma_j' + \gamma_j'') + \frac{C\varepsilon}{n\rho^2} + \frac{\gamma_j' + \gamma_j''}{\rho} h_k.
\]

Summing the inequality over \( j \), we finally get

\[
\left| x_{I_i(t)} - x_{I_i(t')} - \int_{t'}^{t''} \overline{df}(M, N, k, t) \right| dt \right|
\leq \frac{C}{\rho} \eta + \frac{C}{n\rho^2} + \frac{C\varepsilon}{\rho^2} + \frac{C}{\rho} h_k.
\]

Now, for any \( \delta > 0 \), choose an \( \varepsilon' > 0 \) so that \( C\varepsilon'/\rho^2 < \delta/4 \) and let \( \varepsilon_0(\delta) = \min(\varepsilon', \rho/4) \).

Then choose \( n_0 \) so that \( C/(n_0\rho^2) < \delta/4 \). Suppose \( k_2 > 0 \) is so large a constant that \( C h_k/\rho < \delta/4 \) for \( k > k_2 \), and \( \eta_0 > 0 \) is so small a constant that \( C n_0\eta_0/\rho < \delta/4 \), \( K(\delta) = \max(K_1(\eta_0), k_2) \). Then (5.2) holds for \( \varepsilon < \varepsilon_0(\delta) \), \( k > K(\delta) \) and \( t', t'' \in [0, 1] \).

Next, suppose the flux function \( f \) is strictly convex: \( f''(w) > c > 0 \) for some constant \( c \). Consider a sequence \( \{u^k(x, t)\}_{k=1}^\infty \). We then have

**Lemma 5.5** For any fixed \( \alpha > 0 \),

\[
\int_{s-t-\alpha}^{s+t+\alpha} |u^k(x, t) - W(x, t)| dx \to 0 \text{ as } k \to \infty \tag{5.7}
\]

uniformly for all \( t \in [s_\alpha, t_b] \) which is a fixed interval.

**Proof.** Suppose not, then there exists a positive constant \( \eta \), a sequence of integers, \( \{k_j\}_{j=1}^\infty \) with \( k_{j+1} > k_j \) and a sequence \( \{t_j\}_{j=1}^\infty \) in \( [s_\alpha, t_b] \) such that

\[
\int_{s-t-\alpha}^{s+t+\alpha} |u^{k_j}(x, t_j) - W(x, t_j)| dx > \eta. \tag{5.8}
\]
On the other hand,
\[
\| u^{k_j}(t_1) - u^{k_j}(t_2) \|_{L^1} = \sum_{i=\infty}^{\infty} |u^{k_j}_i(t_2) - u^{k_j}_i(t_1)| h_{k_j}
= \sum_{i=\infty}^{\infty} \left| \int_{t_1}^{t_2} h_{k_j} \frac{d u^{k_j}_i(t)}{dt} dt \right|
= \sum_{i=\infty}^{\infty} \left| \int_{t_1}^{t_2} (g^{k_j}_{i+1/2}(t) - g^{k_j}_{i-1/2}(t)) dt \right|
\leq C |t_1 - t_2|
\]
for any \( t_1 \) and \( t_2 \).

Let \( \delta u^{k_j} = u^{k_j} - W \), then
\[
\| \delta u^{k_j}(t_1) - \delta u^{k_j}(t_2) \|_{L^1} \leq C |t_1 - t_2|.
\]

Therefore,
\[
\left| \int_{s^{k_j}(t_2)}^{s^{k_j}(t_1)} |\delta u^{k_j}(t_2)| d\alpha - \int_{s^{k_j}(t_1)}^{s^{k_j}(t_1)} |\delta u^{k_j}(t_1)| d\alpha \right|
\leq \| \delta u^{k_j}(t_1) - \delta u^{k_j}(t_2) \|_{L^1} + C |t_1 - t_2| \tag{5.9}
\]
\[
\leq C |t_2 - t_1|.
\]

(5.8) and (5.9) imply that
\[
\int_{t_a - \eta/C}^{t_b + \eta/C} dt \int_{s^{k_j}(t)}^{s^{k_j}(t)} |\delta u^{k_j}(x, t)| d\alpha \geq \frac{\eta^2}{C}.
\]

This contradicts the \( L^1_{loc} \) convergence of \( u^k \). \( \square \)

Hereafter we use \( W^k_j(t) \) to denote \( W(jh_k, t) \) and \( \overline{W}^k_j(t) \) to denote \( \frac{1}{h_k} \int_{x_{j-1/2}}^{x_{j+1/2}} W(x, t) dx \).

We now turn to estimate the speed of a sequence of extrema where the differences of the values of the solutions from the values of their limit function are uniformly bound away from zero. More precisely, we assume that for each \( k \), on the path \( M^k_{0,1} \) the inequality
\[
V_{M^k_{0,1}}(t) > w_+ + \rho \tag{5.10}
\]
holds, where \( \rho \) is a positive constant. The idea of the estimate is similar to the one which motivates the proof of theorem 5.3. But the arguments turn out to be more involved.
Let 
\[ \Omega_1 = \{(x, t) \in \mathbb{R}^2 : |x - st| < 1, \ 0 \leq t \leq 1\} \]
and
\[ \Omega_2 = \{(x, t) \in \mathbb{R}^2 : |x - st| < 1 + d, \ 0 \leq t \leq 1\} \]
where
\[ d = \frac{1}{\rho}(c_0 + 1) \quad (5.11) \]
and \(c_0\) is a positive constant to be determined. Suppose that \(x_{M_k(t)} = x_{I_k(t)}\) is a \(\varepsilon\)-path of \(M_{0,1}^k\). We have the following preliminary estimate parallel to that in lemma 5.4.

**Lemma 5.6** If \(t' \in [0,1]\) and \((x_{I_k(t')}, t') \in \Omega_1\), then \(\forall \eta \in (0,1], \exists K_2(\eta) > 0\) such that for any \(t'' \in [0,1]\), we have
\[
(x_{I_k(t'')}, t'') \in \Omega_2,
\]
and
\[
|x_{I_k(t'')} - x_{I_k(t')}| \leq \frac{1}{\rho}[C|t'' - t'| + \eta],
\]
provided that \(\varepsilon < \rho/2\) and \(k > K_2(\eta)\).

**Proof** Without loss of generality, suppose \(t'' > t'\) and \(x_{I_k(t'')} > x_{I_k(t')}\). We have
\[
\begin{align*}
\sum_{i = I_k(t')}^{I_k(t'')} & (w_+ + \frac{\rho}{2} - u_i^k(t))_+ h_k \\
& = \sum_{i = I_k(t')}^{I_k(t'')} (w_+ + \frac{\rho}{2} - \tilde{W}_i^k(t) - (u_i^k(t) - \tilde{W}_i^k(t)))_+ h_k \\
& \geq \sum_{i = I_k(t')}^{I_k(t'')} \left(\frac{\rho}{2} - (u_i^k(t) - \tilde{W}_i^k(t))\right)_+ h_k \\
& = \int_{x_{I_k(t')}}^{x_{I_k(t'')}} \left(\frac{\rho}{2} - (u_i^k(t) - \tilde{W}_i^k(t))\right)_+ dx \quad (5.12)
\end{align*}
\]
where \((a)_+ = (a + |a|)/2\). According to lemma 5.5, for any fixed \(c_0 > 0\) and any \(\eta > 0\), a \(K_2(c_0, \eta) > 0\) exists so that for \(k > K_2(c_0, \eta)\) and \(t \in [0,1]\), we have
\[
\int_{|x - st| < 1 + d} |u^k(x, t) - W(x, t)| dx < \frac{\eta}{2}
\]
where \(d\) is given by (5.11).
Next, assume that \( h_k < 1 \) so that
\[
|x_{I^k(u')}_{+1/2} - st| < \frac{3}{2} + |s|.
\]
Let \( d_1 = d - |s| - 1/2 \) and \( b = \min(x_{I^k(u')}_{+1/2} + d_1, x_{I^k(u')}_{+1/2}) \). It follows the last two inequalities that for \( k > K_2(C_0, \eta) \),
\[
\begin{align*}
&\int_{x_{I^k(u')}_{+1/2}}^{x_{I^k(u')}_{+1/2}} \left( \frac{\rho}{2} - (u^k(x, t) - W(x, t)) \right)_{+} dx \\
&\geq \int_{x_{I^k(u')}_{+1/2}}^{b} \left( \frac{\rho}{2} - (u^k(x, t) - W(x, t)) \right)_{+} dx \\
&\geq \frac{\rho}{2} \min(d_1, x_{I^k(u')} - x_{I^k(u')}) - \int_{x_{I^k(u')}_{+1/2}}^{b} |u^k(x, t) - W(x, t)| dx \\
&\geq \frac{\rho}{2} \min(d_1, x_{I^k(u')} - x_{I^k(u')}) - \frac{\eta}{2} \tag{5.13}
\end{align*}
\]
Recalling the definition of \( \epsilon \)-path and the condition (5.10), we find out by applying lemma 3.7 to \( M_{0,1}^k \) that when \( \epsilon < \rho/2 \), for each \( j = I^k(t') + 1, \ldots, I^k(t'') \), there exists a \( t_j \in [t', t''] \) with \( u_j^k(t_j) = u_{j+1}^k(t_j) > w_+ + \rho/2 \). Hence
\[
\int_{t'}^{t''} h_k |\frac{d}{dt} u_j^k(s)| ds \geq (w_+ + \frac{\rho}{2} - u_j^k(t))_{+} h_k \text{ for } t \in [t', t''] \tag{5.14}
\]
(5.12)–(5.14) imply that
\[
\begin{align*}
&\frac{\rho}{2} \min(d_1, x_{I^k(u')} - x_{I^k(u')}) - \frac{\eta}{2} \\
&< \sum_{i=I^k(u')}^{I^k(u'')} \int_{t'}^{t''} h_k |\frac{d}{dt} u_i^k(t)| dt \\
&= \sum_{i=I^k(u')}^{I^k(u'')} \int_{t'}^{t''} |g_{i+1/2}(t) - g_{i-1/2}(t)| dt \\
&\leq C(t'' - t').
\end{align*}
\]
Thus
\[
\min(d_1, x_{I^k(u')} - x_{I^k(u')}) < \frac{C(t'' - t') + \eta}{\rho}.
\]
From now on, we choose \( c_0 = C + \rho(|s| + 1/2) \) which is independent of \( k \). Then \( d_1 \geq (C(t'' - t') + \eta)/\rho \) and, therefore,
\[
x_{I^k(u')} - x_{I^k(u')} < (C(t'' - t') + \eta)/\rho
\]

\[33\]
holds provided that $\varepsilon < \rho/2$ and $k > K_2(\eta) = K_2(c_0, \eta)$. Moreover,

$$\left| x_{I_{k}(\nu')} - st'' \right| \leq \left| x_{I_{k}(\nu')} - x_{I_{k}(\nu')} \right| + \left| x_{I_{k}(\nu')} - st' \right| + \left| s(t'' - t') \right|$$

$$< d_1 + 1 + |s|$$

$$< 1 + d.$$

I.e., $(x_{I_{k}(\nu')}, t'') \in \Omega_2$. □

Based on this lemma, we may assume that $M_{k,1}^k \in \Omega_2$ for all $k$. The following theorem is our second main result on wave speed estimate.

**Theorem 5.7** If for each $k$ the path $M_{k,1}^k$ satisfies (5.10) and $x_{I_{k}(t)} = x_{M_{k}^{\epsilon}}(t)$ is an $\epsilon$-path of $M_{k,1}^k$, then

(i) $\forall \delta > 0, \exists \varepsilon_0(\delta) > 0$ and $K(\delta) > 0$ such that for $t', t'' \in [0, 1],$

$$\left| x_{I_{k}(\nu')} - x_{I_{k}(\nu')} - \int_{t'}^{t''} f(u_{I_{k}(t)}(t) - f(W_{I_{k}(t)}(t))) \frac{d}{dt} \right| < \delta$$

provided that $\varepsilon < \varepsilon_0(\delta)$ and $k > K(\delta)$.

(ii) The same conclusion holds if $u_{I_{k}(t)}(t)$ is replaced by $V_{M_{k}}(t)$ in the above inequality.

**Proof** Firstly, it is clear according to lemma 5.1 that part (ii) is a trivial consequence of part (1). We therefore focus on the proof of part (i). Let $w_{i}^{k}(t) = u_{i}^{k}(t) - W_{i}^{k}(t)$ and

$$G_{i+1/2}^{k}(t) = g_{i+1/2}^{k}(t) - f(W_{i+1/2}^{k}(t)).$$

Then

$$\frac{dw_{i}^{k}(t)}{dt} = - \frac{1}{h_k} (G_{i+1/2}^{k}(t) - G_{i-1/2}^{k}(t)).$$

Suppose that $p_k = [1/h_k] + 1$ where the function $[x]$ is the integer part of $x$:

$$[x] = \max\{m \in \mathbb{Z} : m \leq x\}.$$

Let us consider the quantity

$$S_{i}^{k}(t) = \frac{1}{p_k} \sum_{l=1}^{p_k} \sum_{q = I_{k}(t)+1}^{I_{k}(t)+l} w_{q}^{k}(t) h_k.$$
According to lemma 5.6, we may assume that all the space-time coordinates involved are contained in the compact set

\[\Omega_3 = \{(x, t) \in \mathbb{R}^2 : |x - st| < d + 3, \quad t \in [0, 1]\}\].

Applying lemma 5.5 to the set \(\Omega_3\), we see that \(\forall \eta > 0, \exists K_2(\eta) > 0\), such that the inequality

\[|S^k(t)| < \eta/2\]

holds for all \(t \in [0, 1]\), under the condition:

\[(a) \quad \varepsilon < \rho/2 \text{ and } k > K_2(\eta)\]

which we assume to be always true hereafter.

Again, divide \([0, 1]\) into \(n\) subintervals of equal length:

\[\[0, 1\] = \bigcup_{j=1}^{n} \Gamma_j \text{ where } \Gamma_j = \left[\frac{j-1}{n}, \frac{j}{n}\right].\]

Suppose \(t' \in [t_{j_0-1}, t_{j_0})\) and \(t'' \in [t_{j_1}, t_{j_1+1})\). For convenience, let \(\hat{t}_{j_0-1} = t', \hat{t}_{j_1+1} = t''\) and \(\hat{t}_j = t_j\) for each \(j = j_0, j_0 + 1, \ldots, j_1\). Next for each \(j = j_0, j_0 + 1, \ldots, j_1 + 1\). Let

\[\hat{t}_{j-1} = \tau_0 < \tau_1 < \cdots < \tau_{\nu_j} = \hat{t}_j\]

be such a partition of \([\hat{t}_{j-1}, \hat{t}_j]\) that for \(t \in (\tau_{\nu-1}, \tau_\nu)\), \(x_{t_k(t)} = x_{i_\nu}\) is a constant for each \(\nu = 1, \ldots, \nu_j\). Here, for simplicity, we have omitted the obvious dependence of \(\tau_\nu\) on \(j\) and \(k\) and the dependence of \(\nu_j\) on \(k\). The conservativity of the scheme implies

\[A = S^k_x(\hat{t}_j + 0) - S^k_x(\hat{t}_{j-1} + 0)\]

\[= \int_{\hat{t}_{j-1}}^{\hat{t}_j} (G^k_{t_k(t)+1/2}(t) - \frac{1}{p_k} \sum_{l=1}^{pk} G^k_{t_k(t)+pk+l+1/2}(t)) dt\]

\[+ \sum_{\nu=1}^{\nu_j} \sum_{q=i_{\nu-1}+1}^{i_\nu} \frac{1}{p_k} \sum_{l=1}^{pk} w^k_{q+p_k+l}(\tau_\nu) h_k - \sum_{\nu=1}^{\nu_j} \sum_{q=i_{\nu-1}+1}^{i_\nu} w^k_q(\tau_\nu) h_k\]

\[= A_1 + A_2 - A_3\]

where

\[\sum_{q=i_{\nu-1}+1}^{i_\nu} = \begin{cases} \sum_{q=i_{\nu-1}+1}^{i_\nu} & \text{if } i_\nu \geq i_{\nu-1} \\ -\sum_{q=i_{\nu-1}+1}^{i_\nu} & \text{if } i_\nu < i_{\nu-1} \end{cases}\]
Write $A_1$ as

$$A_1 = \int_{i_{j-1}}^{i_j} G_{I^k(t)+1/2}(t) dt - \int_{i_{j-1}}^{i_j} \frac{1}{p_k} \sum_{l=1}^{p_k} G_{I^l(t)+p_k+l+1/2}(t) dt$$

$$= A_1' + A_1''$$

Clearly, $|A_1'| \leq C\eta/n$. under the condition (a).

Next, rewrite $A_2$ as

$$A_2 = \sum_{q;i_{j-1}}^{i_{j-1}} \frac{1}{p_k} \sum_{l=1}^{p_k} w_{q+p_k+l}(\tau_1) h_k$$

$$+ \sum_{\nu=1}^{\nu_j} \sum_{\mu=1}^{i_\nu} \sum_{q;i_{j-1}}^{i_{j-1}} \int_{\tau_\mu}^{\tau_{\mu+1}} (G_{q+p_k+1/2}(t) - G_{q+2p_k+1/2}(t)) dt$$

$$= A_2' + A_2''$$

An exchange of the summations in $A_2''$ leads to

$$|A_2''| = \left| \frac{1}{p_k} \sum_{\nu=1}^{\nu_j} \sum_{\mu=1}^{i_\nu} \sum_{q;i_{j-1}}^{i_{j-1}} \int_{\tau_\mu}^{\tau_{\mu+1}} (G_{q+p_k+1/2}(t) - G_{q+2p_k+1/2}(t)) dt \right|$$

$$= \left| \frac{1}{p_k} \sum_{\nu=1}^{\nu_j} \sum_{\mu=1}^{i_\nu} \int_{\tau_\mu}^{\tau_{\mu+1}} (G_{q+p_k+1/2}(t) - G_{q+2p_k+1/2}(t)) dt \right|$$

$$\leq \sum_{\mu=1}^{\nu_{\nu_j}} \int_{\tau_\mu}^{\tau_{\mu+1}} \sum_{q;i_{j-1}}^{i_{j-1}} |G_{q+p_k+1/2}(t) - G_{q+2p_k+1/2}(t)| h_k |dt$$

$$\leq C\eta/n.$$  

On the other hand

$$|A_2'| \leq \eta |x_{I^k(i_j)} - x_{I^k(i_{j-1})}|$$

$$\leq \frac{\eta}{\rho} \left( \frac{C}{n} + \eta \right).$$

Summering these estimates, we find that

$$|A_1' - A_3| = \left| \int_{i_{j-1}}^{i_j} G_{I^k(t)+1/2}(t) dt - \sum_{\nu=1}^{\nu_j} \sum_{q;i_{j-1}}^{i_{j-1}} w_{q}(\tau_\nu) h_k \right|$$

$$\leq \eta + \frac{C\eta}{n\rho} + \frac{\eta^2}{\rho}$$

(5.15)
\[ \leq \frac{C\eta}{\rho}. \]

Decompose \( A_3 \) into the following form:

\[
A_3 = \sum_{\nu=1}^{\nu_j} \sum_{q_1i_\nu-1}^{i_\nu} u_q^k(\tau_\nu)h_k - \sum_{\nu=1}^{\nu_j} \sum_{q_1i_\nu-1}^{i_\nu} W_q^k(\tau_\nu)h_k = A'_3 - A''_3 \tag{5.16}
\]

Since \( x_{i_\nu} \) and \( x_{i_\nu-1} \in P_{M^\tau}(\tau_\nu) \), lemma 5.2 can be applied to yield

\[
A'_3 = \sum_{\nu=1}^{\nu_j} (x_{i_\nu} - x_{i_\nu-1})u_{i_\nu}(\tau_\nu)
\]

\[
= (x_{t^k_L(i_j)} - x_{t^k_L(i_{j-1})})u_{t^k_L(i_j)}(i_j) + R_j \tag{5.17}
\]

where

\[
|R_j| \leq \gamma_j \max_{1 \leq \nu \leq \nu_j-1} |x_{i_\nu} - x_{i_\nu-1}|. \tag{5.18}
\]

Since \( W(x, t) \) is a weak solution of the conservation law,

\[
A''_3 = \sum_{q=0}^{i_\nu} W_q^k(\hat{t}_{j-1})h_k - \int_{\hat{t}_{j-1}}^{\hat{t}_j} f(W_{t^k_L(i_j)+1/2}(t))dt + \int_{\hat{t}_{j-1}}^{\hat{t}_j} f(W_{t^k_L(t)+1/2}(t))dt. \tag{5.19}
\]

Substituting (5.16)–(5.19) into (5.15) and applying lemma 5.6 we get

\[
\left| \int_{\hat{t}_{j-1}}^{\hat{t}_j} [g_{t^k_L(t)+1/2}(t) - f(W_{t^k_L(i_j)+1/2}(t))]dt \right.
\]

\[
-[(x_{t^k_L(i_j)} - x_{t^k_L(i_{j-1})})u_{t^k_L(i_j)}(i_j) - \sum_{q=t^k_L(i_{j-1})} W_q^k(\hat{t}_{j-1})h_k]
\]

\[
< \left( \frac{\eta}{\rho} + \frac{C\eta}{\rho} \right)
\]

\[
\leq \frac{C \gamma_j}{\rho \eta} + \frac{C \gamma_j}{\rho \eta}
\]

Applying lemma 5.1 to the first term of the integration in the above inequality, we get

\[
\left| \int_{\hat{t}_{j-1}}^{\hat{t}_j} [f(u_{t^k_L(t)}(t)) - f(W_{t^k_L(i_j)+1/2}(t))]dt \right.
\]

\[
-[(x_{t^k_L(i_j)} - x_{t^k_L(i_{j-1})})u_{t^k_L(i_j)}(i_j) - \sum_{q=t^k_L(i_{j-1})} W_q^k(\hat{t}_{j-1})h_k]
\]

\[
\leq \frac{C \gamma_j}{\rho \eta} + \frac{C \gamma_j}{\rho \eta} + C \frac{\epsilon}{\rho \eta} + \gamma_j h_k, \tag{5.20}
\]

37
and

\[ \int_{t_{i-1}}^{t_i} \left[ f(u^k_{L^k(i_j)}(\hat{t}_{j})) - f(W^k_{L^k(i_j)+1/2}(t)) \right] dt \]

\[-[(x_{L^k(i_j)} - x_{L^k(i_j)})u^k_{L^k(i_j)}(\hat{t}_{j}) + \sum_{q:L^k(i_{j-1})} W^k_q(\hat{t}_{j-1})h_k] \]

\[ \leq \frac{C \gamma_j}{\rho n} + \frac{C}{\rho} \eta + C \frac{\xi}{n} + \gamma_j h_k. \]  

(5.21)

Since \( u^k_{L^k(i_j)}(\hat{t}_{j}) > w_+ + \rho/2, w_- \leq W^k_{L^k(i_j)+1/2}(t) \leq w_+, s = (f(w_-) - f(w_+))/(w_- - w_+) \) and \( f'' > c \), we can write

\[ f(u^k_{L^k(i_j)}(\hat{t}_{j})) - f(W^k_{L^k(i_j)+1/2}(t)) = (s + \sigma_j(t))\left( u^k_{L^k(i_j)}(\hat{t}_{j}) - W^k_{L^k(i_j)+1/2}(t) \right) \]

where \( \sigma_j \geq c_0/4 \). Hence, there exists a constant \( \sigma_j \geq c_0/4 \) with

\[ \int_{t_{i-1}}^{t_i} \left[ f(u^k_{L^k(i_j)}(\hat{t}_{j})) - f(W^k_{L^k(i_j)+1/2}(t)) \right] dt = (s + \sigma_j)\delta W_j(\hat{t}_{j} - \hat{t}_{j-1}) \]  

(5.22)

where

\[ \delta W_j = \int_{t_{i-1}}^{t_i} \frac{u^k_{L^k(i_j)}(\hat{t}_{j}) - W^k_{L^k(i_j)+1/2}(t)}{\hat{t}_{j} - \hat{t}_{j-1}} dt. \]

On the other hand,

\[ (x_{L^k(i_j)} - x_{L^k(i_{j-1})})u^k_{L^k(i_j)}(\hat{t}_{j}) - \sum_{q:L^k(i_{j-1})} W^k_q(\tau_0)h_k \]

\[ = \int_{x_{L^k(i_{j-1})} + \frac{1}{2}}^{x_{L^k(i_j)} + \frac{1}{2}} [u^k_{L^k(i_j)}(\hat{t}_{j}) - W(x, \hat{t}_{j-1})]dx \]

\[ = \int_{x_{L^k(i_{j-1})} + \frac{1}{2}}^{x_{L^k(i_j)} + \frac{1}{2}} [u^k_{L^k(i_j)}(\hat{t}_{j}) - W(x, \hat{t}_{j-1})]dx \]

\[ + \int_{x_{L^k(i_j)} + \frac{1}{2}}^{x_{L^k(i_{j-1})} + \frac{1}{2}} [u^k_{L^k(i_j)}(\hat{t}_{j}) - W(x, \hat{t}_{j-1})]dx \]

\[ = [x_{L^k(i_j)} - x_{L^k(i_{j-1})} - s(\hat{t}_{j} - \hat{t}_{j-1})] \delta W_j + s(\hat{t}_{j} - \hat{t}_{j-1}) \delta W_j \]

where \( \delta W_j \) is the mean of the integrand in the interval between \( x_{L^k(i_{j-1})} + 1/2 \) and \( x_{L^k(i_j)} + 1/2 - s(\hat{t}_{j} - \hat{t}_{j-1}) \). Therefore

\[ |[x_{L^k(i_j)} - x_{L^k(i_{j-1})} - s(\hat{t}_{j} - \hat{t}_{j-1})] \delta W_j - \sigma_j(\hat{t}_{j} - \hat{t}_{j-1}) \delta W_j| \]

\[ \leq \frac{C \gamma_j}{\rho n} + \frac{C}{\rho} \eta + C \frac{\xi}{n} + \gamma_j h_k \]

38
or

\[ |x_{t^k(i_j)} - x_{t^k(i_{j-1})} - (s + \bar{\sigma}_j \frac{\delta W_j}{\delta W_j} (\hat{t}_j - \hat{t}_{j-1}))| \leq \frac{C \gamma_j}{\rho^2 n} + \frac{C \varepsilon}{\rho n} + \frac{2}{\rho} \gamma_j h_k. \]  \hspace{1cm} (5.23)

For any \( i \) and \( l \), if

\[ \sigma_{i,l}(\hat{t}_{i+l-1} - \hat{t}_{i-1}) = \sum_{j=0}^{l-1} \sigma_{i+j} \frac{\delta W_{i+j}}{\delta W_{i+j}} (\hat{t}_{i+j} - \hat{t}_{i+j-1}), \]

then

\[ \sigma_{i,l} \geq c_1 \rho^2 \]

where the positive constant \( c_1 = c/(8B) \). Summing the inequality (5.23) when \( j \) runs over the range \( i \leq j \leq i + l - 1 \), we get

\[ |x_{t^k(i_{i+l-1})} - x_{t^k(i_{i-1})} - (s + \sigma_{i,l})(\hat{t}_{i+l-1} - \hat{t}_{i-1})| \leq \frac{C}{\rho^2 n} + \frac{C l \varepsilon}{\rho n} + \frac{C}{\rho} h_k. \]

Hence

\[ x_{t^k(i_{i+l-1})} - x_{t^k(i_{i-1})} \geq (s + c_1 \rho^2)(\hat{t}_{i+l-1} - \hat{t}_{i-1}) - \left( \frac{C}{\rho^2 n} + \frac{C l \varepsilon}{\rho n} + \frac{C}{\rho} h_k \right). \]

Since \( \hat{t}_{i+l-1} - \hat{t}_{i-1} > (l - 2)/n \), we can choose an \( l_0 \) independent of \( k \) so that \( C/(n \rho^2) < c_1 \rho^2 (\hat{t}_{i+l-1} - \hat{t}_{i-1})/8 \) for \( l \geq l_0 \). Then for any fixed \( n \), we can choose \( \eta_1(n) \), \( \varepsilon_0 \) and \( \tilde{h}(n) \) so that

\[ \frac{C l \eta_1(n)}{\rho^2} \leq \frac{c_1 \rho^2}{8} (\hat{t}_{i+l-1} - \hat{t}_{i-1}), \]

\[ \frac{C l \varepsilon}{\rho n} \leq \frac{c_1 \rho^2}{8} (\hat{t}_{i+l-1} - \hat{t}_{i-1}) \]

and

\[ \frac{C \tilde{h}(n)}{\rho} \leq \frac{c_1 \rho^2}{8} (\hat{t}_{i+l-1} - \hat{t}_{i-1}) \]

for \( l_0 \leq l \leq 2l_0 \). Let \( \varepsilon_1 = \min(\rho/2, \varepsilon_0) \), and let \( K_3(\eta, h) \geq K_2(\eta) \) be so large that \( h_k < h \) for \( k \geq K_3(\eta, h) \). We then have

\[ x_{t^k(i_{i+l-1})} - x_{t^k(i_{i-1})} \geq (s + \frac{c_1}{2} \rho^2)(\hat{t}_{i+l-1} - \hat{t}_{i-1}) \]
for \( l_0 \leq l \leq 2l_0 \) under the condition

\[ \varepsilon < \varepsilon_1 \text{ and } k > K_3(\eta_1(n), \hat{h}(n)). \] (5.24)

This implies that whenever \( t' \) and \( t'' \in [0, 1] \) satisfy that \( [nt''] - [nt'] \geq l_0 \), the inequality

\[ x_{t_k(t'')} - x_{t_k(t')} \geq (s + \frac{c_1 \rho^2}{2})(t'' - t') \]

holds provided that the condition (b) is fulfilled.

**Remark 5.8** From the foregoing arguments, it is easy to see that, if \( (x_{t_k(t)} + 1/2, t) \)
crosses the line \( x = st \) in \([\hat{t}_{i-1}, \hat{t}_i] \), then for \( t \in [\hat{t}_{i-1}, \hat{t}_i] \), the points \((x_{t_k(t)} + 1/2, t \pm (\hat{t}_j - \hat{t}_{j-1})) \)
are all on the right side of the line \( x = st \) if \( j - i > l_0 \); they are all on the left if \( i - j > l_0 \),
provided that all the time variables involved are in \([0, 1]\) and the conditions: \( h_k < 1/n \) and
\( c_1 l_0 \rho^2/2 > s + 1 \) as well as (b) are satisfied. From now on, we assume that these conditions
are indeed satisfied.

Returning to inequality (5.20) and letting

\[ (x_{t_k(t)} - x_{t_k(t-1)})\overline{W}_{j,k} = \sum_{q \in \mathbb{Z}} W_q(t_{j-1})h_k, \]

we have

\[ |x_{t_k(t)} - x_{t_k(t-1)} - \int_{t_{i-1}}^{t_i} \frac{f(u_{t_k(t)}^k(t)) - f(W_{t_k(t)}^k(t))}{u_{t_k(t)}^k(t) - \overline{W}_{j,k}} dt| \] (5.25)

\[ \leq \frac{C \gamma_j}{\rho^2} n + \frac{C \varepsilon}{\rho^2} \eta + \frac{C}{\rho} \eta + \frac{2}{\rho} \gamma_j h_k \]

since \( u_{t_k(t)}^k(t) - \overline{W}_{j,k} > \rho/2 \).

From remark 5.8, we clearly have

\[ \left| \frac{f(u_{t_k(t)}^k(t)) - f(W_{t_k(t)}^k(t))}{u_{t_k(t)}^k(t) - \overline{W}_{j,k}} - \frac{f(u_{t_k(t)}^k(t)) - f(W_{t_k(t)}^k(t))}{u_{t_k(t)}^k(t) - W_{t_k(t)}^k(t)} \right| \]

\[ \leq \begin{cases} \frac{C}{\rho} & |i - j| \leq l_0 \\ \frac{C \gamma_j}{\rho^2} & |i - j| > l_0 \end{cases} \]

Hence

\[ |x_{t_k(t)} - x_{t_k(t-1)} - \int_{t_{i-1}}^{t_i} \frac{f(u_{t_k(t)}^k(t)) - f(W_{t_k(t)}^k(t))}{u_{t_k(t)}^k(t) - W_{t_k(t)}^k(t)} dt| \]
\[
\begin{align*}
\leq \begin{cases}
\frac{C}{\rho^2} \frac{n_i}{n} + \frac{C}{\rho} \eta + \frac{C}{\rho} \frac{1}{n} \gamma_j h_k, & |i - j| \leq l_0 \\
\frac{C}{\rho^2} \frac{n_i}{n} + \frac{C}{\rho} \eta + \frac{C}{\rho} \frac{1}{n} \gamma_j h_k, & \text{otherwise}
\end{cases}
\end{align*}
\]

Summing the inequality for $j$ over the range $j_0 \leq j \leq j_1 + 1$, we get
\[
|x_{t^*}'(v') - x_{t^*}'(v') - \int_{t'}^{v'} \frac{f(u_{t^*}'(t)) - f(W_{t^*}'(t))}{u_{t^*}'(t) - W_{t^*}'(t)} dt| 
\leq \frac{C}{n \rho^2} + \frac{C h_k}{\rho} + \frac{C n \eta}{\rho^2} + \frac{C \varepsilon}{\rho} + \frac{2l_0 + 1}{n \rho} C.
\]

Now, for any $\delta > 0$, Fix an $n$ so that $\frac{C}{n} \left( \frac{1}{\rho^2} + \frac{2l_0 + 1}{\rho} \right) < \frac{\delta}{4}$, then choose $\tilde{h} > 0$ and $\eta(\delta) > 0$ so small that $\frac{C}{\rho^2} (\tilde{h} \rho + n \eta) < \frac{\delta}{2}$. Finally, set $\varepsilon_0(\delta) = \min(\varepsilon_1, \frac{\delta}{4C})$ and $K(\delta) = K_3(\eta(\delta), \tilde{h})$. We then have
\[
|x_{t^*}'(v') - x_{t^*}'(v') - \int_{t'}^{v'} \frac{f(u_{t^*}'(t)) - f(W_{t^*}'(t))}{u_{t^*}'(t) - W_{t^*}'(t)} dt| < \delta
\]
when $k > K(\delta)$, and $\varepsilon < \varepsilon_0(\delta)$. \qed

§6 Non-linear wave analysis.

Define
\[
\Omega_4 = \{(x, t) : |x - st| \leq 1 + |s|, 0 \leq t \leq 1\},
\]
and
\[
\Omega_5 = \{(x, t) : |x - st| \leq 2 + |s|, 0 \leq t \leq 1\}.
\]

Suppose that $\{u^k\} \in \Psi$, $M_{0,1}^k$ and $N_{0,1}^k$ are the paths of a maximum and a minimum of $u^k$ respectively for each $k$. Assume that $V_{M^k}(t) > w^- + \rho$ and $V_{N^k}(t) < w^- - \rho$ hold for $t \in [0, 1]$. We have the following lemma similar to lemma 5.6.

**Lemma 6.1** Suppose that $\{u^k\}$, $M_{0,1}^k$ and $M_{0,1}^k$ are as above, and that $x_{t^*}'(t)$ and $x_{J^k(t)}$ are $\varepsilon$-paths of $M_{0,1}^k$ and $N_{0,1}^k$ respectively. Then there is a constant $d_2 > 2$ independent of $k$ and a constant $K_2$ such that

1. If $(x_{t^*}'(t'), t') \in \Omega_5$ for any $t' \in [0, 1]$, then $x_{t^*}'(t) - st > -(|s| + d_2)$ for $t \in [0, 1]$ provided that $\varepsilon < \rho/2$ and $k > K_3$.

2. If $(x_{J^k(t)}', t') \in \Omega_5$ for any $t' \in [0, 1]$, then $x_{J^k(t)}' - st < |s| + d_2$ for $t \in [0, 1]$ provided that $\varepsilon < \rho/2$ and $k > K_3$. 

41
Proof  The proof is similar to that of lemma 5.6 and is omitted. \qed

To use the lemma conveniently, let us define

$$\Omega_6 = \{(x,t): |x-st| \leq d_2 + |s|, 0 \leq t \leq 1\}.$$  

In this section, we will extensively use similarity transforms and displacements and the similarity invariant property of the schemes. Using the notation introduced in section 4, we say that a map $Q$ of $\mathcal{B}$ into the set of the subsets of $\mathbb{R} \times \mathbb{R}_+$ is similarity invariant if

$$Q(T_{a,b}^h u^h) = (S_{a,b}^u)^{-1}(Q u^h).$$

We then have

**Lemma 6.2** Assume that $\{u^{h_k}\}_{k=1}^\infty \in \Psi$, and that $T$ and $\delta$ be two positive constants.

(i) Suppose $\{K_j\}_{j=1}^\infty$ is a sequence of compact subsets in $\mathbb{R} \times \mathbb{R}_+$ with

$$K_1 \subset K_2 \subset \cdots.$$  

If for each sufficiently large $k$, there exists a $t_k \in [0,T]$ such that the ball

$$B_\delta((st_k,t_k)) \subseteq Q u^{h_k},$$

then $\Psi$ contains a sequence $\{u^{H_j}\}_{j=1}^\infty$ such that

$$Q u^{H_j} \supseteq K_j$$

holds for each $j$.

(ii) Suppose $\tau_1 < \tau_2 < \cdots$ is a sequence of positive numbers. If for each sufficiently large $k$, there exists a $t_k \in [0,T]$ such that the strip

$$\{t_k \leq t \leq t_k + \delta\} \subseteq Q u^{h_k},$$

then $\Psi$ contains a sequence $\{u^{H_j}\}_{j=1}^\infty$ such that

$$Q u^{H_j} \supseteq \{0 \leq t \leq \tau_j\}$$

holds for each $j$.  

42
Proof. The proofs of the two parts are almost identical. We therefore just prove part (i). As in the proof of theorem 4.6, we choose a sequence of positive numbers \( \{\varepsilon_i\}_{i=1}^{\infty} \) such that

\[
(S^{-1}_{a,b})^{-1}B_\varepsilon((a,b)) \supseteq K_i
\]

for any \((a,b) \in \mathbb{R} \times \mathbb{R}_+\). By lemma 4.8, for any fixed \(i\) the sequence \(\{T_{st_j,t_j}^{\varepsilon_i}u^{h_j}\}_{j=1}^{\infty} \in \Psi\). Hence one can choose \(j_i\) such that

\[
\int_{B_2((-0,0))_+} |T_{st_j,t_j}^{\varepsilon_i}u^{h_j} - W|dxdt < 2^{-i}
\]

where \(B_2((-0,0))_+ = \{(x,t) \in B_2((-0,0)) : t \geq 0\}\). The proof is completed by choosing \(u^{H_i} = T_{st_j,t_j}^{\varepsilon_i}u^{h_j}\).

First, we consider a family of strong oscillations as follows: Fix an \(u \in B\). Suppose that \(\rho\) be a positive constant. Let \(\alpha = w_+ + \rho/2\) and \(\beta = w_+ + \rho\). For each \(t > 0\). We divide the set of all subscripts \(j\) with \(u_j(t) > \beta\) or \(u_j(t) < \alpha\) into disjoint subsets:

\[
\{j : u_j(t) > \beta\} \cup \{j : u_j(t) < \alpha\} = \bigcup_{m=1}^{l(t)} J_m(t)
\]

under the conditions:

(a) If \(p \in J_m(t)\) and \(q \in J_{m+1}(t)\), then \(p < q\).

(b) If \(p \in J_m(t)\) and \(u_p(t) > \beta\), then \(u_q(t) > \beta\) for any \(q \in J_m(t)\).

(c) If \(p \in J_m(t)\) and \(u_p(t) > \beta\), then \(u_q(t) < \alpha\) for any \(q \in J_{m-1}(t) \cup J_{m+1}(t)\).

Definition 6.3  (1) A sequence \(\{x_{im(t)}(t)\}_{m=1}^{l(t)}\) is called an \((\alpha, \beta)\)-oscillation if \(i_m(t) \in J_m(t)\) for \(m = 1, \cdots, l(t)\) and \(x_{im(t)}\) is a maximum when \(u_{im(t)}(t) > \beta\); it is a minimum when \(u_{im(t)}(t) < \alpha\).

(2) A belt \([t', t'']\) is said to be \(l\)-regular with respect \(u(x,t)\) if there is a sequence of paths of local extrema: \(\{E_{t',t'}^m\}_{m=1}^l\) such that for any \(t \in [t',t'']\), \(\{x_{vm(t)}(t)\}_{m=1}^l\) is a \((\alpha, \beta)\)-oscillation if \(x_{vm}(t) \in P_{E^m(t)}\) for each \(m = 1, \cdots, l\).

It is easy to see that \(l(t)\) is a decreasing function of \(t\). For one can use theorem 3.2 to construct a path of an extremum, \(E_{0,t}^m\) for each \(m\), such that \(\{i : x_i \in P_{E^m(t)}\} \subseteq J_m(t)\). Then lemma 3.5 implies that for any \(s \in [0,t]\), \(\{i : x_i \in P_{E^m(s)}\}\) belongs to some \(J_{m'}(s)\),
and different \( m \) correspond to different \( m' \). Hence there is a finite partition of \([0, \infty)\):

\[
0 = t'_0 < t'_1 < \cdots < t'_{N'} = \infty
\]

such that \( l(t) = l_\kappa \) is a constant in each subinterval \([t'_{\kappa - 1}, t'_\kappa)\) for \( \kappa = 1, \cdots, N' \). And for any \( \delta > 0 \), the belt \([t'_{\kappa - 1}, t'_\kappa - \delta] \) is \( l_\kappa \)-regular with respect \( u(x, t) \). Obviously, \( N' \leq l_1 \leq 2B/\rho + 1 \). We can let \( \delta \) be so small that there exist at least one \( \kappa \) such that the length of \([t'_{\kappa - 1}, t'_\kappa - \delta] \cap [0, 1] \) larger than \( 1/(N' + 1) \). For each integer \( n \), we define the map \( Q^n \) of \( \mathcal{B} \) into the set of the subsets of \( \mathbb{R} \times \mathbb{R}_+ \) as follows: For any \( u \in \mathcal{B} \), \( Q^n(u) \) is the union of all the closed belt \([t', t'']\) which is \( n \)-regular with respect \( u \). Clearly, \( Q^n \) is similarity invariant.

Therefore, lemma 6.2 implies the following

**Proposition 6.4** If \( \Psi \neq \emptyset \), then there is an integer \( l \) for which \( \Psi \) contains a sequence \( \{u^k(x, t)\}_{k=1}^\infty \) such that the belt \([0, 1]\) is \( l \)-regular with respect to every member in the sequence. We call such a sequence \( l \)-sequence

Let \( \{u^k(x, t)\}_{k=1}^\infty \) be an \( l \)-sequence. For each \( k \), let \( \{E_{0,1}^{k,m}\}_{m=1}^l \) be a collection of the paths of extrema such that \( \{x_{\nu_k,m(t)}\}_{m=1}^l \) is an \((\alpha, \beta)\)-oscillation if

\[
x_{\nu_k,m(t)} \in P_{E_k,m}(t)
\]

for each \( m = 1, \cdots, l \). With a misuse of the subscript \( m \) we denote the subset of the paths of the maxima of \( \{E_{0,1}^{k,m}\}_{m=1}^l \) by \( \{M_{0,1}^{k,m}\}_{m=1}^{l(1)} \) for each \( k \). Here \( l(1) \) is independent of \( k \), which can be achieved by selecting a subsequence of \( \{u^k(x, t)\} \)(see lemma 4.8). For any \( \varepsilon > 0 \), \( k \geq 1 \) and \( m \leq l(1) \), suppose \( x_{l^*,t^*,m}(t) \) is an \( \varepsilon \)-path of \( M_{0,1}^{k,m} \).

Without any harm to our argument, we assume that all \( M_{0,1}^{k,m} \) and \( (x_{l^*,t^*,m(t)}, t) \) belong to \( \Omega_2 \). Since \( \{u^k(x, t)\} \in \Psi \), theorem 5.7 implies that given any \( \eta > 0 \), one can choose \( \varepsilon_0(\eta) \) and \( K(\eta) \) so that whenever \( \varepsilon < \varepsilon_0(\eta) \) and \( k > K(\eta) \), the inequality

\[
x_{l^*,t^*,m(t'')} - x_{l^*,t^*,m(t')} > \int_{t'}^{t''} f(V_{M^*,m}(t)) - f(W_{l^*,t^*,m}(t)) \frac{dt}{V_{M^*,m}(t) - W_{l^*,t^*,m}(t)} - \eta
\]

holds for \( t' \) and \( t'' \in [0, 1] \) with \( t' \leq t'' \) and \( m \leq l(1) \). Hence

\[
(x_{l^*,t^*,m(t'')} - s_{t''}) - (x_{l^*,t^*,m(t')} - s_{t'}) > \frac{c\rho}{2}(t'' - t') - \eta.
\]

44
Let \( \eta = c\rho^2/[16(B + \rho)] \), then for \( t'' - t' \geq \rho/[4(B + \rho)] \),

\[
(x_{t^*_{k,m}(t''')} - st'') - (x_{t^*_{k,m}(t''')} - st') > \eta
\]

holds if \( \varepsilon < \varepsilon_0(\eta) \) and \( k > K(\eta) \). Hence, if \( |x_{t^*_{k,m}(t(m))} - st^{(m)}| < \eta/4 \), then

\[
|x_{t^*_{k,m}(t)} - st| > 3\eta/4, \quad \text{for} \quad |t - t^{(m)}| \geq r(\rho) = \rho/[4(B + \rho)].
\]

According to lemma 5.5, we may let \( K(\eta) \) be so large and \( \varepsilon_0(\eta) \) be so small that when \( \varepsilon < \varepsilon_0(\eta) \) and \( k > K(\eta) \),

\[
|x_i - x_{t^*_{k,m}(t)}| < \eta/2 \quad \text{for each} \quad i \in J_m(t) \quad \text{and} \quad t \in [0, 1].
\]

Hence

\[
|x_i - st| > \eta/4 \quad \text{for} \quad |t - t^{(m)}| \geq r(\rho).
\]

Let

\[
D_m = \{t \in [0, 1] : |t - t^{(m)}| < r(\rho)\}.
\]

Then, \([0, 1] \setminus (\cup_m D_m)\) is divided by these \( D_m \) into at most \( l^{(1)} + 1 \) intervals. Since

\[
\mu([0, 1] \setminus (\cup_m D_m)) \geq 1 - \rho l^{(1)}/[2(B + \rho)]
\]

and

\[
l_0 \leq B/\rho + 1,
\]

at least one of the intervals has a length larger than or equal to \( \rho/[2(B + 2\rho)] \). Suppose the interval is \(|t - t_k| \leq r_k/2\) and let

\[
S_k = \{(x, t) : |x - st| < \eta/4, |t - t_k| \leq r_k/2\}.
\]

Then

\[
u^k(x, t) \leq w_+ + \rho, \quad \text{for} \quad (x, t) \in S_k.
\]

Let \( Q_\rho \) be the map of \( \mathcal{B} \) into the set of the subsets of \( \mathbb{R} \times \mathbb{R}_+ \) such that for any numerical solution \( u(x, t) \in \mathcal{B} \),

\[
Q_\rho(u) = \{(x, t) \in \mathbb{R} \times \mathbb{R}_+ : u(x, t) \leq w_+ + \rho\}.
\]

45
Obviously, $Q_\rho$ is similarity invariant and $Q_\rho(u^k) \supseteq S_k$. Lemma 6.2 implies that $\Psi$ contains a sequence, still denoted by $\{u^k(x, t)\}$, such that

$$u^k(x, t) \leq w_+ + \rho$$

for $(x, t) \in \Omega_6$. By similar arguments, we may also assume that

$$u^k(x, t) \geq w_- - \rho \quad \text{for} \quad (x, t) \in \Omega_6.$$

Hereafter, we use $B_1$ to denote the subset of those $u(x, t) \in B$ such that

$$w_- - \rho \leq u^k(x, t) \leq w_+ + \rho$$

holds for $(x, t) \in \Omega_6$, and for each $k$, and we use $\Psi_1$ to denote the subset of $\Psi$ whose elements are subsets of $B_1$.

Next, let us analyze the waves of the numerical solutions in $B_1$. First, we fix an $u(x, t) \in B_1$ and let $\rho_0 = 2\rho$.

**Definition 6.5.** Let $\rho_0$ be a positive constant. Fix a $t > 0$. A sequence of integers $R_{\rho_0} = \{i_m(t)\}_{m=1}^{n(t)+1}$ for which

$$i_1(t) < \cdots < i_{n(t)+1}(t)$$

is called a $\rho_0$-oscillation of $u(t)$ if the conditions:

(a) All $x_{i_m(t)}$, $m = 1, \cdots, n(t) + 1$ are extrema of $u(t)$.

(b) For $m \leq n(t)$, if $x_{i_m(t)}$ is a maximum (minimum), then $x_{i_{m+1}(t)}$ is a minimum (maximum).

(c) For $m = 1, \cdots, n(t)$,

$$|u_{i_{m+1}(t)}(t) - u_{i_m(t)}(t)| > \rho_0.$$

hold, and

(d) No sequence of more than $n(t) + 1$ integers satisfies all the conditions (a) – (c).

If $|u_i(t) - u_j(t)| \leq \rho_0$ holds for all $i, j$, we set $R_{\rho_0} = \emptyset$ and $n(t) = 0$.

**Lemma 6.6.** Suppose $n(t) > 0$, if $\{\nu_m\}_{m=1}^{n(t)+1}$ is another $\rho_0$-oscillation, then

(1) $u_{\nu_{m+1}(t)} - u_{\nu_m(t)}$ and $u_{i_{m+1}(t)}(t) - u_{i_m(t)}(t)$ have same sign for each $m = 1, \cdots, n(t)$.
(2) For \( m = 1, \cdots, n(t) + 1 \), let \( i_{m-1}(t) < \mu_m < i_{m+1}(t) \) be such that

\[
u_{\mu_m}(t) = \max_{i_{m-1}(t) < j < i_{m+1}(t)} u_j(t)\]

when \( x_{i_m(t)} \) is a maximum; or

\[
u_{\mu_m}(t) = \min_{i_{m-1}(t) < j < i_{m+1}(t)} u_j(t)\]

when \( x_{i_m(t)} \) is a minimum. Then \( \mu_{m-1} < \nu_m < \mu_{m+1} \). Here we let \( i_0(t) = -\infty \) and \( i_{n(t)+2}(t) = +\infty \).

**Proof** (1): It suffices to prove that \( x_{i_1(t)} \) and \( x_{\nu_1} \) are both maxima or both minima. Without loss of generality, suppose \( i_2(t) > \nu_2 \) and \( i_1(t) \) is a maximum. Assume \( \nu_1 \) is a minimum, there may be 3 cases: i) \( \nu_1 < \nu_2 < i_1(t) \), ii) \( \nu_1 < i_1(t) < \nu_2 < i_2(t) \) and iii) \( i_1(t) < \nu_1 < \nu_2 < i_2(t) \). In all the cases, if \( u_{\nu_2}(t) \geq u_{i_1(t)}(t) \) then

\[\{\nu_1, \nu_2, i_2(t), \cdots, i_{n(t)+1}(t)\}\]

satisfies (a), (b) and (c) and therefore contradicts (d). If \( u_{\nu_2}(t) < u_{i_1(t)}(t) \), in the cases i) and ii),

\[\{\nu_1, i_1(t), i_2(t), \cdots, i_{n(t)+1}(t)\}\]

satisfies (a), (b) and (c) and therefore contradicts (d); in the case iii),

\[\{i_1(t), \nu_1, \nu_2, \cdots, \nu_{n(t)+1}\}\]

satisfies (a), (b) and (c) and therefore contradicts (d). This proves (1).

(2): Without loss of generality, let both \( x_{\nu_1} \) and \( x_{i_{m}(t)} \) be maxima and \( \mu_{m-1} < \nu_1 < \mu_{m+1} \). It suffices to show that \( l = m \). We just show that the inequality \( l > m \) leads to a contradiction. The inequality \( l < m \) is excluded similarly.

Actually, if \( u_{\nu_1}(t) \geq u_{i_{m}(t)}(t) \), then

\[\{\nu_1, \cdots, \nu_1, \mu_{m+1}, i_{m+2}(t), \cdots, i_{n(t)+1}(t)\}\]

satisfies (a), (b) and (c) and therefore contradicts (d); if \( u_{\nu_1}(t) < u_{i_{m}(t)}(t) \) and \( \nu_{l-1} < i_{m}(t) \), then

\[\{\nu_1, \cdots, \nu_{l-1}, i_{m}(t), i_{m+1}(t), \cdots, i_{n(t)+1}(t)\}\]
satisfies (a), (b) and (c) and therefore contradicts (d); if \( u_m(t) < u_{i_m(t)}(t) \) and \( \nu_{i-1} > i_m(t) \), then \( u_{\nu_{i-1}}(t) \geq u_{\mu_{m+1}}(t) \) by the definition of \( \mu \), thus \( u_{\nu_{i-1}}(t) - u_{\mu_{m+1}}(t) > \rho_0 \), hence

\[
\{ \nu_1, \cdots, \nu_{i-1}, \nu_i, \mu_{m+1}, i_m(t), i_{m+2}(t), \cdots, i_n(t+1) \}
\]
satisfies (a), (b) and (c) and therefore contradicts (d). \( \square \)

**Lemma 6.7** \( n(t) \) is a decreasing function of \( t \).

**Proof** This is an easy consequence of theorem 3.2 and lemma 3.5.

**Remark 6.8** One can trace the \( \rho_0 \)-oscillations in the following way: From lemma 6.7, there is clearly a finite partition of \( [0, \infty) \):

\[
0 = t_0 < t_1 < \cdots < t_N = \infty
\]
such that \( n(t) + 1 = n_\kappa \) is a constant on each subinterval \([t_{\kappa-1}, t_\kappa]\) for \( \kappa = 1, \cdots, N \). And for any \( \delta > 0 \) one can find a sequence of paths of local extrema: \( \{ E_{t_{\kappa-1}, t_\kappa-\delta}^m \}_{m=1}^{n_\kappa} \) such that for any \( t \in [t_{\kappa-1}, t_\kappa - \delta] \), \( \{ x_{\nu_m} \}_{m=1}^{n_\kappa} \) is a \( \rho_0 \)-wave if \( x_{\nu_m} \in P_{E}^m(t) \) for each \( m = 1, \cdots, n_\kappa \).

In fact, \( \forall \delta \in (0, t_\kappa - t_{\kappa-1}) \), since \( n(t_\kappa - \delta) + 1 = n_\kappa \), there exists a \( \rho_0 \)-oscillation of \( u(t_\kappa - \delta) \):

\[
R_{\rho_0} = \{ i_m(t_\kappa - \delta) \}_{m=1}^{n_\kappa}.
\]

By definition, \( x_{i_m(t_\kappa - \delta)} \) and \( x_{i_{m-1}(t_\kappa - \delta)} \) are two opposite extrema, and \( |u_{i_m(t_\kappa - \delta)}(t_\kappa - \delta) - u_{i_{m+1}(t_\kappa - \delta)}(t_\kappa - \delta)| > \delta \). Theorem 3.2 enables us to find the path of an extremum, \( E_{t_{\kappa}, t_\kappa-\delta}^m \) for each \( m = 1, 2, \cdots, n_\kappa \) and to choose any \( x_{i_m(t_\kappa - \delta)} \in P_{E}^m(t_\kappa - \delta) \). Lemma 3.5 then implies that for any \( t \in [t_{\kappa-1}, t_\kappa - \delta] \), \( \{ x_{\nu_m(t)} \}_{m=1}^{n_\kappa} \) is a \( \rho_0 \)-oscillation if \( x_{\nu_m(t)} \in P_{E}^m(t) \) for each \( m = 1, 2, \cdots, n_\kappa \).

The entropy estimate of §7 requires a more careful consideration in the choice of the \( \rho_0 \)-oscillations. Namely, if \( x_{i_m(t)} \) and \( x_{i_{m+1}(t)} \) are a minimum and a maximum of \( u(t) \) respectively, we required that

\[
 u_p(t) - u_q(t) \leq \rho_0, \quad \text{whenever} \quad i_m(t) \leq p < q \leq i_{m+1}(t).
\]  

(6.1)

This can be achieved in the following way. Let \( \tau^0 = t_\kappa - \delta \). We can construct a sequence \( \{ E_{t_{\kappa-1}, \tau^0}^m \}_{m=1}^{n_\kappa} \) such that \( \{ \nu_m(t) \}_{m=1}^{n_\kappa} \) is a \( \rho_0 \)-oscillation and satisfies (6.1) at \( t = \tau^0 \) if
\( x_{\nu_m(t)} \in P_{E^m}(t) \) for each \( t \in [t_{\kappa-1}, \tau(0)] \) and \( m = 1, \cdots, n_\kappa \). For if (6.1) is not satisfied at \( t = \tau(0) \), then \( p \) and \( q \) exist so that \( p < q \leq i_{m+1}(\tau(0)) \) and \( u_p(\tau(0)) - u_q(\tau(0)) > \rho_0 \). Definition 6.5 implies that either

\[
u_m(\tau(0)) > u_{\nu_{m+1}(\tau(0))}(\tau(0)) \quad \text{and} \quad u_q(\tau(0)) > u_{\nu_m(\tau(0))}(\tau(0)),
\]

and we can assume that \( x_p \) is a maximum of \( u(x, \tau(0)) \), or

\[
u_m(\tau(0)) < u_{\nu_{m+1}(\tau(0))}(\tau(0)) \quad \text{and} \quad u_p(\tau(0)) < u_{\nu_{m+1}(\tau(0))}(\tau(0)),
\]

and we can assume that \( x_q \) is a minimum of \( u(x, \tau(0)) \). In either case,

\[B > ((n_\kappa - 1) + 2)\rho_0.\]

In the case (6.2), we may revise the construction of \( E_{t_{\kappa-1}, \tau(0)}^{m+1} \) so that \( p \) belongs to \( P_{E_{t_{\kappa-1}, \tau(0)}^{m+1}}(\tau(0)) \) and let \( \nu_{m+1}(\tau(0)) = p \); In the case (6.3), we may revise the construction of \( E_{t_{\kappa-1}, \tau(0)}^m \) so that \( q \) belongs to \( P_{E_{t_{\kappa-1}, \tau(0)}^m}(\tau(0)) \), and let \( \nu_m(\tau(0)) = q \). If (6.1) is still not satisfied at \( t = \tau(0) \), we may repeat the process. (6.1) must be satisfied after finite number of repetitions since

\[B > [(n_\kappa - 1) + 2l]\rho_0\]

holds for \( l \) repetitions. Suppose that a sequence \( \{E_{t_{\kappa-1}, \tau(0)}^m\}_{m=1}^{n_\kappa} \) has been constructed and that \( \{\nu_m(t)\}_{m=1}^{n_\kappa} \) is a \( \rho_0 \)-oscillation for each \( t \in [t_{\kappa-1}, \tau(0)] \) which satisfies (6.1) at \( t = \tau(0) \).

We also let that \( x_{\nu_m(t)} \in P_{E^m}(t) \) for each \( t \in [t_{\kappa-1}, \tau(0)] \) and \( m = 1, \cdots, n_\kappa \). Assume \( D_{\kappa}(\tau(0)) \) is the set of \( t \) in \( [t_{\kappa-1}, \tau(0)] \) for which (6.1) is violated. Theorem 3.2 and lemma 3.5 imply that a \( t(0) \) exists so that \( D_{\kappa}(\tau(0)) = [t_{\kappa-1}, t(0)] \subseteq [t_{\kappa-1}, \tau(0)] \). For any \( \delta_0 \in (0, t(0) - \tau_{\kappa-1}) \), let \( \tau(1) = t(0) - \delta_0 \). We can repeat the above process for \( [t_{\kappa-1}, \tau(1)] \) as we have previously done for \( [t_{\kappa-1}, \tau(0)] \) and get a \( t(1) \in (t_{\kappa-1}, \tau(1)] \) and \( \tau(2) \in (t_{\kappa-1}, t(1)) \). Repeating recursively the procedure as many times as possible, we get

\[t_\kappa > t_{\kappa - \delta} = \tau(0) > t(0) > \tau(1) > \cdots > \tau(m_\kappa) > t(m_\kappa) = t_{\kappa-1}\]

(6.5)

For each \( i \), we also get a sequence of paths of extrema \( \{E_{t(i), \tau(i)}^m\}_{m=1}^{n_\kappa} \) so that \( \{\nu_m(t)\}_{m=1}^{n_\kappa} \) is a \( \rho_0 \)-oscillation and satisfies (6.1) if \( x_{\nu_m(t)} \in P_{E^m}(t) \) for each \( m \). The recursive process
must end at some finite \( m_\kappa \) because

\[
(n_\kappa + 1 + 2m_\kappa)\rho_0 < B.
\]

(6.6)

Now we can easily show the following result.

**Proposition 6.9** For any \( \alpha < \frac{2\delta}{(\rho_0 + B)B} \), there exist \( t' \) and \( t'' \) in \([0,1]\) with \( t'' = t' > \alpha \) for which a sequence of paths of extrema, \( \{ E_{t',t''}^m \}_{m=1}^{n(t') + 1} \) exists so that \( \{ \nu_m(t) \}_{m=1}^{n(t') + 1} \) is a \( \rho_0 \)-oscillation which satisfies (6.1) if \( x_{\nu_m(t)} \in P_{E^m}(t) \) for \( t \in [t', t''] \).

**Proof** Since \( \max_\kappa(n_\kappa) = n_1 \geq \max_\kappa(n_\kappa) \), (6.6) implies that \( \kappa \leq B/\rho_0 - 1 \). Hence, there exists a \( \kappa \) such that the length of \([t_\kappa-1, t_\kappa] \cap [0,1] \geq \rho_0/B \). Let us still denote \([t_\kappa-1, t_\kappa] \cap [0,1] \) by \([t_\kappa-1, t_\kappa] \) and form the corresponding sequence (6.5). We may let \( \delta + \sum_{i=0}^{m_\kappa} (t^{(i-1)} - \tau^{(i)}) \) be so small that \( \sum_{i=0}^{m_\kappa} (\tau^{(i)} - t^{(i)}) > \rho_0/B \). There must be an \( i \) with \( 0 \leq i \leq m_\kappa \) such that

\[
\tau^{(i)} - t^{(i)} > \frac{\rho_0}{(m_\kappa + 1)B}.
\]

However, (6.5) implies

\[
\frac{1}{m_\kappa + 1} \geq \frac{2\rho_0}{B + \rho_0}
\]

The theorem is thus proven. \( \Box \)

Next, consider the sequences in \( \Psi_1 \). Following the arguments for proposition 6.4 and applying proposition 6.9, one get the following proposition immediately.

**Proposition 6.10** If \( \Psi_1 \neq \emptyset \), then there exists an integer \( n < B/\rho_0 \) for which \( \Psi_1 \) contains a sequence \( \{ u^k(x, t) \}_{k=1}^\infty \) such that for each \( k \), a collection of extrema of \( u^k \), \( \{ E_{0,1}^k \}_{m=1}^n \) exists so that \( \{ \nu_m^k(t) \}_{m=1}^n \) is a \( \rho_0 \)-oscillation of \( u_k \) which satisfies (6.1) if \( x_{\nu_m^k(t)} \in P_{E^k}(t) \) for each \( t \in [0,1] \) and \( m = 1, \cdots, n \).

Since we are mainly interested in shock regions around the line \( x = st \), we can ignore the paths if they are far away from \( \Omega_4 \) and we may assume that the left most one is a maximum and the right most one is a minimum. We therefore have a sequence of maximum-minimum pairs \( (M_{0,1}^{k,m}, N_{0,1}^{k,m}) \) \( m = 1, \cdots, n \) in \( \{ E_{0,1}^k \}_{m=1}^n \), where \( M_{0,1}^{k,1}(N_{0,1}^{k,1}) \) is the left(right) most one such that \( M_{0,1}^{k,m}(N_{0,1}^{k,m}) \cap \Omega_4 \neq \emptyset \). Let \( x_{J_{\nu}^{k,m}(t)} \) and \( x_{J_{\nu}^{k,m}(t)} \) be \( \varepsilon_k \)-paths of \( M_{0,1}^{k,m} \) and \( N_{0,1}^{k,m} \) respectively, where \( \varepsilon_k \downarrow 0 \) as \( k \to \infty \). We have

\[
I_{\varepsilon_k}^{k,1}(t) < J_{\varepsilon_k}^{k,1}(t) < I_{\varepsilon_k}^{k,2}(t) < J_{\varepsilon_k}^{k,2}(t) < \cdots < I_{\varepsilon_k}^{k,n}(t) < J_{\varepsilon_k}^{k,n}(t).
\]
Define
\[ \alpha_m^k = \inf_{0 \leq t \leq 1} (x_{t_k} - st). \]
We may assume, by selecting subsequence if necessary, that the finite or infinite limit
\[ \alpha_m = \lim_{k \to \infty} \alpha_m^k \]
exist. Suppose \( m_0 \) be the integer with \( \alpha_{m_0} < 0 \) and \( \alpha_{m_0+1} \geq 0 \) (the case that such an \( m_0 \) does not exists can be dealt similarly). Then for sufficiently large \( k \), \( \exists t_k \in [0,1] \), such that
\[ x_{t_k} < \frac{\alpha_{m_0}}{2}. \]
It is easy to see from the proof of lemma 5.6 that the conclusion of the theorem is valid for \( \{x_{t_k}^k \}_{k=1}^\infty \) when
\[ (x_{t_k}^k - st) \leq 0. \]
Therefore there is a constant \( \delta > 0 \) such that \( x > x_{t_k} \) for \( x \in B_\delta((st_k, t_k)) \). To apply lemma 6.2, define the map \( Q \) by
\[ Q(u^k) = \{(x, t) \in \mathbb{R} \times \mathbb{R}_+ : x > x_{t_k} \} \]
which is obviously similarity invariant. The lemma thus implies the existence of a sequence in \( \Psi_1 \) such that if we keep \( \{u_k(x, t)\}_{k=1}^\infty \) and all other corresponding notations for the new sequence, then
\[ \alpha_{m_0+1} \geq 0 \]
and
\[ x > x_{t_k} \]
holds for every \( k \) and \( (x, t) \in \Omega_4. \)
Similarly, for the new sequence, define
\[ \beta_m^k = \sup_{0 \leq t \leq 1} (x_{t_k} - st) \]
and
\[ \beta_m = \lim_{k \to \infty} b_m^k. \]
Repeating the forgoing argument, we see that \( \Psi_1 \) contains a sequence \( \{u^k\} \) for which two integers \( m_0 \) and \( m_1 \) with \( m_0 \leq m_1 \) exist so that

\[
x_{I_k^{m_0}(t)} < x < x_{J_k^{m_1}(t)}
\]

for every \( (x,t) \in \Omega_4 \) and that the inequalities

\[
\alpha_{m_0+1} \geq 0 \quad \text{and} \quad \beta_{m_1-1} \leq 0
\]

hold. This implies that

\[
x_{I_k^{m}(t)} - st \to 0
\]

and

\[
x_{J_k^{m}(t)} - st \to 0
\]

uniformly as \( k \to \infty \) in \( t \in [0,1] \) and for \( m = m_0 + 1, \ldots, m_1 - 1 \).

Now following the proof of theorem 5.7 by considering

\[
S_{M}^{m,k}(t) = \frac{1}{P_k} \sum_{l=1}^{P_k} \sum_{q=I_k^{m}(t)-p_k-l+1}^{I_k^{m}(t)} (u^k_q(t) - w_-)
\]

and

\[
S_{N}^{m,k}(t) = \frac{1}{P_k} \sum_{l=1}^{P_k} \sum_{q=J_k^{m}(t)+1}^{J_k^{m}(t)+p_k+l} (u^k_q(t) - w_+),
\]

one can show that

\[
\int_{t''}^{t'} \frac{f(u_{I_k^{m}(t)}^k(t)) - f(w_-)}{u_{I_k^{m}(t)}^k(t) - w_-} dt \to s(t'' - t') \quad (6.7)
\]

and

\[
\int_{t''}^{t'} \frac{f(u_{J_k^{m}(t)}^k(t)) - f(w_+)}{u_{J_k^{m}(t)}^k(t) - w_+} dt \to s(t'' - t') \quad (6.8)
\]

uniformly in \( 0 \leq t' \leq t'' \leq 1 \) and \( m_0 + 1 \leq m \leq m_1 - 1 \) as \( k \to \infty \). Because \( V_{M+k,m}(t) \) and \( V_{N+k,m}(t) \) are monotone in \( t \) and \( u_{I_k^{m}(t)}^k(t) - V_{M+k,m}(t) \) and \( u_{J_k^{m}(t)}^k(t) - V_{N+k,m}(t) \) converge to 0 uniformly, and because \( f \) is strictly convex, (6.7) and (6.8) imply that \( u_{I_k^{m}(t)}^k(t) \) and
$u^k_{t^k_{s_k}^m(t)}(t)$ converge uniformly in $t \in [1/3, 2/3]$ and $m = m_0 + 1, \ldots, m_1 - 1$ to $w_+$ and $w_-$ respectively. Once again applying lemma 6.2, we get the following proposition.

**Proposition 6.11** If $\Psi \neq \emptyset$, then it contain a sequence such that if we, for simplicity, still denote it by $\{u^k(x,t)\}^\infty_{K=1}$ and keep all the corresponding notations, one can find integers $m_0$ and $m_1$ with $m_0 \leq m_1$, such that

$$x^k_{t^k_{s_k}^{m_0}}(t) < x < x^k_{j^k_{s_k}^{m_1}}(t)$$

for $(x,t) \in \Omega_4$, that $x^k_{t^k_{s_k}^{m}}(t)$ and $x^k_{j^k_{s_k}^{m}}(t)$ converge uniformly in $t \in [0,1]$ and $m = m_0 + 1, \ldots, m_1 - 1$ to 0, and that $u^k_{t^k_{s_k}^{m}}(t)$ and $u^k_{j^k_{s_k}^{m}}(t)$ converge uniformly in $t \in [0,1]$ and $m = m_0 + 1, \ldots, m_1 - 1$ to $w_+$ and $w_-$ respectively.

Suppose that the $m_0$ in the foregoing proposition exists. If there is a positive $\delta$ independent of $k$ such that for $1 - \delta \leq t \leq 1$,

$$u^k_p(t) - u^k_q(t) \leq 2\rho_0$$

whenever $I_{s_k}^{k,m_0}(t) \leq p \leq q \leq J_{s_k}^{k,m_0}(t)$ holds, then applying lemma 6.2 and making an argument similar to the one for proposition 6.10, we may assume that the inequality holds for $0 \leq t \leq 1$. If the $\delta$ with the above property does not exists, we may assume the existence of another pair of extremum $M_{0,1}^{k,(m_0)}$ and $N_{0,1}^{k,(m_0)}$ with $\varepsilon$-paths $x^k_{t^k_{s_k}^{(m_0)}}(t)$ and $x^k_{j^k_{s_k}^{(m_0)}}(t)$ respectively such that

$$x^k_{t^k_{s_k}^{m_0}}(t) \leq x^k_{t^k_{s_k}^{(m_0)}}(t) \leq x^k_{j^k_{s_k}^{(m_0)}}(t) \leq x^k_{j^k_{s_k}^{m_0}}(t)$$

that

$$u^k_{t^k_{s_k}^{(m_0)}}(t) - u^k_{j^k_{s_k}^{(m_0)}}(t) > 2\rho_0,$$

and that

$$u^k_p(t) - u^k_q(t) \leq 2\rho_0$$

whenever $I_{s_k}^{k,m_0}(t) \leq p \leq q \leq I_{s_k}^{k,(m_0)}(t)$. Then, either

1. $x^k_{t^k_{s_k}^{(m_0)}}(t)$ converges to 0 so that $u^k_{t^k_{s_k}^{k,(m_0)}}(t)$ converges uniformly to $w_+$, or...
(2) we may assume (using lemma 6.2 again if necessary)

\[ x_{I^k_{l_k}(m_0)}(t) < x \]

for \((x, t) \in \Omega_5\).

In the case (2), it is guaranteed that for any \(t \in [0, 1]\) and \((st - 1 - |s|)/h_k \leq p \leq q \leq I_{e_k}^{m_0+1}(t)\),

\[ u^k_p(t) - u^k_q(t) \leq 2\rho_0 \]

since otherwise, part (d) of definition 6.5 is clearly violated. We summarize these arguments into the following proposition.

**Proposition 6.12** If \(\Psi \neq \emptyset\), then for any positive constant \(\rho\), either \(\Psi\) contains a sequence \(\{u^k(x, t)\}_{k=1}^{\infty}\) such that for any \(t \in [0, 1]\),

\[ u^k_p(t) - u^k_q(t) \leq \rho \]

provided that \(p < q\) and \((x_p, t)\) and \((x_q, t) \in \Omega_4\); or it contains a sequence \(\{u^k(x, t)\}_{k=1}^{\infty}\) such that for each \(k\), the path \(M^k_{0,1}\) of a maximum and the path \(N^k_{0,1}\) of a minimum exist with \(\varepsilon_k\)-paths \(x_{I^k_{l_k}(t)}\) and \(x_{J^k_{l_k}(t)}\) respectively, so that \(x_{I^k_{l_k}(t)} < x_{J^k_{l_k}(t)}\), that \(\varepsilon_k \downarrow 0\) as \(k \to \infty\), that \(x_{I^k_{l_k}(t)}(t)\) and \(x_{J^k_{l_k}(t)}(t)\) converges to \(st\) uniformly as \(k \to \infty\) and that \(u^k_{I^k_{l_k}(t)}(t)\) and \(u^k_{J^k_{l_k}(t)}(t)\) converges uniformly to \(w_+\) and \(w_-\) as \(k \to \infty\). Moreover, in the later case, for \(t \in [0, 1]\),

\[ u^k_p(t) - u^k_q(t) \leq \rho \]

holds when \((x_p, t)\) and \((x_q, t) \in \Omega_4\) and either \(x_p \leq x_q \leq x_{I^k_{l_k}(t)}\) or \(x_{J^k_{l_k}(t)} \leq x_p \leq x_q\).

§7 Global entropy inequality, the completion of the proof of the main theorem

We will analyze the asymptotic behavior of the approximations by a sequence of numericals in \(\Psi\) to the divergence of the entropy pair \((U,F)\), namely, \(U_t + F_x\) where

\[ U(w) = \frac{w^2}{2}, \quad F(w) = \int_0^w \xi f'(\xi)d\xi. \]

Choose a test function \(\phi \in C^1_0(\Omega_4), \phi \geq 0\) and

\[ \Phi = -\int (U(W)\phi_t + F(W)\phi_x)dxdy > 0. \]
where $W$ is the expansion shock defined by (4.4).

Assume that $\Psi \neq \emptyset$. For any $\{u^k\}_{k=1}^\infty \in \Psi$, we use Osher's discrete divergence form (2.6) with the flux $F_A(u^k_j)$ defined by (2.7). The numerical approximation to (7.1) is

$$\Phi_k = \sum_j h_k \left( \frac{d}{dt} U(u^k_j(t)) + D_A F_A(u^k_j(t)) \right) \phi_{j+\frac{1}{2}}(t) dt$$

where $\phi_{j+\frac{1}{2}}(t) = \phi(x_{j+\frac{1}{2}}, t)$.

On one hand, the assumption of the $L^1_{loc}$ convergence of $u^k$ to $W$ implies that $\Phi_k$ converges to $\Phi$ for every $\{u^k\}_{k=1}^\infty \in \Psi$. On the other hand, we will show that if $\Psi \neq \emptyset$ then it contains sequences for which $\Phi_k$ do not converge to $\Phi$. The contradiction shows that $\Psi$ must be an empty set and our main theorem will thus be proven.

For arbitrary positive constant $\rho$, assume that $\{u^k\} \in \Psi$ satisfies proposition 6.12. We rewrite $\Phi_k$ as

$$\Phi_k = \sum_j \int_0^1 h_k \left( \frac{d}{dt} U(u^k_j) + D_A F_A(u^k_j) \right) \phi_{j+\frac{1}{2}}(t) dt$$

$$= \sum_j \int_0^1 \phi_{j+\frac{1}{2}}(t) dt \int_{u^k_j}^{u^k_{j+1}} (g_{j+\frac{1}{2}} - f(w)) dw$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3$$

where

$$\Sigma_3 = \int_0^1 \sum_{j=I^k_{\infty}(t)}^{I^k_{\infty}(t)+1} h_k \left( \frac{d}{dt} U(u^k_j)^2 + D_A F_A(u^k_j) \right) \phi_{j+\frac{1}{2}}(t) dt$$

$\Sigma_1$ denotes the summation over all the rarefactions (i.e., $u^k_{i+1} > u^k_i$) which are not contained in $\Sigma_3$.

$\Sigma_2$ denotes the summation over all the grid points which are not contained in $\Sigma_1$ and $\Sigma_3$.

Osher[16] proved that

$$\Sigma_1 \leq 0.$$  \hspace{1cm} (7.3)

Obviously,

$$|\Sigma_2| = |\Sigma_2 \int_0^1 \int_{u^k_j}^{u^k_{j+1}} (g_{j+\frac{1}{2}} - f(w)) dw \phi_{j+\frac{1}{2}}(t) dt|$$
\[
\leq \phi_{\text{max}} \Sigma_2 \int_0^1 \left( \int_{u_j^k}^{u_{j+1}^k} |g_{j+\frac{1}{2}} - f(w)| dw \right) dt \\
\leq C \rho \phi_{\text{max}} \int_0^1 \Sigma_2 \int_{u_j^k}^{u_{j+1}^k} dw |dt \\
\leq CB \rho \phi_{\text{max}}. \tag{7.4}
\]

The key estimate is that of \( \Sigma_3 \). We proceed as follows.

\[
\Sigma_3 = \int_0^1 \sum_{j=\ell_{k,t}^h(t)}^{J_k^{\lfloor t \rfloor} - 1} h_k \frac{d}{dt} \left( \frac{(u_j^k)^2}{2} \phi_{j+\frac{1}{2}}(t) \right) dt + \int_0^1 \sum_{j=\ell_{k,t}^h(t)}^{J_k^{\lfloor t \rfloor} - 1} h_k D_+ F_A(u_j^k) \phi_{j+\frac{1}{2}}(t) dt \\
= \Sigma_3' + \Sigma_3''.
\]

Summation by parts of \( \Sigma_3'' \) gives

\[
\Sigma_3'' = -\int_0^1 \left( \sum_{j=\ell_{k,t}^h(t)+1}^{J_k^{\lfloor t \rfloor} - 1} F_A(u_j^k)(\phi_{j+\frac{1}{2}} - \phi_{j-\frac{1}{2}}) \right) dt \\
+ \int_0^1 \left( F_A(u_{\ell_{k,t}^h(t)}^k(t))(\phi_{\ell_{k,t}^h(t)-\frac{1}{2}}(t) - F_A(u_{\ell_{k,t}^h(t)}^k(t))(\phi_{\ell_{k,t}^h(t)+\frac{1}{2}}(t) \right) dt \\
\to \int_{x=st} F(w_-) - F(w_+) \phi(x,t) dt, \quad \text{as } k \to \infty. \tag{7.5}
\]

To estimate \( \Sigma_3' \), we denote by \( \Omega^k \) the closure of the subset of \( \Omega_4 \) bounded by \( \Gamma_1^k = \{(x, t_{\ell_{k,t}^k(t)-1/2}) : t \in [0,1] \} \) and \( \Gamma_r^k = \{(x, t_{\ell_{k,t}^k(t)} - 1/2, t) : t \in [0,1] \} \). For each \( j \) satisfying

\[
\min_{t} \ell_{k,t}^h(t) \leq j \leq \max_{t} (J_{\ell_{k,t}^h(t)}^h - 1),
\]

suppose that from \( t = 0 \) to \( t = 1 \), \( (x_j, t) \) crosses the right boundary of \( \Omega^k 2\theta + 1 \) times at

\[
t = t_1, t_2, \ldots, t_{2\theta + 1}
\]

where \( u_j^k(t) \) resumes the values

\[
u_1, \nu_2, \ldots, \nu_{2\theta + 1}
\]

respectively. Suppose that, during the same time interval, \( (x_j, t) \) crosses the left boundary of \( \Omega^k 2\ell + 1 \) times at

\[
t = \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{2\ell + 1}
\]

56
where \( u_j^k(t) \) resumes the values 
\[
\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{2^d+1}
\]
respectively. Here, we have ignored the dependence of these notations on \( j \) and \( k \).

Consider the following 3 cases separately.

(1) \( s > 0 \). Integration by parts of \( \Sigma'_3 \) gives

\[
\Sigma'_3 = - \int_0^{J^l_k(t)} \sum_{j=2^d_{2^d}} h_k \frac{(u_j^k)^2}{2} \frac{d}{dt} \phi_{j+\frac{3}{2}}(t) dt \\
+ \sum_{\kappa = \min(t^{l_k}(t))}^{\max(J^l_k(t)-1)} h_k \left[ \sum_{\nu=1}^{2^d+1} (-1)^{\nu+1} \bar{u}_\nu^2 \phi_{\kappa+\frac{1}{2}}(t^\nu_{\nu}) - \sum_{\nu=1}^{2^d+1} (-1)^{\nu+1} \mu_{\nu}^2 \phi_{\kappa+\frac{1}{2}}(t^\nu_{\nu}) \right] \\
\rightarrow \frac{w_+^2 - w_-^2}{2} \int_{x=st} \phi(x, t) dt \quad \text{as} \quad k \rightarrow \infty. \tag{7.6}
\]

Here we have used the fact that both \( \bar{u}_\nu \) and \( u_\nu \) are monotone functions of \( \nu \). Hence

\[
\Sigma'_3 + \Sigma''_3 \rightarrow - \int_{w_-}^{w_+} \xi f'(\xi) d\xi - \frac{w_+ + w_-}{2} \int_{w_-}^{w_+} f'(\xi) d\xi \int_{x=st} \phi(x, t) dt \\
\leq - \frac{c}{12} (w_+ - w_-)^3 \int_{x=st} \phi(x, t) dx \quad \text{as} \quad k \rightarrow \infty \tag{7.7}
\]

where \( c \) depends on the minimum of \( f''(\xi) \) only. Noticing the arbitrariness of \( \rho \), substituting (7.3), (7.4) and (7.7) into (7.2), we get a result which clearly contradicts (7.1) for sufficiently small \( \rho \) and sufficiently large \( k \).

(2). \( s < 0 \), the contradiction can be established very similarly. It is not necessary to repeat the details.

(3). \( s = 0 \). In this case (7.6), the estimate of \( J''_3 \) still holds. And \( \Sigma'_3 \) can be written as

\[
\Sigma'_3 = - \int_0^{J^l_k(t)-1} \sum_{j=t^{l_k}(t)} h_k \frac{(u_j^k)^2}{2} \frac{d}{dt} \phi_{j+\frac{3}{2}}(t) dt \\
+ \sum_{\kappa = \min(t^{l_k}(t))}^{\max(J^l_k(t)-1)} h_k \left[ \sum_{\nu=1}^{2^d+1} (-1)^{\nu+1} \omega_1 \bar{u}_\nu^2 \phi_{\kappa+\frac{1}{2}}(t^\nu_{\nu}) - \sum_{\nu=1}^{2^d+1} \mu \omega_2 \mu_{\nu}^2 \phi_{\kappa+\frac{1}{2}}(t^\nu_{\nu}) \right] \tag{7.8}
\]

where \( \omega_1 \) and \( \omega_2 \) are constants which are independent of \( \nu \) and \( \mu \), and which resume the value \( \pm 1 \) only. Since \( \bar{u}_\nu \) and \( u_\mu \) are monotone functions of \( \nu \) and \( \mu \) respectively, it is not
difficult to see that both the integration and the summation on the right sides of the equal
sign of (7.8) tend to zero as k tends to infinity. This means that (7.7) still holds in this
case and leads to the same contradiction. Our main theorem has been proven.

Remark 7.1 From the proof of the theorem, it is clear that the minmod bound
(2.4) is only essential at rarefactions. At shocks, you can safely remove the restriction
on the slopes in the MUSCL reconstruction as long as the resulted scheme remains TVD.
This observation enables us to apply, among others, the artificial compression method
introduced in [29] to the shocks.

§8 Concluding remarks

The convergence of generalized MUSCL schemes which approximates to one space
dimensional scalar conservation laws with strictly convex flux functions is proved.

The theoretical basis of this work lies on an observation of the theory on the BV
solutions of the conservation laws due to Vol'pert[27].

A series of nonlinear wave analyses are developed upon this observations. The center
of these analyses is the estimates of the speed of the waves which is the discretized version
of the Rankine–Hugoniot condition.

These analyses enable us to get an estimate of the divergence of the entropy–entropy
flux on shocks globally. An combination of this estimate and the estimate on rarefactions
due to Osher[16] proves the convergency of the generalized MUSCL schemes.

We conclude this paper with two remarks: First, the observation of the jump points of
the BV functions is valid in multi-dimensional cases. However, since the TVD numerical
schemes in the usual sense can be at most first order accurate, we still need some new
techniques before this observation can be used to show the convergence of high order
schemes in multiple dimensional cases without adding some ingredients useless in practical
computations.

Second, the wave speed analyses are valid for non-convex cases. However, since we need
entropy inequalities for all convex entropy functions, the convergence proof is significant
more difficult and the author is currently working on this subject. My opinion is, some
inside wave analysis is needed to show the instability of the entropy violating waves. This could leads to a complete solution on the convergence of generalized MUSCL schemes for conservation laws with arbitrary fluxes.

Acknowledgement. The author is indebted to Professor Stanley Osher for suggesting this problem and for all his advise and encouragement. Without his pioneering work on and insight into this subject, the present work is impossible. The author also highly grateful to Professor P. L. Lions for showing his interest in this work and his constructive and encouraging comments on an early version of this paper. Thanks also to Professor Bernardo Cockburn and Professor Chi-Wang Shu for many valuable discussions.

References


Recent IMA Preprints

<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>618</td>
<td>L.E. Fraenkel</td>
<td>On a linear, partly hyperbolic model of viscoelastic flow past a plate</td>
</tr>
<tr>
<td>619</td>
<td>Stephen Schechter and Michael Shearer</td>
<td>Undercompressive shocks for nonstrictly hyperbolic conservation laws</td>
</tr>
<tr>
<td>620</td>
<td>Xinfu Chen</td>
<td>Axially symmetric jets of compressible fluid</td>
</tr>
<tr>
<td>621</td>
<td>J. David Logan</td>
<td>Wave propagation in a qualitative model of combustion under equilibrium conditions</td>
</tr>
<tr>
<td>622</td>
<td>M.L. Zeeman</td>
<td>Hopf bifurcations in competitive three-dimensional Lotka-Volterra Systems</td>
</tr>
<tr>
<td>623</td>
<td>Allan P. Fordy</td>
<td>Isospectral flows: their Hamiltonian structures, Miura maps and master symmetries</td>
</tr>
<tr>
<td>624</td>
<td>Daniel D. Joseph, John Nelson, Michael Renardy, and Yuriko Renardy</td>
<td>Two-Dimensional cusped interfaces</td>
</tr>
<tr>
<td>625</td>
<td>Avner Friedman and Bei Hu</td>
<td>A free boundary problem arising in electrophotography</td>
</tr>
<tr>
<td>626</td>
<td>Hamid Bellout, Avner Friedman and Victor Isakov</td>
<td>Stability for an inverse problem in potential theory</td>
</tr>
<tr>
<td>627</td>
<td>Barbara Lee Keyfitz</td>
<td>Shocks near the sonic line: A comparison between steady and unsteady models for change of type</td>
</tr>
<tr>
<td>628</td>
<td>Barbara Lee Keyfitz and Gerald G. Warnecke</td>
<td>The existence of viscous profiles and admissibility for transonic shocks</td>
</tr>
<tr>
<td>629</td>
<td>P. Szmolyan</td>
<td>Transversal heteroclinic and homoclinic orbits in singular perturbation problems</td>
</tr>
<tr>
<td>630</td>
<td>Philip Boyland</td>
<td>Rotation sets and monotone periodic orbits for annulus homeomorphisms</td>
</tr>
<tr>
<td>631</td>
<td>Kenneth R. Meyer</td>
<td>Apollonius coordinates, the N-body problem and continuation of periodic solutions</td>
</tr>
<tr>
<td>632</td>
<td>Chjan C. Lim</td>
<td>On the Poincare–Whitney circuitspace and other properties of an incidence matrix for binary trees</td>
</tr>
<tr>
<td>634</td>
<td>Stanley Minkowitz and Matthew Witten</td>
<td>Periodicity in cell proliferation using an asynchronous cell population</td>
</tr>
<tr>
<td>635</td>
<td>M. Chipot and G. Dal Maso</td>
<td>Relaxed shape optimization: The case of nonnegative data for the Dirichlet problem</td>
</tr>
<tr>
<td>636</td>
<td>Jeffery M. Franke and Harlan W. Stech</td>
<td>Extensions of an algorithm for the analysis of nongeneric Hopf bifurcations, with applications to delay-difference equations</td>
</tr>
<tr>
<td>637</td>
<td>Xinfu Chen</td>
<td>Generation and propagation of the interface for reaction–diffusion equations</td>
</tr>
<tr>
<td>638</td>
<td>Philip Korman</td>
<td>Dynamics of the Lotka–Volterra systems with diffusion</td>
</tr>
<tr>
<td>639</td>
<td>Harlan W. Stech</td>
<td>Generic Hopf bifurcation in a class of integro-differential equations</td>
</tr>
<tr>
<td>640</td>
<td>Stephane Laederich</td>
<td>Periodic solutions of non linear differential difference equations</td>
</tr>
<tr>
<td>641</td>
<td>Peter J. Olver</td>
<td>Canonical Forms and Integrability of BiHamiltonian Systems</td>
</tr>
<tr>
<td>642</td>
<td>S.A. van Gils, M.P. Krupa and W.F. Langford</td>
<td>Hopf bifurcation with nonsemisimple 1:1 Resonance</td>
</tr>
<tr>
<td>643</td>
<td>R.D. James and D. Kinderlehrer</td>
<td>Frustration in ferromagnetic materials</td>
</tr>
<tr>
<td>644</td>
<td>Carlos Rocha</td>
<td>Properties of the attractor of a scalar parabolic P.D.E.</td>
</tr>
<tr>
<td>645</td>
<td>Debra Lewis</td>
<td>Lagrangian block diagonalization</td>
</tr>
<tr>
<td>646</td>
<td>Richard C. Churchill and David L. Rod</td>
<td>On the determination of Ziglin monodromy groups</td>
</tr>
<tr>
<td>647</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>A nonlocal diffusion equation arising in terminally attached polymer chains</td>
</tr>
<tr>
<td>648</td>
<td>Peter Gritzmann and Victor Klee</td>
<td>Inner and outer j- Radii of convex bodies in finite-dimensional normed spaces</td>
</tr>
<tr>
<td>649</td>
<td>P. Szmolyan</td>
<td>Analysis of a singularly perturbed traveling wave problem</td>
</tr>
<tr>
<td>650</td>
<td>Stanley Reiter and Carl P. Simon</td>
<td>Decentralized dynamic processes for finding equilibrium</td>
</tr>
<tr>
<td>651</td>
<td>Fernando Reitich</td>
<td>Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions</td>
</tr>
<tr>
<td>652</td>
<td>Russell A. Johnson</td>
<td>Cantor spectrum for the quasi-periodic Schrödinger equation</td>
</tr>
<tr>
<td>653</td>
<td>Wenhui Liu</td>
<td>Singular solutions for a convection diffusion equation with absorption</td>
</tr>
<tr>
<td>654</td>
<td>Deborah Brandon and William J. Hrusa</td>
<td>Global existence of smooth shearing motions of a nonlinear viscoelastic fluid</td>
</tr>
<tr>
<td>655</td>
<td>James F. Reineck</td>
<td>The connection matrix in Morse–Smale flows II</td>
</tr>
<tr>
<td>656</td>
<td>Claude Baesens, John Guckenheimer, Seunghwan Kim and Robert Mackay</td>
<td>Simple resonance regions of torus diffeomorphisms</td>
</tr>
<tr>
<td>657</td>
<td>Willard Miller, Jr.</td>
<td>Lecture notes in radar/sonar: Topics in Harmonic analysis with applications to radar and sonar</td>
</tr>
</tbody>
</table>
Calvin H. Wilcox, Lecture notes in radar/sonar: Sonar and Radar Echo Structure
Richard E. Blahut, Lecture notes in radar/sonar: Theory of remote surveillance algorithms
D.V. Anosov, Hilbert's 21st problem (according to Bolibruch)
Stephane Laederich, Ray-Singer torsion for complex manifolds and the adiabatic limit
Geneviève Ranghel and George R. Sell, Navier-Stokes equations in thin 3d domains: Global regularity of solutions I
Emanuel Parzen, Time series, statistics, and information
Andrew Majda and Kevin Lamb, Simplified equations for low Mach number combustion with strong heat release
Ju. S. Il'yashenko, Global analysis of the phase portrait for the Kuramoto-Sivashinsky equation
James F. Reineck, Continuation to gradient flows
Mohamed Sami Elbialy, Simultaneous binary collisions in the collinear N-body problem
John A. Jacquez and Carl P. Simon, Aids: The epidemiological significance of two different mean rates of partner-change
Carl P. Simon and John A. Jacquez, Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations
Matthew Stafford, Markov partitions for expanding maps of the circle
Ciprian Foias and Edriss S. Titi, Determining nodes, finite difference schemes and inertial manifolds
M.W. Smiley, Global attractors and approximate inertial manifolds for abstract dissipative equations
M.W. Smiley, On the existence of smooth breathers for nonlinear wave equations
Hitay Özbay and Janos Turi, Robust stabilization of systems governed by singular integro-differential equations
Mary Silber and Edgar Knobloch, Hopf bifurcation on a square lattice
Christophe Golé, Ghost circles for twist maps
Christophe Golé, Ghost tori for monotone maps
Christophe Golé, Monotone maps of $T^n \times R^n$ and their periodic orbits
E.G. Kalnins and W. Miller, Jr., Hypergeometric expansions of Heun polynomials
Víctor A. Pliss and George R. Sell, Perturbations of attractors of differential equations
Avner Friedman and Peter Knabner, A transport model with micro- and macro-structure
E.G. Kalnins and W. Miller, Jr., A note on group contractions and radar ambiguity functions
George R. Sell, References on dynamical systems
Shui-Nee Chow, Kening Lu and George R. Sell, Smoothness of inertial manifolds
Shui-Nee Chow, Xiao-Biao Lin and Kening Lu, Smooth invariant foliations in infinite dimensional spaces
Kening Lu, A Hartman-Grobman theorem for scalar reaction-diffusion equations
Christophe Golé and Glen R. Hall, Poincaré's proof of Poincaré's last geometric theorem
Mario Taboada, Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations
Peter Rejto and Mario Taboada, Weighted resolvent estimates for Volterra operators on unbounded intervals
Joel D. Avrin, Some examples of temperature bounds and concentration decay for a model of solid fuel combustion
Susan Friedlander and Misha M. Vishik, Lax pair formulation for the Euler equation
H. Scott Dumas, Ergodization rates for linear flow on the torus
A. Eden, A.J. Milani and B. Nicolaenko, Finite dimensional exponential attractors for semilinear wave equations with damping
A. Eden, C. Foias, B. Nicolaenko & R. Temam, Inertial sets for dissipative evolution equations
A. Eden, C. Foias, B. Nicolaenko & R. Temam, Hölder continuity for the inverse of Mañe's projection
Michel Chipot and Charles Collins, Numerical approximations in variational problems with potential wells
Huanan Yang, Nonlinear wave analysis and convergence of MUSCL schemes
László Gerencsér and Zsuzsanna Vágó, A strong approximation theorem for estimator processes in continuous time
László Gerencsér, Multiple integrals with respect to L-mixing processes
David Kinderlehrer and Pablo Pedregal, Weak convergence of integrands and the Young measure representation
Bo Deng, Symbolic dynamics for chaotic systems
Charles Collins and Mitchell Luskin, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem
Peter Gritzmann and Victor Klee, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces