DISCONTINUOUS SOLUTIONS OF STEADY STATE, VISCOUS COMPRESSIBLE NAVIER-STOKES EQUATIONS

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DISCONTINUOUS SOLUTIONS OF STEADY STATE, VISCOUS COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We study a steady-state, viscous, compressible Navier-Stokes flow in a rectangle \( \Omega \equiv (0,1) \times (-1,1) \) with the boundary condition \( (u,v) = (1,0) \) for the velocity field \( (u,v) \) and the condition \( p(0,y) = p^0(y) \) for the pressure \( p \) on \( \{0\} \times (-1,1) \) which is the part of the boundary where the stream lines emanate. Under the condition that \( p^0(y) \) has a jump at \( y = 0 \), we establish the existence and uniqueness of the solution having discontinuity along the stream line starting from the origin.

Keywords. Compressible, viscous, Navier-Stokes flow, discontinuous solutions, elliptic systems, hyperbolic equations.


1 Introduction.

The continuity equation for a viscous, compressible Navier-Stokes flow is hyperbolic, so that discontinuities of the pressure (or density) on the boundary may pass to the interior of the flow region along the characteristic curves and therefore produce solutions with interior discontinuities. However, since examples of discontinuous viscous flows are scarcely seen in physics, such existence result is few in the literature. Recently, Kellogg [3] and Chen & Kellogg [1, 2] showed that a linearized and simplified version of the system of viscous, compressible Navier-Stokes equations admits discontinuous solutions. Earlier than their work, Hoff [6, 7] has shown that the one dimensional, time dependent viscous Navier-Stokes flow has discontinuous solutions; however, he also showed that this discontinuity decays exponentially with time. Hence, the question that if there exist discontinuous solutions to the nonlinear, steady state, viscous Navier-Stokes flow is still open. The present paper is just to give an affirmative answer to this question.

Consider the two-dimensional steady-state, viscous, compressible, barotropic flow in a rectangle \( \Omega \equiv (0,1) \times (-1,1) \). The governing equations are

\[-\eta \Delta u - (\zeta + \frac{\eta}{2})(u_x + v_y)_x + \rho uu_x + \rho vu_y + p_x = 0 \quad \text{in } \Omega,
\]

\[1.1\]

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\[-\eta \Delta v - (\zeta + \frac{\eta}{3})(u_x + v_y)y + \rho u v_x + \rho v v_y + p_y = 0 \quad \text{in } \Omega, \quad (1.2)\]
\[u \rho_x + v \rho_y + (u_x + v_y) \rho = 0 \quad \text{in } \Omega \quad (1.3)\]

where \(u\) and \(v\) are the \(x\) and \(y\) components of the velocity vector fields, \(\zeta, \eta > 0\) are the viscosities, and \(p\) and \(\rho\) are, respectively, the pressure and the density. We assume that the pressure \(p\) and the density \(\rho\) are linked by the state relation

\[p = P(\rho) \quad \text{or} \quad \rho = \rho(p) \quad (1.4)\]

where both \(P(\cdot)\) and \(\rho(\cdot)\) are positive and monotone increasing functions.

We impose the following Dirichlet boundary conditions for \(u\) and \(v\):

\[u(x, y) = 1, \quad (x, y) \in \partial \Omega, \quad (1.5)\]
\[v(x, y) = 0, \quad (x, y) \in \partial \Omega \quad (1.6)\]

which means that the flow goes from the left to the right.

Since equation (1.3) is hyperbolic for the function \(\rho\), we need only to give boundary values for \(\rho\) (or \(p\)) on those part(s) of the boundary where the characteristic curves start, i.e., on the set \(\Gamma_{in} = \{(x, y) \in \partial \Omega | n \cdot (u, v) < 0\}\) where \(n\) denotes the outward unit normal to \(\partial \Omega\). From (1.5) and (1.6), we know that \(\Gamma_{in} = \{(0, y)| -1 \leq y \leq 1\}\) so we impose the following boundary condition for \(\rho\):

\[\rho(0, y) = \rho^0(y), \quad y \in [-1, 1]. \quad (1.7)\]

When \(\rho^0(\cdot)\) is small and smooth, the existence of smooth solutions (even in 3-D) has been established by Valli [8]. In this paper, we are only interested in discontinuous solutions, so we assume that \(\rho^0(\cdot)\) has the form

\[\rho^0(y) = \rho^0_c(y) + \delta_0 H(y) \quad (1.8)\]

where \(\delta_0\) is a positive constant, \(\rho^0_c(\cdot)\) is a function in \(C^\alpha[-1, 1]\) with \(\alpha \in (0, 1)\), and \(H(y)\) is the Heaviside function. We shall show that the system (1.1)–(1.7) has a solution with the property that \(p\) is discontinuous across a curve \(\Gamma\) which is the streamline of the flow emanating from the point \((0,0)\). The velocity components \(u\) and \(v\) are continuous across \(\Gamma\) but their normal derivatives have jumps which satisfy certain conditions similar to the Rankine-Hugoniot conditions. In addition, the jump of \(p\) decreases monotonically along \(\Gamma\). Under the condition that

\[\rho^0_c(y) \in C^{0,1}[-1, 1] \quad (1.9)\]

we shall show that the solution is unique in certain functional classes.

The overall strategy of the proof is to convert the solution of (1.1)–(1.7) into a fixed point of a mapping \(T\) defined in the next section and to show that the mapping \(T\) possesses a fixed point via the Schauder fixed point theorem. To do this, we have
to assume that \( \eta \) is suitably large. This condition may be replaced by the condition that either \( \rho^0(\cdot) \) is close to a constant or the width of the rectangle \( \Omega \) is small.

In [1], Chen and Kellogg studied the following simplified version of (1.1)–(1.3):

\[
-\Delta u + P_x = 0 \quad \text{in } \Omega' \equiv (0, a) \times R^1, \quad (1.10)
\]

\[
-\Delta v + P_y = 0 \quad \text{in } \Omega', \quad (1.11)
\]

\[
u \rho_x + v \rho_y = 0 \quad \text{in } \Omega' \quad (1.12)
\]

with the boundary condition (1.5)–(1.7) where \( \rho^0(\cdot) \) is of compact support on \([0, a]\). Under the assumption that \( a \) is sufficiently small, they showed the existence of a weak solution. Notice that by scaling \( \Omega' \) in (1.10)–(1.12) into \((0, 1) \times R^1\), our assumption that \( \eta \) is large is equivalent to the assumption that \( a \) is small in [1].

As pointed out in their paper, dropping the term \( \rho(u_x + v_y) \) in (1.3) to get (1.12) has no physical meaning but is only for the sake of mathematical convenience. This convenience is significant since if one does not drop it, then even with the more delicate estimate we present in this paper, it is still not enough to show the compactness of the mapping constructed in [1], so that the Schauder fixed point theorem may not be applied. To overcome this difficulty, we shall consider this problem as a free boundary problem and show that the essential singularity of \( u_x + v_y \) is the same as that of \( p/\mu \), where

\[
\mu = \frac{4}{3} \eta + \zeta; \quad (1.13)
\]

namely, the difference \( u_x + v_y - p/\mu \) has much weaker singularities than either \( u_x + v_y \) or \( p \). Hence, by writing (1.3) as

\[
u \rho_x + v \rho_y + \rho \frac{P(\rho)}{\mu} = -\rho(u_x + v_y - \frac{p}{\mu}), \quad (1.14)
\]

the singular term \( u_x + v_y \) becomes manageable.

The plan of this paper is as follows. In §2 we present our main results and the scheme of the proof. Then we study the nonlinear elliptic system (1.1), (1.2), (1.5) and (1.6) with given \( p \) and \( \rho \) in §3 and the continuity equation (1.3) and (1.7) in §4. Finally, we establish the existence and uniqueness of the solution in §5.

## 2 Main results.

We say that \((u, v, p, \rho)\) is a solution to (1.1)–(1.3) if

\[
(u, v) \in C(\overline{\Omega}) \cap H^1_{\text{loc}}(\Omega), \quad p \in L^\infty(\Omega) \quad \text{and (1.1)–(1.3) are satisfied in } H^{-1}(\Omega).
\]

Let \((u, v, p, \rho)\) be a solution to (1.1)–(1.7) and assume that \( u > 0 \) and \( v/u \) is Lipschitz in \( y \). Introduce a function \( f \) defined by

\[
\begin{cases}
\frac{df(x)}{dx} = \frac{v(x, f)}{u(x, f)}, & x \in (0, 1), \\
f(0) = 0.
\end{cases}
\quad (2.1)
\]
and introduce sets $\Gamma$, $\Omega^-$, and $\Omega^+$ defined by

$$\Gamma = \{(x, y) \in \Omega \mid y = f(x)\}, \quad \Omega^+ = \{(x, y) \in \Omega \mid y > f(x)\}, \quad \Omega^- = \{(x, y) \in \Omega \mid y < f(x)\}.$$  

(2.2) \hspace{2cm} (2.3) \hspace{2cm} (2.4)

One may notice that $\Gamma$ is a characteristic curve of (1.3). Since we assume that $\rho^0(\cdot)$ is smooth in $[-1, 0^-] \cup [0^+, 1]$, the solution should be smooth in both $\Omega^+$ and $\Omega^-$, so that we can treat $\Gamma$ as a free boundary.

Assume that $(u, v) \in C^{1+\alpha}(\Omega^\pm)$ and $\rho, p \in C^\alpha(\Omega^\pm)$, then the jump conditions for a weak solution of (1.1), (1.2) can be written as

$$[u]_+^+ = [v]_+^+ = 0 \quad \text{on} \quad \Gamma,$$  

(2.5) \hspace{2cm} (2.6) \hspace{2cm} (2.7)

$$\mu[\partial_n u]_+^+ = [p]_+^+ n_x \quad \text{on} \quad \Gamma,$$

$$\mu[\partial_n v]_+^+ = [p]_+^+ n_y \quad \text{on} \quad \Gamma$$

where $[\cdot]^+_-$ represents the jump across $\Gamma$, $n = (n_x, n_y) = (-\frac{f'}{\sqrt{1 + f'^2}}, \frac{1}{\sqrt{1 + f'^2}})$ is the unit normal to $\Gamma$, $\partial_n$ is the normal derivative, and $\mu$ is given by (1.13). Since $u$ and $v$ are continuous, the jump condition for (1.3) is that $\Gamma$ is a characteristic curve; namely, $f$ satisfies (2.1).

Notice that the boundary conditions (1.6), (1.7) do not satisfy the jump condition (2.7) at $(0, 0)$ and $(1, f(1))$, so that we introduce a weight function

$$\omega(x, y) = \min \left\{ \sqrt{x^2 + y^2}, \sqrt{(1 - x)^2 + (y - f(1))^2} \right\}$$  

(2.8)

to take care of the singularities arising from this non-compatibility. Also, it is convenient to introduce a function

$$\bar{\omega}(x, y) = \text{distance}((x, y), \partial \Omega).$$  

(2.9)

In the sequel, we use the notation that

$$\|w f\|_{C^\alpha(\Omega)} \equiv \|w f\|_{C^\alpha(\Omega)}$$

$$\sup_{(x_1, y_1), (x_2, y_2) \in \Omega} \left( \min\{w(x_1, y_1), w(x_2, y_2)\} \frac{|f(x_1, y_1) - f(x_2, y_2)|}{|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha} \right)$$

where $w$ is a weight and $f$ is a function.

**Theorem 1.** Assume that $\rho^0(\cdot)$ has the form of (1.8) with $\delta_0 > 0$ and $\rho_c^0(y) \in C^\alpha[-1, 1]$ ($0 < \alpha < 1$). Then there exists a positive constant $\eta_0$ which depends only on $\rho^0(y)$ such that if $\eta \geq \eta_0$ and $\zeta \geq 0$, then (1.1)–(1.7) has a solution $(u, v, p, \rho) \in C^{1+\alpha}(\Omega^\pm) \times C^\alpha(\Omega^\pm) \times C^\alpha(\Omega^\pm)$ satisfying

$$\|u - 1\|_{C^{1/2}(\Omega)} + \|v\|_{C^{1/2}(\Omega)} + \|\omega^{1/2} \nabla u\|_{C^\alpha(\Omega^\pm)} + \|\omega^{1/2} \nabla v\|_{C^\alpha(\Omega^\pm)}$$

$$+ \frac{1}{\eta} \|p\|_{C^\alpha(\Omega^\pm)} \leq \frac{C_\alpha}{\eta} \left( \|\rho_c\|_{C^\alpha([-1, 1])} + \delta_0 \right)$$  

(2.10)
where $C_\alpha$ depends only on $\alpha$. Moreover, the jumps of $u, v$ and $p$ across $\Gamma$ satisfy (2.5)–(2.7) and $\delta_p(x) \equiv [p]_+^\tau$ is strictly positive and monotone decreasing.

**Remark 2.1.** The fact that $\delta_p(x)$ is monotonically decreasing was first observed by Chen & Kellogg [1].

The proof will be given in the following sections. Here we present the idea of the proof.

To deal with the discontinuity of the pressure, we express $p$ in terms of its jump $\delta(x)$, the location of the jump $y = f(x)$, and the continuous part $p_c$; more precisely, we write $p$ as

$$p(x, y) = p_c(x, y) + \delta(x)H(y - f(x)) \tag{2.11}$$

where $H(s)$ is the Heaviside function taking value 1 when $s > 0$ and value 0 when $s \leq 0$.

We shall prove Theorem 1 by a fixed point argument. For this purpose, we introduce a Banach space $B$ defined by

$$B \equiv C^{\alpha/2}(\Omega) \times C^{\alpha/2}([0, 1]) \times C^{1+\alpha/2}([0, 1])$$

where $\alpha$ is the exponent of the Hölder continuity of $\rho_0^\alpha(\cdot)$. Let $M > 0$ be a fixed constant and $K(M)$ be defined as

$$K(M) = \left\{ (p_c, \delta, f) \in B \mid \|p_c\|_{C^{\alpha/2}(\Omega)} \leq M, \|\delta\|_{C^{\alpha/2}([0, 1])} \leq M, \|f\|_{C^{1+\alpha/2}([0, 1])} \leq \frac{1}{2}, f(0) = f'(0) = f'(1) = 0 \right\} \tag{2.12}$$

Clearly, $K(M)$ is a closed convex subset of $B$.

For any given $(p_c, \delta, f) \in K(M)$, define $p$ as in (2.11) and $\rho$ by

$$\rho(x, y) = \rho(p) \tag{2.13}$$

Consider the elliptic system (1.1), (1.2), (1.5), and (1.6). We shall show that if $\eta$ is large enough, then the system admits a (unique) solution $(u, v) \in C^\alpha(\Omega) \times C^\alpha(\Omega)$; in addition, both $u$ and $v$ are Lipschitz continuous in $y$.

Once we obtained $(u, v)$ from $(p_c, \delta, f) \in K(M)$, we would like to substitute it into (1.3), solve a new $\tilde{p}$ from (1.3), (1.7) by a characteristic method, and then obtain a new $(\tilde{p}_c, \tilde{\delta}, \tilde{f}) \in K(M)$. To do this, we need the condition that $u^2 + v^2 > 0$ and certain regularity properties of the coefficient $u_x + v_y$ of $\tilde{p}$. In fact, we can show that if $\eta$ is large enough, then $u > 1/2$ in $\Omega$. However, we are unable to establish the regularity for $u_x + v_y$ required by the iteration procedure defined below. Nevertheless, we discovered that the essential singularity of $u_x + v_y$ cancels with that of $p/\mu$; more precisely, $u_x + v_y - p/\mu \in C^\beta(\Omega)$ for any $\beta \in (0, 1)$. Hence, we write (1.3) as (1.14); that is, we solve $\tilde{p}$ from the following first order partial differential equation

$$\begin{cases}
    u\tilde{p}_x + v\tilde{p}_y + \tilde{p}P'(\tilde{p})/\mu = -\tilde{p}(u_x + v_y - p/\mu), \\
    \tilde{p}(0, y) = \rho^\beta(y). \tag{2.14}
\end{cases}$$
We shall prove in §4 that \( \bar{\rho} \in C^\alpha(\Omega \cap \overline{\Omega}^c) \).

Next, we shall retrieve \((\bar{p}_c, \bar{\delta}, \bar{f})\) from \(\bar{\rho}\). Let \(\bar{f}(x)\) be the unique solution of the following ordinary differential equation

\[
\begin{aligned}
\frac{d \bar{f}}{dx} &= \frac{v(x, \bar{f})}{u(x, \bar{f})}, & x \in (0, 1), \\
\bar{f}(0) &= 0,
\end{aligned}
\]  
\text{(2.15)}

and set

\[
\begin{aligned}
\bar{p}(x, y) &= P(\bar{p}(x, y)), \\
\bar{\delta}(x) &= \bar{p}(x, \bar{f}^+(x)) - \bar{p}(x, \bar{f}^-(x)), \\
\bar{p}_c(x, y) &= \bar{p} - \bar{\delta}(x)H(y - \bar{f}(x)).
\end{aligned}
\]  
\text{(2.16 - 2.18)}

Here and in the sequel, the superscripts + and - stand for the limits from above and below of \(\Gamma \equiv \{(x, y) \in \Omega \mid y = \bar{f}(x)\}\).

Now, we define a mapping \(T : K(M) \to B\) by

\[T(p_c, \delta, f) = (\bar{p}_c, \bar{\delta}, \bar{f}).\]

One can directly verify that \((p_c, \delta, f)\) is a fixed point of \(T\) if and only if \((u, v, p, \rho)\) is a solution to (1.1)–(1.7) where \((p, \rho)\) is given by (2.11), (2.13) and \((u, v)\) is the solution of (1.1), (1.2), (1.5), (1.6).

We would like to use the Schauder fixed point theorem to show that \(T\) has a fixed point and therefore establish the existence of a solution of (1.1)–(1.7). However, we would confront the difficulty of showing the compactness of \(T\), since the function \(\bar{p}_c\) is not smooth enough near the boundaries \(\{y = \pm 1\}\). To overcome this difficulty, we introduce, for each \(\lambda \in (1/2, 1)\), a mapping \(T_\lambda\) defined by

\[
T_\lambda(p_c, \delta, f) = (G_\lambda \bar{p}_c, \bar{\delta}, \bar{f})
\]

where \((\bar{p}_c, \bar{\delta}, \bar{f}) = T(p_c, \delta, f)\) and \(G_\lambda(p_c)(x, y) = p_c(x, \lambda y)\). We then can use the Schauder fixed point theorem to show that \(T_\lambda\) has a fixed point \((p_{c, \lambda}^\lambda, \delta^\lambda, f^\lambda)\) in \(K(M)\) for some \(M\), which is independent of \(\lambda\). Finally, taking a convergent subsequence \(\{(p_{c, \lambda_j}^\lambda, \delta^\lambda_j, f^\lambda_j)\}_{j=1}^\infty\) (\(\lambda_j \to 1\) as \(j \to \infty\)), we can show that the limit is a fixed point of \(T\). This completes the existence proof.

For the uniqueness, we have the following result:

**Theorem 2.** Assume that \(\rho_c^\theta(\cdot)\) satisfies (1.9). Then if \(\mu\) is large enough, the solution to (1.1)–(1.7) satisfying the estimates in Theorem 1 is unique. Moreover, the unique solution satisfies, for any \(\epsilon > 0\),

\[
|\nabla p(x, y)| \leq C_\epsilon \left( |y - f(x)|^{-\epsilon} + (1 - y^2)^{-\epsilon} \right), \quad (x, y) \in \Omega \setminus (\Gamma \cup \{y = \pm 1\}).
\]  
\text{(2.19)}

The proof will be given in §5.
Remark 2.2. From the proof below, we shall see that the assumption that 
\(\mu\) is large in Theorem 1 and Theorem 2 can be replaced by the assumption that 
\(\rho^\beta(\cdot)\) is closed to a constant or by the assumption that the width of the domain \(\Omega\) is small. This, as one can recall, is equivalent to assume that the Reynolds number (\(\sim\) magnitude of the velocity \(\times\) size of the domain /viscosity) is small.

Remark 2.3. With slight modification, our results can be extended to general domains and to general boundary conditions for \((u,v)\).

Remark 2.4. One can replace the constant \(\zeta + \frac{7}{3}\) by any non-negative constant.

Remark 2.5. The Hölder exponent \(\alpha/2\) on the left hand side of (2.10) can be 
replaced by any constant \(\beta \in (0,\alpha)\). Also, a more accurate estimate of (2.19) is 
\[
|\nabla p(x,y)| \leq C\left( |\ln |y - f(x)|| + |\ln |1 - y^2|| \right), \quad (x,y) \in \Omega \setminus \{\Gamma \cup \{y = \pm 1\}\}.
\]
However, we shall not establish it.

3 The elliptic equations.

In this section, we shall study the nonlinear elliptic problem (1.1), (1.2), (1.5) and (1.6) with given \(p\) and \(\rho\); i.e., we shall prove the following theorem.

Theorem 3.1. Let \((p_c, \delta, f) \in K(M)\) be given and let \((p, \rho)\) be defined as in (2.11) 
and (2.13). Then there exists a positive constant \(\eta_0(M)\) such that if \(\eta \geq \eta_0(M)\) and 
\(\zeta \geq 0\), then the elliptic system (1.1), (1.2), (1.5) and (1.6) has a unique solution 
satisfying 
\[
\|u - 1\|_{H^1_2(\Omega)} + \|v\|_{H^1_2(\Omega)} + \|u - 1\|_{C^\alpha(\partial \Omega)} + \|v\|_{C^\alpha(\partial \Omega)} \leq \frac{1}{2}. \tag{3.1}
\]
In addition, we have the following estimates
\[
\|u - 1\|_{C^\beta(\Omega)} + \|v\|_{C^\beta(\Omega)} + \|\omega^{\alpha/2}\nabla u\|_{C^{\alpha/2}(\Omega^\pm)} + \|\omega^{\alpha/2}\nabla v\|_{C^{\alpha/2}(\Omega^\pm)} \leq \frac{C_{\beta M}}{\eta}, \tag{3.2}
\]
\[
\|\bar{\omega}^{\beta-\alpha/2, -\omega^{\alpha/2}} I\|_{C^\beta(\Omega)} + \|\bar{\omega}^{\beta-\alpha/2, -\omega^{\alpha/2}} I\|_{C^\beta(\Omega)} \leq \frac{C_{\epsilon, \beta M}}{\eta} \tag{3.3}
\]
for any \(\beta \in (\alpha/2, 1)\) and \(\epsilon \in (0, \alpha/2)\) where \(C_{\beta}\) is a constant depending only on \(\beta\), 
\(C_{\epsilon, \beta}\) is a constant depending only on \(\beta\) and \(\epsilon\), \(\omega\) and \(\bar{\omega}\) are defined in (2.8), (2.9), and 
\[
I(x,y) = u_y + v - \frac{p}{\mu}. \tag{3.4}
\]
Proof. We first establish the existence. Set \( X = \{(u, v) \mid u - 1 \in H^1_0(\Omega), v \in H^1_0(\Omega)\} \). For any \((u, v) \in X\), define
\[
\begin{align*}
g & \equiv g(u, v, \rho) \equiv \rho uu_x + \rho uu_y, \quad (3.5) \\
h & \equiv h(u, v, \rho) \equiv \rho vv_x + \rho vv_y \quad (3.6)
\end{align*}
\]
and consider the linear elliptic system
\[
\begin{align*}
\eta \Delta \bar{u} + (\zeta + \frac{\eta}{3})(\bar{u}_x + \bar{v}_y)_x &= g + px \quad \text{in } \Omega, \quad (3.7) \\
\eta \Delta \bar{v} + (\zeta + \frac{\eta}{3})(\bar{u}_x + \bar{v}_y)_y &= h + py \quad \text{in } \Omega, \quad (3.8) \\
\bar{u} - 1 \in H^1_0(\Omega), \quad \bar{v} \in H^1_0(\Omega). \quad (3.9)
\end{align*}
\]
Notice that any (local) minimizer of the functional
\[
J(\bar{u}, \bar{v}) = \int_\Omega \left[ \frac{1}{2} \eta|\nabla \bar{u}|^2 + \frac{1}{2} \eta|\nabla \bar{v}|^2 + \frac{1}{2}(\zeta + \frac{\eta}{3})(\bar{u}_x + \bar{v}_y)^2 + \bar{u}g + \bar{v}h - \bar{u}_x p - \bar{v}_y p \right]
\]
in \( X \) is a solution to (3.7)–(3.9). Since \( g, h \in L^r(\Omega) \) for \( 1 < r < 2 \) and \( p \in L^\infty(\Omega) \), \( J(\bar{u}, \bar{v}) \) is bounded from below. Also \( J \) is convex. Therefore \( J \) possesses a unique minimizer in \( X \). It then follows that (3.7)–(3.9) has a solution in \( X \).

Introduce
\[
Y = \{(u, v) \in X \mid \|u - 1\|_{H^1_0(\Omega)} + \|v\|_{H^1_0(\Omega)} \leq 1\}.
\]
For any \((u_i, v_i) \in X \) \( (i = 1, 2) \), let \((\bar{u}_i, \bar{v}_i)\) be the solution of (3.7)–(3.9) corresponding to \((u_i, v_i)\). Multiplying the difference of equations for \( \bar{u}_1 \) and \( \bar{u}_2 \) by \( \bar{u}_1 - \bar{u}_2 \) and multiplying the difference of equations for \( \bar{v}_1 \) and \( \bar{v}_2 \) by \( \bar{v}_1 - \bar{v}_2 \), adding the resulting equations, and integrating over \( \Omega \), we obtain
\[
\begin{align*}
\eta\|\nabla(\bar{u}_1 - \bar{u}_2)\|^2_2 + \eta\|\nabla(\bar{v}_1 - \bar{v}_2)\|^2_2 + (\zeta + \frac{\eta}{3})\|((\bar{u}_1 - \bar{u}_2)_x + (\bar{v}_1 - \bar{v}_2)_y)\|^2_2 \\
\leq \|g(u_1, v_1, \rho) - g(u_2, v_2, \rho)\|_{4/3} \|\bar{u}_1 - \bar{u}_2\|_4 +
\|h(u_1, v_1, \rho) - h(u_2, v_2, \rho)\|_{4/3} \|\bar{v}_1 - \bar{v}_2\|_4 \\
\leq C\|\rho\|_{\infty}\left(\|u_1\|_{H^1} + \|u_2\|_{H^1} + \|v_1\|_{H^1} + \|v_2\|_{H^1}\right)\left(\|u_1 - u_2\|_{H^1} + \|v_1 - v_2\|_{H^1}\right) \\
\times \left(\|\bar{u}_1 - \bar{u}_2\|_4 + \|\bar{v}_1 - \bar{v}_2\|_4\right).
\end{align*}
\]
Here and also below we use the notation \( \|\cdot\|_r \equiv \|\cdot\|_{r, \Omega} \equiv \|\cdot\|_{L^r(\Omega)} \). Since \((u_i, v_i) \in Y\), it follows that there exists a positive constant \( C \) such that
\[
\|\bar{u}_1 - \bar{u}_2\|_{H^1_0(\Omega)} + \|\bar{v}_1 - \bar{v}_2\|_{H^1_0(\Omega)} \leq \frac{C\|\rho\|_{\infty}}{\eta} \left(\|u_1 - u_2\|_{H^1_0(\Omega)} + \|v_1 - v_2\|_{H^1_0(\Omega)}\right). \quad (3.10)
\]
This inequality shows that the solution to (3.7)–(3.9) is unique.

Now define a mapping \( T : Y \to X \) by \( T(u, v) = (\bar{u}, \bar{v}) \), where \((\bar{u}, \bar{v})\) is the solution of (3.7)–(3.9). One can see from (3.10) that if \( \eta \) is large enough, \( T \) is a contraction in \( Y \).
To show that $T$ maps $Y$ into itself, we multiply equation (3.7) by $\bar{u} - 1$ and equation (3.8) by $\bar{v}$, add the two resulting equations, and integrate over $\Omega$. After integration by parts and using Sobolev’s inequality, we obtain

$$
\|\bar{u} - 1\|_{H^1_0(\Omega)} + \|\bar{v}\|_{H^1_0(\Omega)} \leq \frac{C}{\eta} \left( \|g\|_2 + \|h\|_2 + \|p\|_2 \right)
$$

$$
\leq \frac{C}{\eta} \left( \|\rho\|_\infty \|u\|_{H^1_0(\Omega)}^2 + \|\rho\|_\infty \|v\|_{H^1_0(\Omega)}^2 + \|p\|_\infty \right)
$$

$$
\leq \frac{CM}{\eta}.
$$

Hence, if $\eta$ is large enough, then $T$ maps $Y$ into itself. It follows that $T$ has a unique fixed point in $Y$. Clearly, this fixed point is a solution to the problem (1.1), (1.2), (1.5) and (1.6). This establishes the existence part of Theorem 3.1. The uniqueness follows from (3.10).

Next we study the regularity of the solution $(u, v)$ of the equations (1.1), (1.2), (1.5) and (1.6).

Since $h, g \in L^r(\Omega)$ for any $r \in (1, 2)$ and $p \in L^\infty(\Omega)$, it follows that $u, v \in C^\beta(\overline{\Omega})$ for any $\beta \in (0, 1)$; see [4, pg. 166]. Clearly, if we take $\eta$ large enough, the estimates in (3.1) hold. Consequently, $h, g \in L^2(\Omega)$.

Write $u$ and $v$ as

$$
u = \frac{\phi_x}{\mu} + w + 1, \quad (3.11)$$

$$
v = \frac{\phi_y}{\mu} + z \quad (3.12)$$

where $\phi, w$ and $z$ are, respectively, the solutions to the following problems:

$$
\begin{cases}
-\Delta \phi + p = 0, & (x, y) \in \Omega, \\
\phi(x, \pm 1) = 0, & x \in (0, 1), \\
\phi_x(0, y) = \phi_x(1, y) = 0, & y \in (-1, 1), \\
\eta \Delta w + (\zeta + \frac{\eta}{3})(w_x + z_y)_x &= g(u, v, \rho) \quad \text{in } \Omega, \\
\eta \Delta z + (\zeta + \frac{\eta}{3})(w_x + z_y)_y &= h(u, v, \rho) \quad \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega, \\
z = -\phi_y/\mu & \text{on } \partial \Omega,
\end{cases} \quad (3.13)$$

$$
\begin{cases}
\eta \Delta w + (\zeta + \frac{\eta}{3})(w_x + z_y)_x &= g(u, v, \rho) \quad \text{in } \Omega, \\
\eta \Delta z + (\zeta + \frac{\eta}{3})(w_x + z_y)_y &= h(u, v, \rho) \quad \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega, \\
z = -\phi_y/\mu & \text{on } \partial \Omega,
\end{cases} \quad (3.14)$$

where $g(u, v, \rho)$ and $h(u, v, \rho)$ are given by (3.5) and (3.6). Since $p \in C^{\alpha/2}(\overline{\Omega})$ and $\Gamma \in C^{1+\alpha/2}([0, 1])$, one can use standard potential analysis to show that $\phi \in C^{2+\alpha/2}(\overline{\Omega} \setminus \{(0, 0), (1, f(1))\})$. Differentiating the relations $[\phi_x]^+ = 0$ and $[\phi_y]^+ = 0$ along $\Gamma$ and using $[\Delta \phi]^+ = [p]^+$ yield

$$
[\phi_{xx}]^+ = \frac{(f')^2}{1 + (f')^2}[p]^+ \quad \text{on } \Gamma, \quad (3.15)
$$
\[ [\phi_{xy}]_+^\perp = \frac{-f'}{1 + (f')^2} [p]_+^\perp \quad \text{on } \Gamma, \quad (3.16) \]
\[ [\phi_{yy}]_+^\perp = \frac{1}{1 + (f')^2} [p]_+^\perp \quad \text{on } \Gamma. \quad (3.17) \]
Since \( f'(0) = f'(1) = 0 \), the boundary conditions for \( \phi \) satisfies the compatibility condition \([\phi_{xy}]_+^\perp = 0\) at the points \((0,0)\) and \((1,f(1))\). It then follows that there exists a constant \( C \), depending only on \( \|f\|_{C^{1+\alpha/2}([0,1])} \), such that
\[ \|\phi\|_{C^{2+\alpha/2}(\overline{\Omega})} + \|\phi\|_{C^{2+\alpha/2}(\Omega^c)} \leq C\left(\|p_c\|_{C^{\alpha/2}(\Omega^c)} + \|\delta\|_{C^{\alpha/2}([0,1])}\right) \leq CM. \quad (3.18) \]
To estimate the functions \( w \) and \( z \), we notice that the functions \( h, g \in L^2(\Omega) \), and \( z\big|_{\partial\Omega} = -\frac{1}{\mu} \phi_y \big|_{\partial\Omega} \in C^{1+\alpha/2}(\partial\Omega \cap \overline{\Omega}^\pm) \); it follows from [4, pg. 186] that \( w, z \in C^{1+\alpha/2}(\Omega) \) and
\[
\begin{align*}
\|\omega^{\alpha/2} \nabla w\|_{C^{\alpha/2}(\overline{\Omega})} &+ \|\omega^{\alpha/2} \nabla z\|_{C^{\alpha/2}(\overline{\Omega})} \\
&\leq \frac{C}{\eta} \left(\|g\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} + \|\omega^{\alpha/2} \phi_y\|_{C^{\alpha/2}(\partial\Omega)}\right) \\
&\leq \frac{C}{\eta} \left(\|p_c\|_{C^{\alpha/2}(\overline{\Omega})} + \|\delta\|_{C^{\alpha/2}([0,1])}\right) \leq \frac{C}{\eta} M.
\end{align*}
\]
This estimate, together with (3.18), (3.11) and (3.12), yields (3.2).
Finally we estimate the function \( I \) defined in (3.4). Direct calculation shows that the function \( I \) satisfies
\[
\begin{cases}
\mu \Delta I = g_x + h_y & \text{in } \Omega, \\
I = w_x + z_y & \text{on } \partial\Omega.
\end{cases}
\]
Since \( g, h \in L^2(\Omega) \) and \( \omega^{\alpha/2} I \big|_{\partial\Omega} \in C^{\alpha/2}(\partial\Omega) \), we have, for any \( \beta \in (\alpha/2,1) \),
\[
\begin{align*}
\|\tilde{\omega}^{\beta-\alpha/2} \omega^{\alpha/2} I\|_{C^\beta(\Omega)} &\leq C_\beta \left(\frac{1}{\mu} \|g\|_{L^2(\Omega)} + \frac{1}{\mu} \|h\|_{L^2(\Omega)} + \|\omega^{\alpha/2} I\|_{C^{\alpha/2}(\partial\Omega)}\right) \\
&\leq \frac{C_\beta}{\eta} \left(\|p_c\|_{C^{\alpha/2}(\overline{\Omega})} + \|\delta\|_{C^{\alpha/2}([0,1])}\right) \\
&\leq \frac{C_\beta}{\eta} M, \quad (3.19)
\end{align*}
\]
where the weight \( \tilde{\omega} \) is given by (2.9) and the weight \( \omega \) is given by (2.8). Furthermore, \( h, g \in C^{\alpha/2}(\overline{\Omega}^\pm) \) implies that \( I \in C^{1+\alpha/2}(\Omega \cap \overline{\Omega}^\pm) \). This gives (3.3) and completes the proof of Theorem 3.1.
4 The Continuity Equation.

We shall now study the continuity equation (2.14). First, we study the characteristic curves.

For each $c \in [-1, 1]$, consider the following initial value problem:

\[
\begin{cases}
\frac{dY}{dx} = \frac{v(x, Y)}{u(x, Y)} \equiv k(x, Y), & x \in (0, 1), \\
Y(0, c) = c.
\end{cases}
\]  
(4.1)

Since $k(x, y)$ is continuous in $\overline{\Omega}$ and

\[
\int_0^1 \|k_y(x, \cdot)\|_{L^\infty(-1, 1)} dx \leq C \int_0^1 \sup_{y \in [-1, 1]} \omega^{-\alpha/2}(x, y) dx < \infty,
\]

there exists a unique solution $Y(x, c)$. This solution satisfies $Y(x, \pm 1) = \pm 1$ since $u = 1, v = 0$ on the boundaries $y = \pm 1$. Therefore, the mapping $S : (x, c) \to (x, Y(x, c))$ maps $\Omega$ onto $\Omega$.

One can calculate

\[
\frac{\partial Y}{\partial c} = e^\int_0^x k_s(\xi, Y(\xi, c)) d\xi > 0.
\]

It follows that there is a unique function $c = C(x, y)$ such that

\[
y = Y(x, C(x, y)), \quad (x, y) \in \Omega.
\]  
(4.2)

Using the implicit differentiation, one can compute

\[
\frac{\partial C}{\partial y} = \left(\frac{\partial Y}{\partial c}\right)^{-1} = e^{-\int_0^x k_s(\xi, Y(\xi, C(x, y))) d\xi},
\]

\[
\frac{\partial C}{\partial x} = -\frac{\partial Y}{\partial x} \left(\frac{\partial Y}{\partial c}\right)^{-1} = -k(x, y)e^{-\int_0^x k_s(\xi, Y(\xi, C(x, y))) d\xi}.
\]

In view of Theorem 3.1, we have the following lemma:

**Lemma 4.1.** Let $y = Y(x, c)$ and $c = C(x, y)$ be the functions defined by (4.1) and (4.2). Then the mapping $S : \Omega \to \Omega$ given by $S(x, c) = (x, Y(x, c))$ is a Lipschitz homeomorphism; Moreover, we have

\[
\left\|\frac{\partial S}{\partial x}, \frac{\partial S}{\partial c}, \frac{\partial S^{-1}}{\partial x}, \frac{\partial S^{-1}}{\partial y}\right\|_{L^\infty(\Omega)} \leq 2.  \quad (4.3)
\]

Let $I(x, y)$ be the function defined in (3.4). Then the continuity equation (2.14) becomes

\[
\frac{d}{dx} \bar{\rho}(x, Y(x, c)) = \frac{\bar{\rho}}{u(x, Y(x, c))} \left(\frac{P(\bar{\rho})}{\mu} + I(x, Y(x, c))\right), \quad x \in (0, 1),
\]  
(4.4)

\[
\bar{\rho}(0, Y(0, c)) = \rho^0(c), \quad c \in [-1, 0^-] \cup [0^+, 1].
\]  
(4.5)
By the classical ODE theory, there is a unique solution $\tilde{\rho}$ to the problem (4.4) and (4.5). Since $P(\rho) \geq 0$, we can use a comparison principal to show that the solution $\tilde{\rho}$ satisfies $\rho_1 \leq \tilde{\rho} \leq \rho_2$, where $\rho_1$ and $\rho_2$ are the solutions of

$$
\begin{align*}
\frac{d\rho_1}{dx} &= -\frac{\rho_1}{u} \left[ \frac{P(\rho^0(c))}{\mu} + |I| \right], & x \in (0,1), \\
\frac{d\rho_2}{dx} &= -\frac{\rho_2}{u} I, & x \in (0,1), \\
\rho_1(0) &= \rho_2(0) = \rho^0(c).
\end{align*}
$$

Since $u \geq 1/2$ and $\|I(x,\cdot)\|_{L^\infty(-1,1)} \in L^1(0,1)$, there exist constants $m_1$ and $m_2$ which depends only on $M$ and $\|\rho^0\|_{L^\infty}$ such that

$$m_1 \leq \tilde{\rho}(x,y) \leq m_2, \quad (x,y) \in \Omega.$$

**Lemma 4.2.** Let $u$, $v$ and $I$ be given by Theorem 3.1 and assume that $\eta \geq \max\{\eta(M), C_\alpha M, C_{\alpha,\alpha} M\}$ (so that the right-hand sides of (3.2) and (3.3) are bounded by 1). Then (2.14) (or (4.4), (4.5)) admits a unique solution $\tilde{\rho}$ and there exists a constant $M_0$ depending only on $\|\rho^0\|_{C^\alpha[-1,0]^{-}}$ and $\|\rho^0\|_{C^\alpha[0,1]}$ such that for any $\beta \in [\alpha/2, \alpha]$,

$$\|(1-y)^{\beta-\alpha/2} \tilde{\rho}\|_{C^\beta(D^+)} + \|(1+y)^{\beta-\alpha/2} \tilde{\rho}\|_{C^\beta(D^-)} \leq M_0$$

where $D^+ = \{(x,y) \in \Omega \mid y > \tilde{f}(x)\}$, $D^- = \{(x,y) \in \Omega \mid y < \tilde{f}(x)\}$, and $\tilde{f}(x)$ is the solution of

$$
\begin{align*}
\tilde{f}'(x) &= \frac{v(x,\tilde{f}(x))}{u(x,\tilde{f}(x))}, & x \in (0,1), \\
\tilde{f}(0) &= 0.
\end{align*}
$$

In addition, the function $\tilde{\delta}(x)$ defined by

$$\tilde{\delta}(x) = P(\tilde{\rho}(x,\tilde{f}(x)^+)) - P(\tilde{\rho}(x,\tilde{f}(x)^-))$$

is positive and monotone decreasing and satisfies

$$\|\tilde{\delta}(x)\|_{C^\alpha[0,1]} \leq M_0.$$  

**Proof.** To show (4.6), we need only show the Hölder continuity of the function $\tilde{\rho}(x, Y(x,c))$ in the variable $(x,c)$ in the domain $[0,1] \times [0^+,1]$ and the domain $[0,1] \times [-1,0^-]$ since the mapping $(x,c) \rightarrow (x,Y(x,c))$ is a Lipschitz homeomorphism. By equation (4.4), the function $\tilde{\rho}(x, Y(x,c))$ is Lipschitz continuous in $x$, so that we need only show the Hölder continuity of $\tilde{\rho}(x, Y(x,c))$ in the variable $c$. 

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Let $c_1$, $c_2$ be any two constants with $c_1 > c_2$. Set $\rho_1(x) = \bar{\rho}(x, Y(x, c_1))$ and $\rho_2(x) = \bar{\rho}(x, Y(x, c_2))$. Subtracting the equation satisfied by $\rho_1$ from the equation satisfied by $\rho_2$ gives

\[
\begin{aligned}
\begin{cases}
\frac{d}{dx}(\rho_1(x) - \rho_2(x)) = -A(x)(\rho_1(x) - \rho_2(x)) - B(x)(c_1 - c_2)^\beta \\
\rho_1(0) - \rho_2(0) = \rho^0(c_1) - \rho^0(c_2)
\end{cases}
\end{aligned}
\tag{4.9}
\]

where

\[
A(x) = \frac{P(\rho_1) + \mu I(x, Y(x, c_1))}{\mu u(x, Y(x, c_1))} + \frac{(P(\rho_1) - P(\rho_2))\rho_2}{\mu(\rho_1 - \rho_2)u(x, Y(x, c_1))},
\tag{4.10}
\]

\[
B(x) = D(x)\frac{|Y(x, c_1) - Y(x, c_2)|^\beta}{|c_1 - c_2|^\beta},
\tag{4.11}
\]

and

\[
D(x) = \frac{\rho_2 P(\rho_1)}{\mu} \left( \frac{1}{u(x, Y(x, c_1))} - \frac{1}{u(x, Y(x, c_2))} \right) + \frac{\rho_2 (I(x, Y(x, c_1)) - I(x, Y(x, c_2)))}{|Y(x, c_1) - Y(x, c_2)|^\beta}.
\tag{4.12}
\]

Using Theorem 3.1 and Lemma 4.1, we find that for some positive constant $C$ depending only $\rho^0(y)$,

\[
|A(x)| \leq C(1 + \|I(x, \cdot)\|_{L^\infty(-1,1)}) \leq C(x - x^2)^{-\alpha/2} \in L^1(0, 1),
\]

\[
|B(x)| \leq C\left(\|u_x(x, \cdot)\|_{L^\infty(-1,1)} + \|I(x, \cdot)\|_{L^\infty(-1,1)} + \frac{|I(x, Y(x, c_1)) - I(x, Y(x, c_2))|}{|Y(x, c_1) - Y(x, c_2)|} + \bar{\omega}^\alpha/2 - \beta(x, Y(x, c_1)) + \bar{\omega}^\alpha/2 - \beta(x, Y(x, c_2))\right)
\]

\[
\leq C(\omega^{-\alpha/2}(x, \cdot)) \|I\|_{L^\infty(-1,1)}(1 + \bar{\omega}^\alpha/2 - \beta(x, Y(x, c_1)) + \bar{\omega}^\alpha/2 - \beta(x, Y(x, c_2)))
\]

\[
\leq C\left((x - x^2)^{-\beta} + (x - x^2)^{-\alpha/2}(1 + \bar{\omega}^{-\alpha/2} + \beta)^{-\beta + \alpha/2}\right)
\]

where $\bar{\omega} = \max\{\gamma, \rho^0\}$.

It follows that either for $c_1 > c_2 \geq 0^+$ or for $c_2 < c_1 \leq 0^-$,

\[
|\rho_1 - \rho_2| \leq |\rho_0(c_1) - \rho_0(c_2)| + \int_0^\beta e^{\int_0^\beta A(\xi)d\xi} + (c_1 - c_2)^\beta \int_0^\beta e^{\int_0^\beta A(\xi)d\xi} d\xi
\]

\[
\leq \bar{C}|c_1 - c_2|^\beta \left(1 + \|\rho_0\|_{C^0([-1,0^+])} \right)(1 + (1 - \bar{\omega})^{-\beta + \alpha/2}).
\tag{4.13}
\]

The estimate (4.6) thus follows.

To prove the second assertion of the lemma, we notice that both $u$ and $I$ are continuous in $\Omega$, so that taking $c_1 = 0^+$ in the equation (4.4), dividing both sides by $\bar{\rho}(x, \tilde{f}(x)^+)$, and subtracting the resulting equation from the corresponding equation resulting by taking $c = 0^-$ in (4.4), one obtains

\[
\begin{aligned}
\begin{cases}
\frac{d}{dx}\left(\ln \bar{\rho}^+(x) - \ln \bar{\rho}^-(x)\right) = -\frac{P(\bar{\rho}^+(x)) - P(\bar{\rho}^-(x))}{u(x, Y(x, 0))}, \\
\ln \bar{\rho}^+(0) - \ln \bar{\rho}^-(0) = \ln \rho^0(0^+) - \ln \rho^0(0^-),
\end{cases}
\end{aligned}
\tag{4.14}
\]
where $\tilde{\rho}(x) \equiv \tilde{\rho}(x, Y(x, 0^\pm)) = \tilde{\rho}(x, \tilde{f}(x)^\pm)$. Since $P(\rho)$ is monotone increasing, $P(\rho^+) - P(\rho^-) = \theta (\ln \rho^+ - \ln \rho^-)$ with $\theta \geq 0$, so that the function

$$\delta \rho(x) \equiv \tilde{\rho}^+(x) - \tilde{\rho}^-(x)$$

is strictly positive and monotone decreasing. The same conclusion also holds for the function $\delta(x)$.

The estimate (4.8) follows by using (4.4) with $c = 0^\pm$ and the interpolation $\|\rho^\pm\|_{C^{1-\varepsilon}([0,1])} \leq C \|(x - x^2)^\varepsilon \rho'\|_{L^\infty(0,1)} \leq C (1 + \|\tilde{\omega}I\|_{L^\infty(\Omega)})$. This completes the proof of Lemma 4.2.

5 Existence and uniqueness of the solution.

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** Set $M = M_0$ where $M_0$ is the constant given in lemma 4.2. Let $B$ and $K(M)$ be the Banach space defined in section 2. Using Theorem 3.1 and lemma 4.2, the mapping $T$ defined in the section 2 is well-defined. Then one sees from Theorem 3.1 and Lemma 4.2 that there exists a constant $\eta$ such that if $\eta \geq \eta_0$ and $\zeta \geq 0$, then the mapping $T$ maps $K(M_0)$ into itself.

Note that the mapping $T$ maps $K(M_0)$ into a bounded set in the Banach space

$$\hat{K} \equiv \hat{C}^\alpha(\Omega) \times C^\alpha[0,1] \times C^{1+\alpha}[0,1]$$

where $\hat{C}^\alpha \equiv \{p_c \in C^{\alpha/2}(\bar{\Omega}) \mid \|(1 - y^2)^{\alpha/2} p_c\|_{C^\alpha(\Omega)} < \infty\}$ and

$$\|p_c\|_{\hat{C}^\alpha(\Omega)} \equiv \|p_c\|_{C^{\alpha/2}(\Omega)} + \|(1 - y^2)^{\alpha/2} p_c\|_{C^\alpha(\Omega)}.$$

Since $\hat{K}$ may not be compact in $K(M)$, we shall use the following modification. For any $\lambda \in (1/2, 1)$, let $G_\lambda : \hat{C}^\alpha(\Omega) \to C^\alpha(\bar{\Omega})$ be defined by

$$G_\lambda(p_c)(x, y) = p_c(x, \lambda y), \quad (x, y) \in \Omega.$$

and define $T_\lambda : K(M_0) \to C^\alpha(\bar{\Omega}) \times C^\alpha([0,1]) \times C^{1+\alpha}([0,1])$ by

$$T_\lambda(p_c, \delta, f) = (G_\lambda \tilde{\rho}, \tilde{\delta}, \tilde{f})$$

where $(\tilde{\rho}, \tilde{\delta}, \tilde{f}) = T(p_c, \delta, f)$. One can check that $T_\lambda$ maps $K(M_0)$ into itself (as long as $\eta \geq \tilde{\eta}_0$ where $\tilde{\eta}_0$ is a constant independent of $\lambda$). Clearly, $T_\lambda$ is compact.

We now show that $T_\lambda$ is continuous. Let $\{(p_c^n, \delta^n, f^n)\}_{n=1}^\infty$ be a sequence in $K(M_0)$ such that $(p_c^n, \delta^n, f^n) \to (p_c^\infty, \delta^\infty, f^\infty)$ in $C^{\alpha/2}(\bar{\Omega}) \times C^{\alpha/2}([0,1]) \times C^{1+\alpha/2}([0,1])$ as $n \to \infty$. Let $(u^n, v^n)$ be the solutions to (1.1), (1.2), (1.5), and (1.6) corresponding to $\rho = \rho^n$ and $p = p^n$ and denote by $(\tilde{p}_c^n, \tilde{\delta}^n, \tilde{f}^n) = T(p_c^n, \delta^n, f^n)$, $n = 1, 2, \ldots, \infty$. By the
a priori estimates obtained in §3 and §4, for every subsequence of \((u^n, v^n, G_{\lambda p^n, \tilde{\rho}^n, f^n})\), there exists a subsequence which converges to certain functions \((u, v, \tilde{p}^\lambda, \tilde{\rho}, \tilde{f})\) in \(C^\alpha(\tilde{\Omega}) \times C^\alpha(\tilde{\Omega}) \times C^{\alpha-\varepsilon}(\tilde{\Omega}) \times C^{\alpha-\varepsilon}[0, 1] \times C^{1+\alpha-\varepsilon}[0, 1]\) \((0 < \varepsilon \ll 1)\). Since the solution of the elliptic system (1.1), (1.2), (1.5), and (1.6) depends continuously on \(\rho\) and \(p\), we therefore know that \(u = u^\infty\) and \(v = v^\infty\). Let \(Y^n(x, c)\) be the solution of (4.1) with the right hand side equals to \(k^n = v^n/u^n\). Since any limit point of \(\{Y^n\}\) is a solution to (4.1) with \(k = k^\infty\), which is unique, it follows that \(Y^n \to Y^\infty\). The proof of \(\rho^n(x, Y^n(x, c)) \to \rho^\infty(x, Y^\infty(x, c))\) is similar. This shows that \((\tilde{p}^\lambda, \tilde{\rho}, \tilde{f}) = (G_{\lambda p^\infty}, \tilde{\rho}, \tilde{f}^\infty)\) and therefore \(T_\lambda\) is continuous. Hence, one can apply the Schauder fixed point theorem to deduce that \(T_\lambda\) has a fixed point \((\tilde{p}^\lambda, \tilde{\rho}, \tilde{f}^\lambda) \in K(M_0)\).

Since \(\{p^\lambda, \tilde{\rho}, f^\lambda\}_{\lambda \in (1/2, 1)}\) is compact in \(C^{\alpha/2-\varepsilon}(\tilde{\Omega}) \times C^{\alpha-\varepsilon}[0, 1] \times C^{1+\alpha-\varepsilon}[0, 1]\), there exists a sequence \(\lambda_j \to 1\) such that \((p^{\lambda_j}, \tilde{\rho}^{\lambda_j}, f^{\lambda_j}) \to (p, \tilde{\rho}, f) \in K(M_0)\). Since \(T_\lambda \to T\) as \(\lambda \to 1\), we therefore know that \((p, \tilde{\rho}, f)\) is a fixed point of \(T\). After transferring back to \((u, v, p, \rho)\), this gives a solution of (1.1)–(1.7).

The estimate in Theorem 1 follows from Theorem 3.1 and Lemma 4.2. This completes the proof of Theorem 1.

We shall now assume that \(\rho^0(y)\) satisfies (1.9) and show that the solution to (1.1)–(1.7) is unique. To do this, we need the following lemma concerning the regularity of the solution given in the proof of Theorem 1.

**Lemma 5.1.** Assume that \(\rho^0(y) \in C^{0, \alpha}([-1, 0], C^1[0^+, 1])\), and let \((u, v, p, \rho)\) be the solution given in the proof of Theorem 1. Then for any \(\varepsilon \in (0, 1)\),

\[
|\nabla \rho(x, y)| \leq C_\varepsilon \left( |y - f(x)|^{-\varepsilon} + (1 - y^2)^{-\varepsilon} \right), \quad (x, y) \in \Omega \setminus \left( \Gamma \cup \{y = \pm 1\} \right)
\]

for some constant \(C_\varepsilon\) depending on \(\varepsilon\), \(\eta_0\) and \(\rho^0(y)\). Consequently, \(\nabla p \in L^r(\Omega)\) for any \(r > 1\).

**Proof.** Notice that in the proof of Theorem 1, we can take the exponent \(\alpha/2\) in defining \(K(M)\) to be any exponent less than \(\alpha\) which in the current case is 1. It follows from Theorem 3.1 that

\[
\tilde{\omega}^{2\varepsilon} \omega^{1-\varepsilon} \nabla I \in C^\varepsilon(\Omega).
\]

Taking \(\beta = 1\) in the (4.9) and using (5.2) we can estimate \(B\) in (4.11) by

\[
|B(x)| \leq C \left( \|u(x, \cdot)\|_{C^\varepsilon([0^+, 1])} + \sup_{Y(x, c_2) \leq y \leq Y(x, c_1)} \left( |\nabla u(x, y)| + |\nabla I(x, y)| \right) \right)
\]

\[
\leq C \omega^\varepsilon \omega^{-\varepsilon} (x, \tilde{c})
\]

\[
\leq C (x + \tilde{c})^{\varepsilon-1} (1 - x + c) \varepsilon^{-1} (\varepsilon - 2\varepsilon + (1 - \tilde{c})^{-2\varepsilon}),
\]

where \(c = \min\{|c_1|, |c_2|\}\) and \(\tilde{c} = \max\{|c_1|, |c_2|\}\). This implies that

\[
\int_0^1 |B(\xi)| d\xi \leq C \varepsilon (c^{-\varepsilon} + (1 - \tilde{c})^{-2\varepsilon})
\]
Noting that as \( Y(x, c_1), Y(x, c_2) \to y, c \approx |y - f(x)| \) and \( \dot{c} \approx |1 - y^2| \), the assertion of the lemma thus follows from an estimate similar to (4.13).

We shall now prove the uniqueness of the solution.

Let \((u_1, v_1, p_1, \rho_1)\) and \((u_2, v_2, p_2, \rho_2)\) be two solutions of (1.1)–(1.7) satisfying the estimate (2.10). Without loss of generality, we assume that \((u_2, v_2, p_2, \rho_2)\) is the solution given in the proof of Theorem 1 so that \(\rho_2\) satisfies the estimate (5.1).

Let \(f_i(x)(i = 1, 2)\) be the unique solution of the ordinary differential equation

\[
\frac{df_i(x)}{dx} = \frac{v_i(x, f_i(x))}{u_i(x, f_i(x))}, \quad x \in (0, 1),
\]

\[f_i(0) = 0.\]

Set \(q_i(x, y) = \ln \rho_i(x, y), f = f_1 - f_2\) and \(F(x, y) = f(x)\xi(y), \) where \(\xi = \xi(y) \in C_0^\infty(-1, 1)\) is a function satisfying \(\xi = 1\) if \(|y| \leq 1/2\) and \(\xi = 0\) if \(|y| > 3/4\). Define

\[
\bar{u}_2(x, y) = u_2(x, y - F), \quad u(x, y) = u_1(x, y) - \bar{u}_2(x, y), \quad (5.3)
\]

\[
\bar{v}_2(x, y) = v_2(x, y - F), \quad v(x, y) = v_1(x, y) - \bar{v}_2(x, y), \quad (5.4)
\]

\[
\bar{q}_2(x, y) = q_2(x, y - F), \quad q(x, y) = q_1(x, y) - \bar{q}_2(x, y), \quad (5.5)
\]

\[
\bar{p}_2(x, y) = p_2(x, y - F), \quad p(x, y) = p_1(x, y) - \bar{p}_2(x, y). \quad (5.6)
\]

Clearly, to show the uniqueness of the solution, we need only show that \(u = v = q = p = 0\).

First, we use the continuity equation to estimate \(q\) (or \(\rho\)) in terms of \(u\) and \(v\).

Since \(q_1\) and \(\bar{q}_2\) satisfies

\[
q_{1x} + \frac{v_1}{u_1} q_{1y} = -\frac{u_{1x} + v_{1y}}{u_1} u_1,
\]

\[
\bar{q}_{2x} + \frac{v_1}{u_1} \bar{q}_{2y} = \left(\frac{v_1}{u_1} - \frac{\bar{v}_2}{\bar{u}_2} - F_x - \frac{\bar{v}_2 F_y}{\bar{u}_2}\right) \bar{q}_{2y} - \frac{\bar{u}_{2x} + \bar{v}_{2y}}{\bar{u}_2} - \frac{F_x \bar{u}_{2y} + F_y \bar{v}_{2y}}{\bar{u}_2},
\]

We can subtract the second equation from the first equation to get

\[
q_x + \frac{v_1}{u_1} q_y = \left(\frac{\bar{u}_{2x} + \bar{v}_{2y}}{\bar{u}_2} - \frac{u_{1x} + v_{1y}}{u_1}\right) - \left(\frac{v_1}{u_1} - \frac{\bar{v}_2}{\bar{u}_2} - F_x\right) \bar{q}_{2y} + \left(\frac{F_x \bar{u}_{2y} + F_y \bar{v}_{2y}}{\bar{u}_2} + \frac{\bar{v}_2 F_y \bar{q}_{2y}}{\bar{u}_2}\right)
\]

\[
\equiv A + B + C. \quad (5.7)
\]

Denote by \(y = Y_1(x, c)\) the characteristic curves for the equation satisfying by \(\rho_1\). Equation (5.7) yields

\[
q(x, Y_1(x, c)) = \int_0^x (A + B + C)(\xi, Y_1(\xi, c)) \, d\xi, \quad (x, c) \in (0, 1) \times ([-1, 0^-] \cup [0^+, 1]).
\]

(Notice that the jump of \(\bar{q}_2\) also occurs at \(y = f_1(x)\).) Lemma 4.1 implies that the transformation \((x, c) \to (x, y)\) is a Lipschitz homomorphism, so that the norm
\[\|q(x, y)\|_{p, \Omega} \text{ is equivalent to the norm } \|q(x, Y(x, c))\|_{p, \Omega}. \text{ Hence, for any } p > 1, \text{ we have}
\]
\[\|q(x, y)\|_{p, \Omega} \leq C_p \|q(x, Y_1(x, c))\|_{p, \Omega} \leq \tilde{C}_p \|A + B + C\|_{p, \Omega}. \quad (5.8)\]

Since \(u_i\) and \(v_i\) satisfies the estimates in Theorem 1 and \(q_{2y} \in L^r(\Omega^\pm)\), we have
\[\|A + C\|_{p, \Omega} \leq C_p \left[\|u\|_{W^1_p(\Omega)} + \|v\|_{W^1_p(\Omega)} + \|F\|_{C^{0,1}(\Omega)}\right].\]

Noting that
\[
\frac{df(x)}{dx} = \frac{v_1(x, f_1(x)) - \tilde{v}_2(x, f_1(x))}{u_1(x, f_1(x)) - \tilde{u}_2(x, f_1(x))}, \quad (5.9)
\]
we can write \(B\) as
\[
B = \left. \left\{ \frac{(\frac{v_1}{u_1} - \frac{\tilde{v}_2}{\tilde{u}_2})(x, y) - \xi(y)(\frac{v_1}{u_1} - \frac{\tilde{v}_2}{\tilde{u}_2})(x, f_1(x))}{(y - f_1(x))\xi}\right. \right|_{y = f_1(x)} \left. (y - f_1(x))^{\varepsilon}\tilde{q}_{2y}(x, y - F)\right|_{y = f_1(x)} \]
where \(\varepsilon\) is a small positive constant. It follows by Lemma 5.1 that
\[\|B\|_{L^\infty(\Omega)} \leq C_{\varepsilon} \left\|\frac{v_1}{u_1} - \frac{\tilde{v}_2}{\tilde{u}_2}\right\|_{C^\infty(\Omega)} \leq C C_{\varepsilon} \left(\|u\|_{C^\infty(\Omega)} + \|v\|_{C^\infty(\Omega)}\right).\]

Substituting the estimates for \(A, B, \text{ and } C\) into (5.8), we get, for \(p > 2/(1 - \varepsilon)\),
\[
\|q\|_{p, \Omega} \leq C_{\varepsilon} \left\{\|u\|_{W^1_p(\Omega)} + \|v\|_{W^1_p(\Omega)} + \|F\|_{C^{0,1}(\Omega)} + \|u\|_{C^\infty(\Omega)} + \|v\|_{C^\infty(\Omega)}\right\}
\leq C \left(\|u\|_{W^1_p(\Omega)} + \|v\|_{W^1_p(\Omega)} + \|f\|_{C^{0,1}(0, 1)}\right) \quad (5.10)
\]
by the Sobolev imbedding theorem. Next we estimate \(\|u, v\|_{W^1_p(\Omega)}\). Notice that \(u_1\) and \(v_1\) satisfy (1.1), (1.2) whereas \(\tilde{u}_2\) satisfies
\[
-\eta \Delta \tilde{u}_2 - (\zeta + \frac{\eta}{3})(\tilde{u}_{2x} + \tilde{v}_{2y})x + \tilde{p}_2 \tilde{u}_2 \tilde{v}_2 \tilde{u}_2 \tilde{y} + \tilde{p}_2 \tilde{v}_2 \tilde{u}_2 \tilde{y} = \eta \left(2\tilde{u}_{2x}F_x + \tilde{u}_{2y}(2F_y + F^2_x + F^2_y) + \tilde{u}_{2y}(F_{xx} + F_{yy})\right)
\]
\[
+ (\zeta + \frac{\eta}{3})(\tilde{u}_{2y}F_x + \tilde{v}_{2y}F_y) - \tilde{p}_2 \tilde{u}_2 \tilde{v}_2 \tilde{F}_x - \tilde{p}_2 \tilde{u}_2 \tilde{v}_2 \tilde{F}_y - \tilde{p}_2 \tilde{y}_2 \tilde{F}_x
\]
\[
= -\tilde{p}_2 + \eta \left((2\tilde{u}_{2x}F_x - 2\tilde{u}_{2x}F_{xy} + [\tilde{u}_{2y}(2F_y + F^2_x + F^2_y)]_y\right)

- \tilde{p}_2 [2F_{yy} + 2Fx_{xy} + 2F_{xy}F_{yy}] + [(\tilde{u}_{2y}F_x)_{xy} + \tilde{u}_{2x}F_{xy} + \tilde{u}_{2y}F_{yy}]\right)\]
\[
+ (\zeta + \frac{\eta}{3})[\tilde{u}_{2y}F_x + \tilde{v}_{2y}F_y]_x - \tilde{p}_2 \tilde{u}_2 \tilde{v}_2 \tilde{F}_x - \tilde{p}_2 \tilde{v}_2 \tilde{u}_2 \tilde{F}_y - \tilde{p}_2 \tilde{F}_{xy} = -\tilde{p}_2 + G_1 + (G_2)_x + (G_3)_y.
\]

Subtracting this equation from the equation satisfied by \(u_1\) yields
\[
-\eta \Delta u - (\zeta + \frac{\eta}{3})(u_x + v_y)x + \rho_1 u_1 u_x + \rho_1 v_1 u_y + \rho_1 u \tilde{u}_2 + \rho_1 v \tilde{u}_2
= -\rho_1 - \tilde{G}_1 + (G_2)_x - (G_3)_y
\]

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where $\tilde{G}_1 = G_1 - (\rho_1 - \rho_2)(\tilde{u}_2 \tilde{u}_{2x} + \tilde{v}_2 \tilde{u}_{2y})$. Similarly, one can obtain an equation for $v = v_1 - \tilde{v}_2$.

$$-\eta \Delta v - (\zeta + \frac{\eta}{3})(u_x + v_y) + \rho_1 u_1 v_x + \rho_1 v_1 v_y + \rho_1 u v_{2x} + \rho_1 v v_{2y}$$

$$\equiv - (p)_y - H_1 + (H_2)_x - (H_3)_y.$$

We then obtain from $L^p$-estimate

$$\eta \left( \|u\|_{W^1_p(\Omega)} + \|v\|_{W^1_p(\Omega)} \right) \leq C \left( \|p\|_{L^p(\Omega)} + \|\tilde{G}_1\|_{L^p(\Omega)} + \|G_2\|_{L^p(\Omega)} + \|G_3\|_{L^p(\Omega)} + \|H_1\|_{L^p(\Omega)} + \|H_2\|_{L^p(\Omega)} + \|H_3\|_{L^p(\Omega)} + \|F\|_{C^{0,1}(\Omega)} \right)$$

since $\rho_i \in L^\infty(\Omega)$ and $\|u_i, v_i\|_{W^1_1(\Omega)} \leq C_r / \eta$ for all $r > 1$. Substituting the estimate for $q = \ln p(p)$ in (5.10) into the right-hand side of the last inequality and using (5.9), we obtain

$$\|u\|_{W^1_p(\Omega)} + \|v\|_{W^1_p(\Omega)} \leq \frac{C}{\eta} \left( \|u\|_{W^1_p(\Omega)} + \|v\|_{W^1_p(\Omega)} + \|f\|_{C^{0,1}(\Omega)} \right)$$

$$\leq \frac{C}{\eta} \left( \|u\|_{W^1_p(\Omega)} + \|v\|_{W^1_p(\Omega)} \right).$$

It follows that $u = v = 0$ if $\eta$ is large enough. Theorem 2 thus follows.

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