

**FINITE DIMENSIONAL EXPONENTIAL ATTRACTORS
FOR SEMILINEAR WAVE EQUATIONS WITH DAMPING**

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ABSTRACT. We consider the initial value problem for a class of second order evolution equations that includes, among others, the 3D sine-Gordon equation with damping and the 3D Klein-Gordon type equations with damping. We show the existence of a set with finite fractal dimension that contains the global attractor and attracts all smooth solutions at an exponential rate.

1. **INTRODUCTION.** In this paper we study the initial value problem for a class of semilinear wave equations with damping that includes the 3D sine-Gordon equation with damping, the 3D Klein-Gordon type equations with damping as well as systems of sine-Gordon type equations. The purpose of our work is to characterize the long time behaviour of the solutions for these equations with the help of a finite dimensional set that contains the global attractor and at the same time attracts all smooth solutions at an exponential rate. Such sets are called inertial sets (see [EFNT1] and [EFNT2]). In order to lay the conceptual groundwork for inertial sets, we first recall some of the basic definitions from the theory of (infinite dimensional) dynamical systems. The existence of a bounded set that absorbs all solutions in finite time, i.e. an absorbing set, seems to be the starting point for all the studies of dissipative differential equations. The omega limit set of such an absorbing set is called the global attractor and for most of the dissipative equations the global attractor is known to be compact. However, the manner in which the global attractor becomes compact changes drastically from the first order dissipative evolution equations to the second order damped evolution equations (see [T], [BV] and [H]). The same difference also effects the way one proves the finite dimensionality of the global attractor (see [T]). We will come back to this point later on. Although the attractor is the most obvious set to study the long time dynamics of the underlying evolution equations on, it is not easy to track. The theory of inertial

manifolds is proposed in order to partially resolve this problem ([FST], [CFNT1], [CFNT2] and [FSTi]). An inertial manifold is a Lipschitz manifold that is invariant under the solution operator and attracts all solutions at an exponential rate. At this stage, the existence of inertial manifolds hinges upon a gap condition that is imposed on the eigenvalues of the linear part of the evolution equation. On the one hand, this condition makes the dimension estimates for the inertial manifolds for equations in higher space dimension unreasonably high. On the other hand, for equations like 2D Navier-Stokes equations it leaves the problem of the existence of an inertial manifold open.

By relaxing some of the conditions in the definition of the inertial manifold one can arrive to a more general concept, that of an inertial set. An inertial set is a set of finite fractal dimension which attracts all solutions at an exponential rate (see [EFNT1], [EFNT2]). The existence of inertial sets for a large class of dissipative evolution equations that includes the 2D Navier-Stokes equations with periodic boundary conditions, Kuramoto-Sivashinsky equation, Kolmogorov-Sivashinsky-Spiegel equation, Chaffee-Infante equation in any space dimension and Original Burgers' equations has already been proven ([EFNT1] and [EFNT2]). However, all the equations mentioned above take the form of a first order evolution equation where the linear part is a positive, self-adjoint operator. In contrast, the equations we consider are of the form

$$\varepsilon u_{tt} + u_t + Au + g(u) = f, \quad (1.1)$$

$$u(0) = u_0 \text{ and } u_t(0) = u_1, \quad (1.2)$$

where $\varepsilon > 0$, A is a positive, self-adjoint operator and the non-linearity $g(u)$ satisfies the appropriate conditions (see [T]). When written as a first order evolution equation, in a suitable product of Hilbert spaces, the evolution equation (1.1) can be transformed into

$$U_t + \mathcal{A}U + \mathcal{G}(U) = F, \quad (1.3)$$

$$U(0) = U_0 = [u_1, u_0]^T, \quad (1.4)$$

where

$$U = [u_t, u]^T ; \mathcal{A} = \varepsilon^{-1} \begin{bmatrix} I & A \\ 0 & \varepsilon I \end{bmatrix} \quad (1.5)$$

and

$$\mathcal{G}(U) = [g(u), 0]^T ; F = [f, 0]^T. \quad (1.6)$$

Clearly, the linear part of the evolution equation (1.3), that is \mathcal{A} , is no longer self-adjoint. Moreover, the spectrum of \mathcal{A} contains infinitely many eigenvalues on a vertical line in the complex plane (see [MS]). In terms of the existence of a smooth inertial manifold this particular distribution of eigenvalues causes insurmountable difficulty. Even in one space dimension, for ε large enough the generic situation is the nonexistence of C^1 -inertial manifolds (see [MS]). In contrast, the existence of the compact global attractor is independent of the size of ε (see [T], [H] and [BV]). At the other extreme, that is when ε is very small, at least in one space dimension one can show the existence of C^1 -inertial manifolds under some assumptions on $g(u)$ and f (see [MS] also [CL]). In spite of the difficulties of this set-up we will show that inertial sets still exist. After reviewing the basic result on the existence of inertial sets from [EFNT1] we proceed to verify the key property that allows a construction of an inertial set, that is the *squeezing property*. We only need a simple Lipschitz condition on the non-linearity $g(u)$ to obtain this result. In the final section we give a few equations to which the theory applies.

2. INERTIAL SETS AND SQUEEZING PROPERTY.

Let \mathbb{H} be a separable Hilbert space and B be a compact subset of \mathbb{H} . Let $\{S(t)\}_{t \geq 0}$ be a non-linear continuous semi-group that leaves the set B invariant and set $\mathcal{A} = \bigcap \{S(t)B : t \geq 0\}$, that is \mathcal{A} is the global attractor for $\{S(t)\}_{t \geq 0}$ on B .

Definition 2.1. A set \mathcal{M} is called an *inertial set* for $(\{S(t)\}_{t \geq 0}, B)$ if (i) $\mathcal{A} \subseteq \mathcal{M} \subseteq B$, (ii) $S(t)\mathcal{M} \subseteq \mathcal{M}$ for every $t \geq 0$, (iii) for every u_0 in B , $\text{dist}_{\mathbb{H}}(S(t)u_0, \mathcal{M}) \leq c_1 \exp\{-c_2 t\}$ for all $t \geq 0$, where c_1 and c_2 are independent of u_0 ; and (iv) \mathcal{M} has finite fractal dimension.

A sufficient condition for the existence of inertial sets was furnished in [EFNT1] (see also [EFNT2]). It depends on a dichotomy principle called the squeezing property, we recall this property for flows.

Definition 2.2. A continuous semi-group $\{S(t)\}_{t \geq 0}$ is said to satisfy the *squeezing property* on B if there exists $t_* > 0$ such that $S_* = S(t_*)$ satisfies: there exists an orthogonal projection P of rank N_0 such that if for every u and v in B

$$\|P(S_*u - S_*v)\|_{\mathbb{H}} \leq \|(I - P)(S_*u - S_*v)\|_{\mathbb{H}} \quad (2.1)$$

then

$$\|S_*u - S_*v\|_{\mathbb{H}} \leq \frac{1}{8} \|u - v\|_{\mathbb{H}}. \quad (2.2)$$

Remark 2.2. An implicit assumption that we will make from now on is the Lipschitz condition on the map $(t, u_0) \rightarrow S(t)u_0$ on $[0, t_*] \times B$, which is easy to check for the equations under consideration.

Theorem 2.1. [EFNT1] If $(\{S(t)\}_{t \geq 0}, B)$ satisfies the squeezing property on B and if $S_* = S(t_*)$ is Lipschitz on B with Lipschitz constant L then there exists an inertial set \mathcal{M} for $(\{S(t)\}_{t \geq 0}, B)$ such that

$$d_F(\mathcal{M}) \leq N_0 \max\{1, \ln(16L + 1)/\ln 2\} \quad (2.3)$$

and

$$\text{dist}_{\mathbb{H}}(S(t)u_0, \mathcal{M}) \leq c_1 \exp\{-c_2/t_*\} t \quad (2.4)$$

Clearly, we only need to find the time t_* and the projection P of rank N_0 in order to evaluate the right hand sides of the inequalities (2.3) and (2.4). That is what we proceed to do.

3. SEMILINEAR WAVE EQUATIONS WITH DAMPING: FUNCTIONAL SET-UP AND SQUEEZING PROPERTY.

A. PRELIMINARIES.

We consider the initial value problem for a class of semilinear wave equations on a separable Hilbert space H ,

$$(H^\varepsilon) \quad \varepsilon u_{tt} + u_t + Au + g(u) = f,$$

$$(IV) \quad u(0) = u_0 \text{ and } u_t(0) = u_1.$$

where $\varepsilon > 0$, A is a positive, self-adjoint unbounded operator with compact inverse. Let $V = D(A^{1/2})$ and let $\|\cdot\|_V$, $|\cdot|_H$ and $\|\cdot\|_*$ denote the norms in V , H and V' (dual of V) respectively. We also set $(\cdot, \cdot)_H$ for the inner product on H and $\langle \cdot, \cdot \rangle_{V' \times V}$ for the duality product between V' and V . We denote the eigenvalues of A by

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \dots \rightarrow \infty \quad (3.1)$$

and the corresponding eigenvectors by w_1, w_2, \dots . As for the non-linearly $g(u)$, we assume that g is C^1 -continuous from V into H and satisfies either

$$(N1) \quad |g(u) - g(v)|_H \leq L_1 \|u - v\|_V \quad \text{for all } u, v \text{ in } B^V(0; R)$$

or

$$(N2) \quad \|g(u) - g(v)\|_V \leq L_2 \|u - v\|_V \quad \text{for all } u, v \text{ in } B^{D(A)}(0; R),$$

where $B^V(0; R)$ and $B^{D(A)}(0; R)$ stand for R -balls in the spaces V and $D(A)$ respectively and $L_1 = L_1(R)$ and $L_2 = L_2(R)$ are the corresponding Lipschitz constants.

In order to check the squeezing properties we will take as \mathbb{H} the product space $E_0 = V \times H$; let us recall other related spaces as well. Let

$$E_0 = V \times H \quad \text{furnished with the norm } \|w\|_{E_0}^2 = \|u\|_V^2 + \varepsilon |v|_H^2, \text{ where } w = (u, v), \quad (3.2)$$

$$E_1 = D(A) \times V \quad \text{furnished with the norm } \|w\|_{E_1}^2 = \|u\|_{D(A)}^2 + \varepsilon |v|_V^2, \quad (3.3)$$

and finally,

$$E_{-1} = H \times V' \quad \text{furnished with the norm } \|w\|_{E_{-1}}^2 = \|u\|_H^2 + \varepsilon |v|_{V'}^2. \quad (3.4)$$

Classical existence and regularity results ensure the existence of a solution operator $S^\varepsilon(t)$ for the initial value problem (H^ε) and (IV) on E_i for $i = -1, 0, 1$. (See [T] or [BV].)

Theorem 3.1. (i) For $\varepsilon > 0$, if $f \in \mathcal{E}_b^1(\mathbb{R}^+; H)$ then for every $\{u_0, u_1\}$ in E_0 there exists a unique solution $u(t)$ for the initial value problem (H^ε) and (IV) . Moreover, $u \in \mathcal{E}_b(\mathbb{R}^+; V) \cap \mathcal{E}_b^1(\mathbb{R}^+; H)$.

(ii) If $\{u_0, u_1\}$ is in E_1 and $f \in \mathcal{E}_b^1(\mathbb{R}^+; H)$ then the unique solution $u(t)$ belongs to $\mathcal{E}_b(\mathbb{R}^+; D(A)) \cap \mathcal{E}_b^1(\mathbb{R}^+; V) \cap \mathcal{E}_b^2(\mathbb{R}^+; H)$.

B. THE SQUEEZING PROPERTY FOR SEMILINEAR WAVE EQUATIONS

Now we can proceed to show the squeezing property for the solution operator $\{S^\varepsilon(t)\}_{t \geq 0}$ of (H^ε) on a bounded subset B of E_1 with respect to E_0 -norm. In effect, we take as B the absorbing ball in E_1 , that is guaranteed to exist for all the equations under consideration (see [T] and [BV]), and as our Hilbert space \mathbb{H} we take E_0 .

As before, see 3.1, let $\{\lambda_N: N = 1, 2, \dots\}$ denote the sequence of eigenvalues of A and $\{w_N: N = 1, 2, \dots\}$ be the corresponding eigenvectors, i.e., $Aw_N = \lambda_N w_N$. Let $H_N = \text{span}\{w_1, w_2, \dots, w_N\}$. Then we set

$$p_N: H \rightarrow H_N \text{ be the orthogonal projection onto } H_N; \quad (3.5)$$

and

$$q_N = I - p_N. \quad (3.6)$$

Note that the projections p_N and q_N are orthogonal both in H and in V . It follows easily from the definition of the projection q_N that

$$|u|_H^2 \leq \frac{1}{\lambda_{N+1}} \|u\|_V^2 \quad \text{for all } u \text{ in } q_N V. \quad (3.7)$$

Next we define corresponding product projections on E_0 , namely,

$$P_N: E_0 \rightarrow (p_N V) \times (p_N H) \quad \text{and} \quad Q_N = I - P_N \quad (3.8)$$

in the canonical way by

$$P_N(u, v)^T = (p_N u, p_N v)^T \text{ for } (u, v)^T \text{ in } E_0. \quad (3.9)$$

We propose to show the squeezing property, Definition 2.2, via these orthogonal projections; let us emphasize once again that B is a bounded set in E_1 . The main theorem is the following.

Theorem 3.2. For any $\varepsilon > 0$, there exists $t_* = t_*(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon, t_*)$ such that the squeezing property is satisfied for $(\{S^\varepsilon(t)\}_{t \geq 0}, B)$ in the Hilbert space E_0 with $P = P_{N_0}$, as given in (3.9).

Once t_* and N_0 is specified, the existence of an inertial set follows directly from Theorem 2.1. Moreover, both the fractal dimension of \mathcal{M} and the exponential rate of convergence to \mathcal{M} can be estimated explicitly. We divide the proof of Theorem 3.2 into three steps. First we discuss the equivalency of some norms in E_0 and in $Q_N E_0$. Then considering the difference of two solutions to (H^ε) , we derive some estimates in terms of the norms discussed earlier. The final step involves the correct specification of t_* and N_0 so that the squeezing property is verified. Without loss of generality, we assume that $\varepsilon \leq 1$, otherwise by rescaling the equation (H^ε) we can reduce it to the case $\varepsilon \leq 1$.

Proof (of Theorem 3.2).

Step 1. Some Equivalent Norms on E_0 and $Q_N E_0$:

For $z = \{u, v\}^T$ in E_0 and $\varepsilon > 0$, define the function $N_\varepsilon(z)$ by

$$N^\varepsilon(z) = \frac{1}{2} |u|_H^2 + \varepsilon (u, v)_H + \varepsilon |v|_H^2 + \|u\|_V^2. \quad (3.10)$$

Lemma 3.1. Let $k = \max\{1 + \lambda_1^{-1}, 3/2\}$. If $\varepsilon \leq 1$ then N_ε is an equivalent norm in E_0 ; in fact for all z in E_0 ,

$$N^\varepsilon(z) \geq \|u\|_V^2 \quad \text{and} \quad \frac{1}{2} \|z\|_{E_0}^2 \leq N^\varepsilon(z) \leq k \|z\|_{E_0}^2. \quad (3.11)$$

Proof. Follows directly from the Schwartz inequality. Note that in this argument, the case $\varepsilon > 1$ can also be treated by modifying $\frac{1}{2}$, 1 and k with constants depending on ε . ■

Again for $z = \{u, v\}^T$ in E_0 we define another function by

$$M^\varepsilon(z) = (u, v)_H + \|z\|_{E_0}^2 = \|u\|_V^2 + (u, v)_H + \varepsilon|v|_H^2. \quad (3.12)$$

Lemma 3.2. If $\varepsilon \leq 1$ and N is large enough so that $\lambda_{N+1} \geq \varepsilon^{-1}$ (which is possible since $\lambda_N \rightarrow \infty$) then for all z in $Q_N E_0$ we have

$$\|z\|_{E_0}^2 \leq 2M^\varepsilon(z) \leq 3\|z\|_{E_0}^2. \quad (3.13)$$

Hence $M^\varepsilon(z)$ is equivalent to the usual E_0 -norm on $Q_N E_0$.

Proof. This result is a consequence of Schwartz inequality: note that by (3.6) we have

$$(u, v)_H \leq \frac{1}{2\varepsilon}|u|_H^2 + \frac{\varepsilon}{2}|v|_H^2 \leq \frac{1}{2\varepsilon\lambda_{N+1}}\|u\|_V^2 + \frac{\varepsilon}{2}|v|_H^2 \leq \frac{1}{2}\|z\|_{E_0}^2. \quad \blacksquare$$

Step 2. Estimates on the difference of solutions to (H^ε) :

Let u and \bar{u} be two solutions of (H^ε) in the space $\mathcal{C}_b(\mathbb{R}^+; V) \cap \mathcal{C}_b^1(\mathbb{R}^+; H)$ and let

$$w = u - \bar{u} \text{ and } W = \{w, w_t\}^T, \quad (3.14)$$

so that $W \in \mathcal{C}_b(\mathbb{R}^+; E_0)$. By Gronwall's inequality we can control the growth of $W(t)$:

Lemma 3.3. Let $k = \max\{1 + \lambda^{-1}, 3/2\}$ and $\alpha = L_1^2 + L_1 \lambda_1^{-1}$, where L_1 is as given in (N1), then for all $t > 0$,

$$\|W(t)\|_{E_0}^2 \leq 2ke^{\alpha t} \|W(0)\|_{E_0}^2. \quad (3.15)$$

Proof. Since w solves the equation

$$\varepsilon w_{tt} + w_t + Aw = g(u) - g(\bar{u}), \quad (3.16)$$

by multiplying (3.16) in H by $2w_t$ and w , we obtain that

$$\frac{d}{dt} N^\varepsilon(W) + (2 - \varepsilon)|w_t|_H^2 + \|w\|_V^2 = (g(\bar{u}) - g(u), 2w_t + w)_H \leq$$

$$\leq 2L_1\|w\|_V|w_t|_H + L_1\|w\|_V|w|_H \leq \left(L_1^2 + \frac{L_1}{\lambda_1}\right)\|w\|_V^2 + |w_t|_H^2, \quad (3.17)$$

where we have used (N1) and the definition of $N^\varepsilon(W)$ as given in (3.10). Recalling from (3.10) that $N^\varepsilon(W) \leq \|w\|_V^2$, we deduce from (3.17) that

$$\frac{d}{dt}N^\varepsilon(W) + (1-\varepsilon)|w_t|_H^2 + \|w\|_V^2 \leq \alpha N^\varepsilon(W),$$

whence (3.15) follows from the Gronwall's lemma and from (3.11). ■

Lemma 3.4. Let N be as given in Lemma 3.2, that is $\varepsilon\lambda_{N+1} \geq 1$; also let $w = u - \bar{u}$ as in Lemma 3.3, $\varphi = q_N w$ and $\Phi = \{\varphi, \varphi_t\}^T = Q_N W$. Then $M^\varepsilon(\Phi(t))$ satisfies, for all $t \geq 0$, the differential inequality

$$\frac{d}{dt}M^\varepsilon(\Phi(t)) + \frac{1}{2\varepsilon}M^\varepsilon(\Phi(t)) \leq \frac{6L_2^2}{\lambda_{N+1}}\|w(t)\|_V^2. \quad (3.18)$$

Proof. We apply q_N to (3.16) and note that q_N commutes with A , hence $\varphi = q_N w$ satisfies

$$\varepsilon\varphi_{tt} + \varphi_t + A\varphi = q_N(g(\bar{u}) - g(u)) \equiv \Gamma. \quad (3.19)$$

Multiplying the above equation in H by $2\varphi_t$ and $\frac{1}{\varepsilon}\varphi$, we obtain

$$\begin{aligned} \frac{d}{dt}M^\varepsilon(\Phi) + |\varphi_t|_H^2 + \frac{1}{\varepsilon}\|\varphi\|_V^2 + \frac{1}{2\varepsilon}(\varphi, \varphi_t)_H &= -\frac{1}{2\varepsilon}(\varphi, \varphi_t)_H + \left(\Gamma, 2\varphi_t + \frac{1}{\varepsilon}\varphi\right)_H \\ &\leq \frac{1}{2\varepsilon}|\varphi|_H|\varphi_t|_H + |\Gamma|_H \left(2|\varphi_t|_H + \frac{1}{\varepsilon}|\varphi|_H\right) \equiv S \end{aligned} \quad (3.20)$$

On the other hand, by (3.7) and (N2)

$$|\Gamma|_H = |q_N(g(u) - g(\bar{u}))|_H \leq \frac{1}{\lambda_{N+1}^{1/2}}\|g(\bar{u}) - g(u)\|_V \leq L_2\lambda_{N+1}^{-1/2}\|w\|_V, \quad (3.21)$$

so that S , as defined in (3.20), can be estimated by

$$\begin{aligned} S &\leq \frac{1}{2\varepsilon}|\varphi|_H|\varphi_t|_H + L_2\lambda_{N+1}^{-1/2}\|w\|_V \left(2|\varphi_t|_H + \frac{1}{\varepsilon}|\varphi|_H\right) \\ &\leq \frac{1}{2}|\varphi_t|_H^2 + \frac{3}{8\varepsilon^2}|\varphi|_H^2 + 6L_2^2\lambda_{N+1}^{-1}\|w\|_V^2. \end{aligned} \quad (3.22)$$

Consequently, returning back to (3.20) and using $\varepsilon\lambda_{N_0+1} \geq 1$ combined with the definition of $M^\varepsilon(\Phi)$ in (3.12) we obtain the desired result. \blacksquare

Step 3. Choosing t_* and N_0 .

Let $U = \{u_0, u_1\}$ and $\bar{U} = \{\bar{u}_0, \bar{u}_1\}$ be in E_0 and set $W(t) = S^\varepsilon(t)U - S^\varepsilon(t)\bar{U}$. We will show that there exists t_* such that if

$$\left\| P_{N_0} W(t_*) \right\|_{E_0} \leq \left\| Q_{N_0} W(t_*) \right\|_{E_0} \quad (3.23)$$

holds, then $\left\| W(t_*) \right\|_{E_0} \leq \frac{1}{8} \left\| W(0) \right\|_{E_0}$ holds as well. To this end, we apply Lemma 3.4 to $W(t)$ so that (3.18) implies that

$$\begin{aligned} \frac{d}{dt} M^\varepsilon(\Phi) + \frac{1}{2\varepsilon} M^\varepsilon(\Phi) &\leq 6L_2^2 \lambda_{N_0+1}^{-1} \|w(t)\|_V^2 \leq 6L_2^2 \lambda_{N_0+1}^{-1} \|W(t)\|_{E_0}^2 \\ &\leq 12kL_2^2 \lambda_{N_0+1}^{-1} e^{\alpha t} \|W(0)\|_{E_0}^2. \end{aligned} \quad (3.24)$$

From the above inequality, it follows that

$$\begin{aligned} M^\varepsilon(\Phi(t)) &\leq M^\varepsilon(\Phi(0)) e^{-\frac{1}{2\varepsilon}t} + 12kL_2^2 \lambda_{N_0+1}^{-1} \|W(0)\|_{E_0}^2 e^{-\frac{1}{2\varepsilon}t} \int_0^t \exp\left(\left(\alpha + \frac{1}{2\varepsilon}\right)s\right) ds \\ &\leq M^\varepsilon(\Phi(0)) e^{-\frac{1}{2\varepsilon}t} + 24kL_2^2 \lambda_{N_0+1}^{-1} \|W(0)\|_{E_0}^2 e^{\alpha t} \end{aligned}$$

so that, if N_0 is such that $\varepsilon\lambda_{N_0+1} \geq 1$ then by (3.13), since $\Phi(t)$ is in $Q_N E_0$, we have

$$\left\| \Phi(t) \right\|_{E_0}^2 \leq 6e^{-\frac{1}{2\varepsilon}t} \left\| \Phi(0) \right\|_{E_0}^2 + 48kL_2^2 \lambda_{N_0+1}^{-1} e^{\alpha t} \left\| W(0) \right\|_{E_0}^2.$$

Consequently, using that Q_N is a projection on E_0 , we obtain

$$\left\| Q_N W(t) \right\|_{E_0}^2 \leq 6 \left\| W(0) \right\|_{E_0}^2 \left(e^{-\frac{1}{2\varepsilon}t} + 8kL_2^2 \lambda_{N_0+1}^{-1} e^{\alpha t} \right). \quad (3.25)$$

Now we are ready to choose the values of t_* and N_0 . First, we choose t_* such that $12 \exp\left\{-\frac{t_*}{2\varepsilon}\right\} = \frac{1}{2} \left(\frac{1}{8}\right)^2$, then we choose N_0 large enough so that

$$96kL_2^2 \lambda_{N_0+1}^{-1} e^{\alpha t_*} \leq \frac{1}{2} \left(\frac{1}{8}\right)^2. \quad (3.26)$$

Assume that for this particular choice of t_* (3.23) holds then from (3.25) and (3.26) we deduce that

$$\begin{aligned}\|W(t_*)\|_{E_0}^2 &= \|P_N W(t_*)\|_{E_0}^2 + \|Q_N W(t_*)\|_{E_0}^2 \leq 2\|Q_N W(t_*)\|_{E_0}^2 \\ &\leq 12\|W(0)\|_{E_0}^2 \left(e^{-t_*/2\varepsilon} + 8kL_2^2 \lambda_{N+1}^{-1} e^{\alpha t_*} \right) \\ &\leq \left(\frac{1}{8} \right)^2 \|W(0)\|_{E_0}^2,\end{aligned}$$

With this the proof of Theorem 3.2 is complete: we have chosen

$$t_* = 2\varepsilon \ln\left(\frac{3}{8}\right) \tag{3.27}$$

and $N_0 = N_0(\varepsilon, t_*)$ so large that

$$\lambda_{N_0+1} \geq \max\left\{\frac{1}{\varepsilon}, 3kL_2^2 e^{\alpha t_*}\right\}. \tag{3.28}$$

■

4. APPLICATIONS

The results obtained in the previous section on the abstract equation (H^ε) can be applied to a wide variety of equations. In particular, we mention the sine-Gordon equation and the nonlinear wave equation of relativistic quantum mechanics, i.e. Klein-Gordon type equations, as well as systems of sine-Gordon equations. All these examples are described in [T], chapter IV, to which we refer for further details. As for the non-linearity of type $g(u) = u^3 + p(u)$ in three space dimension we refer the reader to [BV]. In all the examples, Ω is an open bounded subset of \mathbb{R}^3 with a smooth boundary.

1. Sine-Gordon equation:

In this case (H^ε) has the form

$$\varepsilon u_{tt} + u_t - \Delta u + \beta \sin u = f;$$

for the Dirichlet boundary conditions, we choose $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ with the usual norms $\|u\|_V = |\nabla u|$ and $|u|_H = |u|_{L^2(\Omega)}$. The verification of the Lipschitz conditions (N1) and (N2) for $g(u) = \sin u$ is immediate on the absorbing set in E_1 and we obtain that

$$L_1 = \beta \lambda_1^{-1/2} \text{ and } L_2 = \beta(1 + R)$$

where R is the E_1 -radius of the ball containing the E_1 -absorbing set whose existence is guaranteed, see [T] p. 189.

2. An equation from Quantum Mechanics:

(H^ε) now has the form

$$\varepsilon u_{tt} + u_t - \Delta u + u^3 + p(u) = f$$

with $p(u)$ a quadratic polynomial; we choose H and V as before, for Dirichlet boundary conditions. Verification of the Lipschitz conditions (N1) and (N2) is straightforward; in the particular case $g(u) = u^3$ we have

$$|u^3 - v^3|_H \leq |u - v|_{L^6} |u^2 + uv + v^2|_{L^3} \leq 3R^2 \|u - v\|_V,$$

where R is the radius of a ball containing the absorbing set now in E_0 . Likewise, we have

$$\begin{aligned} \|u^3 - v^3\|_V &\leq 3 \left(|u^2 - v^2|_{L^3} |\nabla u|_H + 3 |v^2|_{L^3} |\nabla(u - v)|_H \right) \\ &\leq 3 \left\{ (|u|_{L^\infty} + |v|_{L^\infty}) |\nabla u|_{L^3} |u - v|_{L^6} + |v|_{L^\infty}^2 \|u - v\| \right\} \\ &\leq 3 \left\{ (\|u\|_{H^2} + \|v\|_{H^2}) \|\nabla u\|_V \|u - v\|_V + \|v\|_{H^2}^2 \|u - v\|_V \right\}. \end{aligned}$$

So that we can choose, in (N2), $L_2 = 9R^2$, where R is the radius of an absorbing ball in E_1 (see [BV] and [T]).

3. Systems of Sine-Gordon equations:

We can also treat systems like those considered in [T], chap. IV, examples 4.2 and 4.3. In these cases, (H^ε) have the form, respectively,

$$(A) \quad \begin{cases} \varepsilon u_{tt} + u_t - \Delta u + \sin u + (u - v) = f_1, \\ \varepsilon u_{tt} + v_t - \Delta v + \sin v + (v - u) = f_2; \end{cases}$$

and

$$(B) \quad \begin{cases} \varepsilon u_{tt} + u_t - \Delta u + \sin(u + v) = f_1, \\ \varepsilon u_{tt} + v_t - \Delta v + \sin(u - v) = f_2. \end{cases}$$

In both cases, we choose $V = (H_0^1(\Omega))^2$ and $H = (L^2(\Omega))^2$; the non-linear terms

$$G_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sin u + u - v \\ \sin v + v - u \end{pmatrix}$$

and

$$G_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sin(u + v) \\ \sin(u - v) \end{pmatrix}$$

can also be shown to satisfy both of the Lipschitz conditions (N1) and (N2) in a straightforward way, analogously to the case of the sine-Gordon equation; we obtain respectively

$$L_1 = 6\lambda_1^{-1/2} \text{ and } L_2 = 2(3 + R) \quad \text{for (A)}$$

and

$$L_1 = 4\lambda_1^{-1/2} \text{ and } L_2 = 4(1 + R) \quad \text{for (B)}.$$

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