GLOBAL EXISTENCE, UNIQUENESS AND REGULARITY OF SOLUTIONS TO A VON KÁRMÁN SYSTEM WITH NONLINEAR BOUNDARY DISSIPATION

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1 Introduction

1.1 Statement of the Problem

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$ with boundary $\Gamma$. In $\Omega \times (0, T)$, where $T > 0$ is given, we consider the following von Kármán system with nonlinear boundary conditions:

\begin{align*}
\frac{\partial w_t}{\partial t} + \Delta^2 w &= [\mathcal{F}(w), w] \quad \text{in } Q_T = (0, T) \times \Omega \tag{1.1.a} \\
\Delta w + (1 - \mu) B_1 w &= -h(\frac{\partial}{\partial \nu} w_t) \quad \text{on } \Sigma_T = (0, T) \times \Gamma \tag{1.1.b} \\
\frac{\partial}{\partial \nu} \Delta w + (1 - \mu) B_2 w - w &= g(w_t) \quad \text{on } \Sigma_T = (0, T) \times \Gamma, \tag{1.1.c}
\end{align*}

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and
\[
\Delta^2 \mathcal{F}(w) = -[w, w] \quad \text{in } (0, T) \times \Omega \tag{1.2}
\]
\[
\mathcal{F}|_{\Gamma} = \frac{\partial}{\partial \nu} \mathcal{F}|_{\Gamma} = 0 \quad \text{on } (0, T) \times \Gamma,
\]
where
\[
[u, v] \equiv u_{xx} v_{yy} + u_{yy} v_{xx} - 2u_{xy} v_{xy}.
\]

In (1.1), the boundary operators \(B_1\) and \(B_2\) are given by
\[
\begin{align*}
B_1 w &= 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx}, \\
B_2 w &= \frac{\partial}{\partial \nu} [(n_1^2 - n_2^2) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})],
\end{align*}
\]
where \(0 < \mu < \frac{1}{2}\) is Poisson’s ratio, and the functions \(h\) and \(g\) are differentiable, monotone increasing, real-valued functions and are subject to the following constraints:
\[
\begin{align*}
\begin{cases}
  h(s) s > 0 & \text{for } s \neq 0 \\
g(s) s > 0 & \text{for } s \neq 0 \\
m \leq h'(s) \leq M \\
m \leq g'(s) \leq M(|s|^r + 1),
\end{cases}
\end{align*}
\]
where \(r\) is any positive constant, and
\[
\begin{align*}
(h(x) - h(y), x - y) &\geq \hat{m}|x - y|^\alpha \\
g'(s) &\text{ is locally Lipschitz.}
\end{align*}
\]
This system describes the transversal displacement \(w\) and the Airy-stress function \(\mathcal{F}(w)\) of a vibrating plate, whose boundary is subject to nonlinear damping in the form of moments and forces/shears applied to the edge of the plate.

Questions related to the existence of solutions for the von Kármán system (1.1.a), (1.1.b), (1.2), with the **homogeneous** boundary conditions:
\[
w|_{\Gamma} = \frac{\partial}{\partial \nu} w|_{\Gamma} = 0 \quad \text{on } (0, T) \times \Gamma, \tag{1.4}
\]
have received considerable attention in the literature. Indeed, the existence of global **weak** (i.e., \((w(t), w_t(t)) \in H^2(\Omega) \times L^2(\Omega))\) solutions for (1.1.a), (1.1.b), (1.2), (1.4) has been proven by Faedo-Galerkin methods in
Lions [15] and Vorović [24]. In [22], [23], von Wahl gives a proof of existence and uniqueness of a local solution with higher regularity. Existence of local classical solutions has been established by Stahel in [18]. Arguments of [18] also prove the uniqueness property which is valid on a small time interval. More recently, Koch and Stahel [10] were able to establish appropriate a priori bounds for their classical solutions, hence proving the global existence of classical solutions. Global existence and uniqueness results for strong (i.e., $H^4(\Omega) \times H^2(\Omega)$) solutions to (1.1.a), (1.1.b), (1.2), (1.4) was proven by Chueskov in [3].

As already stated, the results quoted above refer to a situation when the boundary conditions are homogeneous, or, more precisely, of the form as in (1.4). The techniques used in these references can be adapted to treat the case of boundary conditions as in (1.1.c) and (1.1.d), but with linear boundary dissipation, i.e., $g$ and $h$ are linear (this fact was already noted by Lagnese in [11]). Indeed, the presence of nonlinear functions in boundary conditions (1.1.c) and (1.1.d) raises a number of technical difficulties since the methods developed previously are not applicable any longer. Indeed, a standard Faedo-Galerkin method which has been used for proving existence of solutions of von Kármán systems with homogeneous boundary conditions (as in [3], [6], [24]) runs into difficulties (at the limit process) because of the appearance of nonlinearities on the boundary.

To our knowledge, the only results available in the literature and dealing with the nonlinear boundary damping are due to Lagnese and Leugering [12], where the one-dimensional model has been treated, and in [6], [13], where the rotational inertia of the plate, which induces a regularizing effect on the velocity, was accounted for in the model. In this last case, the mathematical nature of the problem is, obviously, very different.

Thus, the main distinctive feature of our paper is that we treat a two dimensional von Kármán system (1.1), (1.2) with fully nonlinear boundary damping. The main result of our paper is a global existence and uniqueness of (i) regular solutions and (ii) weak solutions. We note that the uniqueness result for weak solutions is new even in the case of linear homogeneous boundary conditions. Precise statements of these results are given below.

**Theorem 1.1 (Existence of smooth solutions)** Assume that the functions $g$ and $h$ satisfy hypotheses (H-1)
and (II-2). Then for all initial data, \( w_0 \in H^4(\Omega), w_1 \in H^2(\Omega), \) such that

\[
\Delta w_0 + (1 - \mu)B_1 w_0 = -h(\frac{\partial}{\partial \nu} w_1) \quad \text{on} \Gamma \quad (1.5)
\]

\[
\frac{\partial}{\partial \nu} \Delta w_0 + (1 - \mu)B_2 w_0 - w_0 = g(w_1) \quad \text{on} \Gamma, \quad (1.6)
\]

there exists a solution, \((w, w_1) \in C(0, T; H^3(\Omega) \times H^2(\Omega)), \) where \( T > 0 \) is arbitrary.

**Theorem 1.2 (Uniqueness)** Let \((w, w_1)\) be any solution corresponding to (1.1) which satisfies \((w, w_1) \in C(0, T; H^2(\Omega) \times L^2(\Omega)). \) Then, such a solution is unique.

**Theorem 1.3 (Existence of weak solutions)** In addition to hypotheses (II-1) and (II-2), we assume the following growth condition

\[
g(s) \geq m|s|^{r+1} \quad \text{for} \ |s| > R, \quad (H-3)
\]

where \( R \) is a large number. Then, for all initial data \((w_0, w_1) \in H^2(\Omega) \times L^2(\Omega), \) there exists a unique global solution, \((w, w_1) \in C(0, T; H^2(\Omega)) \times C(0, T; L^2(\Omega)). \)

**Theorem 1.4 (Intermediate solutions)** Assume hypotheses (II-1)-(II-3) hold with the value \( r \leq 1. \) Then for \( 0 \leq \theta \leq 1 \) and all initial data \( w_0 \in H^{2+2\theta}(\Omega), w_1 \in H^{2\theta}(\Omega) \) subject to compatibility conditions (1.5) satisfied for \( \theta > \frac{1}{4} \) and (1.6) satisfied for \( \theta > \frac{3}{4}, \) we have that \((w, w_1) \in C(0, T; H^{2+\theta}(\Omega)) \times C(0, T; H^{2\theta}(\Omega)). \)

The outline of the paper is as follows. Sections 2-4 deal with the existence of regular (smooth) solutions. Here the result of Theorem 1.1 is proven via a suitable application of Schaeffer's Theorem. Section 5 provides an uniqueness result for weak solutions. The proof of this uniqueness (Theorem 1.2) is based on a new “sharp” regularity result for the Airy’s stress function (see Theorem 5.1). Proof of Theorem 5.1 employs pseudodifferential calculus combined with compensated compactness methods and is given in section 8. Section 6 deals with the existence of weak solutions (Theorem 1.3), where the “sharp” regularity of the Airy’s stress function is used again. The existence of the solutions in the intermediate spaces (Theorem 1.4) is obtained in section 7 through an application of a nonlinear interpolation theorem due to Tartar [19].
2 Technical Lemmas

In this section, we shall prove a number of a priori estimates for the following nonlinear equation:

\[
\begin{align*}
\frac{\partial w}{\partial t} + \Delta^2 w &= [f, w] \quad \text{in } Q_T = (0, T) \times \Omega \\
\frac{\partial w}{\partial t} &= 0 \quad \text{on } \Sigma_T = (0, T) \times \Gamma \\
w(0, \cdot) &= w_0 \quad \text{in } \Omega \\
w_t(0, \cdot) &= w_t \quad \text{in } \Omega \\
\Delta w + (1 - \mu) B_1 w &= -h \left( \frac{\partial^2}{\partial y^2} w_t \right) \\
\quad &\text{on } \Sigma_T = (0, T) \times \Gamma
\end{align*}
\]

(2.1)

where the function \( f \) is a given element of \( L_1(0, T; W^2_{\infty}(\Omega)) \).

For the convenience of the reader, we list several known properties of the bracket, \([v, w]\) (see [3], [10], [15]) which will be used throughout our proofs. The constant \( C \) throughout the paper denotes a generic constant.

Properties of \([v, w]\): Let \( 0 < \theta < 1 \). Then

\[
\|[v, w]||_{H^{-1-\theta}(\Omega)} \leq C||v||_{H^2(\Omega)} ||w||_{H^2(\Omega)},
\]

(2.2)

\[
\|[v, w]||_{L_2(\Omega)} \leq C||v||_{H^2(\Omega)} ||w||_{H^2(\Omega)},
\]

(2.3)

where \( \frac{1}{q} + \frac{1}{q} = 1 \),

\[
\|[v, w]||_{H^{-\theta}(\Omega)} \leq C||v||_{H^2(\Omega)} ||w||_{H^2(\Omega)},
\]

(2.4)

\[
\|[v, w]||_{H^{-2\theta}(\Omega)} \leq C||v||_{H^2(\Omega)} ||w||_{H^2(\Omega)},
\]

(2.5)

\[
\|[v, w]||_{H^{-\theta}(\Omega)} \leq C||v||_{H^2(\Omega)} ||w||_{H^2(\Omega)},
\]

(2.6)

\[
\|[v, w]||_{H^{-2\theta}(\Omega)} \leq C||v||_{H^2(\Omega)} ||w||_{H^2(\Omega)},
\]

(2.7)

We shall start the proofs of our a priori estimates with a simple preliminary result.

**Proposition 2.1** For any \( f \in L_1(0, T; W_\infty^2(\Omega)) \), \( g \) and \( h \) subject to hypothesis (H-1), and \( w_0 \in H^2(\Omega) \), \( w_1 \in L_2(\Omega) \), there exists a unique solution, \((w, w_t)\) to (2.1) such that

\[
\|[w(t)]||_{H^2(\Omega)}^2 + ||w_t(t)||_{L^2(\Omega)}^2 + \int_0^T \int_\Gamma h \left( \frac{\partial^2}{\partial y^2} w_t \right) \frac{\partial}{\partial y} w_t d\Gamma dt + \int_0^T \int_\Gamma g(w_t) w_t d\Gamma dt \\
\leq C_T \exp(||f||_{L_1(\Omega; T; W_\infty^2(\Omega))} ||w_0||_{H^2(\Omega)}^2 + ||w_1||_{L^2(\Omega)}^2) \equiv C(f, w_0, w_1) \quad \forall t \leq T.
\]

(2.8)
Proof: It follows from [14] that problem (2.1) with \( f \equiv 0 \) generates a nonlinear semigroup of contractions on \( H^2(\Omega) \times L_2(\Omega) \). To see this, we need to put problem (2.1) into an abstract framework of [14]. We define the following operators: \( A : L_2(\Omega) \to L_2(\Omega) \) is defined by:

\[
Au \equiv \Delta^2 u; \quad D(A) = \{ u \in H^2(\Omega) : \Delta u + (1 - \mu)B_1u|_\Gamma = 0, \frac{\partial}{\partial v} \Delta u + (1 - \mu)B_2u|_\Gamma = 0 \}, \quad (2.9)
\]

\( G_i : L_2(\Gamma) \to L_2(\Omega), i = 1, 2, \) are defined by

\[
G_1g \equiv v \iff \Delta^2 v = 0 \quad \text{in} \Omega \\
\Delta v + (1 - \mu)B_1v = g \quad \text{on} \Gamma, \quad (2.10)
\]

\[
\frac{\partial}{\partial v} \Delta v + (1 - \mu)B_2v = 0 \quad \text{on} \Gamma,
\]

and

\[
G_2g \equiv v \iff \Delta^2 v = 0 \quad \text{in} \Omega \\
\Delta v + (1 - \mu)B_1v = 0 \quad \text{on} \Gamma, \quad (2.11)
\]

\[
\frac{\partial}{\partial v} \Delta v + (1 - \mu)B_2v = g \quad \text{on} \Gamma.
\]

With the above notation, equation (2.1) with \( f = 0 \) can be rewritten (on \( D(A)' \)) as

\[
w_{tt} + Aw + AG_1b(G_1^*Aw_t) + AG_2g(G_2^*Aw_t) = 0, \quad (2.12)
\]

or equivalently,

\[
w_{tt} + Aw + B\partial\Phi(B^*w_t) \ni 0, \quad (2.13)
\]

where \( B : L_2(\Gamma) \times L_2(\Gamma) \to D(A^{1/2})' \) is given by

\[
B(g_1, g_2) = AG_1g_1 + AG_2g_2, \quad (2.14)
\]

and the adjoint of \( B \), defined by

\[
(g, B^*v)_{L_2(\Gamma) \times L_2(\Gamma)} = (Bg, v)_{L_2(\Omega)}, \quad (2.15)
\]

can be written as

\[
B^* = [G_1^*Av, G_2^*Av] = [\frac{\partial}{\partial v} v, v], \quad (2.16)
\]

by using Green’s formula (see [13]).
With $U_0 \equiv L_2(\Gamma) \times L_2(\Gamma)$, $\partial \Phi \equiv [h, g] \in U_0 \times U_0$ and, by monotonicity of $h$ and $g$, is a subgradient of a proper, convex function $\Phi : U_0 \to R$. By using trace theory [16], one easily shows that $B^*$ is bounded and surjective between the spaces $H^2(\Omega) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. Thus, hypothesis (H-1) of [14] is satisfied with $U \equiv H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, $U \subset U_0 \subset U'$ and $D(A^{1/2}) = H^2(\Omega)$ (see [8]). Hence, we are in a position to apply Theorem 2.2 in [14] which claims the generation of a nonlinear semigroup of contractions for the operator

\[
C \equiv \begin{bmatrix}
0 & -I \\
A & B\partial \Phi B^*
\end{bmatrix}
\] (2.17)

on $\overline{D(C)} \subset H^2(\Omega) \times L_2(\Omega)$. One can easily check that, in our case, $\overline{D(C)} = H^2(\Omega) \times L_2(\Omega)$, hence equation (2.1) (written as a system of equations) generates a nonlinear semigroup of contractions on $H^2(\Omega) \times L_2(\Omega)$.

On the other hand, the term $F(w) \equiv [f, w]$ is Lipschitz from $H^2(\Omega) \to L_2(\Omega)$. Indeed,

\[
\|F(w_1) - F(w_2)\|_{L_2(\Omega)} = \|[f, w_1 - w_2]\|_{L_2(\Omega)} \leq C\|f\|_{L^2(\Omega)}\|w_1 - w_2\|_{H^2(\Omega)},
\] (2.18)

where we have used the property (2.3) of the bracket, $[\ , \ ]$.

Applying a standard perturbation theorem for nonlinear semigroups (see Barbu [2]), we obtain the existence and uniqueness of the solutions to (2.1). By standard semigroup estimates applied to (2.1) yield

\[
\|w_t(t)\|^2_{L_2(\Omega)} + a(w(t), w(t)) + 2 \int_0^t \int_\Omega \left(\frac{\partial}{\partial \tau} w_1 \frac{\partial}{\partial \tau} w_1 \right) + 2 \int_0^t \int_\Omega g(w_t)w_t \, d\Omega dt + 2 \int_0^t \int_\Omega \nabla f(\cdot, w)w_t \, d\Omega dt,
\] (2.19)

where

\[
a(w, w) \equiv \int_\Omega w^2 \, d\Omega + \int_\Omega [(\Delta w)^2 + (1 - \mu)(2w_x^2 - 2w_x w_y)] \, d\Omega,
\] (2.20)

and

\[
m_1 \|w\|^2_{H^2(\Omega)} \leq a(w, w) \leq M_1 \|w\|^2_{H^2(\Omega)}.
\] (2.21)

On the other hand, by (2.3),

\[
\|
\int_\Omega [f, w] w \, d\Omega \| \leq \|w_t\|_{L_2(\Omega)} \|w\|_{H^2(\Omega)} \|f\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} [\|w_t\|^2_{L_2(\Omega)} + \|w\|^2_{H^2(\Omega)}].
\] (2.22)
Combining (2.19) with (2.21) and (2.22) and applying Gronwall’s inequality, with $L_1$ kernel, yields the result of our proposition. □

The following higher regularity result is critical.

**Lemma 2.1** Let $w$ be a solution to (2.1). Let $f \in L_1(0, T; W^2_{\infty}(\Omega))$, $f_t \in L_\infty(0,T; H^{2+\epsilon}(\Omega))$, where $\epsilon > 0$ is arbitrarily small, $(w_0, w_1) \in H^{4}(\Omega) \times H^{2}(\Omega)$, and $g$ and $h$ satisfy hypothesis (H-1). Then the following estimate holds for all $t \leq T$:

\[
\begin{align*}
\|w(t)\|_{H^2(\Omega)}^2 + \|w_t(t)\|_{H^1(\Omega)}^2 &+ ||w_{tt}(t)||_{L^2(\Omega)}^2 + \int_0^t \int_{\Gamma} (\frac{\partial}{\partial \nu} w_t)^2 + w_{tt}^2 \, d\Gamma \, dt \\
&\leq C_T(||f||_{L^1(0,T; W^2_{\infty}(\Omega))}, ||f_t||_{L_\infty(0,T; H^{2+\epsilon}(\Omega))}) (||w_0||_{H^4(\Omega)}^2 + ||w_1||_{H^2(\Omega)}^2),
\end{align*}
\]

(2.23)

where $C_T(u,v)$ is a continuous function and $w_0, w_1$ satisfy the following compatibility conditions:

\[
\begin{align*}
\Delta w_0 + (1 - \mu) B_1 w_0 &= -h(\frac{\partial}{\partial \nu} w_1) & &\text{on } \Gamma. \\
\frac{\partial}{\partial \nu} \Delta w_0 + (1 - \mu) B_2 w_0 - w_0 &= g(w_1) & &\text{on } \Gamma.
\end{align*}
\]

**Proof:** We consider the following equation for $z$:

\[
\begin{align*}
z_{tt} + \Delta^2 z &= [f_t, w] + [f, z] & &\text{in } Q_T = (0, T) \times \Omega \\
z(0, \cdot) &= w_0(\cdot) = w_1(\cdot) & &\text{in } \Omega \\
z_t(0, \cdot) &= -\Delta^2 w_0 + [f(0), w_0] & &\text{in } \Omega \\
\Delta z + (1 - \mu) B_1 z &= -h'(\frac{\partial}{\partial \nu} w_1) \frac{\partial}{\partial \nu} z_t & &\text{on } \Sigma_T = (0, T) \times \Gamma \\
\frac{\partial}{\partial \nu} \Delta z + (1 - \mu) B_2 z - z &= g'(w_1)z_t & &\text{on } \Sigma_T = (0, T) \times \Gamma.
\end{align*}
\]

(2.24)

Notice that from inequality (2.8) and hypothesis (H-1), we obtain

\[
\frac{\partial}{\partial \nu} w_t \in L_2(\Sigma_T), \quad w_t \in L_2(\Sigma_T).
\]

(2.25)

Thus $h'(\frac{\partial}{\partial \nu} w_t)$ and $g'(w_t)$ are well defined a.e. functions on $\Sigma_T$. Since

\[
\begin{align*}
h'(\frac{\partial}{\partial \nu} w_t) &\geq 0 \text{ a.e. in } \Sigma, \\
g'(w_t) &\geq 0 \text{ a.e.}
\end{align*}
\]

(2.26)

and problem (2.24) is linear in $z$ with dissipative (linear) boundary conditions, the existence and uniqueness of the solution, $z$, to (2.24) in the space $C(0, T; H^2(\Omega)) \times C(0, T; L_2(\Omega))$ follows from standard linear theory.
Energy methods applied to equation (2.21) yield
\[
\|z(t)\|^2_{L^2(\Omega)} + a(z(t), z(t)) + 2 \int_0^t \int_\Omega \left( \frac{\partial}{\partial \nu} z(t) \right) \left( \frac{\partial}{\partial \nu} z(t) \right)^2 \, d\Omega \, dt + 2 \int_0^t \int_\Omega g'(w(t)) z(t)^2 \, d\Omega \, dt
= \|z_0(t)\|^2_{L^2(\Omega)} + a(z_0(t), z(t)) + 2 \int_0^t \int_\Omega ([f(t), w(t)] z(t) + [f, z(t)]) \, d\Omega \, dt.
\] (2.27)

On the other hand, from (2.3),
\[
\int_\Omega [f, z] z(t) \, d\Omega \leq (\|z_0\|^2_{L^2(\Omega)} + \|z\|^2_{H^2(\Omega)}) \|f\|_{L^2(\Omega)}.
\] (2.28)

For \(\frac{1}{q} + \frac{1}{\delta} = 1\),
\[
\left( \int_\Omega \|f(t, w)\|^2 \, d\Omega \right)^{1/2} \leq C \|w\|_{L^2(\Omega)} \|f(t)\|_{L^2(\Omega)},
\] (2.29)

and by Sobolev’s imbeddings, \(\forall \epsilon > 0, \exists \eta > 1\) such that
\[
\left( \int_\Omega \|f(t, w)\|^2 \, d\Omega \right)^{1/2} \leq C \|w\|_{H^2(\Omega)} \|f(t)\|_{H^2(\Omega)}.
\] (2.30)

Hence,
\[
\int_\Omega [f(t, w)] z(t) \, d\Omega \leq C \|z_0\|_{L^2(\Omega)} \|w\|_{H^2(\Omega)} \|f(t)\|_{H^2(\Omega)},
\] (2.31)

and
\[
\int_0^t \int_\Omega [f(t, w)] z(t) \, d\Omega \, dt \leq C \sup_{\tau \geq 0} \|z(t)\|^2_{L^2(\tau)} \|f(t)\|_{H^2(\tau)} \|w(\tau)\|_{H^2(\tau)} \, d\tau
\]
\[
\leq C \|z(t)\|^2_{L^2(\tau)} \|f(t)\|^2_{H^2(\tau)} \|w(\tau)\|^2_{H^2(\tau)} \, d\tau.
\] (2.32)

Selecting \(\epsilon_0\) suitably small, combining (2.27), (2.28), and (2.32), and recalling (2.20) and hypothesis (H-1) yields
\[
\|z(t)\|^2_{L^2(\Omega)} + \|z(t)\|^2_{H^2(\Omega)} + \int_0^t \|\frac{\partial}{\partial \nu} z(t)\|^2_{L^2(\tau)} \, d\tau + \int_0^t \|z(t)\|^2_{L^2(\tau)} \, dt
\leq C \{\|z_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{H^2(\Omega)} + \sup_{\tau \geq 0} \|f(t)\|^2_{H^2(\tau)} \|w(\tau)\|^2_{H^2(\tau)} \, d\tau
\]
\[
+ \int_0^t \|f(t)\|_{L^2(\tau)} \|z(t)\|^2_{L^2(\tau)} + \|z(t)\|^2_{H^2(\tau)} \, d\Omega\}.
\] (2.33)

Since both \(\frac{\partial}{\partial \nu} z(t), z(t) \in L^2(\Sigma_T)\), one can easily show by using compatibility conditions (1.5) that \(z \equiv w(t)\).

Our next step is to estimate the \(H^2(\Omega)\) norm of \(w(t)\). This is done by using elliptic theory applied to (2.1). Indeed, from (2.1) and [16], we obtain
\[
\|w(t)\|_{H^2(\Omega)} \leq C \{\|w_1(t)\|_{L^2(\Omega)} + \|f(t, w(t))\|_{L^2(\Omega)} + \|h(\frac{\partial}{\partial \nu} w(t))\|_{H^2(\tau)} + \|g(w(t))\|_{H^2(\tau)}\},
\] (2.34)
\begin{equation}
\|f(t), w(t)\|_{L^2(\Omega)} \leq C \|w(t)\|_{H^r(\Omega)} \|f(t)\|_{W^r_{\infty}(\Omega)},
\end{equation}
(2.35)

\begin{equation}
\|g(w_t)\|_{H^{-1/2}(\Gamma)} \leq \|g(w_t)\|_{L^{r+1}(\Gamma)} \leq (\int_\Gamma |g(w_t)| |w_t|^{r+1} d\Gamma)^{\frac{1}{r+1}} = (\int_\Gamma g(w_t) w_t d\Gamma)^{\frac{1}{r+1}},
\end{equation}
(2.36)

where the first inequality follows because

\[ H^{1/2}(\Gamma) \subset L^2(\Gamma) \text{ for any } p \implies L^{p'}(\Gamma) \subset H^{-1/2}(\Gamma) \text{ for any } p' > 1, \]

and the last equality is obtained by choosing \( \epsilon = \frac{1}{r+1} \).

Let \((Hu)(x) \equiv h(u(x))\), where \(h\) satisfies (II-1). We shall show that

\begin{equation}
\|Hu\|_{H^{1/2}(\Gamma)} \leq C\|u\|_{H^{1/2}(\Gamma)},
\end{equation}
(2.37)

Indeed, by the linear growth condition imposed on \(h\), we have that

\begin{equation}
\|Hu\|_{L^2(\Gamma)} \leq C\|u\|_{L^2(\Gamma)}.
\end{equation}
(2.38)

Moreover,

\begin{equation}
\|Hu\|_{H^{1}(\Gamma)} \leq C\|u\|_{H^{1}(\Gamma)}.
\end{equation}
(2.39)

This follows from

\begin{equation}
\|Hu\|_{H^{1}(\Gamma)} = \|h'(u) \frac{\partial}{\partial x_i} u\|_{L^2(\Gamma)} \leq C\|\frac{\partial}{\partial x_i} u\|_{L^2(\Gamma)} \leq C\|u\|_{H^{1}(\Gamma)}.
\end{equation}
(2.40)

Also,

\begin{equation}
\|Hu_1 - Hu_2\|_{L^2(\Gamma)} = \|h(u_1) - h(u_2)\|_{L^2(\Gamma)}
= \|\int_0^1 h'(u_1(s) - (1-s)u_2(s))(u_1 - u_2)ds\|_{L^2(\Gamma)} \leq C\|u_1 - u_2\|_{L^2(\Gamma)}.
\end{equation}
(2.41)

Thus, by a nonlinear interpolation theorem of [19] (see Theorem 2), we conclude that (2.37) holds true.

From (2.37),

\begin{equation}
\|h(\frac{\partial}{\partial y} w_t(t))\|_{H^{1/2}(\Gamma)} \leq C\|\frac{\partial}{\partial y} w_t(t)\|_{H^{1/2}(\Gamma)} \leq C\|w_t(t)\|_{H^r(\Omega)},
\end{equation}
(2.42)

where the last inequality follows by trace theory. Collecting (2.34), (2.35), (2.36), and (2.42),

\begin{equation}
\|w(t)\|_{H^r(\Omega)} \leq C\{\|w(t)\|_{H^r(\Omega)} \|f(t)\|_{W^r_{\infty}(\Omega)} + \|w_t(t)\|_{L^2(\Omega)} + \int_\Gamma g(w_t) w_t d\Gamma + \|w_t(t)\|_{H^r(\Omega)}\},
\end{equation}
(2.43)
and from Proposition 2.1, recalling $w_t = z$,

$$
\int_0^t \|w(t)\|_{H^2(\Omega)} dt \leq C \int_0^t \left( \|z(t)\|_{L^2(\Omega)} + \|w(t)\|_{H^2(\Omega)} \|f(t)\|_{L^2(\Omega)} + \|z(t)\|_{H^2(\Omega)} \right) dt + C \int_0^t \int_\Gamma g(w_t) w_t d\Gamma dt \\
\leq C_T \int_0^t \left( \|z(t)\|_{L^2(\Omega)} + \|z(t)\|_{H^2(\Omega)} \right) dt + (\|f\|_{L^1(0,T; H^2(\Omega))} + 1) C_0(f, w_0, w_1).
$$

(2.44)

Going back to (2.33),

$$
\|z_t(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{H^2(\Omega)}^2 \leq C(\|z(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{H^2(\Omega)}^2) \\
+ C_T \left[ \int_0^t \|f(\tau)\|_{L^2(\Omega)} \|z_t(\tau)\|_{L^2(\Omega)}^2 + \|z(\tau)\|_{H^2(\Omega)}^2 d\tau \right] \\
+ \|f_t\|_{L^\infty(0,T; H^2(\Omega))} \left[ \int_0^t \|z_t(\tau)\|_{L^2(\Omega)}^2 + \|z(\tau)\|_{H^2(\Omega)}^2 d\tau \right] \\
+ \|f\|_{L^1(0,T; w_0^2(\Omega))} C_0(f, w_0, w_1).
$$

(2.45)

Gronwall's inequality applied to (2.45) yields the estimate in (2.23) for $w_t$ and $w_{tt}$. Combining this result with the estimate for $w$ in (2.43) and using the inequality

$$
\int_\Gamma w_t g(w_t) d\Gamma \leq C \int_\Gamma |w_t|^{r+2} d\Gamma \leq C \|w_t\|_{C^{0, 1, T, H^2(\Omega)}}^{r+2},
$$

we obtain the final result of Lemma 2.1. \qed

3 A Priori Bounds

Lemma 3.1 Let $w$ be a solution to (1.1) such that $w_0 \in H^q(\Omega)$, $w_1 \in H^2(\Omega)$ and

$$
w \in C(0, T; H^{3-\epsilon}(\Omega));
$$

$$
w_t \in C(0, T; H^2(\Omega));
$$

$$
w_{tt} \in C(0, T; L^2(\Omega)),
$$

(3.1)

where $0 < \epsilon < \frac{1}{2}$. Then the following a priori bounds hold:

$$
\|w(t)\|_{C(0, T; H^2(\Omega))} + \|w_t(t)\|_{C(0, T; H^2(\Omega))} + \|w_{tt}(t)\|_{C(0, T; L^2(\Omega))} \\
\leq C_T(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{H^2(\Omega)}),
$$

(3.2)

where $C_T$ is bounded for any arbitrary $T > 0$. 

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**Proof:** We first show that energy estimates applied to (1.1) give a priori bounds in $H^2(\Omega) \times L^2(\Omega)$. Notice that with $w \in C(0, T; H^{3-\epsilon}(\Omega))$, we have

\[
\mathcal{F}(w) = -G[w, w] \in H^1(\Omega)
\]

and

\[
[\mathcal{F}(w), w] \in L^2(\Omega),
\]

where $G$ is defined by

\[
Gf = g \iff \Delta^2 g = f \quad \text{in } \Omega
\]

\[
\begin{cases}
  g = 0 \\
  \frac{\partial}{\partial n} g = 0
\end{cases} \quad \text{on } \Gamma.
\]

Applying estimate (2.8) with $f(t) = [\mathcal{F}(w(t)), w(t)]$ yields

\[
a(w(t), w(t)) + \|w_t\|^2_{L^2(\Omega)} + 2 \int_0^1 \int_{\Gamma} h'(\frac{\partial}{\partial n} w_t) \frac{\partial}{\partial n} w_t d\Gamma dt + 2 \int_0^1 \int_{\Gamma} g(w_t) w_t d\Gamma dt
\]

\[
\leq a(w_0, w_0) + \|w_1\|^2_{L^2(\Omega)} + 2 \int_0^1 \int_{\Gamma} [\mathcal{F}(w), w] w_t d\Gamma dt
\]

\[
= a(w_0, w_0) + \|w_1\|^2_{L^2(\Omega)} - 2\|\Delta \mathcal{F}(w(t))\|_{L^2(\Omega)} + 2\|\Delta \mathcal{F}(w(0))\|_{L^2(\Omega)}.
\]

Hence, since $\|\Delta \mathcal{F}(w(0))\|^2_{L^2(\Omega)} \leq C\|w_0\|^2_{H^2(\Omega)}$,

\[
\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L^2(\Omega)} + \int_0^1 \int_{\Gamma} (\|\frac{\partial}{\partial n} w_t\|^2 + w_t^2) d\Gamma dt \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)}),
\]

where hypothesis (H-1) has again been applied.

Our next step is to obtain bounds for higher norms. We return to equation (2.24) which, for a fixed $w_t|_\Gamma$ (well-defined globally by (3.6)), is a linear equation in $z = w_t$. Thus with $f_t(t) \in L^2(\Omega_T)$, by standard energy estimates, which are justified for the linear problem, we have

\[
a(z(t), z(t)) + \|z_t(t)\|^2_{L^2(\Omega)} + 2 \int_0^t \int_{\Gamma} h'(\frac{\partial}{\partial n} w_t) |\frac{\partial}{\partial n} z_t|^2 d\Gamma dt + 2 \int_0^t \int_{\Gamma} g'(w_t) z_t^2 d\Gamma dt
\]

\[
= a(z(0), z(0)) + \|z_t(0)\|^2_{L^2(\Omega)} + 2 \int_0^t \int_{\Omega} f_t(t) z_t d\Omega dt.
\]

Let $f_t(t) \equiv \frac{d}{dt}[\mathcal{F}(w(t)), w(t)]$. Then $f_t(t)$ can be rewritten as:

\[
f_t(t) = [\frac{d}{dt} \mathcal{F}(w(t)), w(t)] + [\mathcal{F}(w(t)), w_t(t)]
\]

\[
= -2[G[w, w_t], w(t)] - [G[w, w], w_t].
\]

Clearly, if $w_t \in C(0, T; H^2(\Omega))$, $w \in C(0, T; H^{3-\epsilon}(\Omega))$, then, by the properties of the bracket and elliptic regularity, $f_t(t) \in C(0, T; L^2(\Omega))$, therefore (3.7) holds with $f_t(t)$ defined as above.
To simplify the integral in (3.7) corresponding to \( f(t) \), the following estimates are useful:

\[
\int_{\Omega} G[w, w(t)] z(t) d\Omega = \int_{\Omega} G[w, w(t)] z(t) d\Omega
\]

\[
= \int_{\Omega} G[w, z] \frac{d}{dt} [z, w] d\Omega - \int_{\Omega} [z, z] G[w, z] d\Omega
\]

\[
= \int_{\Omega} G[w, z] \frac{d}{dt} \Delta^2 G[w, z] d\Omega - \int_{\Omega} [z, z] G[w, z] d\Omega
\]

\[
= \frac{1}{2} \int_{\Omega} |\Delta G[w, z]|^2 d\Omega - \int_{\Omega} [z, z] G[w, z] d\Omega, \tag{3.9}
\]

where the last line follows by using the boundary conditions associated with the operator \( G \). In addition, we have

\[
\int_{\Omega} [F(w), w_t] z_t d\Omega = \frac{d}{dt} \int_{\Omega} [F(w), z] z d\Omega - \int_{\Omega} [F(w), z_t] z d\Omega - \int_{\Omega} [F(w), z] z_t d\Omega \tag{3.10}
\]

\[
= \frac{d}{dt} \int_{\Omega} [F(w), z] z d\Omega - \int_{\Omega} [F(w), z_t] z d\Omega + 2 \int_{\Omega} [G[w, z], z] z d\Omega.
\]

Hence, rearranging terms in the above identity,

\[
\int_{\Omega} [F(w), w_t] z_t d\Omega = \frac{1}{2} \frac{d}{dt} \int_{\Omega} [F(w), z] z d\Omega + \int_{\Omega} [z, z] G[w, z] d\Omega. \tag{3.11}
\]

Combining (3.8), (3.9), and (3.11) yields

\[
\int_{\Omega} f(t) z_t d\Omega = -\frac{d}{dt} \int_{\Omega} |\Delta G[w, z]|^2 d\Omega + 3 \int_{\Omega} [z, z] G[w, z] d\Omega - \frac{1}{2} \frac{d}{dt} \int_{\Omega} [F(w), z] z d\Omega. \tag{3.12}
\]

Substituting (3.12) into (3.7) gives

\[
a(z(t), z(0)) + \|z(t)\|^2_{L^2(\Omega)} + 2 \|\Delta G[w, z](t)\|^2_{L^2(\Omega)} \\
\leq a(z(0), z(0)) + \|z_t(0)\|^2_{L^2(\Omega)} + 2 \|\Delta G[w, z](0)\|^2_{L^2(\Omega)} + 6 \int_{\Omega} [z, z] G[w, z] d\Omega \tag{3.13}
\]

\[
+ \int_{\Omega} [F(w(t)), z(t)] z(t) d\Omega + \int_{\Omega} [F(w(0)), z(0)] z(0) d\Omega.
\]

To bound the terms involving \( F \) and \( G \) on the right-hand side of (3.13), we will need the following estimates:

**Proposition 3.1**

\[
|\int_{\Omega} [F(w(t)), z(t)] z(t) d\Omega| \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)})(\|z(t)\|_{H^2(\Omega)})^2 \tag{3.14}
\]

\[
|\int_{\Omega} [z(t), z(t)] G[w(t), z(t)] d\Omega| \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)})(\|z(t)\|_{H^2(\Omega)})^2 \tag{3.15}
\]

**Proof of Proposition 3.1: Step 1: Proof of (3.14):** We already know (see [11])

\[
\|F(w)\|_{H^2(\Omega)} \leq C \|w\|_{H^2(\Omega)}^2. \tag{3.16}
\]
Therefore,
\[
| \int_\Omega [\mathcal{F}(w), z] d\Omega | = | \int_\Omega [z, z] \mathcal{F}(w) d\Omega |
\]
\[
\leq C \| \mathcal{F}(w) \|_{H^2_0(\Omega)} \| [z, z] \|_{H^{-2}(\Omega)}
\]
\[
\leq C \| \mathcal{F}(w) \|_{H^2_0(\Omega)} \| z \|_{H^{2-\theta}(\Omega)} \| z \|_{H^{1+\theta}(\Omega)},
\]
where \( 0 < \theta < 1 \) and the last inequality follows by property (2.6) of the bracket. By interpolation inequalities,
\[
\| z \|_{H^s(\Omega)} \leq C \| z \|_{L^2(\Omega)} \| z \|_{H^{s/2}(\Omega)}^{1-\theta/2} \| z \|_{H^{s/2}(\Omega)}^{\theta/2}, \quad 0 < \theta < 2.
\] (3.18)

Thus,
\[
\| z \|_{H^{2-\theta}(\Omega)} \leq C \| z \|_{L^2(\Omega)}^{1-\theta/2} \| z \|_{H^{2}(\Omega)}^{\theta/2}
\]
and
\[
\| z \|_{H^{1+\theta}(\Omega)} \leq C \| z \|_{L^2(\Omega)}^{1/2} \| z \|_{H^{2}(\Omega)}^{\theta/2}.
\] (3.19)

Hence,
\[
\| z \|_{H^{2-\theta}(\Omega)} \| z \|_{H^{1+\theta}(\Omega)} \leq C \| z \|_{L^2(\Omega)}^{1/2} \| z \|_{H^{2}(\Omega)}^{3/2}.
\] (3.20)

Collecting equations (3.17) and (3.20), we find that
\[
| \int_\Omega [\mathcal{F}(w), z] d\Omega | \leq C \| w \|_{H^2(\Omega)} \| z \|_{L^2(\Omega)}^{1/2} \| z \|_{H^{2}(\Omega)}^{3/2}
\]
\[
= C \| w \|_{H^2(\Omega)} \| w \|_{L^2(\Omega)} \| z \|_{H^{2}(\Omega)}^{3/2}.
\] (3.21)

Now (3.14) follows from (3.21) combined with (3.6).

Step 2: Proof of (3.15): Recall from (2.6),
\[
\| [z, w] \|_{H^{-1-\theta}(\Omega)} \leq C \| z \|_{H^{2-\theta}(\Omega)} \| w \|_{H^{2}(\Omega)} \quad 0 < \theta < 1.
\] (3.22)

Hence,
\[
\| G[z, w] \|_{H^{2-\theta}(\Omega)} \leq C \| z \|_{H^{2-\theta}(\Omega)} \| w \|_{H^{2}(\Omega)}.
\] (3.23)

By property (2.4) of the bracket and (3.23),
\[
\| [G[z, w], z] \|_{H^{-\theta}(\Omega)} \leq C \| z \|_{H^{2}(\Omega)} \| G[z, w] \|_{H^{2-\theta}(\Omega)} \leq C \| z \|_{H^{2}(\Omega)} \| z \|_{H^{2-\theta}(\Omega)} \| w \|_{H^{2}(\Omega)}.
\] (3.24)

Therefore,
\[
| \int_\Omega [z, z] G[w, z] d\Omega | = | \int_\Omega [G[w, z], z] d\Omega |
\]
\[
\leq C \| [G[w, z], z] \|_{H^{-\theta}(\Omega)} \| z \|_{H^{2}(\Omega)} \leq C \| z \|_{H^{2}(\Omega)} \| w \|_{H^{2}(\Omega)} \| z \|_{H^{2-\theta}(\Omega)} \| z \|_{H^{\theta}(\Omega)} \| z \|_{H^{\theta}(\Omega)}
\]
\[
\leq C \| z \|_{H^{2}(\Omega)} \| w \|_{H^{2}(\Omega)} \| z \|_{L^2(\Omega)},
\] (3.25)
where the last inequality follows by application of the interpolation inequality,
\[ \|z\|_{L^2(\Omega)} \le C \|z\|_{H^s(\Omega)} \|z\|_{H^L(\Omega)}. \] (3.26)

Combining (3.6) with (3.26) yields (3.15).

**Proof of Lemma 3.1 (con't):** From (3.13), (3.14), and (3.15) applied at \( t = 0 \) and Gronwall's inequality, we find
\[ \|z(t)\|_{H^s(\Omega)}^2 + \|z_t(t)\|_{L^2(\Omega)}^2 \le C_0(C_1 + \int_{\Omega} \mathcal{F}(w(t)) z(t) \, d\Omega) e^{C_0 t}, \] (3.27)

where the constants depend on the following norms:
\[ C_1 \equiv C_1(\|w_0\|_{H^s(\Omega)} + \|w_1\|_{H^L(\Omega)}), \]
\[ C_0 \equiv C_0(\|w(t)\|_{H^s(\Omega)}, \|w_t(t)\|_{L^2(\Omega)}) \le C_0(\|w_0\|_{H^s(\Omega)}, \|w_1\|_{L^2(\Omega)}). \] (3.28)

Inserting (3.14) into (3.27), we find
\[ \|z(t)\|_{H^s(\Omega)}^2 + \|z_t(t)\|_{L^2(\Omega)}^2 \le C_0 C_1 (1 + \|z(t)\|_{H^s(\Omega)}^{3/2}) \le C_0 (C_1 + \|z(t)\|_{H^s(\Omega)}^{3/2}) e^{C_0 t}, \] (3.29)

by the a priori bound in (3.6). Since the dependence on \( \|z(t)\|_{H^s(\Omega)} \) is subquadratic, it follows that
\[ \|z(t)\|_{H^s(\Omega)}^2 + \|z_t(t)\|_{L^2(\Omega)}^2 \le C_0 C_1 \quad t \le T. \] (3.30)

To achieve the desired result of Lemma 3.1, elliptic regularity results applied to
\[ \begin{align*}
\Delta^2 w &= -w_{tt} + \mathcal{F}(w), w \quad \text{in } \Omega \\
\Delta w + (1 - \mu) B_1 w &= -h(\frac{\partial}{\partial n}) \quad \text{on } \Gamma \\
\frac{\partial}{\partial n} \Delta w + (1 - \mu) B_2 w - w &= g(w_1) \quad \text{on } \Gamma,
\end{align*} \] (3.31)
yields, with \( \epsilon < \frac{1}{2}, \)
\[ \|w(t)\|_{H^{s+\epsilon}(\Omega)} \le \{ \|w_{tt}(t)\|_{H^{-s+\epsilon}(\Omega)} + \|\mathcal{F}(w(t), w_t(t))\|_{H^{-s+\epsilon}(\Omega)} + \|h(\frac{\partial}{\partial n} w_t)\|_{H^{s+\epsilon}(\Gamma)} + \|g(w_t)\|_{H^{-s+\epsilon}(\Gamma)} \} \]
\[ \le C \{ \|w_{tt}(t)\|_{H^{-s+\epsilon}(\Omega)} + \|w(t)\|_{H^{s+\epsilon}(\Omega)} + \|w_t(t)\|_{H^{s+\epsilon}(\Omega)} + \|w_t\|_{L^{r+1}(\Gamma)} \}, \] (3.32)

where we have used \( L^{r+1}(\Gamma) \subset H^{-1/2}(\Gamma) \). By Sobolev's embeddings, trace theory, interpolation inequalities and (3.30) with (3.6),
\[ \|w_t\|_{L^{r+1}(\Gamma)} \le C \|w_t\|_{H^{s+\epsilon}(\Omega)} \le C \|w_t\|_{H^{s+\epsilon}(\Omega)} \|w_t\|_{L^2(\Omega)} \]
\[ \le (C_0 C_1)^{\frac{r+1}{4}} C_0^{\frac{r+1}{4}}. \] (3.33)

Combining (3.30), (3.32)-(3.33) and (3.6) proves the desired result of Lemma 3.1. \( \Box \)
4 Existence of Regular Solutions: Proof of Theorem 1.1

We shall use Schaeffer's Theorem (see Kesavan [9], pg. 221). To accomplish this, we construct a map

\[ v \mapsto T v, \tag{4.1} \]

defined on a Banach space, \( X \), where

\[ X \equiv C(0,T; H^{3-\epsilon}(\Omega)) \cap C^1(0,T; H^{2-\epsilon}(\Omega)), \tag{4.2} \]

\( 0 < \epsilon < \frac{1}{2} \) is a fixed number, and \( T v \) is defined as the solution to (2.1) with \( f(t) \) given by

\[ f(t) \equiv \mathcal{F}(v(t)). \tag{4.3} \]

**Proposition 4.1** With reference to (4.3), \( \forall 0 < \epsilon < \frac{1}{2} \), we have

\[ \| f(t) \|_{H^{3-\epsilon}(\Omega)} \leq C \| v(t) \|_{H^{2-\epsilon}(\Omega)} \| v(t) \|_{H^{2}(\Omega)}, \tag{4.4} \]

\[ \| f_t(t) \|_{H^{2+\epsilon}(\Omega)} \leq C \| v(t) \|_{H^{2-\epsilon}(\Omega)} \| v_t(t) \|_{H^{2-\epsilon}(\Omega)}. \tag{4.5} \]

**Proof:** From property (2.4) of the bracket,

\[ \| [v, v] \|_{H^{3-\epsilon}(\Omega)} \leq C \| v(t) \|_{H^{2-\epsilon}(\Omega)} \| v(t) \|_{H^2(\Omega)}. \tag{4.6} \]

By elliptic regularity and (4.6),

\[ \| \mathcal{F}(v) \|_{H^{3-\epsilon}(\Omega)} \leq \| G[v, v] \|_{H^{3-\epsilon}(\Omega)} \leq C \| [v, v] \|_{H^{3-\epsilon}(\Omega)} \leq C \| v \|_{H^{2-\epsilon}(\Omega)} \| v \|_{H^2(\Omega)}, \tag{4.7} \]

proving (4.4). As for (4.5), we use property (2.5) of the bracket to find

\[ \| [v, v_t] \|_{H^{2+\epsilon}(\Omega)} \leq C \| v_t \|_{H^{2-\epsilon}(\Omega)} \| v \|_{H^{2-\epsilon}(\Omega)}, \tag{4.8} \]

which, in turn, by elliptic regularity gives

\[ \| G[v, v_t] \|_{H^{2+\epsilon}(\Omega)} \leq C \| v_t \|_{H^{2-\epsilon}(\Omega)} \| v \|_{H^{2-\epsilon}(\Omega)}, \tag{4.9} \]

which, in particular, proves (4.5). \( \square \)
Since, by Sobolev imbeddings (see [1])

$$H^{4-\epsilon}(\Omega) \subset W^2_{\infty}(\Omega), \quad \forall 0 < \epsilon < \frac{1}{2},$$ \hspace{1cm} (4.10)

the result of Lemma 2.1 together with Proposition 4.1 assert that the map \(T\) is bounded from \(X\) into

$$X_1 \equiv L^\infty(0, T; H^3(\Omega)) \cap W^1_\infty(0, T; H^2(\Omega)) \cap W^2_\infty(0, T; L^2(\Omega)).$$

Thus, by the compactness of imbeddings, \(H^3(\Omega) \subset H^{3-\epsilon}(\Omega)\) and \(H^2(\Omega) \subset H^{2-\epsilon}(\Omega)\), and by the result due to Simon ([17], Corollary 4), we obtain that

$$T : X \to X$$

is compact. \hspace{1cm} (4.11)

Our next step is to show that the map \(T : X \to X\) is continuous. Denote \(F \equiv L_1([0, T]; W^2_\infty(\Omega)) \cap L^\infty([0, T], H^{2+\epsilon}(\Omega)).\) By the same arguments as those in Proposition 4.1, one easily shows that the map

$$F : X \to F$$

is continuous. Thus, it suffices to prove the continuity of the map \(f \to (w, w_t)\) from \(F \to X\), where \((w, w_t)\) is a solution to (2.1) corresponding to \(f\). Let \(f_n \in F\) be such that \(f_n \to f\) in \(F\). We need to show that

\((w_n, w_{n,t}) \to (w, w_t)\) in \(X\), where \(w_n\) is a solution to (2.1) corresponding to \(f_n\). Consider the system solved by \(\hat{w} \equiv w_n - w\). Multiplying the result by \(\hat{w}_t\) and integrating by parts, we find

\[
\begin{align*}
||\hat{w}(t)||_{H^2(\Omega)}^2 + ||\hat{w}_t(t)||_{L^2(\Omega)}^2 &+ \int_0^t \int_{\Omega} b\left(\frac{\partial w_n}{\partial v}, \frac{\partial w_t}{\partial v}\right) \frac{\partial \hat{w}_t}{\partial v} dx dt + \int_0^t \int_{\Omega} \left[ g(w_{n,t}) - g(w_t) \right] \hat{w}_t dx dt \\
&\leq C \int_0^t \int_{\Omega} [f_n - f, w_n] - [f, w] \hat{w}_t dx dt \\
&= C \int_0^t \int_{\Omega} [(f_n - f, w_n) + [f, \hat{w}]] \hat{w}_t dx dt \\
&\leq C \{ \|f_n - f, w_n\|_{L^2(\Omega)} + \|f, \hat{w}\|_{L^2(\Omega)} \} \|\hat{w}_t\|_{L^2(\Omega)} dt \\
&\leq C \{ \int_0^t \|f_n - f\|_{H^2(\Omega)} \|w_n\|_{H^2(\Omega)} + \|f, \hat{w}\|_{L^2(\Omega)} \} \|\hat{w}_t\|_{L^2(\Omega)} dt \\
&+ \int_0^t \|f\|_{H^2(\Omega)} \|\hat{w}\|_{H^2(\Omega)} + \|\hat{w}_t\|_{L^2(\Omega)} \} dt \}.
\end{align*}
\] \hspace{1cm} (4.12)

Applying Proposition 2.1 and Gronwall's inequality and noting that the two integral terms on the left-hand side of (4.12) are positive by the monotonicity of \(b\) and \(g\), we obtain

$$||\hat{w}(t)||_{H^2(\Omega)}^2 + ||\hat{w}_t(t)||_{L^2(\Omega)}^2 \leq C \{ \|f\|_{L^2(\Omega)}, \|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)} \} \int_0^t \|f_n - f\|_{H^2(\Omega)} dt.$$ \hspace{1cm} (4.13)
Hence, as \( \|f_n - f\|_r \to 0 \), \( \|\tilde{w}(t)\|_H^2 + \|\tilde{w}_t(t)\|_L^2 \to 0 \).

Next, we consider the continuity of the higher norms. Returning to equation (2.24), we proceed as before.

Let \( z_n \) denote the solution of equation (2.24) with \( f \) replaced by \( f_n \). Consider the system solved by \( \tilde{z} \equiv z_n - z \).

Multiplying the result by \( \tilde{z}_t \), integrating by parts, and recalling hypothesis (H-1), we find

\[
\|\tilde{z}(t)\|^2_{L^2} + \|\tilde{z}_t(t)\|^2_{L^2} + \int_0^t \int_{\Omega} \left( \frac{\partial}{\partial t} \tilde{z}_t \right)^2 \mathrm{d}\Gamma \mathrm{d}t + \int_0^t \int_{\Omega} \left( \tilde{z}_t \right)^2 \mathrm{d}\Gamma \mathrm{d}t \\
\leq C \int_0^t \int_{\Omega} \left[ h\left( \frac{\partial}{\partial t} w_{n,t} \right) - h\left( \frac{\partial}{\partial t} w_{n,t} \right) \right] \frac{\partial}{\partial t} \tilde{z}_t \mathrm{d}\Gamma \mathrm{d}t + C \int_0^t \int_{\Omega} \left[ g\left( w_{n,t} \right) - g\left( w_{n,t} \right) \right] \tilde{z}_t \mathrm{d}\Gamma \mathrm{d}t \\
+ C \int_0^t \int_{\Omega} \left[ \int_{\Omega} f_n \left[ f_n - f, w_{n,t} \right] + \left[ f_n - f, w_{n,t} \right] + \left[ f - z_n - z, z_n - z \right] \tilde{z}_t \mathrm{d}\Omega \mathrm{d}t \right]
\]

\[
\leq C_{T,\rho} \int_0^t \int_{\Omega} \left[ h\left( \frac{\partial}{\partial t} w_{n,t} \right) - h\left( \frac{\partial}{\partial t} w_{n,t} \right) \right] \frac{\partial}{\partial t} \tilde{z}_t \mathrm{d}\Gamma \mathrm{d}t \\
+ \int_0^t \int_{\Omega} \left[ h\left( \frac{\partial}{\partial t} w_{n,t} \right) - h\left( \frac{\partial}{\partial t} w_{n,t} \right) \right] \tilde{z}_t \mathrm{d}\Gamma \mathrm{d}t \\
+ C \int_0^t \int_{\Omega} \left[ \int_{\Omega} f_n \left[ f_n - f, w_{n,t} \right] + \left[ f_n - f, w_{n,t} \right] + \left[ f - z_n - z, z_n - z \right] \tilde{z}_t \mathrm{d}\Omega \mathrm{d}t \right],
\]

(4.14)

where \( \rho > 0 \) can be taken arbitrarily small.

Using the properties of the bracket, (2.2)-(2.7), we obtain the following bounds:

\[
\int_0^t \int_{\Omega} \left[ f_n - f, w_{n,t} \right] \tilde{z}_t \mathrm{d}\Omega \mathrm{d}t \leq C \| \tilde{z}_t \|_{L^1(0, T; L^2(\Omega))} \| w_{n,t} \|_{L^1(0, T; H^2(\Omega))} \| f_n - f \|_{L^1(0, T; H^2(\Omega))}.
\]

(4.15)

On the other hand, by using elliptic theory to estimate \( \|\tilde{w}(t)\|_{H^2(\Omega)} \), we obtain

\[
\|\tilde{w}(t)\|_{H^2(\Omega)} \leq C \left\{ \|\tilde{w}_t(t)\|_{L^2(\Omega)} + \| w_{n,t}(t) \|_{L^2(\Omega)} + \| f - \tilde{w}(t) \|_{L^2(\Omega)} \right\}
\]

(4.16)

By an application of Theorem 4 in [19] and arguments similar to those in (2.37)-(2.42), combined with trace theory, we obtain

\[
\| h\left( \frac{\partial}{\partial t} w_{n,t}(t) \right) - h\left( \frac{\partial}{\partial t} w(t) \right) \|_{H^{1/2}(\Gamma)} \leq C \| \tilde{w}_t(t) \|_{H^{1/2}(\Gamma)} \leq C \|\tilde{w}(t)\|_{H^2(\Omega)}.
\]

(4.17)

The local Lipschitz property of the function \( g \), hypothesis (H-1), and Sobolev's imbeddings give us

\[
\| g(w_{n,t}(t)) - g(w(t)) \|_{H^{1/2}(\Gamma)} \leq C \left\{ \| w_{n,t}(t) \|_{H^2(\Omega)}, \| w(t) \|_{H^2(\Omega)} \right\} \|\tilde{w}(t)\|_{H^2(\Omega)}
\]

(4.18)
where we have used the result of Lemma 2.1.

Combining (4.16)-(4.18), yields, for all $t \leq T$,

$$
\|\tilde{w}(t)\|_{H^2(\Omega)} \leq C_T(\|f\|_{\mathcal{F}} \cdot \|w_0\|_{H^2(\Omega)} \cdot \|w_1\|_{H^2(\Omega)} + \|\tilde{w}_0(t)\|_{L^2(\Omega)} + \|\tilde{w}_1(t)\|_{H^2(\Omega)} + \|f_n(t) - f(t)\|_{W^2(\Omega)}).
$$

(4.19)

Going back to (4.14), using (4.15), (4.19), Gronwall's inequality and selecting $\rho$ in (4.14) suitably small yields

$$
\|\tilde{w}(t)\|^2_{H^2(\Omega)} + \|\tilde{w}_0(t)\|^2_{L^2(\Omega)} + \|\tilde{w}_1(t)\|^2_{L^2(\Omega)}
\leq C_T(\|f\|_{\mathcal{F}} \cdot \|w_0\|_{H^2(\Omega)} \cdot \|w_1\|_{H^2(\Omega)} + \|f_n - f\|^2_{\mathcal{F}} + C\rho \int_0^t \int_T |h'(\frac{\partial}{\partial \nu} w_{n,t}) - h'(\frac{\partial}{\partial \nu} w_t)|^2 |\frac{\partial}{\partial \nu} z_t|^2 d\Gamma dt
\quad + \int_0^t \int_T |g'(w_{n,t}) - g'(w_t)|^2 |\tilde{z}_t|^2 d\Gamma dt).
$$

(4.20)

Next, we must analyze the effects of the boundary integrals in (4.20).

**Analysis of $\mathcal{A}_n \equiv \int_0^t \int_T \left| h'(\frac{\partial}{\partial \nu} w_{n,t}) - h'(\frac{\partial}{\partial \nu} w_t) \right| |\frac{\partial}{\partial \nu} z_t|^2 d\Gamma dt$:** Notice that from Lemma 2.1 and hypothesis (H-1), the integrand satisfies the following estimate uniformly in $n$.

$$
|h'(\frac{\partial}{\partial \nu} w_{n,t}) - h'(\frac{\partial}{\partial \nu} w_t)| |\frac{\partial}{\partial \nu} z_t|^2 \leq C |\frac{\partial}{\partial \nu} z_t|^2 \in L_2(\Sigma_T),
$$

(4.21)

therefore Lebesgue's dominated convergence theorem applies, provided the integrand goes to zero as $f_n \to f$ in $F$. Recall the estimate for $\tilde{w}$, (4.12). We know

$$
\int_0^t \int_T \left| h(\frac{\partial}{\partial \nu} w_{n,t}) - h(\frac{\partial}{\partial \nu} w_t) \right| |\frac{\partial}{\partial \nu} \tilde{w}_t| d\Gamma dt \leq C_0(f, w_0, w_1) \int_0^t \|f_n - f\|_{W^2(\Omega)} dt,
$$

(4.22)

therefore, by hypothesis (H-2).

$$
\int_0^t \int_T \left| \frac{\partial}{\partial \nu} (w_{n,t} - w_t) \right| d\Gamma dt \to 0 \quad \text{a.e.,} \quad \frac{\partial}{\partial \nu} \tilde{w} \to 0 \quad \text{a.e.,}
$$

(4.23)

hence,

$$
h'(\frac{\partial}{\partial \nu} w_{n,t}) - h'(\frac{\partial}{\partial \nu} w_t) \to 0 \quad \text{a.e.,}
$$

(4.24)

and Lebesgue's dominated convergence theorem applies, allowing us to conclude $\mathcal{A}_n \to 0$ when $f_n \to f$ in $F$. 

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Analysis of $B_n \equiv \int_0^t \int_\Gamma |g'(w_t) - g'(w_{n,t})|^2 z_t^2 d\Gamma dt$: We bound $B_n$ as follows:

\[
\int_0^t \int_\Gamma |g'(w_t) - g'(w_{n,t})|^2 z_t^2 d\Gamma dt \\
\leq C||g'(w_t) - g'(w_{n,t})||_{C^0(0,T;\Gamma)}^2 ||z_t||_{L^2(0,T;\Gamma)}^2 \\
\leq C_T(||f||_F, ||w_0||_{H^s(\Omega)}, ||w_1||_{L^2(\Omega)} ||g'(w_t) - g'(w_{n,t})||_{C^0(0,T;\Gamma)}^2 \\
\leq C_T(||f||_F, ||w_0||_{H^s(\Omega)}, ||w_1||_{H^r(\Omega)} ||w_{n,t} - w_t||_{L^\infty(0,T;H^r(\Omega))} ||w_{n,t} - w_t||_{C^0(0,T;\Gamma)}^2 \\
\leq C_T(||f||_F, ||w_0||_{H^s(\Omega)}, ||w_1||_{H^r(\Omega)} ||w_{n,t} - w_t||_{L^\infty(0,T;H^r(\Omega))} ||w_{n,t} - w_t||_{C^0(0,T;\Gamma)}^{2(1-\theta)} + \theta ||w||_{C^0(0,T;H^s(\Omega))}^2,
\]

(4.25)

where the second inequality follows from Lemma 2.1, the third by the local Lipschitz property of $g'$, Sobolev's imbeddings and, again, Lemma 2.1, and the fifth by interpolation inequality with $0 < \theta < 1$. Let $\theta = \frac{1}{2}$. Using

\[
abla \leq \epsilon_0 \nabla^\theta + C_n \nabla^\theta,
\]

where \( \frac{1}{\theta} + \frac{1}{\nabla} = 1 \),

we obtain that $\forall \epsilon_0 > 0$,

\[
B_n \leq C_T(||f||_F, ||w_0||_{H^s(\Omega)}, ||w_1||_{H^r(\Omega)} ||w_{n,t} - w_t||_{C^0(0,T;H^s(\Omega))}^2 + \epsilon_0 ||w_{n,t} - w_t||_{C^0(0,T;H^s(\Omega))}^2.
\]

(4.26)

By (4.20), (4.24), and (4.26), after taking $\epsilon_0$ small enough, we obtain

\[
||\tilde{\omega}(t)||_{H^s(\Omega)}^2 + ||\tilde{\omega}_t(t)||_{L^2(\Omega)}^2 + ||\tilde{\omega}_{tt}(t)||_{L^2(\Omega)}^2 \\
\leq C_T(||f||_F, ||w_0||_{H^s(\Omega)}, ||w_1||_{H^r(\Omega)} ||f_n - f||_F^2 + A_n + ||\tilde{\omega}_t||_{C^0(0,T;L^2(\Omega))}^2 \\
\to 0 \text{ as } n \to \infty,
\]

(4.27)

where the last conclusion follows from (4.24) and (4.13). This concludes the proof of the continuity of the map, $T$, i.e., as $||f_n - f||_F \to 0$, $||\tilde{z}(t)||_{H^r(\Omega)}^2 + ||\tilde{z}_t(t)||_{L^2(\Omega)}^2 \to 0$. Thus, the map $T : X \to X$ is continuous.

The a priori bounds of Lemma 3.1 imply that if $w$ is a solution to $w = \delta T(w)$, where $\delta < 1$, then

\[
||w||_X \leq C_0(||w_0||_{H^s(\Omega)}, ||w_1||_{H^r(\Omega)}).
\]

(4.28)

This, together with (4.11) and the continuity of $T$ allows for an application of Schaeffer's Theorem ([9], pg. 221) to conclude that there is a solution, $w$, to (1.1) which belongs to the space $X$. Once again
using the result of Lemma 3.1 allows us to “boost” (by $\epsilon$) the regularity of the solution $w$ from $X$ to $C(0,T;H^3(\Omega)) \cap C^1(0,T;H^2(\Omega))$. \hfill \Box

5 Uniqueness: Proof of Theorem 1.2

The following result announced in [20] is critical to the proof.

**Theorem 5.1** The map $(u, v) \rightarrow G[u, v]$ is bounded from $H^2(\Omega) \times H^2(\Omega) \rightarrow W_{\infty}^{0}(\Omega)$.

As the proof of Theorem 5.1, based on the compensated compactness method, is technical and is relegated to section 8.

**Remark 5.1:** Notice that the result of Theorem 5.1 improves by “$\epsilon$” a known regularity result stating that this map is bounded from $H^{2+\epsilon}(\Omega) \times H^{2}(\Omega) \rightarrow W_{\infty}^{0}(\Omega)$. As we shall see later, this improvement by “$\epsilon$” is critical.

Let $w_1$ and $w_2$ be two solutions of (1.1). Set $\tilde{w} \equiv w_1 - w_2$. Then $\tilde{w}$ satisfies

\[
\begin{aligned}
\tilde{w}_{tt} + \Delta^2 \tilde{w} &= \mathcal{F}(w_1) - \mathcal{F}(w_2), \quad \text{in } Q_T \\
\Delta \tilde{w} + (1 - \mu)B_{1}\tilde{w} &= -[h(\frac{\partial}{\partial \nu} w_{1,t}) - h(\frac{\partial}{\partial \nu} w_{2,t})], \quad \text{on } \Sigma_T \\
\frac{\partial}{\partial \nu} \Delta \tilde{w} + (1 - \mu)B_{2}\tilde{w} &= g(w_{1,t}) - g(w_{2,t}), \quad \text{on } \Sigma_T.
\end{aligned}
\]

(5.1)

Energy estimates applied to (5.1) yield

\[
\begin{aligned}
||\tilde{w}(t)||_{H^2(\Omega)}^2 + ||\tilde{w}_t(t)||_{L^2(\Omega)}^2 &+ \int_0^t \int_{\Omega} [h(\frac{\partial}{\partial \nu} w_{1,t}) - h(\frac{\partial}{\partial \nu} w_{2,t})](\frac{\partial}{\partial \nu} w_{1,t} - \frac{\partial}{\partial \nu} w_{2,t}) d\Gamma dt \\
&+ \int_0^t \int_{\Omega} [g(w_{1,t}) - g(w_{2,t})](w_{1,t} - w_{2,t}) d\Gamma dt \\
&= \int_0^t \int_{\Omega} [(\mathcal{F}(w_1) - \mathcal{F}(w_2), w_1)\tilde{w}_t + (\mathcal{F}(w_2), \tilde{w})\tilde{w}_t] d\Omega dt.
\end{aligned}
\]

(5.2)

Indeed, the formal computation leading to (5.2) can be made rigorous by treating the nonlinear bracket, $[w, \chi(w)]$ as a perturbation of a nonlinear semigroup of contractions generated by (2.17) and applying the usual (see [2]) energy estimates in the context of nonlinear semigroup theory.

From monotonicity of $h$ and $g$ and using the Cauchy-Schwartz inequality, we obtain

\[
\begin{aligned}
||\tilde{w}(t)||_{H^2(\Omega)}^2 + ||\tilde{w}_t(t)||_{L^2(\Omega)}^2 &\leq C \int_0^t \left( ||\mathcal{F}(w_1) - \mathcal{F}(w_2), w_1||_{L^2(\Omega)}^2 + ||\mathcal{F}(w_2), \tilde{w}||_{L^2(\Omega)}^2 \right) dt.
\end{aligned}
\]

(5.3)
From Theorem 5.1,
\begin{equation}
\|\mathcal{F}(w_2)\|_{W^2_\infty(\Omega)} \leq C\|w_2\|_{H^2(\Omega)}\|w_2\|_{H^2(\Omega)}.
\end{equation}

Moreover,
\begin{equation}
\begin{align*}
\|\mathcal{F}(w_2), \tilde{u}\|_{L^2(\Omega)} & \leq C\|\tilde{w}\|_{H^2(\Omega)}\|\mathcal{F}(w_2)\|_{W^2_\infty(\Omega)} \\
& \leq C\|\tilde{w}\|_{H^2(\Omega)}\|w_2\|_{H^2(\Omega)}\|w_2\|_{H^2(\Omega)}.
\end{align*}
\end{equation}

We can rewrite \(\mathcal{F}(w_1) - \mathcal{F}(w_2)\) in the following way:
\begin{equation}
\mathcal{F}(w_1) - \mathcal{F}(w_2) = G([w_1, w_1] - [w_2, w_2]) = G[\tilde{w}, w_1 + w_2].
\end{equation}

Hence, again from Theorem 5.1,
\begin{equation}
\begin{align*}
\|G[\tilde{w}, w_1 + w_2]\|_{W^2_\infty(\Omega)} & \leq C\|\tilde{w}\|_{H^2(\Omega)}\|w_1\|_{H^2(\Omega)} + \|w_2\|_{H^2(\Omega)},
\end{align*}
\end{equation}

\(\implies\)
\begin{equation}
\begin{align*}
\|\mathcal{F}(w_1) - \mathcal{F}(w_2), w_1\|_{L^2(\Omega)} & \leq C\|w_1\|_{H^2(\Omega)}\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_{W^2_\infty(\Omega)} \\
& \leq C\|w_1\|_{H^2(\Omega)}\|\tilde{w}\|_{H^2(\Omega)}\|w_1\|_{H^2(\Omega)} + \|w_2\|_{H^2(\Omega)}.
\end{align*}
\end{equation}

Combining (5.2), (5.5), and (5.8) proves that the solution must be unique. \(\Box\)

6 Existence of Weak Solutions: Proof of Theorem 1.3

Lemma 6.1 (Local existence) Assume hypothesis (H1). Then there exists \(T_0 > 0\) such that for any initial data \(w_0 \in H^2(\Omega), w_1 \in L^2(\Omega)\), there exists a unique solution to (1.1), \(w \in C(0,T_0;H^2(\Omega)), w_t \in C(0,T_0;L^2(\Omega))\).

Proof: It suffices to construct a unique fixed point for the map \(T\) introduced in (4.1) and defined on \(C(0,T_0;B_R)\), where \(B_R \equiv \{w \in H^2(\Omega) \times L^2(\Omega) : \|w\|_{H^2(\Omega) \times L^2(\Omega)} \leq R\}\).

We shall prove that for sufficiently small values of \(T_0\) and sufficiently large values of \(R\), the map \(T\) is a contraction on \(C(0,T_0;B_R)\). To accomplish this, we first note that by virtue of Theorem 5.1, the Airy's stress function \(\mathcal{F}(w)\) is locally Lipschitz: \(H^2(\Omega) \rightarrow W^2_\infty(\Omega)\). Indeed, from Theorem 5.1, it follows that
\begin{equation}
\begin{align*}
\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_{W^2_\infty(\Omega)} & = \|G[w_1 - w_2, w_1 + w_2]\|_{W^2_\infty(\Omega)} \\
& \leq C\|w_1 - w_2\|_{H^2(\Omega)}\|w_1\|_{H^2(\Omega)} + \|w_2\|_{H^2(\Omega)}.
\end{align*}
\end{equation}
Applying (2.19) with \( w \equiv T v \), \( v \in C(0, T; B_R) \), and \( f \equiv F(v) \), we obtain for \( t \leq T_0 \),
\[
\| T v(t) \|_{L^2(\Omega)}^2 + \frac{d}{dt} T v(t)\|_{L^2(\Omega)}^2 \leq C\{ \| w_0 \|_{H^2(\Omega)}^2 + \| w_1 \|_{L^2(\Omega)}^2 + \int_0^t \| F(v(\tau)) , w(\tau) \|_{L^2(\Omega)}^2 d\tau \}
\]
\[
\leq C\{ \| w_0 \|_{H^2(\Omega)}^2 + \| w_1 \|_{L^2(\Omega)}^2 + \int_0^t \| v(\tau) \|^4_{H^2(\Omega)} \| T v(\tau) \|^2_{H^2(\Omega)} d\tau \}
\]
\[
\leq C\{ \| w_0 \|_{H^2(\Omega)}^2 + \| w_1 \|_{L^2(\Omega)}^2 + CR^4 T_0 \| T v \|^2_{C(0, T_0; B_R)} \}
\]  
\tag{6.2}
where the first inequality follows from Theorem 5.1 and (2.3). Selecting \( R \) so that
\[
\| w_0 \|^2_{H^2(\Omega)} + \| w_1 \|^2_{L^2(\Omega)} \leq \frac{R^2}{4C},
\]
and taking \( T_0 \) sufficiently small yields
\[
T(C(0, T_0; B_R)) \subset C(0, T_0; B_R).
\]  
\tag{6.3}

On the other hand, estimate (4.12) applied with \( f_n \equiv F(v_1), f \equiv F(v_2) \) gives
\[
\| (T v_1 - T v_2)(t) \|^2_{H^2(\Omega)} \leq \| T v_1 - T v_2 \|^2_{L^2(\Omega)} + \frac{d}{dt} (T v_1 - T v_2)(t)\|_{L^2(\Omega)}^2
\]
\[
\leq C \int_0^t \{ \| F(v_1) - F(v_2) \|_{W^2_0(\Omega)}^2 \| T v_1 \|^2_{H^2(\Omega)} + \| F(v_2) \|_{W^2_0(\Omega)}^2 \| T v_1 - T v_2 \|^2_{H^2(\Omega)} d\tau \}
\]
\[
\leq C R^4 T_0 \{ \| v_1 - v_2 \|^2_{C(0, T_0; H^2(\Omega))} + \| T v_1 - T v_2 \|^2_{L^2(\Omega)} \}
\]  
\tag{6.4}
where we have used (5.1), (5.2) and Theorem 5.1. Hence,
\[
\| T v_1 - T v_2 \|_{C(0, T_0; H^2(\Omega))} \leq \frac{C R^4 T_0}{1 - C R^4 T_0} \| v_1 - v_2 \|^2_{C(0, T_0; H^2(\Omega))}. 
\]  
\tag{6.5}
Taking \( T_0 \) small enough yields contraction property for the map \( T \). The result of Lemma 6.1 now follows from the Contraction Mapping Principle. \( \Box \)

To complete the proof of Theorem 1.3, it suffices to establish the following a priori bound.

**Lemma 6.2 (A priori bounds)** Assume hypothesis (H.1)-(H.3) hold. Let \( (w, w_t) \) be any local solution to (1.1) such that \( w \in C(0, T_0; H^2(\Omega)), w_t \in C(0, T_0; L^2(\Omega)) \). Then the following a priori bound holds.
\[
\| w(t) \|^2_{H^2(\Omega)} + \| w_t(t) \|^2_{L^2(\Omega)} \leq C\{ \| w_0 \|^2_{H^2(\Omega)} + \| w_1 \|^2_{L^2(\Omega)} \} \quad t \geq 0.
\]  
\tag{6.6}

**Proof:** The a priori bound in (6.6) was already proven for "smooth" solutions (see (3.6)). We need to extend this bound to hold for all weak solutions. To this end, we select a suitable approximation of the initial data.
\( w_0 \in H^2(\Omega), w_1 \in L^2(\Omega) \) such that
\[
\begin{align*}
H^2(\Omega) & \ni w_{0m} - w_0 \quad \text{in } H^2(\Omega) \\
H^2(\Omega) & \ni w_{1m} - w_1 \quad \text{in } L^2(\Omega),
\end{align*}
\]
(6.7)
and \( w_{0m}, w_{1m} \) satisfy compatibility conditions (1.5). Let \( w_m(t) \) denote a solution to (1.1) corresponding to initial data \( (w_{0m}, w_{1m}) \).

From Theorem 1.1, we infer that \( w_m(t) \) satisfies the regularity properties listed in (3.1). Hence, inequality (3.6) applies and
\[
\|w_m(t)\|_{H^2(\Omega)}^2 + \|w_m(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Gamma} h(\frac{\partial}{\partial n} w_{m_t}) \frac{\partial}{\partial t} w_{m_t} d\Gamma dt + \int_0^t \int_{\Omega} g(w_{m_t}) w_{m_t} d\Omega dt \leq C(\|w_0\|_{L^2(\Omega)}, \|w_1\|_{L^2(\Omega)}).
\]
(6.8)
We shall show that
\[
\begin{align*}
w_m - w & \quad \text{in } C(0, T_0; H^2(\Omega)) \\
w_{m,t} - w_t & \quad \text{in } C(0, T_0; L^2(\Omega)),
\end{align*}
\]
(6.9)
where, we recall, \( w \) is a weak solution to (1.1).

Estimate (4.13) applied with \( f_n \equiv \mathcal{F}(w_n), f \equiv \mathcal{F}(w_m) \) yields
\[
\frac{1}{2} \left( \|w_n - w_m(t)\|_{H^2(\Omega)}^2 + \|w_{n,t} - w_{m,t}(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Gamma} h(\frac{\partial}{\partial n} w_{n,t}) \frac{\partial}{\partial t} w_{n,t} d\Gamma dt + \int_0^t \int_{\Omega} g(w_{n,t}) - g(w_{m,t})(w_{n,t} - w_{m,t}) d\Omega dt \right)
\leq C\{\|w_{0n} - w_{0m}\|_{L^2(\Omega)}^2 + \|w_{1n} - w_{1m}\|_{L^2(\Omega)}^2 \}
\]
\[
+ C \int_0^t \|\mathcal{F}(w_n) - \mathcal{F}(w_m)\|_{H^2(\Omega)}^2 \|w_n\|_{H^2(\Omega)}^2 d\tau
\]
\[
+ C \int_0^t \|\mathcal{F}(w_m)\|_{H^2(\Omega)}^2 \|w_n - w_m\|_{H^2(\Omega)}^2 d\tau.
\]
(6.10)
From (6.1) and (6.8),
\[
\int_0^t \|\mathcal{F}(w_n) - \mathcal{F}(w_m)\|_{H^2(\Omega)}^2 \|w_n\|_{H^2(\Omega)}^2 d\tau \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)}) \int_0^t \|w_n(\tau) - w_m(\tau)\|_{H^2(\Omega)}^2 d\tau.
\]
(6.11)
From Theorem 5.1 and (6.8),
\[
\int_0^t \|\mathcal{F}(w_m(\tau))\|_{H^2(\Omega)}^2 \|w_n(\tau) - w_m(\tau)\|_{L^2(\Omega)}^2 d\tau \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)}) \int_0^t \|w_n(\tau) - w_m(\tau)\|_{H^2(\Omega)}^2 d\tau.
\]
(6.12)
Inserting (6.11) and (6.12) into (6.10) and applying Gronwall’s inequality with (6.7) gives

\[
\begin{cases}
    w_m \to w^* \quad \text{in } C(0, T; H^2(\Omega)) \\
    w_{m,t} \to w_t^* \quad \text{in } C(0, T; L_2(\Omega)),
\end{cases}
\]  
(6.13)

\[
\lim_{n,m \to \infty} \int_0^T \int_\Gamma \left[ h\left( \frac{\partial}{\partial v} w_{n,t} \right) - h\left( \frac{\partial}{\partial v} w_{m,t} \right) \right] \frac{\partial}{\partial v} (w_{n,t} - w_{m,t}) d\Gamma dt + \int_0^T \int_\Gamma \left[ g(w_{n,t}) - g(w_{m,t}) \right] (w_{n,t} - w_{m,t}) d\Gamma dt \to 0.
\]  
(6.14)

We need to show that \( w^* \) coincides with \( w \), the weak solution to (1.1).

From (6.1), (2.3), and (6.13), it follows that

\[
[F(w_m(t), w_m(t))] - [F(w^*(t), w^*(t))] \text{ in } L_2(\Omega).
\]

This result allows us to deduce (by standard arguments) that \( w^* \) satisfies (1.1.a), (1.1.b) in the sense of distributions. In order to show that \( w^* \) also satisfies boundary conditions (1.1.c) and (1.1.d), it suffices to prove that

\[
\begin{cases}
    h\left( \frac{\partial}{\partial v} w_{m,t} \right) \to h\left( \frac{\partial}{\partial v} w_t^* \right) \quad \text{in } L_2(0, T_0; \Gamma) \\
    g(w_{m,t}) \to g(w_t^*) \quad \text{in } L_2(0, T_0; \Gamma)
\end{cases}
\]
(6.15)

Indeed, once (6.15) is established, the uniqueness of solutions to (1.1) in \( C(0, T_0; H^2(\Omega)) \times C(0, T_0; L_2(\Omega)) \)
asserts that \( w \equiv w^* \), as desired for (6.9). To show (6.15), we return to (6.8) which, in particular, implies

\[
\| \frac{\partial}{\partial v} w_{m,t} \|_{L_2(0, \tau_0; \Gamma)} + \| w_{m,t} \|_{L_{r+2}(0, \tau_0; \Omega)} \leq C(\| w_0 \|_{H^2(\Omega)}, \| w_1 \|_{L_2(\Omega)}).
\]
(6.16)

Hence,

\[
\begin{cases}
    \frac{\partial}{\partial v} w_{m,t} \to \frac{\partial}{\partial v} w_t^* \quad \text{weakly in } L_2(0, T_0; \Gamma) \\
    w_{m,t} \to w_t^* \quad \text{weakly in } L_{r+2}(0, T_0; \Gamma).
\end{cases}
\]  
(6.17)

On the other hand, by recalling hypothesis (II-1), we obtain

\[
\begin{cases}
    h(\frac{\partial}{\partial v} w_{m,t}) \to h_0 \quad \text{weakly in } L_2(0, T_0; \Gamma) \\
    g(w_{m,t}) \to g_0 \quad \text{weakly in } L_{\frac{r}{r+1}}(0, T_0; \Gamma).
\end{cases}
\]  
(6.18)

Applying Lemma 1.3 in [2] after recalling the monotonicity of \( h \) and \( g \) yields

\[
h_0 \equiv h(\frac{\partial}{\partial v} w_t^*), \quad g_0 \equiv g(w_t^*).
\]
(6.19)
Thus, $w^* \equiv w$ is a unique solution to (1.1).

Convergence in (6.9) together with passage to the limit on inequality (6.8) yields the desired a priori bound in Lemma 6.2. □

7 Proof of Theorem 1.4

We shall use interpolation Theorem 2 due to Tartar [19]. To this end, we define a nonlinear map

$$
\mathcal{K}(w_0, w_1) = (w(t), w_1(t)), \text{ where } w(t) \text{ solves (1.1).}
$$

Denote

$$
A_1 \equiv H^2(\Omega) \times L^2(\Omega)
$$

$$
B_1 \equiv C(0, T; H^2(\Omega) \times L^2(\Omega))
$$

$$
A_2 \equiv \{ w = (w_0, w_1) \in H^4(\Omega) \times H^2(\Omega) : (w_0, w_1) \text{ satisfy compatibility relations (1.5)} \}
$$

$$
B_2 \equiv C(0, T; H^3(\Omega) \times L^2(\Omega))
$$

By virtue of Theorems 1.1 and 1.3, we know that $\mathcal{K}$ is well-defined from $A_1 \to B_1$ and from $A_2 \to B_2$.

Moreover, estimate (3.6) yields the following bound:

$$
\|\mathcal{K}(w_0, w_1)\|_{A_1} \leq C\left(\|w_0\|_{H^2(\Omega)} \|w_0\|_{H^2(\Omega)} + \|w_1\|_{L^2(\Omega)}\right)
$$

$$
\quad \leq C\left(\|w_0\|_{A_1} \|w_0, w_1\|_{A_1}\right).
$$

(7.1)

By tracing the constants in estimates (3.30)-(3.33) and using the fact that $r \leq 1$, we also obtain

$$
\|\mathcal{K}(w_0, w_1)\|_{A_2} \leq C\left(\|w_0\|_{H^2(\Omega)} \|w_1\|_{L^2(\Omega)}(\|w_0\|_{H^2(\Omega)} + \|w_1\|_{H^2(\Omega)})\right)
$$

$$
\quad \leq C\left(\|w_0\|_{A_1} \|w_0, w_1\|_{A_2}\right).
$$

(7.2)

In order to apply nonlinear interpolation Theorem 2 from [19], we need to show that $\mathcal{K}$ is locally Lipschitz from $A_1 \to B_1$. To accomplish this, we use inequality (4.12) applied with (i) $\hat{w} = w - v, w_n = w$, where $w$ (respectively, $v$) is a solution to (1.1) corresponding to initial data $(w_0, w_1)$ (respectively, $(v_0, v_1)$, (ii)
\( f_n = F(w), f = F(v) \). This yields

\[
\|w(t) - v(t)\|_{H^2(\Omega)}^2 + \|w(t) - v(t)\|_{L^2(\Omega)}^2 \\
\leq C \left\{ \int_0^t \left\{ |F(w(\tau)) - F(v(\tau)), w(\tau)|^2 + |F(v(\tau)), v(\tau)|^2 \right\} d\Omega d\tau \\
+ \|w_0 - v_0\|_{H^2(\Omega)}^2 + \|w_1 - v_1\|_{L^2(\Omega)}^2 \right\} \\
\leq C \left\{ \int_0^t \|w(\tau) - v(\tau)\|_{H^2(\Omega)}^2 + \|v(\tau)\|_{H^2(\Omega)}^2 d\tau \\
+ \|w_0 - v_0\|_{H^2(\Omega)}^2 + \|w_1 - v_1\|_{L^2(\Omega)}^2 \right\} \\
\leq C \left\{ \|w_0 - v_0\|_{H^2(\Omega)}^2 + \|w_1 - v_1\|_{L^2(\Omega)}^2 \\
+ C(\|w_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)}, \|v_1\|_{L^2(\Omega)}) \int_0^t \|w(\tau) - v(\tau)\|_{H^2(\Omega)}^2 d\tau \right\},
\]

(7.3)

where we have used (6.1), Theorem 5.1, and Lemma 6.2. Gronwall's inequality applied to (7.3) yields

\[
\|w(t) - v(t)\|_{H^2(\Omega)}^2 + \|w(t) - v(t)\|_{L^2(\Omega)}^2 \leq C \left\{ \|w_0 - v_0\|_{H^2(\Omega)}^2 + \|w_1 - v_1\|_{L^2(\Omega)}^2 \right\},
\]

(7.4)

Hence,

\[
\|\mathcal{K}(w_0, w_1) - \mathcal{K}(v_0, v_1)\|_{\mathfrak{H}_1} \leq C \left\{ \|w_0, w_1\|_{\mathfrak{H}_1}, \|v_0, v_1\|_{\mathfrak{H}_1} \right\}(w_0, w_1) - (v_0, v_1)\|_{\mathfrak{H}_1}.\]

(7.5)

Now, by virtue of (7.1), (7.2) and (7.5), the conclusion of Theorem 2 in [19] applies to yield the final result of Theorem 1.4. \( \square \)

8 Proof of Theorem 5.1

Let \( f = G[u, v] \), i.e.,

\[
\Delta^2 f = [u, v]; \quad f = 0, \quad \frac{\partial f}{\partial n} = 0 \text{ on } \Gamma.
\]

(8.1)

By the Closed Graph Theorem, it suffices to prove that for \( u, v \in H^2(\Omega), f \in W^2_\infty(\Omega). \) By using partition of unity, it is enough to consider the case when both \( u \) and \( v \) are supported in the neighborhood of the point \( x_0 \in \bar{\Omega}. \) We shall consider two cases: (i) \( x_0 \in \text{int}(\Omega), (ii) x_0 \in \Gamma. \)

Case (i): We introduce a bilinear continuous operator \( B \) from \( C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2) \) into \( D'(\mathbb{R}^2) \) defined as

\[
B(u, v) \equiv [\Delta^{-1} u, \Delta^{-1} v],
\]
where $\Delta^{-1}$ denotes the inverse of the Laplacian on $\mathbb{R}^2$. Let $b(\xi, \eta)$ be a symbol associated with $B$ (see [4], page 29) defined by
\[
B(e^{i\xi \cdot x} e^{i\eta \cdot x}) = b(\xi, \eta) e^{i(\xi + \eta \cdot x)}. \quad \xi, \eta \in \mathbb{R}^2, x \in \mathbb{R}^2.
\]
Then
\[
b(\xi, \eta) = \frac{\xi_2 \eta_2 + \xi_1 \eta_1 - 2\xi_1 \xi_2 \eta_1 \eta_2}{(\xi_1^2 + \xi_2^2)(\eta_1^2 + \eta_2^2)}. \quad (8.2)
\]
Since $b(\xi, -\xi) = 0$ for all $\xi \neq 0$, the result of Theorem VI in [4] applies and tells us that $B$ is bounded from $L_2(\mathbb{R}^2) \times L_2(\mathbb{R}^2) \to H_1(\mathbb{R}^2)$, where $H_1$ is a real Hardy space (see [5]). This means that for $u \in H^2(\Omega)$, $v \in H^2(\Omega)$ (where, we recall, $u$ and $v$ are supported away from $\Gamma$),
\[
[u, v] \in H_1. \quad (8.3)
\]
Let $f_0 \equiv G_0[u, v]$, where $G_0 \in OPS^{-4}$ with a symbol
\[
y_0 = \frac{1}{(\xi_1^2 + \xi_2^2)^{1/4}}.
\]
By $OPS^n$, we denote, as usual (see [21]), a class of pseudodifferential operators of order $n$.

Since $[u, v]$ is supported away from $\Gamma$, $f_0$ differs from $f$ by a $C^\infty$ function. From
\[
D^\alpha G_0 \in OPS^0 \text{ for } |\alpha| \leq 4. \quad (8.4)
\]
where $D^\alpha$ stands for the differential operator of order $|\alpha|$, we obtain, by Theorem 26 of [5] (see page 121), (8.3) and (8.4), that
\[
D^\alpha G_0[u, v] \in L_1(\mathbb{R}^2). \text{ for } |\alpha| \leq 4. \quad (8.5)
\]
By Sobolev's Imbedding (see [1], page 97, (7)),
\[
f_0 \in W_1^4(\Omega) \subset W_\infty^2(\Omega) \text{ and } f \in W_\infty^2(\Omega). \quad (8.6)
\]

Case (ii): Let $x_0 \in \Gamma$. By a local change of coordinates, we flatten the boundary and consider the extended functions $u, v \in H^2(\mathbb{R}^2)$ with compact support (i.e., supported in the neighborhood of $x_0$). We denote by $\mathcal{L}$ the differential operator (with variable coefficients) defined on $\mathbb{R}^2$ which corresponds to $\Delta^2$ in new coordinates. Writing $[u, v]$ in new coordinates produces a symbol whose principal part satisfies the conditions of Theorem VI in [4] and the lower-order terms are the products of $D^\alpha u D^\beta v, |\alpha| + |\beta| \leq 3$. Thus,
the contribution of lower-order terms is in $L_{1+\epsilon}(R^2)$, where $\epsilon > 0$. Therefore, a straightforward modification (to account for variable coefficients) of the compensated compactness result of [4] applied to the principal part of the symbol of $[u, v]$ gives

$$[u, v] = w + z, \ w \in H_1, \ z \in L_{1+\epsilon}(R^2).$$

(8.7)

It follows from the definition of $H_1$ space and boundedness of the Riesz transform on $L_p(R^2)$ ($p > 1$) spaces, that the element $z \in L_{1+\epsilon}(R^2)$ with compact support can be written as $z = z_1 + z_2$, where $z_1 \in H_1, z_2 \in C^\infty$. Thus, without loss of generality, we may write that

$$[u, v] = w + z, \ w \in H_1, \ z \in C^\infty.$$  

(8.8)

We denote by $W$ a pseudodifferential operator in $OPS^{-4}$ whose symbol is compactly supported in $R^2$ and is an approximation of $L^{-1}$ in a small neighborhood of $x_0$. This is to say that $(W - I)u \in C^\infty$ whenever $u$ is supported in a neighborhood of $x_0$. (Recall $L$ represents the biharmonic operator written in the new coordinates and extended to $R^2$.) We define

$$g \equiv W[u, v].$$

(8.9)

By (8.8),

$$g = W[u, v] = Ww + Wz \in Ww + C^\infty, \ w \in H_1.$$  

(8.10)

From the construction of $W$ and $L$, it follows that

$$Lf - Ly \in C^\infty(R^2).$$

(8.11)

We shall show that

$$g \in W^2_\infty(R^2).$$

(8.12)

Indeed, since $W \in OPS^{-4}, D^\alpha W \in OPS^0$ for $|\alpha| \leq 4$. Theorem 4 in [7] then gives

$$D^\alpha W \in h_1, \ |\alpha| \leq 4, \ n \geq 4,$$

(8.13)
where $h_1$ is a local Hardy space (see [7]). By using Lemma 4 in [7] together with (8.13), we obtain

$$D^\alpha \nabla w \in H_1 + C^\infty, \ |\alpha| \leq 4.$$  \hfill (8.14)

Combining (8.10) with (8.14) gives

$$D^\alpha y \in H_1 + C^\infty, \ |\alpha| \leq 4.$$  \hfill (8.15)

This, in particular, implies

$$y \in W^4_1(R^2) \subset W^\infty_\infty(R^2).$$  \hfill (8.16)

which proves (8.12).

We next consider the following elliptic problem:

$$\begin{cases}
\mathcal{L}h = 0 & \text{in } \Omega_{x_0} \\
h = y_1 & \text{on } \Gamma_{x_0} \\
\frac{\partial h}{\partial n} = y_2 & \text{on } \Gamma_{x_0}.
\end{cases}$$  \hfill (8.17)

where $g_1 \equiv -g|_\Gamma$, $g_2 \equiv -\frac{\partial}{\partial v}g|_\Gamma$, and $\Omega_{x_0}$ denotes $\Omega \cap U(x_0)$, $\Gamma_{x_0} = \Gamma \cap U(x_0)$. From (8.15) we infer $D^\alpha g \in L_1(R^2)$, $|\alpha| \leq 4$, and from (8.10), $D^\alpha R_\gamma g \in L_1(R^2)$, $|\alpha| \leq 4$, where $R_\gamma \in OPS^0$ denotes the Riesz transform in the tangential direction (see [5]). Hence, both $g$, $R_\gamma g \in W^4_1(R^2)$. Applying trace theory on $L_1$ spaces yields $g|_\Gamma \in W^3_1(R)$, $R_\gamma g|_\Gamma \in W^3_1(R)$, $\frac{\partial}{\partial v} g|_\Gamma \in W^3_1(R)$, $\frac{\partial}{\partial v} R_\gamma g|_\Gamma \in W^3_1(R)$. From the definition of Hardy's spaces [5], we infer

$$\begin{cases}
D^\alpha g_1 \in H_1(R^1) + C^\infty(R^1), \ |\alpha| \leq 3 \\
D^\alpha g_2 \in H_1(R^1) + C^\infty(R^1), \ |\alpha| \leq 2.
\end{cases}$$  \hfill (8.18)

**Proposition 8.1** Let $h$ satisfy (8.17) with $y_1$, $y_2$ subject to (8.18). Then $h \in C^2(\Omega)$.

**Proof:** We shall use a decoupling procedure as in [21], Chapter 5. Let $x$ represent the normal outward direction to the boundary $\Gamma$ and $y$ represent the tangential direction. By the collar neighborhood theorem, it suffices to consider $x \in [0, 1]$, $y \in R^1$.

By Proposition 2.1 in [21], the solution $h$ to (8.17) can be written as

$$h(x, y) = A_1(x, y, D_y)P^\infty g_1 + A_2(x, y, D_y)P^\infty g_2 + h_0,$$  \hfill (8.19)
where $h_0 \in C^\infty(\Omega_{x_0})$ and
\[(P^{cx}g_i)(y) \equiv \int_{\mathbb{R}^n} e^{-\xi \cdot y} e^{i\xi \cdot \xi} g_i(\xi) d\xi,\]
$\hat{g}_i$ are Fourier transforms of $g_i$, and the constant $c \geq 0$ is determined from the ellipticity constant of $\mathcal{L}$.

The pseudodifferential operators, $A_1, A_2$ satisfy (see Proposition 2.1 in [22])
\[
\begin{align*}
D_x A_1(x, y, D_y) &\in OPS^j \\
D_x A_2(x, y, D_y) &\in OPS^{j-1}
\end{align*}
\]  
(8.20)

Taking derivatives up to the order 3 in (8.19) yields
\[
D^\alpha h = B_1(x, y, D_y) P^{cx} D_y^3 y_1 + B_2(x, y, D_y) P^{cx} D_y^2 y_2, \quad |\alpha| \leq 3,
\]
(8.21)
and $B_i(x, y, D_y) \in OPS^0$.

From (8.18) and the structure of the operator $P^{cx}$, it follows that
\[
\begin{align*}
P^{cx} D_y^3 y_1 &\in C([0, 1]; H_1(\Gamma^1)) \\
P^{cx} D_y^2 y_2 &\in C([0, 1]; H_1(\Gamma^1)).
\end{align*}
\]  
(8.22)

By Theorem 26 in [5], and (8.21), (8.22), we infer
\[
D^\alpha h \in C([0, 1]; L_1(\Gamma^1)), \quad |\alpha| \leq 3.
\]
(8.23)

Hence, by standard Sobolev’s Imbeddings, $D^2 h \in C([0, 1]; C(\Gamma_x))$, which implies $h \in C^2(\Omega_{x_0})$, as desired. □

From (8.11), it follows that
\[
\begin{align*}
\mathcal{L}(f - g) &\in C^\infty(\Omega_{x_0}) \\
(f - g)|_{\Gamma} = g_1 &\quad \text{on } \Gamma_{x_0} \\
\frac{\partial}{\partial n}(f - g)|_{\Gamma} = g_2 &\quad \text{on } \Gamma_{x_0}.
\end{align*}
\]  
(8.24)
Hence, by the result of Proposition 8.1 and standard elliptic regularity,
\[
f - g \in C^2(\Omega_{x_0}).
\]
(8.25)
Combining (8.25) with (8.12) yields $f \in W^{2}_{\infty}(\Omega_{x_0})$. □
References


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