A HARTMAN-GROBMAN THEOREM
FOR SCALAR REACTION-DIFFUSION EQUATIONS

By

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§1. Introduction.
Consider the following scalar parabolic equation

(1.1) \[ u_t = u_{xx} + f(u), \quad 0 < x < \pi, \quad t > 0 \]

with the Dirichlet boundary condition

(1.2) \[ u = 0 \quad \text{at} \quad x = 0, \pi, \]

where \( f : \mathbb{R} \to \mathbb{R} \) is a smooth function. Precise conditions on \( f \) will be given later.

Let \( \phi(x) \) be a stationary solution of problem (1.1), (1.2). Suppose that \( \phi(x) \) is hyperbolic, i.e., the operator \( \frac{\partial^2}{\partial x^2} + f'(\phi(x)) \) has no zero eigenvalue under the boundary condition (1.2). One of the fundamental problems in dynamical systems is the following: Is the nonlinear problem (1.1), (1.2) topologically conjugate to a linear problem in a neighborhood of the stationary solution \( \phi(x) \)? In other words, is the flow nearby the hyperbolic stationary solution \( \phi(x) \) structurally stable? Can the nonlinear problem (1.1), (1.2) be linearized by a homeomorphism in a neighborhood of \( \phi \)?

The theory of linearization for ordinary differential equations and diffeomorphisms near a fixed point has been widely studied. The literature on this subject is extensive; in this paper we will mention only some of it. For simplicity, we assume that 0 is a fixed point of ordinary differential equations. Then the ordinary differential equations near the fixed point 0 can be written as

(1.3) \[ \dot{y} = By + f(y) \]

where \( y \in \mathbb{R}^n, B \) is an \( n \times n \) matrix, \( f : U \subset \mathbb{R}^n \to \mathbb{R}^n \) is a smooth function with \( f(0) = 0 \) and \( f'(0) = 0 \), where \( U \) is an open neighborhood of 0. In the theory of linearization one is concerned with the existence of smooth maps

\[ z = \Phi(y) \]

where \( \Phi \) has a smooth inverse, defined for small \( |y| \) and transforming (1.3) to

\[ \dot{z} = Bz. \]

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Analogously, the diffeomorphism $G : U \subset \mathbb{R}^n \to \mathbb{R}^n$ near the fixed point 0 can be written as

\[(1.4) \quad G(y) = Cy + g(y)\]

where $y \in \mathbb{R}^n$, $C$ is an nonsingular $n \times n$ matrix, and $g : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function with $g'(0) = 0$. One is concerned with the existence of a $C^k$ map $\Phi(y)$ defined for small $|y|$ such that $\Phi \circ G \circ \Phi^{-1} = C$. In the analytic case, this problem has been considered by Poincaré [27]; Birkhoff [7]; Siegel [32],[33]; Arnold [3]; Moser [23]; Meyer [21]; Zehnder [38], [39] and others. The general, nonanalytic case has been studied by Sternberg [34], [35]; Naguma and Isé [26]; Chen [8]; Hartman [16], [17], [18]; Nelson [24]; Pugh [29]; Takens [37] and others. Recently, more delicate conditions showing that the nonlinear system (1.3) (resp. (1.4)) admits a $C^s$-smooth linearization have been obtained by Sell [30],[31] and Beliskii [4], [5]. Stowe [36] gives a complete classification for two dimensional systems. The other conditions under which the ordinary differential equations admit a $C^k$ linearization have been obtained by Sami Elbialy [12]. The point we want to emphasize here is that the eigenvalues of $B$ (resp. $C$) have to satisfy certain nonresonance conditions in order to have a $C^s(s \geq 1)$ linearization, even if the nonlinear terms are analytic. Otherwise, it is false; see, for example, Hartman [16] and Sell [28]. More generally, if the fixed point 0 is hyperbolic, i.e., $B$ (resp. $C$) has no eigenvalue on the imaginary axis (resp. unit circle), then the nonlinear system (1.3) (resp. (1.4)) can be $C^0$ linearized. This is the Hartman-Grobman theorem. Proofs can be found in Hartman [18], Grobman [14], Chow and Hale [9], and Palmer [27]. It is also known that there are the ordinary differential equations with nonhyperbolic $B$ which do not admit a $C^0$ linearization. Pugh [29] proves the Hartman-Grobman theorem for diffeomorphisms in a Banach Space. However, this result can’t be applied to parabolic equations since the time-1 map is not a diffeomorphism. Mora and Sola-Morales [22] proved a $C^1$ linearization theorem for damped wave equations.

More recently, Nikolenco [25] and Zehnder [40] prove the Siegel’s theorem for the following evolution equations in a Banach space

\[(1.5) \quad \dot{u} = Au + f(u)\]

where $A$ is diagonalizable and generates a $C^0$ semigroup, $f$ is analytic with $f(u) = O(|u|^2)$ as $|u| \to 0$. It is proved that if the eigenvalues of $A$ satisfy certain “small-divisor” conditions, then (1.5) can be transformed to the linear equation $\dot{v} = Av$ by an analytic transformation near the identity operator. The proofs are essentially based on the KAM accelerated convergence method. However their approaches are different. Foias and Saut [13] obtain a certain kind of analytic linearization theorem for a class of Navier-Stokes equations provided the eigenvalues satisfy certain nonresonance conditions.

However, in applications, the eigenvalues of $A$ may not satisfy the “small-divisor” conditions nor even certain nonresonance conditions. Another difficulty we face here is that we can’t solve the parabolic equations backwards. In other words, the corresponding
time-1 map is not a diffeomorphism since 0 is the accumulative point of the eigenvalues.  
The known methods for proving the $C^0$ linearization for the ordinary differential equations 
do not work for the problem (1.1) and (1.2).

In this paper, we study the $C^0$-linearization theory for the problem (1.1), (1.2) near 
the hyperbolic stationary solution $\phi(x)$. The problem (1.1), (1.2) in a neighborhood of 
$\phi(x)$ can be written as

\begin{align*}
(1.6) & \quad v_t = v_{xx} + a(x)v + g(x,v), \quad 0 < x < \pi, \ t > 0 \\
(1.7) & \quad v = 0 \quad \text{at} \quad x = 0, \ \pi
\end{align*}

where $a$ is a smooth function and $g$ is a smooth function with $g(x,0) = 0$ and $g'_u(x,0) = 0$. 
Let $g^\varepsilon(v) = g(x,v)$ for each $v \in H_0^1(0,\pi)$. Our main result is

**Theorem 1.1.** Assume that $a \in C^0((0,\pi))$ and $g^\varepsilon : U \to H_0^1(0,\pi)$ is a $C^1$ mapping 
with $g^\varepsilon(0) = 0$ and $Dg^\varepsilon(0) = 0$, where $U$ is an open neighborhood of 0 in $H_0^1(0,\pi)$ and $D$ 
is the differentiation operator. Then there exists an open neighborhood $V$ of 0 in $H_0^1(0,\pi)$ 
and a homeomorphism $\Phi : V \to \Phi(V) \subset H_0^1(0,\pi)$ such that if $v(t,x)$ is a solution of (1.6), 
(1.7) and $v(t,\cdot) \in V$, then $\Phi(v(t,x))$ is a solution of the linear equation

\begin{align*}
(1.8) & \quad w_t = w_{xx} + a(x)w, \quad 0 < x < \pi, \ t > 0 \\
(1.9) & \quad w = 0 \quad \text{at} \quad x = 0, \pi,
\end{align*}

and if $w(t,x)$ is a solution of (1.8), (1.9) and $w(t,\cdot) \in \Phi(V)$, then $\Phi^{-1}(w(t,x))$ is a solution 
of (1.6), (1.7).

We note that in this theorem we only need that $g^\varepsilon$ is a Lipschitz continuous function 
with a small Lipschitz constant. The space $H_0^1(0,\pi)$ can be replaced by other spaces, which 
depends on the nonlinear function $g^\varepsilon$. We believe that this result should hold for more 
general equations which include the case in which $A$ is a sectorial operator and $g^\varepsilon$ is a $C^1$ 
function from a fractional power space to a larger one. This is going to be discussed some 
where else.

**Remark.** This result holds for general evolution equations (1.5) in a Hilbert space 
provided that the spectrum $\sigma(A)$ of $A$ satisfies the following conditions:

(i) $\sigma(A)$ is bounded from below.

(ii) There are two sequences $\{a_n\}$, $\{b_n\}$, and constants $C > 0$ and $\beta > \frac{1}{2}$ such that

$$a_1 < b_1 < a_2 < b_2 < \ldots,$$

$$a_n \geq Cn^\beta, \quad \text{for large} \quad n,$$

$$\inf_{n \geq 1} \{a_{n+1} - b_n\} > 0$$
and \( \sigma(A) \subset \cup_{n \geq 1} (a_n, b_n) \).

For example, let \( A = -\Delta \) with the domain \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \), where \( \Omega = (0, \pi)^d \) for \( d \leq 3 \). It is known that the spectrum of \( A \) consists only of eigenvalues \( \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \) with finite multiplicity and the eigenvalues satisfy the following asymptotic property

\[
\lambda_n \geq C_d n^{2\frac{d}{d+1}}, \quad \text{for large } n,
\]

where \( C_d \) is a positive constant, and satisfy \( \lambda_{n+1} - \lambda_n \geq 1 \) for \( n \geq 1 \). See, for example, Courant and Hilbert [11]. This implies that \( A \) satisfy the above properties (i) and (ii).

The method we use here is based on the invariant manifold theory and the invariant foliation theory. We will show the existence of infinitely many one-dimensional invariant manifolds and the existence of invariant foliations. Using these one-dimensional invariant manifolds as new axes and using the invariant foliations to trace new coordinates, we can completely decouple the problem (1.6), (1.7) into an infinitely many one-dimensional system. Then we linearize each one-dimensional problem, put them together and get a \( C^0 \) linearization theory for (1.6), (1.7).

We organize this paper as follows: in Section 2 we formulate problems and introduce notations; in Section 3, we prove the existence of infinitely many one-dimensional invariant manifolds and the existence of n-dimensional invariant manifolds for each integer \( n \geq 1 \); in Section 4, we prove the existence of invariant foliations; in Section 5, we study transformations for flows on the n-dimensional manifold; in Section 6 we prove theorem 1.1.

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**§2. Hypotheses and Notation.**

Consider the following scalar parabolic equation

\[
(2.1) \quad u_t = u_{xx} + a(x)u + f(x, u), \quad 0 < x < \pi, \; t > 0
\]

with the Dirichlet boundary condition

\[
(2.2) \quad u = 0 \quad \text{at} \quad x = 0, \pi
\]

where \( a \) is \( C^0 \) and \( f \) is \( C^1 \), \( f(x, 0) = 0 \) and \( f'(x, 0) = 0 \).

Let \( X = L^2(0, \pi) \) and \( Au = -\left( \frac{\partial^2}{\partial x^2} + a(x) \right) u \) for smooth \( u \) which vanishes at 0 and \( \pi \). Then \( A \) can be extended to a self-adjoint densely defined operator in \( X \) with the domain \( D(A) = H^2(0, \pi) \cap H^1_0(0, \pi) \). Let \( E = H^1_0(0, \pi) \). Define \( f^*(u) = f(x, u) \) for each \( u \in E \).
Assumption: $f^e : U \to E$ is $C^1$ with $f^e(0) = 0$ and $Df^e(0) = 0$, where $U$ is an open neighborhood of $0$ in $E$ and $D$ is the differentiation operator.

Let $\theta$ be a $C^\infty$ cut-off function from $[0, \infty]$ to $[0, 1]$ with

$$\theta(s) = 1 \text{ for } 0 \leq s \leq 1, \quad \theta(s) = 0 \text{ for } s \geq 2, \quad \sup_{s \geq 0} |\theta'(s)| \leq 2,$$

and we set $\theta_\rho(s) = \theta\left(\frac{s}{\rho}\right)$ for $\rho > 0$, and $F_\rho(u) = \theta_\rho(||u||)f^e(u)$, where $||.||$ is the usual norm of the space $H^1_0(0, \pi)$. Clearly $F_\rho$ is $C^1$ from $E$ to $E$, $\max_{u \in E} ||F_\rho(u)|| \to 0$ as $\rho \to 0$ and $\max_{u \in E} ||DF(u)|| \to 0$ as $\rho \to 0$. Let $\text{Lip}F_\rho$ denote the Lipschitz constant of $F_\rho$. Consider the following modified equation

$$(2.3) \quad \dot{u} = -Au + F_\rho(u).$$

Clearly, the problem (2.1), (2.2) is the same as equation (2.3) in the ball $B_\rho = \{u \in E ||u|| \leq \rho\}$.

From now on, we will study equation (2.3) instead of (2.1), (2.2) because we study local properties.

By the Sturm-Liouville theory, we have that the spectrum $\sigma(A)$ of $A$ consists only of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ and the corresponding eigenfunctions $e_1, \cdots, e_n, \cdots$ form a basis of $E$. Moreover

$$(2.4) \quad \lambda_n = n^2 + o(1) \text{ as } n \to \infty.$$

Let

$$(2.5) \quad \alpha_n = \min\left\{\frac{\lambda_{n+1} - \lambda_n}{2}, \frac{\lambda_n - \lambda_{n-1}}{2}\right\} \quad \text{for } n > 1$$

and

$$(2.6) \quad \alpha = \min_{n > 1}\{\alpha_n\}$$

Thus we have that $\frac{\lambda_n}{\alpha_n} = O(1)$ as $n \to \infty$ and $\alpha > 0$. Let

$$E_n = \text{span}\{e_n\},$$
$$E_n^\perp = \text{span}\{e_1, \cdots, e_{n-1}, e_{n+1}, \cdots\},$$
$$E_m^n = \text{span}\{e_n, \cdots, e_m\}, \quad \text{for } m > n,$$

and let $P_n, P_n^\perp$ and $P_m^n$ be the corresponding projections. Let $A_m^n = A|_{E_m^n}$ and write $u \in E$ as $u = \sum_{n=1}^\infty u_n$, where $u_n \in E$. 

§3. Invariant manifolds.

By choosing suitable $\rho$, we are going to show that there exist infinitely many one-dimensional invariant manifolds for equation (2.3) and that there exists an $n$-dimensional invariant manifolds for equation (2.3) for each $n$. We also show the existence of invariant manifolds for flows on the $n$-dimensional manifold.

**Theorem 3.1.** Let $\rho > 0$ such that $\frac{\lambda}{\alpha} \text{Lip}F_\rho < 1$. Then for each $n$ there is a one-dimensional invariant manifold for (2.3) which is given by

$$M_n = \{u_n + h_n(u_n)|u_n \in E_n\}$$

where $h_n : E_n \rightarrow E_n^\perp$ is $C^{0,1}$ (Lipschitz continuous) and satisfies $h_n(0) = 0$ and

$$\text{Lip} \ h_n \leq \frac{2 \text{Lip}F_\rho}{\alpha_n - 3 \text{Lip}F_\rho} < 1$$

**Remark.** In fact, it can be shown that $h_n$ is $C^1$. However, for our purposes, we don’t need that. The interested reader may consult Chow-Lu [10].

**Proof.** Let us define the following Banach space

$$C_{\alpha_n} = \left\{ f : \mathbb{R} \rightarrow E \text{ is continuous and } \sup_{t \in \mathbb{R}} ||e^{\lambda_n t - \alpha_n |t|} f(t)|| < \infty \right\}$$

with the norm $|f|_{\alpha_n} = \sup_{t \in \mathbb{R}} ||e^{\lambda_n t - \alpha_n |t|} f(t)||$, where $\alpha_n$ is given by (2.5).

Denote by $u(t, u^0)$ the solution of (2.3) with initial data $u(0, u^0) = u^0$. Let

$$M_n = \{u^0 | u(t, u^0) \text{ is defined for all } t \in \mathbb{R} \text{ and } u(\cdot, u^0) \in C_{\alpha_n} \}.$$

Then clearly, $M_n$ is nonempty since $0 \in M_n$ and is invariant under the flow of equation (2.3). We are going to show that $M_n$ is given by the graph of a $C^{0,1}$ function over $E_n$.

**Claim.** $u^0 \in M_n \iff u(\cdot) \in C_{\alpha_n}$ with $u(0) = u^0$ and satisfies

$$u = e^{-\lambda_n t} \xi + \int_0^t e^{-\lambda_n (t-s)} P_n F_\rho(u) ds$$

$$+ \int_0^t e^{-A_n^{n-1}(t-s)} P_n^{n-1} F_\rho(u) ds$$

$$+ \int_{-\infty}^t e^{-A_n^{n+1}(t-s)} P_n^{n+1} F_\rho(u) ds,$$

where $\xi = P_n u^0$.  

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First we prove "⇒". By using the variation of constants formula, we have

\begin{equation}
\tag{3.3}
P_n u(t, u^0) = e^{-\lambda nt} P_n u^0 + \int_0^t e^{-\lambda_n(t-s)} P_n F_\rho(u) ds,
\end{equation}

\begin{align}
\tag{3.4}
P_n^{-1} u(t, u^0) &= e^{-A_n^{-1}(t-\tau)} P_n^{-1} u(\tau, u^0) \\
&+ \int_{\tau}^t e^{-A_n^{-1}(t-s)} P_n^{-1} F_\rho(u) ds,
\end{align}

\begin{equation}
\tag{3.5}
P_{n+1}^\infty u(t, u^0) = e^{-A_{n+1}^\infty(t-\tau)} P_{n+1}^\infty u(\tau, u^0) \\
&+ \int_{\tau}^t e^{-A_{n+1}^\infty(t-s)} P_{n+1}^\infty F_\rho(u) ds.
\end{equation}

Since \( u \in C_{\alpha_n} \), we have that for \( t < \tau, 0 < \tau \)

\begin{align*}
||e^{-A_n^{-1}(t-\tau)} P_n^{-1} u(\tau, u^0)|| \\
&\leq e^{-\lambda_{n-1}(t-\tau)} e^{-\lambda_n \tau + \alpha_n \tau} |u|_{\alpha_n} \\
&\leq e^{-\lambda_{n-1} t} e^{-\alpha_n \tau} |u|_{\alpha_n} \to 0 \quad \text{as} \quad \tau \to +\infty.
\end{align*}

Taking the limit \( \tau \to +\infty \) in (3.4), we have

\begin{equation}
\tag{3.6}
P_n^{-1} u(t, u^0) = \int_\infty^t e^{-A_n^{-1}(t-s)} P_n^{-1} F_\rho(u) ds.
\end{equation}

Similarly, we have for \( \tau < t, \quad \tau < 0 \)

\begin{align*}
||e^{-A_{n+1}^\infty(t-\tau)} P_{n+1}^\infty u(\tau, u^0)|| \\
&\leq e^{-\lambda_{n+1}(t-\tau)} e^{-\lambda_n \tau - \alpha_n \tau} |u|_{\alpha_n} \\
&= e^{-\lambda_{n+1} t} e^{(\lambda_{n+1} - \lambda_n - \alpha_n) \tau} |u|_{\alpha_n} \\
&\leq e^{-\lambda_{n+1} t} e^{\alpha_n \tau} |u|_{\alpha_n} \to 0 \quad \text{as} \quad \tau \to -\infty.
\end{align*}

Taking the limit \( \tau \to -\infty \) in (3.5), we have

\begin{equation}
\tag{3.7}
P_{n+1}^\infty u(t, u^0) = \int_{-\infty}^t e^{-A_{n+1}^\infty(t-s)} P_{n+1}^\infty F_\rho(u) ds.
\end{equation}

Putting (3.3), (3.6) and (3.7) together, we have (3.2). "≤" follows from the variation of constants formula, see, for example, Henry [19].
Let $J_n(u, \xi)$ be the right hand side of equality (3.2). It is not hard to see that $J_n$ is well-defined from $C_{\alpha_n} \times E_n$ to $C_{\alpha_n}$. For each $u, \bar{u} \in C_{\alpha_n}$, we have

$$|J_n(u, \xi) - J_n(\bar{u}, \xi)|_{\alpha_n}$$

$$\leq \sup_{t \in \mathbb{R}} \left\{ e^{\lambda_n t - \alpha_n |t|} \left( \left\| \int_0^t e^{-\lambda_n (t-s)} P_n(F_\rho(u) - F_\rho(\bar{u})) ds \right\| + \int_{-\infty}^t e^{-A_{n-1}^{n-1} (t-s)} P_1^{n-1} (F_\rho(u) - F_\rho(\bar{u})) ds \right. 

+ \left. \int_{-\infty}^t e^{-A_{n+1}^{\infty} (t-s)} P_{n+1}^{\infty} (F_\rho(u) - F_\rho(\bar{u})) ds \right\| \right\}$$

$$\leq \frac{3}{\alpha_n} \text{Lip}_\rho \bar{u} - u||_{\alpha_n}$$

Obviously $J_n$ is Lipschitz continuous in $\xi$. By the assumption of this theorem, we have

$$\frac{3 \text{Lip}_\rho}{\alpha_n} < 1.$$ Hence $J_n$ is a uniform contraction with respect to the parameter $\xi$. By the uniform contraction mapping principle, we have that for each $\xi \in E_n$, $J_n(\cdot, \xi)$ has a unique fixed point $u(\cdot; \xi) \in C_{\alpha_n}$ and $u(\cdot; \cdot)$ is Lipschitz from $E_n$ to $C_{\alpha_n}$ and satisfies

$$|u(\cdot; \xi) - u(\cdot; \bar{\xi})|_{\alpha_n} \leq \frac{\alpha_n}{\alpha_n - 3 \text{Lip}_\rho} ||\xi - \bar{\xi}||.$$ (3.8)

In other words, equation (3.2) has the unique solution $u(\cdot, \xi) \in C_{\alpha_n}$ which satisfies (3.8).

Let

$$h_n(\xi) = P_n^{+1} u(0; \xi)$$

$$= \int_0^\infty e^{A_{n+1}^{\infty} s} P_{n+1}^{\infty} F_\rho(u(s; \xi)) ds$$

$$+ \int_{-\infty}^0 e^{A_{n}^{\infty} s} P_n^{n-1} F_\rho(u(s; \xi)) ds.$$ Then $h_n(0) = 0$, $h_n$ is Lipschitz continuous and satisfies

$$||h_n(\xi) - h_n(\bar{\xi})|| \leq \frac{2 \text{Lip}_\rho}{\alpha_n - 3 \text{Lip}_\rho} ||\xi - \bar{\xi}||.$$ Using (3.2), we have

$$M_n = \{ u_n + h_n(u_n) | u_n \in E_n \}.$$ This completes the proof. \[\Box\]
Theorem 3.2. Choose $\rho$ such that $\frac{3}{\alpha} \text{Lip} F_\rho < 1$. Then for each $n$ there exists an $n$-dimensional invariant manifold for (2.3) which is given by

$$M_1^n = \{ p + h_1^n(p) | p \in E_1^n \}$$

where $h_1^n : E_1^n \to E_{n+1}^\infty$ is Lipschitz continuous and satisfies

$$h_1^n(0) = 0$$

$$\text{Lip } h_1^n \leq \frac{\text{Lip} F_\rho}{\alpha_n - 2 \text{Lip} F_\rho}$$

and

$$||h_1^n(p)|| \leq \frac{1}{\lambda_{n+1}} \sup_{u \in E} ||F_\rho(u)||, \text{ if } \lambda_{n+1} > 0.$$ (3.9)

Proof. Define the following Banach Space

$$C_{\alpha_n}^- = \left\{ f | f : \mathbb{R}^- \to E \text{ is continuous and } \sup_{t \leq 0} ||e^{\lambda_n t + \alpha_n t} f(t)|| < \infty \right\}$$

with the norm $|f|_{\alpha_n}^- = \sup_{t \leq 0} ||e^{\lambda_n t + \alpha_n t} f(t)||$. Let

$$M_1^n = \{ u^0 | u(t, u^0) \text{ is defined for all } t \leq 0 \text{ and } u(\cdot, u^0) \in C_{\alpha_n}^- \}.$$  

Then $M_1^n$ is nonempty and invariant under the flow of equation (2.3). We will prove that $M_1^n$ is given by the graph of a $C^{0,1}$ function over $E_1^n$. To see this, we first claim

Claim: $u^0 \in M_1^n \iff u(\cdot) \in C_{\alpha_n}^-$ with $u(0) = u^0$ and satisfies

$$u(t) = e^{-A_1^n t} p + \int_0^t e^{-A_1^n (t-s)} P_1^n F_\rho(u) ds$$

$$+ \int_{-\infty}^t e^{-A_{n+1}^\infty (t-s)} P_{n+1}^\infty F_\rho(u) ds.$$ (3.11)

where $p = P_1^n u^0$. The proof of this claim is similar to that in Theorem 3.1. Now let $J_1^n(u, p)$ be the right hand side of equality (3.11). We have

$$||J_1^n(u, p) - J_1^n(u, \bar{u})||_{\alpha_n}^- \leq \frac{2}{\alpha_n} \text{Lip} F_\rho |u - \bar{u}|_{\alpha_n}^-$$

and

$$||J_1^n(u, p) - J_1^n(u, \bar{p})||_{\alpha_n}^- \leq ||p - \bar{p}||.$$
By the assumption of Theorem 3.2, we have that $\frac{2}{\alpha_n} \text{Lip} F_\rho < 1$. Using the uniform contraction mapping principle, we have that for each $p \in E_1^n$ equation (3.11) has a unique solution $u(\cdot; p) \in C_{\alpha_n}^-$, which is Lipschitz continuous in $p$. Furthermore

$$|u(\cdot; p) - u(\cdot; \bar{p})|_{\alpha_n} \leq \frac{\alpha_n}{\alpha_n - 2 \text{Lip} F_\rho} ||p - \bar{p}||.$$ 

Let

$$h_1^n(p) = P_{n+1}^\infty u(0; p) = \int_{-\infty}^{0} e^{\Lambda_{n+1}^{\infty}s} P_{n+1}^\infty F_\rho(u(s; p))ds.$$ 

Then $h_1^n(0) = 0$ and $h_1^n$ is Lipschitz continuous from $E_1^n$ to $E_1^\infty$ with

$$\text{Lip} h_1^n \leq \frac{\text{Lip} F_\rho}{\alpha_n - 2 \text{Lip} F_\rho} < 1.$$ 

If $\lambda_{n+1} > 0$, then

$$||h_1^n(p)|| \leq \frac{1}{\lambda_{n+1}} \sup_{u \in E_1^n} ||F_\rho(u)||.$$ 

By the claim, we have

$$M_1^n = \{ p + h_1^n(p) | p \in E_1^n \}.$$ 

This completes the proof. ■

Now let us consider flows on the $n$-dimensional invariant manifold $M_1^n$. The flows on $M_1^n$ are described by the following equation

$$(3.12) \quad \dot{p} = -A_1^n p + P_1^n F_\rho(p + h_1^n(p)),$$ 

where $p \in E_1^n$.

Analogously, we have the following theorems for equation (3.12).

**Theorem 3.3.** Choose $\rho$ such that $\frac{10}{\alpha} \text{Lip} F_\rho < 1$. Then for each $k, 1 \leq k \leq n$, there exists a one-dimensional invariant manifold for (3.12) which is given by

$$M_k^{k,n} = \{ \xi + l_k^{k,n}(\xi) | \xi \in E_k \}$$

where $l_k^{k,n}(0) = 0$, $l_k^{k,n}$ is Lipschitz continuous from $E_k$ to $E_1^{k-1} \oplus E_1^n$ and satisfies

$$\text{Lip} l_k^{k,n} \leq \frac{4 \text{Lip} F_\rho}{\alpha_k - 6 \text{Lip} F_\rho} < 1.$$
Note \( \text{Lip} (P^n_1 F_\rho(\cdot + h^n_2(\cdot))) \leq 2 \text{Lip}_\rho. \)

**Theorem 3.4.** Choose \( \rho \) such that \( \frac{k}{\alpha} \text{Lip}_\rho < 1. \) Then for each \( k, 1 \leq k \leq n, \) there exists a \( k \)-dimensional invariant manifold for (3.12) which is given by

\[
M^k_1 = \{ \xi + l^k_1(\xi) | \xi \in E^k_1 \}
\]

where \( l^k_1(0) = 0, l^k_1 \) is Lipschitz continuous from \( E^k_1 \) to \( E^{n}_{k+1} \) and satisfies

\[
\text{Lip} (l^k_1) \leq \frac{2 \text{Lip}_\rho}{\alpha_k - 4 \text{Lip}_\rho}
\]

and

\[
||l^k_1(\xi)|| \leq \frac{1}{\lambda_{k+1}} \sup_{u \in E} ||F_\rho||, \text{ if } \lambda_{k+1} > 0.
\]

**Proposition 3.5.** Let \( n \geq k. \) Then the following statements holds

i) Let \( u^0_k \in E_k. \) Then \( u_k(t, u^0_k) \) is a solution of

\[
\dot{u}_k = -\lambda_k u_k + P_k F_\rho(u_k + h_k(u_k))
\]

if and only if \( u_k(t, u^0_k) \) satisfies

\[
\dot{u}_k = -\lambda_k u_k + P_k F_\rho(u_k + l^k_1(u_k) + h^n_1(u_k + l^k_1(u_k))).
\]

Moreover \( u_k + h_k(u_k) = u_k + l^k_1(u_k) + h^n_1(u_k + l^k_1(u_k)). \)

ii) Let \( p^0 \in E^k_1. \) Then \( p(t, p^0) \) is a solution of

\[
\dot{p} = -A^k_1 p + P^k_1 F_\rho(p + h^k_1(p))
\]

if and only if \( p(t, p^0) \) satisfies

\[
\dot{p} = -A^k_1 p + P^k_1 F_\rho(p + l^k_1(p) + h^n_1(p + l^k_1(p))).
\]

Moreover

\[
p + h^k_1(p) = p + l^k_1(p) + h^n_1(p + l^k_1(p)).
\]

**Proof.** Let \( u_k(t, u^0_k) \) be the solution of (3.16) with the initial data \( u_k(0, u^0_k) = u^0_k. \)

Since \( M^k_1 \) is invariant, we have \( u(t) = u_k(t, u^0_k) + h_k(u_k(t, u^0_k)) \) is a solution of (2.3) and \( u(\cdot) \in C_\alpha. \)

Since \( C_\alpha \subseteq C_\alpha, \) we have \( u(0) \in M^n_1. \)

Hence there exists a \( p^0 \in E^k_1 \) such that \( u(0) = p^0 + h^n_1(p^0). \)

Let \( p(t, p^0) \) is the solution of (3.12) with the initial data \( p(0, p^0) = p^0. \)

By the invariance of the manifold \( M^n_1, \) we have that \( p(t, p^0) + h^n_1(p(t, p^0)) \) is a solution.
of (2.3). By the uniqueness of solutions we have that \( u(t) = p(t, p^0) + h_1^n(p(t, p^0)) \). In the meantime, \( u(\cdot) \in C_{\alpha_k} \) implies \( p^0 \in M^{k,n}_k \). Hence there exists \( p^0_k \in E_k \) such that \( p^0 = p^0_k + l^{k,n}_k(p^0_k) \). Since \( M^{k,n}_k \) is invariant, by the uniqueness of solutions, we have that \( p(t, p^0) = p_k(t, p^0_k) + l^{k,n}_k(p_k(t, p^0_k)) \), where \( p_k(t, p^0_k) \) is the solution of (3.17) with the initial data \( p_k(0) = p^0_k \). Therefore

\[
  u(t) = p(t, p^0) + h_1^n(p(t, p^0)) \\
  = p(t, p^0) + l^{k,n}_k(p_k(t, p^0_k)) + h_1^n(p_k(t, p^0_k) + l^{k,n}_k(p_k(t, p^0_k)))
\]

This implies \( u_k(t, u^0_k) = p_k(t, p^0_k) \). The converse follows from the same arguments. Using the same argument we can show statement (ii). This completes the proof. \( \square \)
§4. Invariant foliations.

In this section we will show the existence of invariant foliations for equation (2.3) and for equation (3.12).

**Theorem 4.1.** Let $\rho$ such that $\frac{3 \operatorname{Lip} F_\rho}{\alpha} < 1$. Then there exists an invariant foliation for (2.3) whose leaf is given by

$$W_{n+1}^\infty(u^0) = \{ \phi_{n+1}^\infty(q, u^0) + q | q \in E_{n+1}^\infty \},$$

where $u^0 \in E, \phi_{n+1}^\infty : E_{n+1}^\infty \times E \to E_1^n$ is continuous in both variables and uniform Lipschitz continuous in $q$ with

$$\operatorname{Lip} \phi_{n+1}^\infty (\cdot, u^0) \leq \frac{\operatorname{Lip} F_\rho}{\alpha - 2 \operatorname{Lip} F_\rho} < 1.$$  

Furthermore, $\phi_{n+1}^\infty(q, u^0) = P_1^n u^0 + R_{n+1}^\infty(q, u^0)$ and $R_{n+1}^\infty$ satisfies

$$\||R_{n+1}^\infty(q, u^0)|| \leq \frac{\operatorname{Lip} F_\rho}{\alpha - 2 \operatorname{Lip} F_\rho} ||q||,$$

$$\||R_{n+1}^\infty(q, u^0)|| \leq \frac{2}{|\lambda_n|} \sup_{u \in E} ||F_\rho(u)||, \text{ if } \lambda_n < 0$$

and $W_{n+1}^\infty(u^0) \cap M_1^n$ contains a unique point $\xi + h_1^n(\xi)$, where $\xi \in E_1^n$ is uniquely determined by $u^0$ and $\xi = \xi(u^0)$ is continuous.

**Remark.** In fact, we can have that each leaf is a $C^1$ submanifold and is transversal to the invariant manifold $M_1^n$.

For the definition of invariant foliations reader may consult Hirsch, Pugh and Shub [20].

**Proof.** Define the following Banach space

$$C_{\alpha_n}^+ = \left\{ f | f : \mathbb{R}^+ \to E \text{ is continuous and } \sup_{t \geq 0} ||e^{\lambda_n t + \alpha_n t} f(t)|| < \infty \right\}$$

with the norm $|f|_{\alpha_n} = \sup_{t \geq 0} ||e^{\lambda_n t + \alpha_n t} f(t)||$.

Given a solution $u(t, u^0)(t \geq 0)$ of (2.3). We are looking for all solutions $u(t)$ of (2.3) such that

$$w(t) = u(t) - u(t, u^0) \in C_{\alpha_n}^+.$$
Equivalently, \( w(t) \) satisfies the following equation

\[
  w(t) = e^{-A_{n+1}t} q + \int_0^t e^{-A_{n+1}(t-s)P_{n+1}^\infty} \left( F_\rho(w + u(s,u^0)) - F_\rho(u(s,u^0)) \right) \, ds \\
  + \int_0^t e^{-A_n(t-s)P_1^n} (F_\rho(w + u(s,u^0)) - F_\rho(u(s,u^0))) \, ds,
\]

where \( q = P_{n+1}^\infty w(0) \).

We claim that for each \((q,u^0) \in E_{n+1}^\infty \times E\), equation (4.4) has a unique solution in \( C^+_{\alpha_n} \). To see this, let \( J_{n+1}^\infty(w,q,u^0) \) be the right hand side of (4.4). It is easy to see that \( J_{n+1}^\infty \) is well-defined from \( C^+_{\alpha_n} \times E_{n+1}^\infty \times E \) to \( C^+_{\alpha_n} \). For any \( w, \bar{w} \in C^+_{\alpha_n} \) we have

\[
  |J_{n+1}^\infty(w,q,u^0) - J_{n+1}^\infty(\bar{w},q,u^0)|_{\alpha_n}^+ \leq \frac{2 \text{Lip}_{\rho} F_\rho}{\alpha_n} |w - \bar{w}|_{\alpha_n}^+.
\]

It is clear that \( J_{n+1}^\infty \) is Lipschitz continuous in \( q \). However it is not known that \( J_{n+1}^\infty \) is continuous in \( u^0 \) because of the lack of compactness. Using the uniform contraction principle, we have that for each \((q,u^0) \in E_{n+1}^\infty \times E\), equation (4.4) has a unique solution \( w(\cdot; q,u^0) \in C^+_{\alpha_n} \) which is Lipschitz continuous in \( q \) and satisfies

\[
  |w(\cdot; q,u^0) - w(\cdot; \bar{q},u^0)| \leq \frac{\alpha_n}{\alpha_n - 2 \text{Lip}_{\rho} F_\rho} ||q - \bar{q}||.
\]

To see that \( w \) is continuous in \( u^0 \), we choose \( \gamma > 0 \) so small that

\[
  0 < \gamma < \alpha_n \quad \text{and} \quad \frac{2 \text{Lip}_{\rho} F_\rho}{\alpha_n - \gamma} < 1.
\]

We have that for each \( w, \bar{w} \in C^+_{\alpha_n + \gamma} \)

\[
  |J_{n+1}^\infty(w,q,u^0) - J_{n+1}^\infty(\bar{w},q,u^0)|_{\alpha_n+\gamma}^+ \\
  \leq \frac{2 \text{Lip}_{\rho} F_\rho}{\alpha_n - \gamma} |w - \bar{w}|_{\alpha_n+\gamma}^+.
\]

By the uniform contraction mapping principle, we have that for each \((q,u^0) \in E_{n+1}^\infty \times E\) equation (4.4) has a unique solution \( w_\gamma(\cdot; q,u^0) \in C^+_{\alpha_n + \gamma} \). Since \( C^+_{\alpha_n + \gamma} \subset C^+_{\alpha_n} \), by the uniqueness of solutions of (4.4), we have that \( w = w_\gamma \). Hence \( w(\cdot; q,u^0) \in C^+_{\alpha_n + \gamma} \). To show that \( w(\cdot; q,\cdot) \) is continuous from \( E \) to \( C^+_{\alpha_n} \) for each fixed \( q \), it is sufficient to show that for given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( ||\bar{u}^0 - u^0|| < \delta \), then

\[
  |w(\cdot; q,\bar{u}^0) - w(\cdot; q,u^0)|_{\alpha_n} \leq \varepsilon.
\]
We write
\[
 w(t; q, \bar{u}^0) - w(t; q, u^0) \\
= \int_0^t e^{-A_{n+1}^\infty(t-s)} P_{n+1}^\infty[F_\rho(w(s; q, \bar{u}^0) + u(s, \bar{u}^0)) \\
- F_\rho(w(s; q, u^0) + u(s, u^0))]ds \\
+ \int_\infty^t e^{-A_1^\infty(t-s)} P_1^n[F_\rho(w(s; q, \bar{u}^0) + u(s, \bar{u}^0)) \\
- F_\rho(w(s; q, u^0) + u(s, u^0))]ds \\
+ I_1 + I_2,
\]
where
\[
I_1 = \int_0^t e^{-A_{n+1}^\infty(t-s)} P_{n+1}^\infty[F_\rho(w(s; q, u^0) + u(s, \bar{u}^0)) - F_\rho(u(s, \bar{u}^0)) \\
- F_\rho(w(s; q, u^0) + u(s, u^0)) + F_\rho(u(s, u^0))]ds
\]
\[
I_2 = \int_\infty^t e^{-A_1^\infty(t-s)} P_1^n[F_\rho(w(s; q, u^0) + u(s, \bar{u}^0)) - F_\rho(u(s, u^0)) \\
- F_\rho(w(s; q, u^0) + u(s, u^0)) + F_\rho(u(s, u^0))]ds.
\]
Since \( \frac{2 \text{Lip}_F}{\alpha_n} < 1 \), we have
\[
|w(\cdot; q, \bar{u}^0) - u(\cdot; q, u^0)|_{\alpha_n^+}^+ \\
\leq \frac{\alpha_n}{\alpha_n - 2 \text{Lip}_F}(|I_1|_{\alpha_n^+}^+ + |I_2|_{\alpha_n^+}^+).
\]
We first show that there exists \( \delta_1 > 0 \) such that if \( ||\bar{u}^0 - u^0|| \leq \delta_1 \), then \( |I_1|_{\alpha_n^+} \leq \frac{\alpha_n - 2 \text{Lip}_F}{2\alpha_n} \varepsilon. \)
Let \( T > 0 \) so large that
\[
(4.7) \quad \frac{2 \text{Lip}_F}{\alpha_n}|w(\cdot; q, u^0)|_{\alpha_n^+}^+ e^{-\gamma T} \leq \frac{\alpha_n - 2 \text{Lip}_F}{4\alpha_n} \varepsilon.
\]
If \( 0 \leq t \leq T \), then \( I_1 \) is an integral on the compact interval \([0, t]\). By the continuity of \( u(t, u^0) \), we have that there exists \( \delta_1^1 = \delta_1^1(T) \) such that if \( ||\bar{u}^0 - u^0|| \leq \delta_1^1 \), then
\[
|e^{(\lambda_n + \alpha_n)t}I_1| \leq \frac{\alpha_n - 2 \text{Lip}_F}{2\alpha_n} \varepsilon.
\]
If \( t > T \), we write \( I_1 \) as \( I_1 = I_1^1 + I_1^2 \), where
\[
I_1^1 = \int_0^T e^{-A_{n+1}^\infty(t-s)} P_{n+1}^\infty[F_\rho(w(s; q, u^0) + u(s, \bar{u}^0)) - F_\rho(u(s, \bar{u}^0)) \\
- F_\rho(w(s; q, u^0) + u(s, u^0)) + F_\rho(u(s, u^0))]ds
\]
\[ I_1^2 = \int_T^t e^{-\lambda_{n+1}(t-s)} P_{n+1}^\infty [F_\rho(w(s; q, u^0) + u(s, \bar{u}^0)) - F_\rho(u(s, \bar{u}^0))] - F_\rho(w(s; q, u^0) + u(s, u^0)) + F_\rho(u(s, u^0))] ds \]

For \( I_1^1 \) we have

\[ \| e^{(\lambda_n + \alpha_n) t} I_1^1 \| \leq \int_0^T e^{\lambda_n + \alpha_n t} \| [F_\rho(w(s; q, u^0) + u(s, \bar{u}^0)) - F_\rho(u(s, \bar{u}^0))] - F_\rho(w(s; q, u^0) + u(s, u^0)) + F_\rho(u(s, u^0))] \| ds. \]

The last integral is on the compact interval \([0, T]\). By the continuity of \( u(t, u^0) \), we have that there exists \( \delta_2^2 > 0 \) such that if \( \| \bar{u}^0 - u^0 \| \leq \delta_2^2 \) then

\[ \| e^{\lambda_n + \alpha_n t} I_1^1 \| \leq \frac{\alpha_n - 2 \text{Lip} F_\rho \varepsilon}{4 \alpha_n}. \]

We have from (4.7)

\[ \| e^{\lambda_n + \alpha_n t} I_2^2 \| \leq \frac{2 \text{Lip} F_\rho |w(\cdot; q, u^0)|_{\alpha_n^+} e^{-\gamma T}}{\alpha_n} \leq \frac{\alpha_n - 2 \text{Lip} F_\rho \varepsilon}{4 \alpha_n}. \]

Taking \( \delta_1 = \min\{\delta_1^1, \delta_1^2\} \), we have that if \( \| \bar{u}^0 - u^0 \| \leq \delta_1 \) then \( |I_1|_{\alpha_n^+} \leq \frac{\alpha_n - 2 \text{Lip} F_\rho \varepsilon}{2 \alpha_n} \).

Similarly, there exists \( \delta_2 > 0 \) such that if \( \| \bar{u}^0 - u^0 \| \leq \delta_2 \), then

\[ |I_2|_{\alpha_n^+} \leq \frac{\alpha_n - 2 \text{Lip} F_\rho \varepsilon}{2 \alpha_n}. \]

This implies that if \( \| \bar{u}^0 - u^0 \| \leq \delta = \min\{\delta_1, \delta_2\} \), then

\[ |w(\cdot; q, \bar{u}^0) - w(\cdot; q, u^0)|_{\alpha_n^+} \leq \varepsilon. \]

Let \( \phi_{n+1}^\infty(q, u^0) = P^n_1 u^0 + P^n_1 w(0; q - P_n^\infty u^0, u^0) \) and

\[ R_{n+1}^\infty(q, u^0) = P^n_1 w(0; q - P_n^\infty u^0, u^0) = \int_0^\infty e^{\lambda_n s} P^n_1 [F_\rho(w(s; q - P_n^\infty u^0, u^0) + u(s, u^0))] - F_\rho(u(s, u^0))] ds. \]
Then $\phi_{n+1}\infty$ and $R_{n+1}\infty$ are continuous.

Using (4.6), we have

\[ \text{Lip}\phi_{n+1}\infty(\cdot,u^0) = \text{Lip}R_{n+1}\infty(\cdot,u^0) \]
\[ \leq \frac{\text{Lip}F_\rho}{\alpha_n - 2 \text{Lip}F_\rho} < 1, \]
\[ ||R_{n+1}\infty(q,u^0)|| \leq \frac{\text{Lip}F_\rho}{\alpha_n - 2 \text{Lip}F_\rho} ||q|| \]

and

\[ ||R_{n+1}\infty(q,u^0)|| \leq \frac{2}{|\lambda|} \sup_{u \in E} ||F_\rho(u)||, \text{ if } \lambda < 0. \]

Let

\[ W_{n+1}\infty(u^0) = \{ \bar{u}^0 | u(\cdot,\bar{u}^0) - u(\cdot,u^0) \in C_\alpha^+ \}. \]

By using (4.4), we have

\[ W_{n+1}\infty(u^0) = \{ \phi_{n+1}\infty(q,u^0) + q | q \in E_{n+1}\infty \}. \]

Next we show that

\[ W_{n+1}\infty(u^0) \cap M_1^n \text{ contains a unique point}. \]

Suppose $u \in W_{n+1}\infty(u^0) \cap M_1^n$. Then there exist $p \in E_1^n$ and $q \in E_{n+1}\infty$ such that

\[ u = p + h_1^n(p) \text{ and } u = \phi_{n+1}\infty(q,u^0) + q. \]

This implies

\[ p = \phi_{n+1}\infty(q,u^0) \text{ and } q = h_1^n(p). \]

On the other hand, since \text{Lip} $\phi_{n+1}\infty(\cdot,u^0) < 1$ and \text{Lip} $h_1^n < 1$,

\[ p = \phi_{n+1}\infty(h_1^n(p),u^0) \]

has a unique solution $p \in E_1^n$, and $p = p(u^0)$ is continuous in $u^0$ by the uniform contraction principle. This implies $W_{n+1}\infty(u^0) \cap M_1^n$ contains a unique point. It is easy to see that

\[ \text{graph} \phi_{n+1}\infty(\cdot,u^0) = \text{graph} \phi_{n+1}\infty(\cdot,p + h_1^n(p)) \]

and

\[ E = \bigcup_{p \in E_1^n} \text{graph} \phi_{n+1}\infty(\cdot,p + h_1^n(p)) \]

is a $C^0$ foliation. Finally, we prove that this foliation is invariant under the flows of (2.3). To see this, taking any leaf $= \text{graph} \phi_{n+1}\infty(\cdot,p^0 + h_1^n(p^0))$, we show that the time $\tau$-map $u(\tau,\cdot)$ maps this leaf to a leaf. Taking any point $u^0 \in \text{graph} \phi_{n+1}\infty(\cdot,p^0 + h_1^n(p^0))$, we have that
\( u(\cdot, u^0) - u(\cdot, p^0 + h_1^n(p^0)) \in C_{a_n}^+ \). Since equation (2.3) is autonomous and \( M_1^n \) is invariant,
\( u(\cdot, u(\tau, u^0)) - u(\cdot, p(\tau) + h_1^n(p(\tau))) \in C_{a_n}^+ \), where \( p(t) \) is the solution of (3.12) with the initial data \( p(0) = p^0 \). This implies \( u(\tau, u^0) \in \) graph \( \phi_{n+1}^\infty(\cdot, p(\tau) + h_1^n(p(\tau))) \) for all \( u^0 \in \) graph \( \phi_{n+1}^\infty(\cdot, p^0 + h_1^n(p^0)) \). This completes the proof. \( \square \)

Now we set up invariant foliations for the flows on the \( n \)-dimensional manifold \( M_1^n \). The flows on \( M_1^n \) are described by the following equation

\[
\dot{p} = -A_1^n p + P_1^n F_\rho(p + h_1^n(p)),
\]

where \( p \in E_1^n \). For simplicity, let \( F_\rho^n(p) = P_1^n F_\rho(p + h_1^n(p)) \) and write

\[
(4.8) \quad \dot{p} = -A_1^n p + F_\rho^n(p).
\]

**Theorem 4.2.** Choose \( \rho \) such that \( \frac{6 \text{ Lip} F_k}{\alpha} < 1 \). Then for each \( 1 \leq k < n \) there exists an invariant foliation for (3.12) whose leaf is given by

\[
W_{k+1}^{n,n}(p^0) = \{ \phi_{k+1}^{n,n}(\eta, p^0) + \eta|\eta \in E_{k+1}^n \},
\]

where \( p^0 \in E_1^n, \phi_{k+1}^{n,n} : E_{k+1}^n \times E_1^n \to E_1^k \) is continuous in both variables and Lipschitz continuous in \( \eta \) with

\[
(4.9) \quad \text{Lip } \phi_{k+1}^{n,n}(\cdot, p^0) < \frac{2 \text{ Lip} F_\rho}{\alpha_k - 4 \text{ Lip} F_\rho} < 1.
\]

Furthermore, \( \phi_{k+1}^{n,n}(\eta, p^0) = P_1^k p^0 + R_{k+1}^{n,n}(\eta, p^0) \) and \( R_{k+1}^{n,n} \) satisfies

\[
(4.10) \quad ||R_{k+1}^{n,n}(\eta, p^0)|| \leq \frac{2 \text{ Lip} F_\rho}{\alpha_k - 4 \text{ Lip} F_\rho} ||\eta||, \\
(4.11) \quad ||R_{k+1}^{n,n}(\eta, p^0)|| \leq \frac{2}{|\lambda_k|} \sup_{u \in E} ||F_\rho(u)||, \text{ if } \lambda_k < 0,
\]

and \( W_{k+1}^{n,n}(p^0) \cap M_1^{k,n} \) contains a unique point \( \xi^k + \xi_{k+1}^k(\xi^k) \), where \( \xi \in E_1^k \) is uniquely determined by \( p^0 \) and \( \xi^k = \xi^k(p^0) \) is continuous. Let \( H_{k+1}^{n,n}(\eta) = \phi_{k+1}^{n,n}(\eta, 0) \). Then

\[
M_{k+1}^{n,n} = \{ H_{k+1}^{n,n}(\eta) + \eta|\eta \in E_{k+1}^n \}
\]

is an invariant manifold for (4.8).

The proof is analogous to theorem 4.1. We omit it.
Theorem 4.3. Let $\rho$ such that $\frac{6}{\alpha} \text{Lip}F_\rho < 1$. Then for each $1 \leq k < n$ there exists an invariant foliation for (3.12) whose leaf is given by

$$W_{1,k}^{k,n}(p^0) = \{\phi_{1,k}^{k,n}(\xi, p^0) + \xi | \xi \in E_1^k\},$$

where $p^0 \in E_1^n, \phi_{1,k}^{k,n} : E_1^k \times E_1^n \to E_{k+1}^n$ is continuous in both variables and Lipschitz continuous in $\xi$ with

$$\text{Lip}\phi_{1,k}^{k,n}(\cdot, p^0) \leq \frac{2 \text{Lip}F_\rho}{\alpha_k - 4 \text{Lip}F_\rho} < 1. \quad (4.12)$$

Furthermore: $\phi_{1,k}^{k,n}(\xi, p^0) = P_{k+1}^n p^0 + R_{1,k}^{k,n}(\xi, p^0)$ and $R_{1,k}^{k,n}$ satisfies

$$||R_{1,k}^{k,n}(\xi, p^0)|| \leq \frac{2 \text{Lip}F_\rho}{\alpha_k - 4 \text{Lip}F_\rho} ||\xi|| \quad (4.13)$$

$$||R_{1,k}^{k,n}(\xi, p^0)|| \leq \frac{2}{\lambda_{k+1}} \sup_{u \in E} F_\rho(u), \text{ if } \lambda_{k+1} > 0, \quad (4.14)$$

and $W_{1,k}^{k,n}(p^0) \cap M_{k+1}^{n,n}$ contains a unique point $H_{k+1}^{n,n}(\eta) + \eta$ and $\eta = \eta(p^0)$ is continuous.

Proof. Let $p(t, p^0)$ be the solution of (4.8). We are looking for all solutions $p(t)$ of (4.8) such that $w(t) = p(t) - p(t, p^0) \in C_{\alpha_k}^-$. Equivalently, $w(t)$ satisfies the following equation

$$w(t) = e^{-A_k^p(t-s)}\xi + \int_0^t e^{-A_k^p(t-s)} P_{\rho}^k(F_{\rho}^n(w + p(s, p^0)) - F_{\rho}^n(p(s, p^0))) ds$$

$$+ \int_{-\infty}^t e^{-A_{k+1}^\rho(t-s)} P_{k+1}^n(F_{\rho}^n(w + p(s, p^0)) - F_{\rho}^n(p(s, p^0))) ds \quad (4.15)$$

Let $J_k^n(w, \xi, p^0)$ be the right hand side of (4.15). For each $w, \bar{w} \in C_{\alpha_k}^-$ we have

$$|J_k^n(w, \xi, p^0) - J_k^n(\bar{w}, \xi, p^0)|_{\alpha_k} \leq \frac{4 \text{Lip}F_\rho}{\alpha_k} |w - \bar{w}|_{\alpha_k}^-.$$ 

Obviously $J_k^n$ is Lipschitz in $\xi$. By the uniform contraction principle, we have that for each $(\xi, p^0) \in E_1^k \times E_1^n$ equation (4.15) has a unique solution $w(\cdot; \xi, p^0) \in C_{\alpha_k}$ and satisfies

$$|w(\cdot; \xi, p^0) - w(\cdot; \bar{\xi}, p^0)|_{\alpha_k} \leq \frac{\alpha_k}{\alpha_k - 4 \text{Lip}F_\rho} ||\xi - \bar{\xi}||. \quad (4.16)$$

Using the same arguments as in theorem 4.1, we can show that $w(\cdot; \xi, \cdot)$ is continuous from $E_1^k$ to $C_{\alpha_k}^-$. Let

$$\phi_{1,k}^{k,n}(\xi, p^0) = P_{k+1}^n p^0 + P_{k+1}^n w(0; \xi - P_{1,k}^n p^0, p^0)$$
and
\[ R^k_n(\xi, p^0) = P^n_{k+1} w(0; \xi - P^k_1 p^0, p_0). \]

Then \( \phi^k_n \) and \( R^k_n \) are continuous and
\[
\text{Lip} \phi^k_n(\cdot, p^0) = \text{Lip} R^k_n(\cdot, p^0) \\
\leq \frac{2 \text{Lip} F_\rho}{\alpha_n - 4 \text{Lip} F_\rho}.
\]

Furthermore
\[
||R^k_n(\xi, p^0)|| \leq \frac{2}{\lambda_{k+1}} \sup_{u \in E} F_\rho(u) \quad \text{if} \quad \lambda_{k+1} > 0.
\]

Let \( W^k_n(p^0) = \{ p^0 | p(\cdot, p^0) - p(\cdot, p^0) \in C_{\alpha_k} \} \). Then we have
\[
W^k_n(p_0) = \text{graph} \phi^k_n(\cdot, p^0).
\]

Using the same arguments as in Theorem 4.1, we have
\[
W^k_n(p^0) \cap M^{n,n}_{k+1}
\]
contains a unique point \( H^{n,n}_{k+1}(\eta) + \eta \) and \( \eta = \eta(p^0) \) is continuous, and
\[
E_1^n = \bigcup_{\eta \in E^n_{k+1}} \text{graph} \phi^k_n(\cdot, H^{n,n}_{k+1}(\eta) + \eta)
\]
is a \( C^0 \) invariant foliation. This completes the proof. \( \square \)
§5. Transformation on \( M^n \).

Consider the following differential equation which describes the flows of (2.3) on the invariant manifold \( M^n \)

\[
\dot{p} = -A^1 \rho + P^1 \rho(p + h^1(p))
\]

(5.1)

**Theorem 5.1.** Let \( \rho > 0 \) such that \( \frac{10}{\alpha} \text{Lip} \rho \rho < 1 \). Then there exists a homeomorphism \( \Phi_n \) from \( E^n_1 \) onto \( E^n_1 \) such that \( \Phi_n \) maps a solution of (5.1) to the solution of

\[
\begin{align*}
\dot{v}_1 &= -\lambda_1 v_1 + P_1 \rho(v_1 + h_1(v_1)) \\
&\vdots \\
\dot{v}_n &= -\lambda_n v_n + P_n \rho(v_n + h_n(v_n)).
\end{align*}
\]

(5.2)

Conversely, \( \Phi^{-1}_n \) maps a solution of (5.2) to the solution of (5.1). Moreover \( \Phi_n(0) = 0 \) and

\[
|\Phi_n(p) - p| \leq C,
\]

(5.3)

where \( C \) is a constant which is independent of \( n \).

**Proof.** We prove this theorem in four steps.

**Step 1. Construction of \( \Phi_n \).**

Let \( 1 \leq k \leq n \) and \( p^0 \in E^n_1 \). Denote by \( p(t, p^0) \) the solution of equation (5.1) with the initial data \( p(0) = p^0 \) and by \( v_k(t, v_k^0) \) the solution of the following equation

\[
\dot{v}_k = -\lambda_k v_k + P_k \rho(v_k + h_k(v_k))
\]

with the initial data \( v_k(0) = v_k^0 \).

(i) \( k = 1 \). By Theorem 4.2, \( W_2^{n,n}(p^0) \cap M_1^{1,n} \) contains a unique point \( v_1^0 + l_1^{1,n}(v_1^0) \), where \( v_1^0 \in E_1 \), \( v_1^0 = v_1(p^0) \) is continuous and satisfies

\[
v_1^0 = P_1 p^0 + R_2^{n,n}(l_1^{1,n}(v_1^0), p^0).
\]

(5.4)

By the invariance of the foliations and manifolds, we have \( v_1(t, v_1^0) = \phi_{2}^{n,n}(l_1^{1,n}(v_1(t, v_1^0)), p(t, p^0)) \).

(ii) \( k = n \). By Theorem 4.3, \( W_1^{n-1,n}(p^0) \cap M_n^{n,n} \) contains a unique point \( H_n^{n,n}(v_n^0) + v_n^0 \), where \( v_n^0 \in E_n \), \( v_n^0 = v_n(p^0) \) is continuous and satisfies

\[
v_n^0 = P_n p^0 + R_1^{n-1,n}(h_n^{n,n}(v_n^0), p^0).
\]

(5.5)

Moreover we have by the invariance of the foliations and invariant manifolds

\[
v_n(t, v_n^0) = \phi_{1}^{n-1,n}(h_n^{n,n}(v_n(t, v_n^0)), p(t, p^0)).
\]
(iii) $1 < k < n$. By Theorem 4.2, $W_{k+1}^n(p^0) \cap M^k_{1,n}$ contains a unique point $\xi + l_1^{k,n}(\xi)$, where $\xi \in E_1^k$, and $\xi = \xi^k(p^0)$ is continuous and satisfies

$$\xi^k = P_{k+1}^n p^0 + R_{k+1}^{n,n}(l_1^{k,n}(\xi), p^0).$$

By Theorem 4.3, $W_{1}^{k-1,n}(\xi) \cap M^k_{1,k}$ contains a unique point $H_{k}^{k,k}(v_k^0) + v_k^0$, where $v_k^0 \in E_k$, $v_k^0 = v_k^0(\xi)$ is continuous and satisfies

$$v_k^0 = P_k \xi + R_{1}^{k-1,k}(H_{1}^{k,k}(v_k^0), \xi^k).$$

Therefore, $v_k^0 = v_k^0(\xi^k(p^0))$ is continuous and satisfies

$$v_k^0 = P_k p^0 + P_k R_{k+1}^{n,n}(l_1^{k,n}(\xi^k), p^0) + R_{1}^{k-1,k}(H_{1}^{k,k}(v_k^0), \xi^k).$$

It follows from the invariance of the foliations and the manifolds that

$$v_k(t, v_k^0) = \phi_{1}^{k-1,k}(H_{1}^{k,k}(v_k(t, v_k^0)), \xi^k(p(t, p^0))).$$

Define

$$\Phi_n(p^0) = \sum_{k=1}^{n} v_k^0.$$ 

Clearly $\Phi_n$ is well-defined and continuous and $\Phi_n(0) = 0$. Denote $\sum_{k=1}^{n} v_k^0$ by $v^n$. By (5.4), (5.5) and (5.8), we have

$$\Phi_n(p^0) = v^n = p^0 + R_{2}^{n,n}(l_1^{1,n}(v_1^0), p^0) + R_{1}^{n-1,n}(H_{n}^{n,n}(v_n^0), p^0)$$

$$+ \sum_{k=2}^{n-1} [P_k R_{k+1}^{n,n}(l_1^{k,n}(\xi^k), p^0) + R_{1}^{k-1,k}(H_{1}^{k,k}(v_k^0), \xi^k)].$$

By using (3.15), (4.10), (4.11), (4.13) and (4.14), we have

$$|\Phi_n(p^0) - p^0| \leq \sum_{k=1}^{\infty} \frac{4}{|\lambda_k|} \sup_{u \in E} F_{\rho}(u) = C.$$ 

Note this infinite series is convergent since $\lambda_k = k^2 + o(1)$. Clearly, we have that $\Phi$ maps a solution of (2.3) to the solution of (6.1).

**Step 2. $\Phi_n$ is one-one.**

First, we claim that for $m > n$

$$P_1^n \Phi_n(p^0 + l_1^{n,m}(p^0)) = \Phi_n(p^0).$$
Proof of the claim.

Let \( P_1^n \Phi_m(p^0 + l^{
}_{1,m}(p^0)) = \sum_{k=1}^{n} v^0_k \) and \( \Phi_n(p^0) = \sum_{k=1}^{n} v^0_k \). We are going to show \( \bar{v}^0_k = v^0_k \).

(i) \( k = 1 \). From the constructions of \( \Phi_m \) and \( \Phi_n \) we know \( \bar{v}^0_1 + l^{
}_{1,m}(\bar{v}^0_1) \in W_2^{m,m}(p^0 + l^{
}_{1,m}(p^0)) \cap M_1^{1,m} \) and \( v^0_1 + l^{
}_{1,m}(v^0_1) \in W_2^{m,n}(p^0) \cap M_1^{1,n} \). Using proposition 4.4, we have \( v^0_1 + l^{
}_{1,m}(v^0_1) + l^{
}_{1,n}(v^0_1) \in M_1^{1,m} \). In the meantime \( v^0_1 + l^{
}_{1,n}(v^0_1) + l^{
}_{1,m}(v^0_1) \in W_2^{m,m}(p^0 + l^{
}_{1,m}(p^0)) \). Thus \( \bar{v}^0_1 = v^0_1 \) since \( W_2^{m,m}(p^0 + l^{
}_{1,m}(p^0)) \cap M_1^{1,m} \) contains a unique point.

(ii) \( k = n \). This is trivial.

(iii) \( 1 < k < n \). From the construction of \( \Phi_m \) and \( \Phi_n \) we have \( \bar{v}^0_k = v^0_k \). Therefore \( P_n \Phi_m(p^0 + l^{
}_{1,n}(p^0)) = \Phi_n(p^0) \). This completes the proof of the claim.

Now let us prove that \( \Phi_n \) is one-to-one. Let \( p^0 \neq \bar{p}^0 \). We consider the following three cases:

1. if \( p(\cdot, p^0) - p(\cdot, \bar{p}^0) \notin C_{\alpha_k}^+ \) for \( k = 1, \ldots, n \). This means that there is no leaf on which \( p^0 \) and \( \bar{p}^0 \) stay simultaneously. This implies \( v^0_k \neq \bar{v}^0_k \).
2. if \( p(\cdot, p^0) - p(\cdot, \bar{p}^0) \in C_{\alpha_{n-1}}^+ \), then \( v^0_n \neq \bar{v}^0_n \) since \( p^0 \neq \bar{p}^0 \).
3. \( p(\cdot, p^0) - p(\cdot, \bar{p}^0) \in C_{\alpha_k}^+ \) but \( \notin C_{\alpha_{k+1}}^+ \) for \( 1 \leq k < n - 1 \). Then by Theorem 4.2 we have

\[ \xi^{k+1}(p^0) \neq \xi^{k+1}(\bar{p}^0) \in E_1^{k+1}, \]

where \( \xi^k \) is given by (5.6). By using (2), we have that \( P_{k+1} \Phi_{n+1}(\xi^{k+1}(p^0)) \neq P_{k+1} \Phi_{k+1}(\xi^{k+1}(\bar{p}^0)) \).

Using (5.10), we have that \( P_{k+1} \Phi_n(\xi^{k+1}(p^0) + l^{
}_{1,k+1,n}(\xi^{k+1}(p^0))) \neq P_{k+1} \Phi_n(\xi^{k+1}(\bar{p}^0) + l^{
}_{1,k+1,n}(\xi^{k+1}(\bar{p}^0))) \). Namely \( v^0_{k+1} \neq \bar{v}^0_{k+1} \). Therefore \( \Phi_n \) is one-to-one.

Step 3. \( \Phi_n \) is onto.

We prove this by induction.

(i) \( n = 1 \). It is trivial.

(ii) Suppose that it is true for \( n - 1 \)

(iii) We show that \( \Phi_n \) is onto. Given any \( v^n = \sum_{k=1}^{n} v^0_k \), by the hypotheses of induction, we have that there exists a unique \( p^{n-1} \in E_1^{n-1} \) such that

\[ \Phi_{n-1}(p^{n-1}) = \sum_{k=1}^{n-1} v_k = v^{n-1} \]

By using the contraction mapping principle, we have

\[ W_1^{n-1,n}(v_n + H_n^{n,n}(v_n)) \cap W_n^{n,n}(p^{n-1} + l^{n-1,n}(p^{n-1})) \]

contains a unique point \( p^n \). Using (5.10) and construction of \( \Phi_n \) we have \( \Phi_n(p^n) = v^n \). Hence \( \Phi_n \) is onto.
Step 4. $\Phi_n^{-1}$ is continuous.

Since $E^n$ is a finite dimensional space and $\Phi_n$ is continuous and satisfies (5.3), by the inverse function theorem we have that $\Phi_n^{-1}$ is continuous. By the uniqueness of solutions, $\Phi_n^{-1}$ maps a solution of (5.2) to the solution of (5.1). This completes the proof. ☐
§6. Transformation for Equation (2.3).

In this section we prove Theorem 1.1.

**Theorem 6.1.** Let $\rho > 0$ such that $\frac{10}{\alpha} \text{Lip} F_\rho < 1$. Then there exists a homeomorphism $\Phi$ from $E$ onto $E$ such that $\Phi$ maps a solution of (2.3) to the solution of the following equation

\[
\begin{align*}
\dot{v}_1 &= -\lambda_1 v_1 + P_1 F_\rho(v_1 + h_1(v_1)) \\
\vdots \\
\dot{v}_n &= -\lambda_n v_n + P_n F_\rho(v_n + h_n(v_n)) \\
\vdots
\end{align*}
\]  

(6.1)

and $\Phi^{-1}$ maps a solution of (6.1) to the solution of (2.3). Moreover $||\Phi(u) - u|| \leq C$, where $C$ is a constant.

**Proof.** We prove this theorem in four steps.

**Step 1. Construction of $\Phi$.**

Let $u \in E$. For each integer $n > 0$, by Theorem 4.1, we have that $W_{n+1}^\infty(u) \cap M^n_1$ contains the unique point $p^n + h^n_1(p^n)$, where $p^n \in E^n_1$, $p^n = p^n(u)$ is continuous and satisfies

\[
p^n = \varphi_{n+1}^\infty(h^n_1(p^n), u) \\
= P^n_1 u + R_{n+1}^\infty(h^n_1(p^n), u).
\]  

(6.2)

Note that $p^n$ is uniquely determined by $u$.

(i) $n = 1$. Let $v_1 = p^1(u)$

(ii) $n > 1$. By theorem 4.3, $W_1^{n-1,n}(p^n) \cap M_n^{n,n}$ contains a unique point $v_n + H_n^{n,n}(v_n)$, where $v_n \in E_n$, $v_n = v_n(p^n)$ is continuous and satisfies

\[
v_n = \varphi_1^{n-1,n}(H_n^{n,n}(v_n), p^n) \\
= P_n p^n + R_1^{n-1,n}(H_n^{n,n}(v_n), p^n).
\]

Therefore, we have that $v_n = v_n(p^n(u))$ is continuous in $u$ and satisfies

\[
v_n = P_n u + P_n R_{n+1}^\infty(h^n_1(p^n(u)), u) + R_1^{n-1,n}(H_n^{n,n}(v_n), p^n(u)),
\]  

(6.3)

**Claim.**

\[
\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} v_n(p^n(u))
\]

is convergent and continuous in $u$. 

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Proof of this claim: By Theorem 3.2, Theorem 4.1 and Theorem 4.3, we have

$$||R_{n+1}^\infty|| \leq \frac{2}{|\lambda_n|} \sup_{u \in E} ||F_\rho(u)||$$

and

$$||R_1^{n-1,n}|| \leq \frac{2}{|\lambda_n|} \sup_{u \in E} ||F_\rho(u)||$$

The fact that $\lambda_n = n^2 + 0(1)$ as $n \to \infty$ implies that

$$\sum_{n=1}^\infty P_n R_{n+1}^\infty(h_1^n(p^n(u)), u)$$

and

$$\sum_{n=2}^\infty R_1^{n-1,n}(H_n^{n,n}(v_n(p^n(u)), p^n(u)))$$

are uniformly convergent. By using (6.3), we have that $\sum_{n=1}^\infty v_n$ is convergent. Let $v = \sum_{n=1}^\infty v_n$. Then $v$ satisfies

$$v = u + \sum_{n=1}^\infty P_n R_{n+1}^\infty(h_1^n(p^n(u)), u) + \sum_{n=2}^\infty R_1^{n-1,n}(H_n^{n,n}(v_n(p^n(u)), p^n(u)))$$

Since $\sum_{n=1}^\infty P_n R_{n+1}^\infty$ and $\sum_{n=2}^\infty R_1^{n-1,n}$ are uniformly convergent and $P_n R_{n+1}^\infty$ and $R_1^{n-1,n}$ are continuous in $u$, $v = v(u)$ is continuous. This completes the proof of the claim.

Define $\Phi(u) = v(u)$.

Then $\Phi$ is well-defined, continuous and satisfies

$$||\Phi(u) - u|| \leq C = \sum_{n=1}^\infty \frac{4}{|\lambda_n|} \sup_{u \in E} ||F_\rho(u)||.$$  

By the invariance of the foliations and the manifolds, we have that $\Phi$ maps a solution of (2.3) to the solution of (6.1).

**Step 2.** $\Phi(u)$ is one-to-one.

First we claim

**Claim.**

(6.4)  

$$P_1^n \Phi(u) = \Phi_n(p^n(u)).$$

Proof of the claim: Let $P_1^n \Phi(u) = \sum_{k=1}^n v_k, \Phi_n(p^n(u)) = \sum_{k=1}^n \bar{v}_k$. We want to show $v_k = \bar{v}_k$ for $1 \leq k \leq n$. 

(1) $k = 1$. By Theorem 4.1, $W_2^\infty(u) \cap M_1$ contains a unique point $v_1 + h_1^n(v_1)$; by Theorem 4.2, $W_2^{n,\infty}(p^n(u)) \cap M_1^n$ contains a unique point $\bar{v}_1 + l_1^n(\bar{v}_1)$. Using Proposition 3.5, we have $\bar{v}_1 + l_1^n(\bar{v}_1) + h_n^n(\bar{v}_1 + l_1^n(\bar{v}_1)) \in M_1$. In the meantime, $\bar{v}_1 + l_1^n(\bar{v}_1) + h_n^n(\bar{v}_1 + l_1^n(\bar{v}_1)) \in W_2^\infty(u)$. By the uniqueness, we have that $v_1 = \bar{v}_1$.

(2) $k = n$. $v_n = \bar{v}_n$ follows from the construction of $\Phi$.

(3) $1 < k < n$. Using the same arguments as in theorem 5.1, we have $v_k = \bar{v}_k$. This completes the proof of the claim. Now we show that $\Phi$ is one-to-one.

Suppose $u \neq \bar{u}$.

(i) If $u(\cdot, \bar{u}) - u(\cdot, u) \notin C^+_{\alpha_n}$ for all $n$, then from the construction of $\Phi$, we have $P_1 \Phi(u) \neq P_1 \Phi(\bar{u})$.

(ii) There exists $n > 0$ such that

$$u(\cdot, \bar{u}) - u(\cdot, u) \in C^+_{\alpha_n} \text{ but } \notin C^+_{\alpha_{n+1}}.$$ 

From the construction of $\Phi$, we have

$$p^{n+1}(u) \neq p^{n+1}(\bar{u})$$

and

$$p^n(u) = p^n(\bar{u}).$$

Using (6.4) and Theorem 5.1, we have that $P_{n+1} \Phi(u) \neq P_{n+1} \Phi(\bar{u})$.

(iii) $u(\cdot, \bar{u}) - u(\cdot, u) \in C^+_{\alpha_n}$ for all $n$ is impossible unless $\bar{u} = u$. This statement follows from the well-known fact that linear parabolic equations have no exponentially small solutions. For example, see Agmon [1], Hale [15], and Angenent [2]. Therefore $\Phi$ is one to one.

**Step 3.** $\Phi$ is onto.

For each $v = \sum_{k=1}^\infty v_k \in E$, let $v^n = \sum_{n=1}^n v_k$. By Theorem 5.1, we have that there exists a unique $\xi^n \in E_1^n$ such that $\Phi_n(\xi^n) = v^n$. We claim that

$$\xi^n + h_1^n(\xi^n)$$

is a Cauchy sequence in $E$. For $m > n$, let $\xi^m = p^m + q^m$ where $p^m \in E_1^m$ and $q^m \in E_{n+1}^m$.

Then we have

$$\xi^m = \phi_1^{m,m}(p^m, \xi^m), \quad \eta^m = \phi_1^{m,m}(H_{n+1}^m(\eta^m), \xi^m),$$

$$p^m = \phi_{n+1}^{m,m}(q^m, \xi^m), \quad \xi^n = \phi_{n+1}^{m,m}(h_1^n(\xi^n), \xi^m),$$

where $\eta^m \in E_{n+1}^m$. From (5.9) we have

$$v^m = \xi^m + R_2^{m,m}(l_1^m(v_1), \xi^m) + R_1^{m-1,m}(H_m^m(v_m), \xi^m)$$

$$+ \sum_{k=2}^{m-1} [P_k R_{k+1}^{m,m}(l_1^k(\xi^k), \xi^m) + R_1^{k-1,k}(H_k^k(v_k), \xi^k)].$$
and
\begin{equation}
(6.5) \quad \sum_{k=n+1}^{m} v_k = q^m + \sum_{k=n+1}^{m-1} \left[ P_k R_{k+1}^{m,m}(i_{k,m}^{k,m}(\xi^k), \xi^m) + R_{1}^{k-1,k}(H_{k}^{k,k}(v_k), \xi^k) \right] + R_{1}^{m-1,m}(H_{m}^{m,m}(v_m), \xi^m).
\end{equation}

Now let us look at $\xi^m - \xi^n$. We have
\begin{align*}
||\xi^m - \xi^n||^2 &= ||q^m||^2 + ||p^m - \xi^n||^2 \\
&\leq ||q^m||^2 + ||q^m - h_1^n(\xi^n)||^2.
\end{align*}

Using Theorem 3.2, (3.15), (4.10), (4.11), (4.13), (4.14) and (6.5), we have that for $\varepsilon > 0$ there exists $N > 0$ such that if $m \geq n > N$, then
\begin{equation}
||q^m|| \leq \frac{1}{5} \varepsilon \text{ and } ||h_1^n(\xi^n)|| \leq \frac{1}{5} \varepsilon.
\end{equation}

This implies
\begin{equation}
||\xi^m + h_1^n(\xi^m) - (\xi^n + h_1^n(\xi^n))|| \leq \varepsilon.
\end{equation}

Therefore $\xi^n + h_1^n(\xi^n)$ is Cauchy. Let
\begin{equation}
\lim_{n \to \infty} (\xi^n + h_1^n(\xi^n)) = u.
\end{equation}

We claim $\Phi(u) = v$. Since $\Phi$ is continuous, we have
\begin{equation}
P_1^n(\Phi(u) - \Phi(\xi^m + h_1^n(\xi^m))) \to 0 \text{ as } m \to \infty.
\end{equation}

On the other hand, by using (6.4), we have
\begin{equation}
P_1^n(\Phi(u) - \Phi(\xi^m + h_1^n(\xi^m))) = P_1^n(\Phi(u) - \sum_{k=1}^{n} v_k), \text{ for } m > n.
\end{equation}

This implies $P_1^n \Phi(u) = \sum_{k=1}^{n} v_k$. Therefore $\Phi(u) = v$.

**Step 4.** $\Phi^{-1}$ is continuous.

For fixed $\nu$, it is sufficient to show that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $||\nu - \bar{\nu}|| < \delta$ for $\nu, \bar{\nu} \in E$, then
\begin{equation}
||\Phi^{-1}(\nu) - \Phi^{-1}(\bar{\nu})|| \leq \varepsilon
\end{equation}

Using Theorem 3.2, (3.15), (4.10), (4.11), (4.13), (4.14) and (6.5), we have that for $\varepsilon > 0$ there exist $N > 0$ such that if $m \geq n > N$, then
\[ ||\xi^m(v^m) - \xi^n(v^n)|| \leq \varepsilon \frac{1}{4}, \]

\[ ||\xi^m(\bar{v}^m) - \xi^n(\bar{v}^n)|| \leq 2||P_{n+1}^m \bar{v}^m|| + \frac{1}{8} \varepsilon. \]

\[ ||P_n^m v^m|| \leq \varepsilon \frac{1}{4}. \]

Let \( m \to \infty \). Then we have

\[ ||\Phi^{-1}(v) - \xi^n(v^n)|| \leq \varepsilon \frac{1}{4}, \]

\[ ||\Phi^{-1}(\bar{v}) - \xi^n(\bar{v}^n)|| \leq ||P_{n+1}^\infty \bar{v}|| + \frac{1}{8} \varepsilon \]

\[ \leq ||P_{n+1}^\infty \bar{v} - P_{n+1}^\infty v|| + ||P_{n+1}^\infty v|| + \frac{1}{8} \varepsilon \]

\[ \leq ||v - \bar{v}|| + \frac{1}{4} \varepsilon + \frac{1}{8} \varepsilon \]

\[ \leq ||v - \bar{v}|| + \frac{3}{8} \varepsilon. \]

Let \( n > N \) be fixed. From the continuity of \( \xi^n \) we have that there exists a \( \delta_1 \) such that if \( ||v^n - \bar{v}^n|| \leq \delta_1 \), then

\[ ||\xi^n(v^n) - \xi^n(\bar{v}^n)|| \leq \varepsilon \frac{1}{4}. \]

Let \( \delta = \min\{\delta_1, \varepsilon \frac{1}{8}\}. \) Then we have that if \( ||v - \bar{v}|| \leq \delta \) then

\[ ||\Phi^{-1}(v) - \Phi^{-1}(\bar{v})|| \]

\[ \leq ||\Phi^{-1}(v) - \xi^n(v) + \xi^n(v) - \xi^n(\bar{v}) + \xi^n(\bar{v}) - \Phi^{-1}(\bar{v})|| \]

\[ \leq \varepsilon. \]

Therefore \( \Phi^{-1} \) is continuous. It is easy to see that \( \Phi^{-1} \) maps a solution of (6.1) to the solution of (2.3). This completes the proof of this theorem. \( \Box \)

**Theorem 6.2.** There exists a homeomorphism \( \Psi \) from \( E \) onto \( E \) such that \( \Psi \) maps a solution of (6.1) to a solution of

\[ \dot{w} = -Aw \]
and $\Psi^{-1}$ maps a solution of (6.6) to a solution of (6.1).

Proof. Let us look at the following equation

\begin{equation}
\dot{v}_n = -\lambda_n v_n + P_n F_\rho(v_n + h_1^n(v_n))
\end{equation}

This is a one dimensional ordinary differential equation. Since $\lambda_n \neq 0$, by using the theorem [27], we have that there exists a homeomorphism $\psi_n$ from $E_n$ to $E_n$ which is given by

$$
\psi_n(v_n) = v_n + H_n(v_n),
$$

where $H_n$ satisfies $\|H_n\| \leq \frac{2}{|\lambda_n|} \sup_{u \in E} F_\rho(u)$, and transforms (6.7) to

\begin{equation}
\dot{w}_n = -\lambda_n w_n.
\end{equation}

Since $\lambda_n = n^2 + o(1)$ as $n \to \infty$, we have that $\sum_{n=1}^\infty H_n(v_n)$ is uniformly convergent. The continuity of $H_n$ implies that $\sum_{n=1}^\infty H_n(P_n v)$ is continuous in $v$. This implies

$$
\sum_{n=1}^\infty \psi_n(P_n v)
$$

is convergent and continuous in $v$. Let

$$
w = \Psi(v) = \sum_{n=1}^\infty \psi_n(P_n v).
$$

Since $\psi_n$ is the homeomorphism given by (6.8), we have that $\sum_{n=1}^\infty \psi_n^{-1}(P_n w)$ is convergent and is continuous in $w$, which is the inverse of $\Psi$.

This completes the proof. \[ \]

Proof of Theorem 1.1. Let $V = B_\rho(0)$ where $\rho$ is a number such $\frac{10}{\alpha} \text{Lip} F_\rho < 1$. By Theorem 6.1 and Theorem 6.2, letting $\Phi^* = \Psi \cdot \Phi$, then $\Phi^*$ is a homeomorphism from $V$ to $\Phi^*(V)$ and transforms (1.6) and (1.7) to (1.8) and (1.9). This completes the proof. \[ \]

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