ON THE EXISTENCE OF A LEAST AND
NEGATIVE-SEMIDEFINITE SOLUTION OF THE
DISCRETE-TIME ALGEBRAIC RICCATI EQUATION

By

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On the existence of a least and negative-semidefinite solution of the discrete-time algebraic Riccati equation

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Abstract.

The Riccati equation \( X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0 \) is studied and the existence of a least (and negative-semidefinite) solution \( X_\ast \) is investigated. In the case where \( F \) is singular there is a discrepancy between the results on \( X_\ast \) and the corresponding ones for a greatest (and positive-semidefinite) solution.

Keywords: discrete-time algebraic Riccati equation, least solution, negative-semidefinite solutions.

AMS Subject Classifications: 15A24, 93C55.
1 Introduction

This paper deals with the discrete-time algebraic Riccati equation (DARE)

\[ X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0 \]  \hspace{1cm} (1.1)

and its hermitian solutions. Here \( F, G \) and \( H \) are complex matrices of size \( n \times n, n \times p \) and \( q \times n \) respectively. - To motivate our study let us first consider the continuous-time algebraic Riccati equation (CARE)

\[ F^*X + XF - XGG^*X + H^*H = 0. \]  \hspace{1cm} (1.2)

If \( (F, G) \) is controllable then it is well known that there exists a greatest solution \( X_+ \) and a least solution \( X_- \) of (1.2) such that \( X_+ \geq 0 \) and \( X_- \leq 0 \), and \( F_{X_+} = F - GG^*X_+ \) and \( F_{X_-} = F - GG^*X_- \) have their spectra \( \sigma(F_{X_+}) \) and \( \sigma(F_{X_-}) \) in the closed left and right halfplane respectively. In the case of the DARE a corresponding result holds for a greatest solution \( X_+ \) and its associated closed loop matrix

\[ F_{X_+} = F - G(I + G^*X_+)^{-1}G^*X_+F = (I + GG^*X_+)^{-1}F. \]

But only if \( F \) is nonsingular a complete analogy between (1.1) and (1.2) can be established which involves a least solution \( X_- \) which is negative-semidefinite and where \( |\lambda| \geq 1 \) holds for all \( \lambda \in \sigma(F_{X_-}) \). It has been emphasized by E. Jonckheere [2] that in the discrete-time case as far as existence of negative-semidefinite and antistabilizing solutions is concerned the theory of algebraic Riccati equations is not complete. This note is intended to make a contribution to that problem area.

The standard assumption will be the condition that the unimodular eigenvalues should be \( G \)-controllable, i.e. that

\[ \text{rank}(F - \lambda I, G) = n \hspace{1cm} \text{if} \hspace{1cm} |\lambda| = 1. \]  \hspace{1cm} (A)

Our main results are the following two theorems.

**Theorem 1.1** Assume Condition (A). Let \( X_- \) be a solution of (1.1) where the spectrum of the associated closed loop matrix \( F_{X_-} = (I + GG^*X_-)^{-1}F \) has the property

\[ \sigma(F_{X_-}) \subseteq \{ \lambda \mid \lambda = 0 \hspace{1cm} \text{or} \hspace{1cm} |\lambda| \geq 1 \}. \]  \hspace{1cm} (1.3)
(1) Then $X_-$ is a least solution, i.e. $X_- \leq X$ holds for all solutions $X$ of (1.1).

(2) We have $X_- \leq 0$ if and only if

$$\text{Ker } F^n \subseteq \text{Ker } H.$$  \hspace{1cm} (1.4)

**Theorem 1.2** There exists a solution $X_-$ with the property (1.3) if and only if

$$0 < |\lambda| \leq 1 \quad \text{implies rank}(F - \lambda, G) = n.$$  \hspace{1cm} (1.5)

The paper is organized as follows. In Section 2 we adapt a general existence result of [5] in order to prove Theorem 1.2. Section 3 contains auxiliary results. In Section 4 we focus on $\text{Ker } F^n$ and its intersection with the $(F, H)$-unobservable subspace, and give a proof of Theorem 1.1.

## 2 An existence result

To prove Theorem 1.2 we make use of the symplectic pencil

$$M - sL = \begin{pmatrix} F & 0 \\ -Q & I \end{pmatrix} - s \begin{pmatrix} I & \Gamma \\ 0 & F^* \end{pmatrix}, \quad Q = H^* H, \quad \Gamma = G G^*,$$

which is associated to (1.1). It is well known that

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} I + \Gamma X & 0 \\ F^* X & I \end{pmatrix} \begin{pmatrix} F X - sI & -sD \\ 0 & I - sF_X^* \end{pmatrix},$$  \hspace{1cm} (2.1)

$$D = (I + \Gamma X)^{-1} \Gamma.$$

if $X$ is a solution of (1.1). Therefore

$$\det(M - sL) = c \det(F_X - sI) \det(I - sF_X^*).$$  \hspace{1cm} (2.2)

Let

$$f(s) = \prod_{\nu=1}^{m} (\lambda_\nu - s).$$
be a complex polynomial. Put
\[ \tilde{f}(s) = \prod_{\nu=1}^{n}(1 - \tilde{\lambda}_\nu s). \]
Because of (2.1) a solution \( X \) gives rise to a factorization \( \det(M - sL) = cg(s)\tilde{g}(s), \quad c \in \mathbb{C}, \) where \( g(s) = \det(sI - F_X). \) A more precise statement is given in the following lemma, which can be recovered from [6] in the case where \( \det(M - \lambda L) \neq 0 \) for \( |\lambda| = 1. \)

**Lemma 2.1** Assume Condition (A). Let
\[ \det(M - sL) = cg(s)\tilde{g}(s), \quad c \in \mathbb{C}. \]  
(2.3)
be a factorization with the property that
\[ g(\lambda) = \tilde{g}(\lambda) = 0 \text{ implies } |\lambda| = 1. \]  
(2.4)
Then there exists a unique solution \( X \) of (1.1) with
\[ \det(sI - F_X) = g(s) \]  
(2.5)
if and only if
\[ g(\lambda) = 0 \text{ and } \lambda \neq 0 \text{ imply } \text{rank}(F - \tilde{\lambda}^{-1}I, G) = n. \]  
(2.6)

**Proof.** It is no loss of generality to assume
\[ F = \begin{pmatrix} F_1 & 0 \\ F_2 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix} \]
where the pair \((F_2, G_2)\) is controllable. Put \( h(s) = \det(sI - F_1). \) Note that
\[ (h, \tilde{g}) = 1 \]  
(2.7)
holds if and only if \( \tilde{g}(\mu) = 0 \) implies \( \text{rank}(F - \mu I, G) = n, \) which is equivalent to (2.6). It was shown in [5, Theorem 1.2] that (2.4) and (2.7) ensure the existence of a unique solution \( X \) with (2.5).

The converse is fairly obvious. Let \( X \) be a solution such that (2.5) holds. Suppose \( \lambda \neq 0 \) and
\[ \text{rank}(F - \tilde{\lambda}^{-1}I, G) < n. \]  
(2.8)
Then $\tilde{\lambda}^{-1}$ is an eigenvalue of $F_X$, i.e. $g(\tilde{\lambda}^{-1}) = 0$. Because of (A) only $|\lambda| \neq 1$ is possible in (2.8). From (2.4) follows $g(\lambda) \neq 0$ which proves (2.6).

\[ \Box \]

**Proof of Theorem 1.2.** It is known (see e.g. [3]) that under the assumption (A) there exists a factorization (2.3) such that $g(\lambda) \neq 0$ if $0 < |\lambda| < 1$ and $\tilde{g}(\lambda) \neq 0$ if $|\lambda| > 1$. Now let $g$ have the properties above. Note that $|\lambda| \geq 1$ together with $g(\lambda) \neq 0$ implies

$$\text{rank}(F - \tilde{\lambda}^{-1}I, G) = n.$$ \hspace{1cm} (2.9)

Otherwise (2.8) would imply $|\lambda| \neq 1$ and

$$0 = \det(M - \tilde{\lambda}^{-1}L) = cg(\tilde{\lambda}^{-1})\tilde{g}(\tilde{\lambda}^{-1}) = c\tilde{g}(\lambda)g(\lambda).$$

Because of $g(\lambda) \neq 0$ we would obtain $\tilde{g}(\lambda) = 0$, which is impossible for $|\lambda| > 1$. Therefore (2.6) holds if and only if the rank condition (2.9) is satisfied whenever $|\lambda| \geq 1$, which is equivalent to the condition (1.5) of Theorem 1.2.

\[ \Box \]

### 3 Auxiliary results

If $X$ and $Y$ are solutions of (1.1) then [1] the matrix $\Delta = X - Y$ satisfies

$$F_Y^* \Delta F_Y - \Delta = F_Y^* \Delta G(I + G^*XG)^{-1} G^* \Delta F_Y.$$ \hspace{1cm} (3.1)

Also note that (1.1) can be written in an equivalent form as

$$X - F_Y^*XF_X = F_Y^*XF_X + H^*H.$$ \hspace{1cm} (3.2)

Since (3.1) can be regarded as a discrete-time Lyapunov equation we can relate the inertia of $\Delta$ to the location of $\sigma(F_X)$ as soon as we know that the righthand side of (3.1) is semidefinite, in particular that $I + G^*XG > 0$. -

For the proof of the following lemma I am indebted to a referee.

**Lemma 3.1** If $X$ is a solution of (1.1) then we have $I + G^*XG > 0$. 

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Proof. Let $X$ be a hermitian $n \times n$ matrix. For $w \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ define
\[ P(w, y) = w^*(I + G^*XG)w - (F_Xy + Gw)^*X(F_Xy + Gw) + y^*Xy. \]
Then
\[ P(w, y) = w^*w - y^*F_X^*XGw - w^*G^*XF_Xy + y^*(X - F_X^*XF_X)y. \]
If $X$ is a solution of (1.1) then (3.2) implies
\[ P(w, y) = (w^*y^*) \begin{pmatrix} I & -G^*XF_X \\ -F_X^*XG & F_X^*XG + XF_X \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} + y^*Hy = \]
\[ = (w^*y^*) \begin{pmatrix} I \\ -F_X^*XG \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} + y^*Hy \geq 0. \]
Now choose $\lambda$ such that $|\lambda| = 1$ and $\lambda \notin \sigma(F_X)$. If $y = (\lambda I - F_X)^{-1}Gw$ then
\[ y^*Xy = (F_Xy + Gw)^*X(F_Xy + Gw) \]
and $P(w, y) = w^*(I + G^*XG)w$. By assumption $I + G^*XG$ is nonsingular. Hence $P(w, y) \geq 0$ yields $I + G^*XG > 0$.

We recall some facts on unimodular eigenvalues of $F_X$ and their generalized eigenspaces. Define
\[ E_\pm(F) = \{ \text{Ker} (F - \lambda I)^n, \ |\lambda| = 1 \}. \]
Let
\[ V(F, H) = \text{Ker} \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{pmatrix}. \]
be the unobservable subspace associated to $(F, H)$. Put
\[ V_\pm = V_\pm(F, H) = V(F, H) \cap E_\pm(F). \]

**Lemma 3.2** [5] *Assume Condition (A). Let $X$ be a solution of (1.1). Then we have $E_\pm(F_X) = V_\pm$ and $V_\pm \subseteq \text{Ker} X$ and $F_X = F$ on $V_\pm$.**
4 A least solution

The inclusions (1.3) and (1.4) point at a special role of zero if $0 \in \sigma(F)$. Put $E_0(F) = \text{Ker } F^n$.

**Lemma 4.1** [4] Let $X$ be a solution of (1.1). Put $U_0 = E_0(F_X)$. Then each solution $W$ of (1.1) satisfies

1. $E_0(F_W) = U_0$,
2. $F_W = F_X$ on $U_0$,
3. $U_0 \subseteq \text{Ker } (X - W)$.

Define

$$V_0 = V(F, H) \cap E_0(F)$$

such that $V_0$ is the largest $F$-invariant subspace in Ker $H$ where the restriction of $F$ is nilpotent.

**Lemma 4.2** For each solution $X$ of (1.1) we have $V_0 \subseteq \text{Ker } X$ and $F = F_X$ on $V_0$ and $V_0 \subseteq E_0(F_X)$.

**Proof.** Let a basis of $C^n$ be chosen such that

$$V_0 = \text{Im} \begin{pmatrix} I_{n-n_1} \\ 0 \end{pmatrix}.$$  \hfill (7.1)

Then

$$F = \begin{pmatrix} F_0 & F_{01} \\ 0 & F_1 \end{pmatrix}, \quad H = (0, H_1)$$  \hfill (7.2)

and $\sigma(F_0) = \{0\}$ and

$$\text{rank} \begin{pmatrix} F_1 - \lambda I \\ H_1 \end{pmatrix} = n_1 \text{ for } \lambda = 0.$$  \hfill (7.3)

If we partition

$$X = \begin{pmatrix} X_0 & * \\ X_{10} & * \end{pmatrix}$$  \hfill (7.4)
according to (4.2) and write (1.1) as 
\[ X - F_X^*XF + H^*H = 0 \]
then we obtain
\[
\begin{pmatrix} X_0 \\ X_{10} \end{pmatrix} - F_X^* \begin{pmatrix} X_0 \\ X_{10} \end{pmatrix} F_0 = 0. \tag{4.3}
\]
As \( F_0 \) is nilpotent the equation (4.3) has only a trivial solution \( X_0 = 0, X_{10} = 0 \). Hence \( V_0 \subseteq \text{Ker } X \). From \( X = \text{diag} (0, X_1) \) follows
\[
F_X = \begin{pmatrix} F_0 & \ast \\ 0 & (I + \Gamma_1X_1)^{-1}F_1 \end{pmatrix}. \tag{4.4}
\]
\[
\square
\]

**Lemma 4.3** Let \( X \) be a solution of (1.1). The we have
\[
v^*Xv \geq 0 \text{ for all } v \in E_0(F_X)\tag{4.5}
\]
and
\[
E_0(F_X) \cap \text{Ker } X = V_0. \tag{4.6}
\]
Furthermore
\[
E_0(F_X) \subseteq \text{Ker } X
\]
holds if and only if \( E_0(F) = V_0 \) or equivalent if and only if
\[
E_0(F) \subseteq \text{Ker } H.
\]

**Proof.** Let \( F_X \) be given as
\[
F_X = \text{diag} (\Phi_0, \Phi_2) \tag{4.8}
\]
where \( \Phi_0 \) is nilpotent and \( \Phi_2 \) is nonsingular. Let
\[
X = \begin{pmatrix} X_0 & X_{20}^* \\ X_{20} & X_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_0 & Q_{20}^* \\ Q_{20} & Q_2 \end{pmatrix} \tag{4.9}
\]
be partitioned conforming to (4.3). Then (3.2) yields
\[
X_0 - \Phi_0^*X_0\Phi_0 = \Phi_0^*(X_0 \cdot X_{20}^*) \Gamma \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} \Phi_0 + Q_0 \tag{4.10}
\]

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and
\[ X_{20} - \Phi_2^* X_{20} \Phi_0 = \Phi_2^* (X_{20} \quad X_2) \Gamma \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} \Phi_0 + Q_{20}. \]  \hfill (4.11)

For later use we also note
\[ X_2 - \Phi_2^* X_2 \Phi_2 = S_2, \quad S_2 \geq 0. \]  \hfill (4.12)

Because of $\Gamma \geq 0$, $Q_0 \geq 0$ and since $\Phi_0$ is nilpotent it is obvious from (4.10) that $X_0 \geq 0$ holds, which proves (4.5).

Now put $D_0 = E_0(F_X) \cap \text{Ker} \ X$. Take $y \in D_0$. Then
\[ y = \begin{pmatrix} y_0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} y_0 = 0. \]

From (4.10) and $X_0 \geq 0$ we conclude that
\[ X_0 \Phi_0 y_0 = 0. \]  \hfill (4.13)

and
\[ \Gamma \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} \Phi_0 y_0 = 0, \quad Q_0 y_0 = 0. \]

Then $Q \geq 0$ implies $Q_{20} y_0 = 0$ and (4.11) yields $\Phi_2^* X_{20} \Phi_0 y_0 = 0$. Recall that $\Phi_2$ is nonsingular. Hence $X_{20} \Phi_0 y_0 = 0$, which together with (4.13) shows that
\[ X F_X y = \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} \Phi_0 y_0 = 0. \]  \hfill (4.14)

Hence $D_0$ is invariant under $F_X$. From $F = (I + \Gamma X) F_X$ and (4.14) follows $F y = F_X y$. We already know that $Q y = 0$. Hence $D_0$ is an $F$-invariant subspace in $\text{Ker} \ H \cap E_0(F)$, which implies $D_0 \subseteq V_0$. From Lemma 4.2 follows $V_0 \subseteq E_0(F_X) \cap \text{Ker} \ X$, which completes the proof of (4.6).

It is immediate from (4.6) that (4.7) is equivalent to
\[ E_0(F_X) = V_0. \]  \hfill (4.15)

Now (4.1) shows that (4.15) holds if and only if $F_1$ is nonsingular which is
equivalent to \( \text{Ker } F^n \subseteq \text{Ker } H \).

\[
\text{Proof of Theorem 1.1.} \ (1) \text{ Let } X \text{ be a solution of (1.1). Put } \Delta = X - X_- \text{. From Lemma 3.2 and Lemma 4.1 we obtain}
\]

\[
E_x(F_{X_-}) \oplus E_0(F_{X_-}) \subseteq \text{Ker } (X - X_-).
\]

(4.16)

Because of (1.3) we can assume that \( F_{X_-} = \text{diag } (\Phi_1, \Phi_2) \) and that \( \lambda = 0 \) or \( |\lambda| = 1 \) if \( \lambda \in \sigma(\Phi_1) \) and

\[
|\lambda| > 1 \text{ if } \lambda \in \sigma(\Phi_2).
\]

(4.17)

Recall (3.1), i.e.

\[
F_{X_-}^* \Delta F_{X_-} - \Delta = F_{X_-}^* \Delta G(I + G^*XG)^{-1}G^* \Delta F_{X_-}.
\]

Then (4.16) implies \( \Delta = \text{diag } (0, \Delta_2) \), and \( \Delta_2 \) satisfies \( \Phi_2^* \Delta_2 \Phi_2 - \Delta_2 = T_2 \). According to Lemma 3.1 we have \( I + G^*XG > 0 \). Hence \( T_2 \geq 0 \) and (4.17) yields \( \Delta_2 \geq 0 \). Therefore \( \Delta \geq 0 \), which shows that \( X_- \) is indeed a least solution.

(2) Take \( X = X_- \) in Lemma 4.3 and its proof and let \( X \) and \( F_X \) be given as in (4.9) and (4.8) such that \( E_0(F_X) = \text{Im } (f) \). From \( E_x(F_X) \subseteq \text{Ker } X \) follows \( E_x(\Phi_2) \subseteq \text{Ker } X \). The assumption (1.3) for \( X_- \) implies \( |\lambda| \geq 1 \) if \( \lambda \in \sigma(\Phi_2) \). Hence (4.12) yields \( X \leq 0 \). Now (4.5) shows that we have \( X_- \leq 0 \) if and only if \( E_0(F_{X_-}) \subseteq \text{Ker } X_- \), which is the condition (4.7) of Lemma 4.3.

\[
\square
\]

References


