NONLINEAR BOUNDARY STABILIZATION OF PARALLELLY CONNECTED KIRCHHOFF PLATES

By

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Abstract
A system of nonlinearly coupled Kirchhoff plates is considered. It is shown that by applying nonlinear dissipation on the boundary, the energy of the system decays to zero at a uniform rate.

Keywords: Kirchhoff plates, nonlinear coupling, nonlinear boundary dissipation, uniform decay rates.

1 Introduction
As more results become available for single plates, attention has naturally shifted to more complex problems, such as those involving connected plates. Because one of the motivating factors in considering plate equations is to model more complicated vibrating flexible structures, our goal is to consider a coupled system, where the actions of one plate are allowed to affect those of the other.

The problem we consider is the following. Let $\Omega$ be an open bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary, $\Gamma$. In $\Omega$, we consider the following system of nonlinearly coupled Kirchhoff’s equations.

\begin{align*}
v_{tt} - \gamma^2 \Delta v_{tt} + \Delta^2 v = k_1(u - v) & \quad \text{in } Q_\infty = (0, \infty) \times \Omega \\
u_{tt} - \gamma^2 \Delta u_{tt} + \Delta^2 u = k_2(v - u) & \quad \text{in } Q_\infty = (0, \infty) \times \Omega \tag{1.1.a}
\end{align*}

with initial conditions

\begin{align*}
v(0, \cdot) = v_0 & \in H^2(\Omega), \quad v_t(0, \cdot) = v_1 & \in H^1(\Omega) \\
u(0, \cdot) = u_0 & \in H^2(\Omega), \quad u_t(0, \cdot) = u_1 & \in H^1(\Omega) \tag{1.1.b}
\end{align*}

and boundary conditions on $\Sigma_\infty \equiv \Gamma \times (0, \infty)$,

\begin{align*}
\Delta v + (1 - \mu)B_1 v &= -f_1(\frac{\partial}{\partial \tau} v_t) \\
\frac{\partial}{\partial \tau} \Delta v + (1 - \mu)B_2 v &= -\gamma^2 \frac{\partial}{\partial \tau} v_{tt} - v + \frac{\partial}{\partial \tau} h_1(\frac{\partial}{\partial \tau} (v - u)) - l_1(v - u) = g(v_t) - \frac{\partial}{\partial \tau} f_2(\frac{\partial}{\partial \tau} v_t) \tag{1.1.c}
\end{align*}

\begin{align*}
\Delta u + (1 - \mu)B_1 u &= -f_1(\frac{\partial}{\partial \tau} u_t) \\
\frac{\partial}{\partial \tau} \Delta u + (1 - \mu)B_2 u &= -\gamma^2 \frac{\partial}{\partial \tau} u_{tt} - u + \frac{\partial}{\partial \tau} h_2(\frac{\partial}{\partial \tau} (u - v)) - l_2(u - v) = g(u_t) - \frac{\partial}{\partial \tau} f_2(\frac{\partial}{\partial \tau} u_t). \tag{1.1.d}
\end{align*}

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Here, $\gamma > 0$ is a constant proportional to the thickness of the plate, $0 < \mu < \frac{1}{2}$ is Poisson’s ratio, the operators $B_1$ and $B_2$ are given by

$$
B_1 w = 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx} \\
B_2 w = \frac{\nu}{\gamma} [(n_1^2 - n_2^2) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})].
$$

(1.2)

In the above equations, $\nu = (n_1, n_2)$ and $\tau$ represent the normal and tangential directions to the boundary. The functions $k_i(s), h_i(s), l_i(s)$ are assumed to be differentiable with the property that for $s \in R$,

$$
\begin{align*}
&k_i(s) \geq 0 \\
&h_i(s) \geq 0 \\
l_i(s) \geq 0 \\
|h(s)| \leq C|s|^K & \text{ for some } K > 0 \text{ and } |s| > 1.
\end{align*}
$$

(H - 1)

The main goal in this paper is to show that the boundary dissipation acting via moments and shears applied to the edge of the plate and represented by the nonlinear functions, $f_i$ and $g$, produces uniform decay of the energy. To accomplish this, we shall assume the following hypotheses imposed on the functions $f_i$ and $g$. The functions $g$ and $f_i$ are continuous, monotone, zero at the origin and subject to the following growth conditions

$$
\begin{align*}
m |s| \leq |f_i(s)| \leq M |s| & \text{ for } |s| > 1, \ i = 1, 2 \\
g(s) \leq M |s|^r & \text{ for } |s| > 1.
\end{align*}
$$

(H - 2)

for some positive constants $m, M$ and $r$.

The control model for a single Kirchhoff plate (without coupling) was introduced and analyzed in [4]. Systems of equations of the type as in (1.1) are motivated by problems arising in the modelling of parallelly coupled plates (see [10], where a system of two one-dimensional wave equations with linear boundary conditions was considered). Indeed, parallel connection of plates introduces a force acting upon the system which is proportional to the difference of the displacement. If the plates are connected by, for example, springs, then linear theory gives the constants $k$ and $l$. In the above system, a more general situation is assumed. Coupling of the plates is allowed to produce nonlinear forces. Of particular interest is the case when $k(s) = l(s) = s^3$, which results from an application of a nonlinear version of Hooke’s Law.

To define the energy of the system, we introduce the bilinear form

$$
a(w, v) = \int_{\Omega} (\Delta w \Delta v + (1 - \mu)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}))d\Omega + \int_{\Gamma} wvd\Gamma.
$$

(1.3)

The energy functional is defined by $E(t) \equiv E_w(t) + E_u(t)$, where

$$
E_w(t) = \frac{1}{2} \int_{\Omega} \{|w_t|^2 + \gamma^2 |\nabla w_t|^2\}d\Omega + \frac{1}{2} a(w, w).
$$

(1.4)

Note that $a(w, w)$ is equivalent to the $H^2(\Omega)$ norm.

The main result of our paper is the following.

**Theorem 1.1** Assume (H-1) and (H-2) hold.

I. For any initial data $v_0, u_0 \in H^2(\Omega), v_1, u_1 \in H^1(\Omega)$, there exists a solution $(v, u) \in C(0, \infty; H^2(\Omega)) \cap C^1(0, \infty; H^1(\Omega))$ such that

$$
\nabla v_t|_\Gamma \in L_2(\Sigma_\infty), \ \nabla u_t|_\Gamma \in L_2(\Sigma_\infty).
$$

(1.5)

II. Assuming, in addition, that

$$
\begin{align*}
m &\leq f_i'(s) \leq M \\
g'(s) &\leq M(|s|^{r-1} + 1),
\end{align*}
$$

(H - 3)

then there exists constants $C, \omega > 0$ such that

$$
E(t) \leq Ce^{-\omega t}, \quad t \geq 0,
$$

(1.6)

where the constants $C, \omega$ may depend on $E(0)$, but they are independent of $t > 0$ and $\gamma > 0$. 

2
Exponential decay rates for a system of one dimensional wave equations with distributed linear coupling and linear boundary dissipation was proven recently in [9] and [10]. The methods of [9] and [10] rely on spectral analysis which is typically restricted to one dimensional problems. Moreover, [9] and [10] state explicitly that other methods such as multipliers (see [3]) or direct methods, which are tailored to multimensional stability analysis, fail even in the case of linear coupling of a one dimensional wave equation with linear dissipation. We shall see that a different and more general approach based on intrinsic comparison of asymptotic behavior of the energy functional with the solution to a certain nonlinear ordinary differential equation problem (introduced in [6]) prove successful in this more complicated situation.

A critical role in the proof of Theorem 1.1 is played by a result on stabilizability of a semilinear Kirchhoff plate with nonlinear boundary conditions. Since this result is of independent interest, we shall state it below.

Let \( a_i \) (resp. \( b \)) be given elements in \( L_2(\Sigma_{\infty}) \) (resp. \( b \in C(0, \infty; L_2(\Gamma)) \)). Consider the following equation of a nonautonomous semilinear Kirchhoff plate.

\[
\begin{cases}
  w_{tt} - \gamma^2 \Delta w_{tt} + \Delta^2 w = -k(w) & \text{in } Q_{\infty} \\
  w(0) = w_0 \in H^2(\Omega); \quad w_t(0) = w_1 \in H^1(\Omega) & \text{in } \Omega \\
  \Delta w + (1 - \mu) \partial_\nu w = -[f_1(\partial_\nu w_t + \alpha_1) - f_2(\alpha_1)] & \text{on } \Sigma_{\infty} \\
  \frac{\partial}{\partial \nu} \Delta w + (1 - \mu) \partial_\nu w_t - w + \frac{\partial}{\partial \tau} [f_2(\partial_\nu w_t + \alpha_2) - f_2(\alpha_2)] & \text{on } \Sigma_{\infty}.
\end{cases}
\tag{1.7}
\]

Under the assumption that the functions \( k, h, l \) satisfy (H-1) and the functions \( g, f_i \) satisfy (H-2), one can show (see [5]), that for all initial data \( w_0 \in H^2(\Omega), w_1 \in H^1(\Omega) \), there exists a solution \( w \) (not necessarily unique) to problem (1.7) such that

\[
E_w(t) < C(E_w(0)),
\tag{1.8}
\]

\[
\nabla w_t|_{r} \in L_2(\Sigma_{\infty}).
\tag{1.9}
\]

**Theorem 1.2** Assume that (H-1) and (H-3) hold true. Then there exist constants \( C, \omega > 0 \) such that every solution to (1.7) of finite energy satisfies

\[
E_w(t) \leq Ce^{-\omega t}.
\tag{1.10}
\]

The constants \( C \) and \( \omega \) may depend on the initial energy \( E_w(0), \|b\|_{C(0, \infty; L_2(\Gamma))} \) and the constants \( m, M \).

In the special case when \( a_i = b = 0 \) (or they are constant), a more general version of Theorem 1.2 holds true.

**Theorem 1.3** Let \( a_i = b = 0 \). Assume (H-1) and (H-2). Then every solution of finite energy which satisfies (1.9) obeys the following estimate. For some \( T_0 > 0 \),

\[
E_w(t) \leq S(t) \frac{t}{T_0} - 1 \text{ for } t > T_0,
\tag{1.11}
\]

where \( S(t) \to 0 \text{ as } t \to \infty \) and is the solution (contraction semigroup) of the differential equation

\[
\frac{d}{dt} S(t) + q(S(t)) = 0
\]

\[
S(0) = E_w(0),
\tag{1.12}
\]

and \( q(x) \) is a strictly monotone function constructed in terms of \( g \) and \( f_i \) (see Remark below).

**Remark 1.1:** To construct the function \( q(x) \), we proceed as follows: Let the function \( h(x) \) be defined by:

\[
h(x) \equiv h_1(x) + h_2(x).
\tag{1.13}
\]

where \( h_i(x) \) are concave, strictly increasing functions with \( h_3(0) = 0 \) such that

\[
h_i(s f_i(s)) \geq s^2 + f_i^2(s) \quad |s| \leq 1 \quad i = 1, 2.
\tag{1.14}
\]
(Such functions can be easily constructed. See [6].) Then \( h(x) \) enjoys the same properties, i.e., it is concave, strictly increasing, and \( h(0) = 0 \). Define

\[
\tilde{h}(x) \equiv h\left( \frac{x}{\text{mes} \Sigma_T} \right).
\] (1.15)

Since \( \tilde{h} \) is monotone increasing, for every \( c \geq 0 \), \( cI + \tilde{h} \) is invertible. Setting

\[
p(x) \equiv (cI + \tilde{h})^{-1}(Kx),
\] (1.16)

where \( c = \text{mes} \Sigma_T^{-1}(m^{-1} + M) \), the constant \( K \) will generally depend on \( E_w(0) \) unless \( r = 1 \) and \( h, k, l \) are linearly bounded. We then define \( q(x) \) by

\[
q(x) \equiv x - (I + p)^{-1}(x), \quad x > 0.
\] (1.17)

Remark 1.2: In the special case when the growth at the origin of the nonlinear boundary feedbacks is specified, one can compute explicitly the governing decay rates for the energy function, \( E_w(t) \). Indeed, this can be easily accomplished by constructing the appropriate function \( q \) and solving the nonlinear ODE problem, (1.12). For instance, if the nonlinear functions \( f_t \) and \( g \) have a linear growth at the origin, then (1.11) specializes to

\[
E_w(t) \leq C(E_w(0))e^{-\omega t} \quad \text{for some } \omega > 0.
\] (1.18)

If, instead, these nonlinearities are of polynomial growth at the origin (e.g. \( \sim x^p, p > 1 \)), then \( E_w(t) \leq C(E_w(0))t^{-\frac{1}{2}} \).

Remark 1.3: The result of Theorem 1.3 should be compared to a recent result on uniform decay rates obtained in [11] (and references therein) for a linear Euler-Bernoulli model with nonlinear boundary dissipation. Indeed, in [11], the uniform (polynomial) decay rates were obtained under the following hypotheses: (i) \( \Omega \) is star-shaped, (ii) nonlinear functions \( f_t \) and \( g \) are subject to a linear growth at infinity and polynomial growth at the origin (hypotheses (1.25), (1.26) in [11]), (iii) \( k = 0, h = 0, (iv) l \) is linear. Under the above hypotheses, it is shown in [11] that smooth solutions decay polynomially to zero.

In view of this, Theorem 1.3 extends the above result in several directions: (i) it does not require any geometric hypotheses to be imposed on \( \Omega \), (ii) it remains valid for all weak solutions, (iii) it allows for the presence of nonlinear terms \( h, l \) on the boundary together with the nonlinear interior term \( k \), and, finally, (iv) it does not require any growth conditions to be imposed on the functions \( f_t \) and \( g \) at the origin.

Needless to say, the techniques leading to the proof of Theorem 1.3 are very different (unfortunately more complicated) from those in [11]. Indeed, Lyapunov’s function method used in [11] runs into well recognized difficulties when dealing with the level of generality presented in Theorem 1.3.

The outline of our paper is as follows. In section 2, we prove the result of Theorem 1.2 which is critical to the proof of Theorem 1.1 in section 4. The proof of Theorem 1.3 is relegated to section 3.

2 Proof of Theorem 1.2

2.1 Preliminaries

Our goal is to prove energy decay rates for problem (1.7). In order to do this, one needs to perform certain partial differential equation calculations on the problem. These calculations require regularity of the solutions higher than is available. Since our nonlinear problem may not have a sufficiently regular solution (even if the initial data are smooth), we resort to an approximation argument (this argument was used in the context of wave equations in [6]). In fact, the idea here is to approximate solutions to the nonlinear problem (1.7) by solutions to different (linear) problems. Since this linear problem admits regular solutions for smooth initial data, the partial differential equation calculations can be performed on this problem. Final passage to the limit on the approximation problem allows us to obtain needed energy identities for the original nonlinear problem.

\footnote{A smoothness assumption was not stated in [11], but implicitly assumed (to validate arguments leading to decay estimates).}
To follow our program, we start by defining appropriate approximations for the quantities in equation (1.7). To do this, we introduce the following notation:

\[
\hat{f}_i(s) \equiv f_i(s + a_i) - f_i(a_i); \quad \hat{g}(s) \equiv g(s + b) - g(b).
\]  

(2.1)

Note that by virtue of hypothesis (H-3), \(\hat{f}_i\) and \(\hat{g}\) are strictly monotone functions and

\[
m \leq \hat{f}_i(s) \leq M.\]  

(2.2)

**Corollary 2.1** Let \(w\) be a solution to (1.7) such that (1.8) and (1.9) hold true. Let \(T > 0\). Then

\[
k(w) \in L_2(0, T; L_2(\Omega)),\]  

(2.3)

\[
\frac{\partial}{\partial t} h\left(\frac{\partial}{\partial t} - w\right) \in L_2(0, T; H^{-1}(\Gamma)); \quad l \in L_2(0, T; L_2(\Gamma)),\]  

(2.4)

\[
\hat{f}_1\left(\frac{\partial}{\partial \nu} w_i\right) \in L_2(0, T; L_2(\Gamma)),\]  

(2.5)

\[
\hat{g}(w_i) - \frac{\partial}{\partial \tau} \hat{f}_2\left(\frac{\partial}{\partial \tau} w_i\right) \in L_2(0, T; H^{-1}(\Gamma)).\]  

(2.6)

**Proof of Corollary 2.1:** Regularity in (2.3), (2.4) follows from (1.8), (1.9), (H-1) and Sobolev’s Imbeddings. Hypothesis (H-3) together with (1.9) implies

\[
\begin{align*}
\hat{f}_1\left(\frac{\partial}{\partial \nu} w_i\right) & \in L_2(\Sigma_T) \\
\hat{f}_2\left(\frac{\partial}{\partial \tau} w_i\right) & \in L_2(\Sigma_T).
\end{align*}
\]  

(2.7)

Hence,

\[
\frac{\partial}{\partial \tau} \hat{f}_2\left(\frac{\partial}{\partial \tau} w_i\right) \in L_2(0, T; H^{-1}(\Gamma)).\]  

(2.8)

On the other hand, with \(\phi \in L_2(0, T; H^1(\Gamma))\),

\[
\begin{align*}
\int_0^T \int_{\Gamma} |g(w(t, x))\phi(t, x)| dx dt & \leq C \int_0^T \|\phi(t)\|_{H^{1}(\Gamma)} \int_{\Gamma} \|w_i(t, x)\|^2 dx dt \\
& \leq C \int_0^T \|\phi(t)\|_{H^{1}(\Gamma)} \left(\|w_i(t)\|_{L^2(\Gamma)}^2 + 1\right) dt \\
& \leq C \int_0^T \|\phi(t)\|_{H^{1}(\Gamma)} E_w(t)^{1/2} dt \\
& \leq C \|E_w(0)^{1/2} \int_0^T \|\phi(t)\|_{H^{1}(\Gamma)} dt,
\end{align*}
\]  

(2.9)

where the first inequality follows from hypothesis (H-2), the second and third follow from Sobolev Imbeddings and the boundedness of \(\Gamma\), and the fourth from trace theory. Hence,

\[
g(w_i) \in L_2(0, T; H^{-1}(\Gamma)).\]  

(2.10)

By similar arguments,

\[
\|\hat{g}(w_i)\|_{L_2(0, T; H^{-1}(\Gamma))} \leq C\|E_w(0)^{1/2} + \|b\|_{C(0, \infty; L_2(\Gamma))}.\]  

(2.11)

This, together with (2.8) proves (2.6). \(\square\)

Let \(w\) be the solution of the original problem (1.7). By using the regularity properties in (1.9), (2.3)-(2.6), along with density of approximations (see below) Sobolev spaces, we are in a position to define

\[
f_n \in H^{1,1}(\Omega_T); \quad \|f_n + k(w)\|_{L_2(0, T; L_2(\Omega))} \to 0\]  

(2.12)

\[
f_{1n} \in H^{1,1}(\Sigma_T); \quad \|f_{1n} - \hat{f}_1\left(\frac{\partial}{\partial \nu} w_i\right)\|_{L_2(\Sigma_T)} \to 0\]  

(2.13)

\[
f_{2n} \in H^{1,1}(\Sigma_T); \quad \|f_{2n} - \left[\hat{g}(w_i) - \frac{\partial}{\partial \tau} \hat{f}_2\left(\frac{\partial}{\partial \tau} w_i\right) - \frac{\partial}{\partial \tau} h\left(\frac{\partial}{\partial \tau} w\right) + l(w)\right]\|_{L_2(0, T; H^{-1}(\Gamma))} \to 0\]  

(2.14)

\[
\alpha_n \in H^{1,1}(\Sigma_T); \quad \|\alpha_n - \frac{\partial}{\partial \nu} w_i\|_{L_2(\Sigma_T)} \to 0\]  

(2.15)

\[
\beta_n \in H^{1,1}(\Sigma_T); \quad \|\beta_n - \left(w_i - \frac{\partial^2}{\partial \tau^2} w_i\right)\|_{L_2(0, T; H^{-1}(\Gamma))} \to 0,\]  

(2.16)
where \( Q_T \equiv \Omega \times (0, T) \) and \( \Sigma_T \equiv \Gamma \times (0, T) \). We consider the following approximating problem:

\[
\begin{align*}
\begin{cases}
  w_{n,t} - \gamma^2 \Delta w_{n,t} + \Delta^2 w_n = f_n \\
  w_n(0) = w_{n,0}; \quad w_{n,t}(0) = w_{n,1} \\
  \Delta w_n + (1 - \mu) B_1 w_n + \frac{\partial}{\partial v} w_{n,t} = -f_n + \alpha_n \\
  \frac{\partial}{\partial v} \Delta w_n + (1 - \mu) B_2 w_n - \gamma^2 \frac{\partial}{\partial v} w_{n,t} = -w_n - w_{n,t} + \frac{\partial^2}{\partial \tau^2} w_{n,1} = f_{2n} - \beta_n,
\end{cases}
\end{align*}
\]

where

\[
\|w_{n,0} - w_0\|_{H^2(\Omega)} \to 0; \quad \|w_{n,1} - w_1\|_{H^1(\Omega)} \to 0,
\]

and \((w_{n,0}, w_{n,1}) \in \mathcal{D}\), where \( \mathcal{D} \), as dense set of \( \mathcal{H} \), consists of \( w_{n,0} \in H^4(\Omega), w_{n,1} \in H^3(\Omega) \), where \( w_{n,0}, w_{n,1} \) satisfy the appropriate compatibility conditions on the boundary. By standard linear semigroup methods, one easily shows that the linear problem, (2.17), admits a classical solution,

\[
w_n \in C(0, T; H^3(\Omega)) \cap C^1(0, T; H^3(\Omega)).
\]

The following proposition plays a critical role in our development.

**Proposition 2.1** Let \( w_n \) (respectively, \( w \)) be a solution of (2.17) (respectively, (1.7)). Then as \( n \to \infty \), the following convergence holds.

\[
w_n \to w \text{ in } C(0, T; H^2(\Omega)) \cap C^1(0, T; H^1(\Omega))
\]

\[
\nabla w_{n,t}|_{\Gamma} \to \nabla w_t \text{ in } L_2(\Sigma_T).
\]

**Proof:** Consider the equation satisfied by the difference \( w_n - w_m \). Taking the inner product of this equation with \( w_{n,t} - w_{m,t} \) and integrating the result from 0 to \( t \) yields

\[
E_{w_n - w_m}(t) + \int_0^t \int_\Omega |\nabla w_n|^2 \partial_t \partial \eta dt + \int_0^t \int_\Omega (f_n - f_m) \partial w_n \partial \eta dt + \int_0^t \int_\Omega |\nabla \partial w_n|^2 \partial \eta dt
\]

\[
= \int_0^t \int_{\Gamma}(f_n - f_m \partial w_n + \alpha_n - f_{1m} - \alpha_m) \partial \eta dt + \int_0^t \int_{\Gamma}(f_n - f_m \partial w_n + \beta_n - f_{2m} - \beta_m) \partial \eta dt + E_{w_n - w_m}(0),
\]

where \( \partial w \equiv w_n - w_m \). Hence,

\[
C_0 \||\nabla \partial w(t)||^2_{H^2(\Omega)} + \||\nabla \partial w(t)||^2_{L_2(\Sigma_T)} + \||\nabla \partial w(t)||^2_{L_2(\Sigma_T)}
\]

\[
\leq \||f_n - f_m||^2_{L_2(\partial \Omega, H^{-1}(\Omega))} + \||\nabla \partial w(t)||^2_{H^2(\Omega)} dt + \||f_n - f_m||^2_{L_2(\partial \Omega, H^{-1}(\Omega))} + \||\nabla \partial w(t)||^2_{L_2(\Sigma_T)}
\]

\[
+ \||f_n - f_m||^2_{L_2(\partial \Omega, H^{-1}(\Omega))} + \||\nabla \partial w(t)||^2_{L_2(\Sigma_T)}
\]

\[
\to 0 \text{ as } n \to \infty,
\]

where the limit follows by using (2.12)-(2.16). Thus, by (2.8) and Corollary 2.1,

\[
w_n \to w^* \text{ in } C(0, T; H^2(\Omega)) \cap C(0, T; H^1(\Omega))
\]

\[
\nabla w_{n,t}|_{\Gamma} \to \nabla w^*_t |_{\Gamma} \text{ in } L_2(\Sigma_T).
\]

This allows us to pass with the limit on the linear equation, (2.17). Indeed, from (2.25), (2.12)-(2.16), we obtain

\[
\begin{align*}
\begin{cases}
  w^*_{tt} - \gamma^2 \Delta w^*_t + \Delta^2 w^* + k(w) = 0 \\
  w^*(0) = w_0; \quad w^*_t(0) = w_1 \in H^2(\Omega) \\
  \Delta w^* + (1 - \mu) B_1 w^* + \frac{\partial}{\partial v} w^*_t = -f_1(\partial \eta w_t) + \frac{\partial}{\partial v} w_t \\
  \frac{\partial}{\partial v} \Delta w^* + (1 - \mu) B_2 w^* - \gamma^2 \frac{\partial}{\partial v} w^*_t = -w^* - w^*_t + \frac{\partial^2}{\partial \tau^2} w^*_t |_{\Gamma}
\end{cases}
\end{align*}
\]
Since \( w \) satisfies (2.26) and the solution to (2.26) is unique, we infer that \( w \equiv w^* \) and
\[
w_n \rightharpoonup w \text{ in } C(0, T; H^2(\Omega)) \cap C^1(0, T; H^1(\Omega))
\]
\[
\nabla w_n \rightharpoonup \nabla w \text{ in } L_2(\Sigma T),
\]
(2.27)
as desired. \( \square \)

Now we are in a position to prove the fundamental energy relation for problem (1.7).

**Lemma 2.1** (Energy Identity) Let \( w \) be the solution to (1.7). Then the following energy identity holds.
\[
\hat{E}_w(T) - \hat{E}_w(0) + \int_{\Sigma T} [\hat{g}(w_1)w_1 + \hat{f}_1(\frac{\partial}{\partial \nu} w_1) \frac{\partial}{\partial \nu} w_1 + \hat{f}_2(\frac{\partial}{\partial \tau} w_1) \frac{\partial}{\partial \tau} w_1] d\Gamma dt = 0,
\]
(2.28)
where
\[
\hat{E}_w(t) \equiv E_w(t) + \int_{\Omega} K(w(t)) d\Omega + \int_{\Gamma} [H(\frac{\partial}{\partial \tau} w(t)) + L(w(t))] d\Gamma,
\]
and \( K, H, \) and \( L \) are the antiderivatives of \( k, h, \) and \( l. \) In particular, \( \hat{E}_w(t) \leq \hat{E}_w(s) \) for \( t \geq s. \)

**Proof:** We first prove this energy identity for the solution, \( w_n, \) of the approximation problem (2.17).
Indeed, by applying a standard energy argument (valid due to the smoothness of solutions \( w_n \)) to (2.17), we obtain
\[
\frac{1}{2} (E_{w_n}(T) - E_{w_n}(0)) + \int_{\Sigma T} \|\frac{\partial}{\partial \nu} w_n, t\|^2 d\Gamma dt + \int_{\Sigma T} \|\frac{\partial}{\partial \tau} w_n, t\|^2 d\Gamma dt + \int_{\Sigma T} \|\frac{\partial}{\partial \tau} w_n, t\|^2 d\Gamma dt
\]
\[
= \int_0^T \int_{\Omega} f_n w_n, t d\Omega dt - \int_{\Sigma T} (f_{2n} - \beta_n) w_n, t d\Gamma dt.
\]
(2.29)

Using convergence properties (2.12)-(2.16) and the result of Proposition 2.1, we obtain
\[
\frac{1}{2} (E_w(T) - E_w(0)) + \int_{\Sigma T} \|\frac{\partial}{\partial \nu} w, t\|^2 d\Gamma dt + \int_{\Sigma T} \|\frac{\partial}{\partial \tau} w, t\|^2 d\Gamma dt + \int_{\Sigma T} \|\frac{\partial}{\partial \tau} w, t\|^2 d\Gamma dt
\]
\[
= -\int_0^T \int_{\Omega} k(w) w, t d\Omega dt + \int_{\Sigma T} \left[\frac{\partial}{\partial \nu} w_1 \frac{\partial}{\partial \tau} w_1\right] d\Gamma dt
\]
\[
- \int_{\Sigma T} \hat{g}(w_1) - \frac{\partial}{\partial \nu} f_2 \frac{\partial}{\partial \tau} w_1] w_1 d\Gamma dt + \int_{\Sigma T} \left(\frac{\partial}{\partial \tau} w_1^2 + \frac{\partial}{\partial \tau} w_1^2\right) d\Gamma dt
\]
\[
+ \int_{\Sigma T} \left(l(w, t) + h(\frac{\partial}{\partial \tau} w) \frac{\partial}{\partial \tau} w\right) d\Gamma dt.
\]
(2.30)

After canceling boundary terms and taking into account the definition of \( \hat{E}_w(t) \), we obtain (2.28). \( \square \)

### 2.2 A Priori Estimates

To prove Theorem 1.2, we first show the following inequality holds.

**Theorem 2.1** Assume (H-1) and (H-3) hold. Let \( w \) be the solution to (1.7) and \( T \) be sufficiently large. Then there exists a constant, \( C_T(E_w(0)) > 0 \), such that the following inequality holds:
\[
\int_0^T \hat{E}_w(t) dt \leq C_T(E_w(0), \|b\|_{C(0, \infty; L_{2}(\Gamma))}) \int_{\Sigma T} \left(\hat{g}(w_1) w_1 + \hat{f}_1 \left(\frac{\partial}{\partial \nu} w_1\right) \frac{\partial}{\partial \nu} w_1 + \hat{f}_2 \left(\frac{\partial}{\partial \tau} w_1\right) \frac{\partial}{\partial \tau} w_1\right) d\Gamma dt.
\]
(2.31)

#### 2.2.1 Estimates for the Approximated Problem

To prove Theorem 2.1, we begin by using a multiplier method on the approximation problem (2.17) to prove the following preliminary estimate.

**Lemma 2.2** Let \((w_0, w_1) \in D \) and \( 0 < \alpha < T/2 \). Then the energy of system (2.17) satisfies the following estimate:
\[
\int_0^T \hat{E}_{w_n}(t) dt = \left[ C_1 \left(1 + \gamma^2\right) E_{w_n, 1} (T - \alpha) \right] - C_2 \left(1 + \gamma^2\right) E_{w_n, 1}(\alpha)
\]
\[
\leq C_T, \alpha, \epsilon \left\{ \|f_1\|^2_{L_2(\Sigma T)} + \|f_2\|^2_{L_2(\Sigma T)} \right\} + \|\hat{g}\|^2_{L_2(\Sigma T)} + \|\beta_n - w_n, t\|^2_{L_2(\Sigma T), H^{-1}(\Gamma)}
\]
\[
+ \|w_n, t\|^2_{L_2(\Sigma T)} + (1 + \gamma^2)\|
\]
\[
\left. \right\} \left\{ \|f_n\|^2_{L_2(\Sigma T), H^{-1}(\Omega)} + \|f_n\|^2_{L_2(\Sigma T), H^{-1}(\Omega)} + \|f_n\|^2_{L_2(\Sigma T), H^{-1}(\Omega)} + \|f_n\|^2_{L_2(\Sigma T), H^{-1}(\Omega)} \right\}.
\]
(2.32)
where
\[ l.o.(w_n) \equiv \|w_n\|_{L^2(\Omega)}^2, \] (2.33)
and \( 0 < \epsilon < 1/2. \)

**Proof of Lemma 2.2:** Step 1: Identities. From [4] (p. 84, (4.5.17)), with adjustments to take both the nonhomogeneous right-hand side of (2.17) and the boundary conditions into account, we have

\[
\begin{align*}
\int_0^T E_{w_n,1}(t) \, dt & \leq -\frac{1}{2} \left[ (w_n, \nabla \beta_n, \nabla \beta_n)_{L^2(\Omega)} + \gamma^2 (\nabla w_n, \nabla (\beta_n - \nabla w_n))_{L^2(\Omega)} \right] + \frac{1}{2} \left[ (w_n, w_n)_{L^2(\Omega)} + \gamma^2 (\nabla w_n, \nabla w_n)_{L^2(\Omega)} \right] \nonumber \\
& \quad + \frac{1}{2} \left[ (\partial_{\omega} w_n, f_1)_{L^2(\Omega)} + \frac{1}{2} (w_n, f_2)_{L^2(\Omega)} \right] \nonumber \\
& \quad + \frac{1}{2} \left[ \left( \alpha_n - \frac{\partial}{\partial \nu} w_n, t \right)_{L^2(\Sigma_T)} + \frac{1}{2} \left( w_n, \beta_n - w_n, t + \frac{\partial^2}{\partial \nu^2} w_n, t \right)_{L^2(\Sigma_T)} \right] \nonumber \\
& \quad - \int_{\Sigma_T} \left[ \frac{\partial}{\partial \nu} \nabla w_n \right] f_1 + (\nabla w_n) f_2 \, d\Gamma \, dt \nonumber \\
& \quad + \int_{\Sigma_T} \left( \alpha_n - \frac{\partial}{\partial \nu} w_n, t \right) \, d\Gamma \, dt \nonumber \\
& \quad - \frac{1}{2} \int_{\Sigma_T} \nabla w_n, \nabla (\beta_n - w_n, t + \frac{\partial^2}{\partial \nu^2} w_n, t) \, d\Gamma \, dt - \frac{1}{2} \int_{\Sigma_T} f_n \, d\Omega \, dt.
\end{align*}
\] (2.34)

Notice that the regularity of the solution given by (2.19) allows us to justify the calculations in [4].

Step 2: Bounding Linear Terms. All terms which need to be evaluated at 0 and \( T, \) including the first and second lines and the last term on the right-hand side of (2.34), can be bounded by

\[ C_1 (1 + \gamma^2) E_{w_n,1}(T) + C_2 (1 + \gamma^2) E_{w_n,1}(0). \] (2.35)

Finally, by using duality to split the terms involving \( \alpha_n \) and \( \beta_n, \) noting that all boundary terms involving second derivatives of the solution can be bounded by second-order traces of the solution, and taking into account the above estimates, we obtain:

\[
\begin{align*}
\int_0^T E_{w_n}(t) \, dt & \leq C \left( (1 + \gamma^2) E_{w_n}(T) + (1 + \gamma^2) E_{w_n}(0) + \|f_1\|_{L^2(\Sigma_T)}^2 + \|f_2\|_{L^2(\Omega)}^2 \right) \\
& \quad + \|\alpha_n - \frac{\partial}{\partial \nu} w_n, t\|_{L^2(\Sigma_T)}^2 + \|\beta_n - w_n, t + \frac{\partial^2}{\partial \nu^2} w_n, t\|_{L^2(\Omega)}^2 \\
& \quad + \int_{\Sigma_T} \left( \frac{\partial^2}{\partial \nu^2} w_n \right]^2 \, d\Gamma \, dt + \int_{\Sigma_T} f_n \, d\Omega \, dt \\
& \quad + \int_{\Omega} \left( \frac{\partial^2}{\partial \nu^2} w_n \right)^2 \, dx + \|\nabla w_n\|^2 \, dx + \frac{1}{2} \int_{\Omega} f_n \, dx \, dt \nonumber \\
& \quad + \frac{1}{2} \int_{\Omega} f_n \, dx \, dt.
\end{align*}
\] (2.36)

Step 3: Bounding Second-Order Traces. Our next step is to estimate the second-order traces of the function \( w_n \) on the boundary. To accomplish this, it is critical to use the following “regularity” result obtained by microlocal analysis methods.

**Proposition 2.2 ([7], Theorem 2.1)** Let \( p(t, x) \) be a solution to the following linear problem (in the sense of distributions)

\[
\begin{align*}
\partial_t p - \gamma^2 \Delta p + \Delta^2 p & = F & \text{in } Q_T \\
\partial_t p(0, \cdot) & = p_0; \quad p(t, \cdot) & = p_1 & \text{in } \Omega \\
\Delta p + (1 - \mu) B_1 p & = g_1 & \text{on } \Sigma_T \\
\frac{\partial}{\partial \nu} \Delta p + (1 - \mu) B_2 p - \gamma^2 \frac{\partial}{\partial \nu} p & = g_2 & \text{on } \Sigma_T.
\end{align*}
\] (2.37)

For every \( T > \alpha > 0 \) and \( \frac{1}{2} > \epsilon > 0, \) the following estimate holds:

\[
\begin{align*}
\int_0^{T-\alpha} \int_{\Sigma_T} \left( \frac{\partial^2}{\partial \nu^2} w_n \right)^2 + \left| \frac{\partial^2}{\partial \nu^2} \right|^2 + \left| \frac{\partial^2}{\partial \nu^2} \right|^2 \, d\Gamma \, dt & \leq C T \alpha \left( \|F\|_{L^2(\Omega)}^2 + \|g_1\|_{L^2(\Sigma_T)}^2 + \|g_2\|_{L^2(\Sigma_T)}^2 \right) \\
& \quad + \|\nabla p_1\|_{L^2(\Sigma_T)}^2 + \|p\|_{L^2(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \right) \}.
\] (2.38)

Using the result of Proposition 2.2, we shall prove
Proposition 2.3 Let \( w_n \) be the solution to (2.17). Then for any \( T/2 > \alpha > 0 \) and \( \frac{1}{2} > \epsilon > 0 \), \( w_n \) satisfies the following inequality:

\[
\int_0^{T-\alpha} \int_T (|\frac{\partial^2 w_n}{\partial \tau^2}|^2 + |\frac{\partial w_n}{\partial \tau}|^2 + |\frac{\partial \tau w_n}{\partial \tau}|^2) dt + \int_0^{T} \int_T (|\frac{\partial^2 w_n}{\partial \tau^2}|^2 + |\frac{\partial w_n}{\partial \tau}|^2 + |\frac{\partial \tau w_n}{\partial \tau}|^2) dt + \int_0^{T} \alpha \{||f_n||_{L^2(0,T;H^{-\epsilon}(\Omega))} + ||\alpha_n - \frac{\partial}{\partial \tau} w_n||_{L^2(0,T;H^{-\epsilon}(\Omega))} + ||f_1||_{L^2(0,T;H^{-\epsilon}(\Omega))} + ||f_2||_{L^2(0,T;H^{-\epsilon}(\Omega))} + ||\nabla w_n||_{L^2(\partial \Gamma)} + l.o.w. \}. \tag{2.39}
\]

**Proof:** We apply the result of Proposition 2.2 to system (2.17) with

\[
F \equiv f_n, \\
g_1 \equiv \alpha_n - \frac{\partial}{\partial \tau} w_n \quad f_{1n}, \\
g_2 \equiv -\beta_n + w_n - \frac{\partial}{\partial \tau} w_n + f_{2n}. \tag{2.40}
\]

Completion of Proof of Lemma 2.2: Now the result of Lemma 2.2 follows by combining (2.36) applied on \([\alpha, T-\alpha] \) with Proposition 2.3 and using duality to bound the terms involving \( f_n \). \( \square \)

### 2.2.2 Estimates for the Original Problem

Recalling Proposition 2.1, we take the limit of (2.32) as \( n \to \infty \) to obtain a similar inequality for the solution to (1.7), which we state in the following lemma.

**Lemma 2.3** Let \((w_0, w_1) \in H\) and let \( T \) be sufficiently large. Then the energy of system (1.7) satisfies the following estimate:

\[
\int_0^T \bar{E}_w(t) dt \leq C_{T, \epsilon}(E_w(0), \|b\|_{C(\alpha, T; L^\epsilon(\Gamma))} + \|w_2\|_{L^2(\Omega)} + |1 + \lambda^2| \|\nabla w_2\|_{L^2(\Gamma)} + \int_0^{T} g(w(t)) dt + l.o.w. \). \tag{2.41}
\]

**Proof of Lemma 2.3:** **Step 1:** Approximation Results. Taking the limit as \( n \to \infty \) in (2.32), by virtue of (2.12)-(2.16) and Proposition 2.1, we find

\[
\int_0^{T-\alpha} \int_T E_w(t) dt - C_1(1 + \lambda^2) \int_0^{T} E_w(\alpha) + C_2(1 + \lambda^2) \int_0^{T} \bar{E}_w(\alpha) \leq C_{T, \epsilon} , \|\hat{f}(\frac{\partial}{\partial \tau} w_1)||_{L^2(\Gamma)} + ||\hat{f}(w_1) - \frac{\partial}{\partial \tau} f_2||_{L^2(\Gamma)} \|w_2||_{L^2(\Omega)} + |1 + \lambda^2| \|\nabla w_2||_{L^2(\Gamma)} + ||h(\frac{\partial}{\partial \tau} w)\|_{L^2(\Gamma)} + l.o.w. \}. \tag{2.42}
\]

**Step 2:** The following bounds can be obtained in a straightforward manner by using Sobolev’s Imbeddings and (H-1).

\[
\|k(w)\|_{L^2(\Omega)} \leq C(\|w\|_{H^{3/2+\epsilon}(\Omega)}) \|w\|_{L^2(\Omega)}, \tag{2.43}
\]
\[
\|l(w)\|_{L^2(\Gamma)} \leq C(\|w\|_{H^{3/2+\epsilon}(\Omega)}) \|w\|_{L^2(\Gamma)}, \tag{2.44}
\]
\[
\|h(\frac{\partial}{\partial \tau} w)\|_{L^2(\Gamma)} \leq C(\|w\|_{H^2(\Omega)}) \|\frac{\partial}{\partial \tau} w\|_{L^2(\Gamma)}. \tag{2.45}
\]

From Lemma 2.1 and (2.43)-(2.45),

\[
\|k(w)\|^2_{L^2(\Omega)} + \|l(w)\|^2_{L^2(\Omega)} + \|h(\frac{\partial}{\partial \tau} w)\|^2_{L^2(\Gamma)} \leq C(E_w(0)) l.o.w. \). \tag{2.46}
\]

**Step 3:** To estimate the first two boundary terms in (2.42), we notice that by virtue of (2.2),

\[
\|\hat{f}(\frac{\partial}{\partial \tau} w_1)\|_{L^2(\Gamma)} \leq M \|\frac{\partial}{\partial \tau} w_1\|^2_{L^2(\Gamma)}, \tag{2.47}
\]
\[
\|\frac{\partial}{\partial \tau} \hat{f}(\frac{\partial}{\partial \tau} w_1)\|_{L^2(\Omega)} \leq M \|\frac{\partial}{\partial \tau} w_1\|^2_{L^2(\Gamma)}. \tag{2.48}
\]
As for the term with $\hat{g}(w_t)$, we use hypothesis (H-3). Indeed, from the imbedding $H^{-1}(\Gamma) \subset L^1(\Gamma)$,

\[
\|\hat{g}(w_t)\|_{L^2(0,T,H^{-1}(\Gamma))}^2 \leq C\|\hat{g}(w_t)\|_{L^2(\Omega)}^2 \leq C\int_0^T \int_\Gamma |w_t|^\gamma \leq \int_\Gamma (|w_t|^\gamma + |\nabla w_t|^2) d\Gamma \leq C\left( T\int_\Omega |w_t|^\gamma + \int_\Omega |\nabla w_t|^2 \right) \text{d}t,
\]

where we have used the imbedding $H^{1/2}(\Gamma) \subset L^r(\Gamma)$ together with the Trace Theorem and the result of Lemma 2.2.

\textbf{Step 4:} Noticing that

\[
\int_0^{T-\alpha} \dot{E}_w(t) dt = \int_0^{T-\alpha} E_w(t) dt + \int_0^{T-\alpha} \int_\Gamma \mathcal{L}(w) + \mathcal{H}(\frac{\partial}{\partial t} w) \text{d}\Gamma dt + \int_0^{T-\alpha} \int_\Gamma K(w) d\Omega dt,
\]

where $\mathcal{K}, \mathcal{L}, \mathcal{H}$ also satisfy (2.43)-(2.45), and applying estimates (2.46)-(2.49) to (2.42) yields

\[
\int_0^{T-\alpha} \dot{E}_w(t) dt \leq C_1 \dot{E}_w(T-\alpha) - C_2 \dot{E}_w(\alpha)
\]

\[
\leq C_{T,\alpha}(E_w(0), \|b\|_{L^2(0,T,L^r(\Gamma))}) \{ \int_{\Sigma_T} (w_t + \|\nabla w_t\|_{L^2(\Sigma_T)})^2 \text{d}t + l.o.(w) \}. \tag{2.51}
\]

From (2.28) and (H-3),

\[
\int_0^{T-\alpha} \dot{E}_w(t) dt + \int_0^{T-\alpha} \dot{E}_w(t) dt \leq 2\alpha \dot{E}_w(0). \tag{2.52}
\]

Applying now the result of Lemma 2.1 to the terms $\dot{E}_w(T-\alpha)$ and $\dot{E}_w(\alpha)$ together with (2.52) yields

\[
\int_0^T \dot{E}_w(t) dt \leq C_1 (1 + \gamma^2) \dot{E}_w(0)
\]

\[
\leq C_{T,\alpha}(E_w(0), \|b\|_{L^2(0,T,L^r(\Gamma))}) \{ \|w_t\|_{L^2(\Sigma_T)}^2 + (1 + \gamma^2)\|\nabla w_t\|_{L^2(\Sigma_T)}^2 \} \tag{2.53}
\]

From Lemma 2.1,

\[
\dot{E}_w(0) \leq \dot{E}_w(T) + C \int_{\Sigma_T} (\hat{g}(w_t) w_t + |\nabla w_t|^2) |\text{d}t. \tag{2.54}
\]

Hence, for a sufficiently large $T$,

\[
\int_0^T \dot{E}_w(t) dt \leq C_1 (1 + \gamma^2) \dot{E}_w(0)
\]

\[
\geq \frac{1}{2} \int_0^T \dot{E}_w(t) dt + T \dot{E}_w(T) - C_1 (1 + \gamma^2) \dot{E}_w(0)
\]

\[
- C \int_{\Sigma_T} (\hat{g}(w_t) w_t + |\nabla w_t|^2) |\text{d}t \tag{2.55}
\]

which implies the conclusion of Lemma 2.3. \hfill \Box

\textbf{2.2.3 Nonlinear Compactness-Uniqueness Argument}

\textbf{Lemma 2.4} Let $T > 0$ be sufficiently large. Then

\[
l.o.(w) \leq C_T(E_w(0), \|b\|_{C(0,T,L^r(\Gamma))}) \int_{\Sigma_T} |\nabla w_t|^2 |\text{d}t. \tag{2.56}
\]

Here, the function $C_T(E_w(0), \|b\|_{C(0,T,L^r(\Gamma))})$ does not depend on $\gamma > 0$.

\textbf{Proof:} Identical to that in [2]. \hfill \Box

\textbf{2.2.4 Completion of the Proof of Theorem 2.1}

We combine the result of Lemma 2.3 with that of Lemma 2.4 and use (2.2) together with $\|w\|_{L^2(\Gamma)} \leq C\|\frac{\partial}{\partial \nu} w\|_{L^2(\Gamma)}$. \hfill \Box
2.3 Final Estimates: Proof of Theorem 1.2

Denoting \( \mathcal{F} = \int_{\Sigma_T} \left\{ \hat{g}(w_t)w_t + \hat{f}_1(\frac{\partial}{\partial v} w_t) \frac{\partial}{\partial v} w_t + \hat{f}_2(\frac{\partial}{\partial \theta} w_t) \frac{\partial}{\partial \theta} w_t \right\} d\Gamma dt \), we obtain from Theorem 2.1,

\[
\int_0^T \dot{E}_w(t) dt \leq C_{T, \alpha, \epsilon} \left( E_w(0), \|b\|_{C(0, \infty; L_r(\Gamma))} \right) \mathcal{F},
\]

(2.57)

and by Lemma 2.1,

\[
\int_0^T \dot{E}_w(t) dt \leq C_{T, \epsilon} \left( E_w(0), \|b\|_{C(0, \infty; L_r(\Gamma))} \right) \mathcal{F}
\rightarrow T \dot{E}_w(T) \leq C_{T, \epsilon} \left( E_w(0), \|b\|_{C(0, \infty; L_r(\Gamma))} \right) \mathcal{F}
\rightarrow \dot{E}_w(T) \leq C_T \left( E_w(0), \|b\|_{C(0, \infty; L_r(\Gamma))} \right) \mathcal{F}.
\]

(2.58)

Hence, recalling (2.28),

\[
\frac{\dot{E}_w(T)}{C_T \left( E_w(0), \|b\|_{C(0, \infty; L_r(\Gamma))} \right)} \leq \mathcal{F} = E_w(0) - \dot{E}_w(T).
\]

(2.59)

Setting

\[
p(s) \equiv \frac{s}{C_T \left( E_w(0), \|b\|_{C(0, \infty; L_r(\Gamma))} \right)},
\]

(2.60)

we have proven the following proposition.

**Proposition 2.4** Let \( w \) be the solution to (1.1) and \( E_w(t) \) be the corresponding energy at time \( t \). If \( T \) is sufficiently large, then

\[
p(\dot{E}_w(T)) + \dot{E}_w(T) \leq \dot{E}_w(0).
\]

(2.61)

To arrive at the conclusion of Theorem 1.2, we apply a (much more general than needed here) result of Lemma 3.3 in [6].

**Lemma 2.5 ([6], Lemma 3.3)** Let \( p \) be a positive, increasing function such that \( p(0) = 0 \). Since \( p \) is increasing, we can define a function \( q \) such that \( q(x) = x - (I + p)^{-1}(x) \). Notice that \( q \) is also an increasing function. Consider a sequence \( s_n \) of positive numbers which satisfy:

\[
s_{m+1} + p(s_{m+1}) \leq s_m.
\]

(2.62)

Then \( s_\infty \leq S(m) \), where \( S(t) \) is a solution of a differential equation

\[
\begin{cases}
\frac{d}{dt} S(t) + q(S(t)) = 0 \\
S(0) = s_0.
\end{cases}
\]

(2.63)

Moreover, if \( p(x) > 0 \) for \( x > 0 \), then \( \lim_{t \to \infty} S(t) = 0 \).

Applying the result of Proposition 2.4 and noticing that the energy \( \dot{E}_w(t) \) is decreasing, we obtain

\[
\dot{E}_w(m(T + 1)) + p(\dot{E}_w(m(T + 1))) \leq \dot{E}_w(mT),
\]

(2.64)

for \( m = 0, 1, ... \) Thus, applying Lemma 2.5 with

\[
s_m \equiv \dot{E}_w(mT),
\]

(2.65)

yields

\[
\dot{E}_w(mT) \leq S(m), \ m = 0, 1, ...
\]

(2.66)

Setting \( t = mT + \tau, \ 0 \leq \tau < T \),

\[
\dot{E}_w(t) \leq \dot{E}_w(mT) \leq S(m) \leq S\left(\frac{t-\tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right) \text{ for } t > T,
\]

(2.67)

and noting that in our case since \( q(x) \) is linear, \( S(t) \sim e^{-\omega t} \), completes the proof of Theorem 1.2. (Note that \( \dot{E}_w(t) \) and \( E_w(t) \) are topologically equivalent.) \( \square \)
3 Proof of Theorem 1.3

By arguments similar to those used in section 2.2 (see also the proof of Theorem 2.1 in [2]), we obtain the following counterpart of Theorem 2.1.

**Theorem 3.1** Assume (H-1), (H-2) and \( a_i = b = 0 \). Let \( w \) be a solution to (1.7) with regularity properties (1.8), (1.9) and let \( T > 0 \) be sufficiently large. Then

\[
\int_0^T \dot{E}_w(t)dt \leq C_T(E_w(0))\left\{ \int_{\Sigma_T} |w_i|^2 + |\nabla w_i|^2 + g(w_i)w_i d\Gamma dt \right\}. \tag{3.1}
\]

To proceed with the proof of Theorem 1.3, let the functions \( h(x), h_i(x), i = 1, 2, \) and \( \tilde{h}(x) \) be defined as in (1.13) (1.14). By the hypotheses imposed on functions \( h_i(x) \), we obtain

\[
\int_{\Sigma_T} \left| f_1\left( \frac{\partial}{\partial v} w_i \right) \right|^2 d\Gamma dt = \int_{\Sigma_{A_1}} \left| f_1\left( \frac{\partial}{\partial v} w_i \right) \right|^2 d\Gamma dt + \int_{\Sigma_{B_1}} \left| f_1\left( \frac{\partial}{\partial v} w_i \right) \right|^2 d\Gamma dt, \tag{3.2}
\]

where \( \Sigma_{A_1} \equiv \{ (t, x) \in \Sigma_T : \frac{\partial}{\partial v} w_i(x) \leq q_1 \} \) and \( \Sigma_{B_1} \equiv \Sigma_T \setminus \Sigma_{A_1} \). Using hypothesis (H-2) on \( \Sigma_{B_1} \), we find

\[
\int_{\Sigma_T} \frac{\partial}{\partial v} w_i^2 d\Gamma dt + \int_{\Sigma_T} \left| f_1\left( \frac{\partial}{\partial v} w_i \right) \right|^2 d\Gamma dt \\
\leq \int_{\Sigma_{A_1}} \left| f_1\left( \frac{\partial}{\partial v} w_i \right) \right|^2 d\Gamma dt + \int_{\Sigma_{B_1}} \left| f_1\left( \frac{\partial}{\partial v} w_i \right) \right|^2 d\Gamma dt + (M + \frac{1}{m}) \int_{\Sigma_{B_1}} f_1\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i d\Gamma dt \\
\leq \int_{\Sigma_{A_1}} h_1\left( \frac{\partial}{\partial v} w_i f_1\left( \frac{\partial}{\partial v} w_i \right) \right) d\Gamma dt + (M + \frac{1}{m}) \int_{\Sigma_T} f_1\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i d\Gamma dt. \tag{3.3}
\]

Similarly, the same argument applied to \( f_2 \) yields

\[
\int_{\Sigma_T} \frac{\partial}{\partial v} w_i^2 d\Gamma dt + \int_{\Sigma_T} \left| f_2\left( \frac{\partial}{\partial v} w_i \right) \right|^2 d\Gamma dt \\
\leq \int_{\Sigma_{A_2}} h_2\left( \frac{\partial}{\partial v} w_i f_2\left( \frac{\partial}{\partial v} w_i \right) \right) d\Gamma dt + (M + \frac{1}{m}) \int_{\Sigma_T} f_2\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i d\Gamma dt. \tag{3.4}
\]

Recall

\[
\tilde{h}_i(x) \equiv h_i\left( \frac{x}{\text{mes } \Sigma_T} \right). \tag{3.5}
\]

Then, by Jensen's inequality,

\[
\int_{\Sigma_T} \{|w_i|^2 + |\nabla w_i|^2 + g(w_i)w_i| d\Gamma dt \\
\leq C_1 \int_{\Sigma_T} \left\{ g(w_i)w_i + f_1\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i + f_2\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i \right\} d\Gamma dt \\
+ C_2 \left[ \tilde{h}_1\left( \int_{\Sigma_T} f_1\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i d\Gamma dt \right) + \tilde{h}_2\left( \int_{\Sigma_T} f_2\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i d\Gamma dt \right) \right] \tag{3.6}
\]

where the last inequality follows from the monotonicity of the functions \( \tilde{h}_i \).

Denoting \( \mathcal{F} \equiv \int_{\Sigma_T} \left\{ g(w_i)w_i + f_1\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i + f_2\left( \frac{\partial}{\partial v} w_i \right) \frac{\partial}{\partial v} w_i \right\} d\Gamma dt \), we obtain from Theorem 3.1 and (3.6),

\[
\int_0^T \dot{E}_w(t)dt \leq C_T(E_w(0))[\mathcal{F} + \tilde{h}\mathcal{F}], \tag{3.7}
\]

and by Lemma 2.1,

\[
\int_0^T \dot{E}_w(t)dt \leq C_T(E_w(0))[\mathcal{F} + \tilde{h}\mathcal{F}] \\
\Rightarrow T\dot{E}_w(T) \leq C_T(E_w(0))[\mathcal{F} + \tilde{h}(\mathcal{F})] \tag{3.8}
\]

Hence, recalling (2.28),

\[
(I + \tilde{h})^{-1}\left( \frac{\dot{E}_w(T)}{C_T(E_w(0))} \right) \leq \mathcal{F} = E_w(0) - \dot{E}_w(T). \tag{3.9}
\]
Setting
\[ p(s) \equiv (I + \hat{h})^{-1}\left(\frac{s}{C_T(E_w(0))}\right), \] (3.10)
we have the result of Proposition 2.4 valid with the above function \( p(s) \).

As in section 2.3, applying the result of Proposition 2.4, we obtain
\[ \hat{E}_w(m(T + 1)) + p(\hat{E}_w(m(T + 1))) \leq \hat{E}_w(mT), \] (3.11)
for \( m = 0, 1, \ldots \) Thus, applying Lemma 2.5 with
\[ s_m \equiv E_w(mT), \] (3.12)
yields
\[ \hat{E}_w(mT) \leq S(m), \quad m = 0, 1, \ldots \] (3.13)
Setting \( t = mT + \tau, 0 \leq \tau < T \),
\[ \hat{E}_w(t) \leq \hat{E}_w(mT) \leq S(m) \leq S\left(\frac{t}{T} - 1\right) \leq S\left(\frac{t}{T}\right) \] (3.14)
for \( t > T \), which, in view of the topological equivalence of \( \hat{E}_w(t) \) and \( E_w(t) \), completes the proof of Theorem 1.3. \( \square \)

4 Proof of Theorem 1.1

The first part of Theorem 1.1 (existence) follows by applying the result of the main theorem in [5] within the framework described in the first section of [1] where a system of coupled plate equations is considered. We shall concentrate on the second part of Theorem 1.1, i.e., estimate (1.6).

4.1 Change of Variables

We introduce a change of variables, \( w \equiv v - u \). Then \( w \) satisfies
\[
\begin{cases}
\frac{\partial^2 w}{\partial t^2} - \gamma^2 \Delta w_{tt} + \Delta^2 w = k_1(-w) - k_2(w) \\
w(0, \cdot) = v_0 - u_0; \quad w_t(0, \cdot) = v_1 - u_1 \\
\frac{\partial w}{\partial \nu} + (1 - \mu)B_1 w = -\left[f_1\left(\frac{\partial}{\partial \nu} w_t + \frac{\partial}{\partial \tau} u_t\right) - f_1\left(\frac{\partial}{\partial \tau} u_t\right)\right] \\
\frac{\partial}{\partial \nu} \Delta w + (1 - \mu)B_2 w - \gamma^2 \Delta w_{tt} - w + \frac{\partial}{\partial \nu} h_1\left(\frac{\partial}{\partial \nu} w_t\right) - \frac{\partial}{\partial \tau} h_2\left(-\frac{\partial}{\partial \tau} w\right) = l_1(w) + l_2(-w) \\
= g(w_t + u_t) - g(u_t) - \frac{\partial}{\partial \nu} f_2\left(\frac{\partial}{\partial \nu} w_t\right) + \frac{\partial}{\partial \tau} f_2\left(\frac{\partial}{\partial \tau} u_t\right).
\end{cases}
\] (4.1)

By virtue of (1.5),
\[
\frac{\partial}{\partial \nu} u_t, \quad \frac{\partial}{\partial \tau} u_t \in L_2(\Sigma_\infty).
\] (4.2)

Moreover,
\[
\|u_t\|_{C(0, \infty; H^1(\Omega))} \leq C(E_u(0), E_v(0)).
\] (4.3)

Hence, by Sobolev's Imbeddings, for any \( r \geq 1 \),
\[
\|u_t\|_{C(0, \infty; L^2_r(\Gamma))} \leq C(E_u(0), E_v(0)).
\] (4.4)

Setting \( a_1 \equiv \frac{\partial}{\partial \nu} u_t, \quad a_2 \equiv \frac{\partial}{\partial \tau} u_t, \quad b \equiv u_t |_\Gamma \), We conclude that \( a_t \in L_2(\Sigma_\infty) \) and \( b \in C(0, \infty; L_r(\Gamma)) \). On the other hand, functions defined by
\[
k(s) \equiv k_2(s) - k_1(-s)
\]
\[
l(s) \equiv l_1(s) - l_2(-s)
\]
\[
h(s) \equiv h_1(s) - h_2(-s)
\]
comply with hypothesis (H-1).

Thus we are in a position to apply the result of Theorem 1.2 to equation (4.1). This yields

**Theorem 4.1** Let \( w(t) \) be any solution of finite energy corresponding to system (4.1). Then there exist constants \( C, \omega > 0 \) such that
\[
\|(u - v)(t)\|_{H^2(\Omega)} + \|(u_t - v_t)(t)\|_{H^1(\Omega)} \leq Ce^{-\omega t},
\] (4.5)
where \( C, \omega \) may depend on \( ||u_0||_{H^2(\Omega)}, ||v_0||_{H^2(\Omega)}, ||u_1||_{H^1(\Omega)}, ||v_1||_{H^1(\Omega)} \).
4.2 Analysis of "u" Equation

Using the variable w, we see that the equation for u (see (1.1)) is equivalent to

\[
\begin{align*}
\begin{cases}
    u_{tt} - \gamma^2 \Delta u_{tt} + \Delta^2 u = k_2(u) & \text{in } Q_\infty \\
    u(0, \cdot) = u_0; \quad u_t(0, \cdot) = u_t & \text{in } \Omega \\
    \Delta u + (1 - \mu) B_1 u = -f_1(\frac{\partial}{\partial t}, u_t) & \text{on } \Sigma_\infty \\
    \frac{\partial}{\partial v} \Delta u + (1 - \mu) B_2 u - \gamma^2 \frac{\partial}{\partial v} u_{tt} - u = g(u_t) - \frac{\partial}{\partial t} f_2(\frac{\partial}{\partial t}, u_t) + l_2(-w) - \frac{\partial}{\partial t} h_2(\frac{\partial}{\partial t}(-w)) & \text{on } \Sigma_\infty,
\end{cases}
\end{align*}
\]

where from Theorem 4.1, w(t) satisfies

\[ ||w(t)||_{H^2(\Omega)} \leq Ce^{-\omega t}, \tag{4.7} \]

with the constants C, \omega depending on E_0(0), E_u(0).

**Proposition 4.1** Let E_0(0) \leq R, E_u(0) \leq R. Then there exist constants C, \omega > 0 such that for all t \geq 0,

\[ ||k_2(w)(t)||_{L^2(\Omega)} \leq Ce^{-\omega t} \tag{4.8} \]

\[ ||l_2(w)(t)||_{L^2(\Gamma)} \leq Ce^{-\omega t} \tag{4.9} \]

\[ ||\frac{\partial}{\partial t} h_2(\frac{\partial}{\partial t}(-w(t)))||_{H^{-1}(\Gamma)} \leq Ce^{-\omega t}. \tag{4.10} \]

**Proof:** Inequalities (4.8) and (4.9) follow direct from (4.7), hypothesis (H-1) and the imbedding \( H^2(\Omega) \subset C(\Omega). \) As for (4.10), we have

\[
\begin{align*}
||\frac{\partial}{\partial t} h_2(\frac{\partial}{\partial t}(-w(t)))||_{H^{-1}(\Gamma)}^2 & \leq C||h_2(\frac{\partial}{\partial t}(-w(t)))||_{L^2(\Gamma)}^2 \\
& \leq C \int_{\Gamma} \left( ||\frac{\partial}{\partial t} w(t)||^2 + ||\frac{\partial}{\partial t} w(t)||^2 \right) d\Gamma \\
& \leq C \left( ||w(t)||_{H^2(\Omega)}^2 + ||w(t)||_{H^2(\Omega)}^2 \right).
\end{align*}
\]

where the second inequality follows from the imbedding \( H^{1/2}(\Gamma) \subset L^{2k}(\Gamma) \) followed by the Trace Theorem. \( (4.7) \) together with \( (4.11) \) implies \( (4.10). \)

We next consider the following nonautonomous linear problem.

\[
\begin{align*}
\begin{cases}
    u_{tt} - \gamma^2 \Delta u_{tt} + \Delta^2 u = k(t, x) & \text{in } Q_\infty \\
    u(0, \cdot) = u_0; \quad u_t(0, \cdot) = u_t & \text{in } \Omega \\
    \Delta u + (1 - \mu) B_1 u = -f_1(t, x) \frac{\partial}{\partial t} u_t & \text{on } \Sigma_\infty \\
    \frac{\partial}{\partial v} \Delta u + (1 - \mu) B_2 u - \gamma^2 \frac{\partial}{\partial v} u_{tt} - u = g(t, x) u_t - \frac{\partial}{\partial t} f_2(t, x) \frac{\partial}{\partial t} u_t + l(t, x) & \text{on } \Sigma_\infty,
\end{cases}
\end{align*}
\]

where

\[ ||k(t)||_{L^2(\Omega)} \leq C_0 e^{-\omega_0 t}, \quad t \geq 0 \tag{4.13} \]

\[ ||l(t)||_{H^{-1}(\Gamma)} \leq C_0 e^{-\omega_0 t}, \quad t \geq 0 \tag{4.14} \]

\[ m \leq f_1(t, x) \leq M, \quad (t, x) \in \Sigma_\infty \tag{4.15} \]

\[ g(t, x) \geq 0, \quad (t, x) \in \Sigma_\infty \tag{4.16} \]

\[ ||g(t, \cdot)||_{L^2(\Gamma)} \leq M, \quad t \geq 0 \tag{4.17} \]

We shall first show that a homogeneous system (4.12) with \( k \equiv l \equiv 0 \) is exponentially stable. Then by using linear evolution methods, we will be able to prove that this stability is preserved for a system with nonhomogeneous, but exponentially decaying terms. This is stated below.

**Theorem 4.2** Let u be a solution to (4.12) subject to assumptions (4.13)-(4.17). Then there exist constants C, \omega > 0 depending on C_0, \omega_0, M, m such that for all t \geq 0,

\[ ||u(t)||_{H^2(\Omega)} + ||u_t(t)||_{H^1(\Omega)} \leq Ce^{-\omega t}(||u_0||_{H^2(\Omega)} + ||u_1||_{H^1(\Omega)} + C_0). \tag{4.18} \]
We note that the solution $u$ to (4.6) satisfies (4.12) with
\[
k(t, x) \equiv k_2(w(t, x)), \quad l(t, x) \equiv l_2(-w(t, x)) - \frac{\partial}{\partial \tau} h_2(-\frac{\partial}{\partial \tau} w(t, x)) \tag{4.19}
\]
\[
f_1(t, x) \equiv \frac{f_1(\frac{\partial}{\partial v} u_1(t, x))}{\frac{\partial}{\partial v} u_1(t, x)} \tag{4.20}
\]
\[
f_2(t, x) \equiv \frac{f_2(\frac{\partial}{\partial v} u_1(t, x))}{\frac{\partial}{\partial v} u_1(t, x)} \tag{4.21}
\]
\[
g(t, x) \equiv \frac{g(u_1(t, x))}{u_1(t, x)}. \tag{4.22}
\]

Proof of Theorem 4.2 is relegated to section 5.

Remark 4.1: Though the result of Theorem 4.2 appears very plausible, there are technical difficulties in the proof due to the presence of boundary inhomogeneities. This, in turn, gives rise to evolutions with unbounded forcing terms.

Hypotheses assumed in (H-1) and (H-3) together with the result of Proposition 4.1 imply that (4.13)-(4.17) hold true. Thus we are in a position to apply the result of Theorem 4.2 to the solution $u$ of equation (4.1). This yields

**Theorem 4.3** Let $u$ be a solution to (4.5). Then there exist constants $C$, $\omega > 0$ depending on $E_v(0)$, $E_u(0)$ such that
\[
\|u(t)\|_{H^2(\Omega)} + \|u_1(t)\|_{H^1(\Omega)} \leq Ce^{-\omega t}. \tag{4.23}
\]

Combining the results of Theorem 4.1 and Theorem 4.3 and recalling $v \equiv w + u$ yields the final conclusion, (1.6) in Theorem 1.1. □

## 5 Proof of Theorem 4.2

Consider the following linear nonhomogeneous equation in a variable $z$.
\[
z_{tt} - \gamma^2 \Delta z_{tt} + \Delta^2 z = 0 \quad \text{in } \Omega \times (s, \infty)
\]
\[
z(t = s, \cdot) = z_0 \in H^2(\Omega), \quad z_1(t = s, \cdot) = z_1 \in H^1(\Omega)
\]
\[
\Delta z + (1 - \mu)B_1 z = -f_1(t, \cdot) \frac{\partial}{\partial v} z_t \quad \text{on } \Gamma \times (s, \infty)
\]
\[
\frac{\partial}{\partial v} \Delta z + (1 - \mu)B_2 z - \gamma^2 \frac{\partial}{\partial v} z_{tt} - z = g(t, \cdot)z_t - \frac{\partial}{\partial v}[f_2(t, \cdot) \frac{\partial}{\partial v} z_t] \quad \text{on } \Gamma \times (s, \infty), \tag{5.1}
\]

where $f_1, f_2$, and $g$ satisfy conditions (4.15)-(4.17).

By linear evolution methods, one can easily show that $(z(t), z_t(t)) \equiv Z(t, s)(z_0, z_1)$, where $Z(t, s)$ is an evolution operator on $H^2(\Omega) \times H^1(\Omega)$ associated with (5.1).

**Lemma 5.1** There exist constants $\omega_1 > 0, C_1 > 0$, such that
\[
\|Z(t, s)\|_{C^0(H)} \leq C_1 e^{-\omega_1(t-s)}, \quad t \geq s. \tag{5.2}
\]
The constants $\omega_1$ and $C_1$ depend only on $m$, $M$ (see (4.15)-(4.17)).

**Proof:** Step 1: Computations identical to those used for the proof of Lemma 2.3 give the following estimate.
\[
\int_s^{T+s} E_z(t)dt \leq C_T(M, m)\left\{\int_s^{T+s} \int_\Gamma |\nabla z_t|^2 d\Gamma dt + \int_s^{T+s} \|g(t, \cdot)z_t(t)\|_{H^2(\Gamma)}^2 dt + l.o.(z)\right\}, \tag{5.3}
\]
where $l.o.(z) \leq C \int_s^{T+s} \|z(t)\|_{H^2(\Gamma)}^2 dt, 0 < \epsilon < \frac{1}{2}$.

Noting that
\[
\|g z_t\|_{H^2(\Gamma)}^2 \leq C \|g z_t\|_{L^2(\Gamma)}^2 \leq C \|g\|_{L^2(\Gamma)}^2 \|z_t\|_{L^2(\Gamma)}^2,
\]

\[2\]
we obtain
\[
\int_s^{T+s} \|g(t, \cdot)z(t)\|_{H_{\infty}^{-1}(\Gamma)}^2 dt \leq \sup_{t \in [s, T+s]} \|g(t)\|_{Z_{\infty}^2(\Gamma)}^2 \int_s^{T+s} \|z(t)\|_{L_\infty^2(\Gamma)}^2 dt
\]
\[
\leq M \int_s^{T+s} \|z(t)\|_{L_\infty^2(\Gamma)}^2 dt.
\]
(5.4)

Combining this with (5.3),
\[
\int_s^{T+s} E_z(t) dt \leq C_T(M, m) \int_s^{T+s} \int_\Gamma |\nabla z|^2 d\Gamma dt + l.o.(z).
\]
(5.5)

**Step 2:** By applying a compactness-uniqueness argument together with the uniqueness property for the equation
\[
\begin{align*}
\tau z_{tt} - \gamma^2 \Delta z_{tt} + \Delta^2 z &= 0 & \text{in } \Omega \times (s, T + s) \\
\Delta z + (1 - \mu)B_1 z &= 0 & \text{on } \Gamma \times (s, T + s) \\
\frac{\partial}{\partial \nu} \Delta z + (1 - \mu)B_2 z - z &= 0 & \text{on } \Gamma \times (s, T + s) \\
z_t &= 0, & \frac{\partial}{\partial \nu} z_t = 0 & \text{on } \Gamma \times (s, T + s),
\end{align*}
\]
(5.6)

we can absorb lower order terms. Hence, taking $T$ sufficiently large,
\[
l.o.(z) \leq C_T(M, m) \int_s^{T+s} |\nabla z_t|^2_{L_\infty^2(\Gamma)} dt.
\]
(5.7)

**Step 3:** From (4.15), we estimate the boundary term in (5.5).
\[
\int_s^{T+s} \int_\Gamma |\nabla z_t|^2 d\Gamma dt \leq \frac{1}{m} \int_s^{T+s} \int_\Gamma \{f_1(t, x) |\frac{\partial}{\partial \nu} z_t|^2 + f_2(t, x) \frac{\partial}{\partial \tau} z_t|^2 + g(t, x) z_t^2\} d\Gamma dt.
\]
(5.8)

Combining (5.5), (5.7), and (5.8) gives
\[
\int_s^{T+s} E_z(t) dt \leq C_T(M, m) \int_s^{T+s} \int_\Gamma \{f_1(t, x) |\frac{\partial}{\partial \nu} z_t|^2 + f_2(t, x) \frac{\partial}{\partial \tau} z_t|^2 + g(t, x) z_t^2\} d\Gamma dt.
\]
(5.9)

**Step 4:** From the energy identity derived in an identical manner as in Lemma 2.1, we infer the identity,
\[
E_z(t) + \int^t_s \int_\Gamma \{f_1(t, x) |\frac{\partial}{\partial \nu} z_t|^2 + f_2(t, x) \frac{\partial}{\partial \tau} z_t|^2 + g(t, x) z_t^2\} d\Gamma dt = E_z(s), \quad t \geq s.
\]
(5.10)

Applying (5.9) and (5.10) with $t = T + s$ gives
\[
\int_s^{T+s} E_z(t) dt \leq C_T(M, m) \{E_z(s) - E_z(T + s)\},
\]
and since the energy is nonincreasing,
\[
TE_z(T + s) \leq C_T(M, m) \{E_z(s) - E_z(T + s)\}.
\]
(5.11)

\[
\iff \frac{T}{C_T(M, m)} E_z(T + s) + E_z(T + s) \leq E_z(s) \quad \forall s \geq 0.
\]
(5.12)

**Step 5:** Reiterating the above argument on the intervals
\[
(lT + s, (l + 1)T + s), \quad l = 1, 2, ..., \]
we obtain
\[
E_z((l + 1)T + s) + pE_z((l + 1)T + s) = E_z(lT + s), \quad l = 0, 1, ..., \]
(5.13)

where $p \equiv \frac{T}{C_T(M, m)}$ does not depend on $s > 0$. Thus, we are in a position to apply Lemma 3.3 in [6]
with \( s_i \equiv E_x(s + lT) \). From the conclusion of Lemma 3.3 in [6], we infer
\[
E_x(s + t) \leq CS(t),
\]
(5.14)
where \( \mathcal{S}(t) \) satisfies the differential equation
\[
\mathcal{S}_t(t) + \frac{p}{p+1} \mathcal{S}(t) = 0 \\
\mathcal{S}(0) = E_x(s).
\]
(5.15)
Since \( \frac{p}{p+1} \equiv \omega_1 > 0 \), we obtain from (5.14),
\[
||Z(s + t,)(z_0, z_1)||_{H^2(\Omega) \times H^1(\Omega)} \leq C e^{-\omega_1 t} ||(z_0, z_1)||_{H^2(\Omega) \times H^1(\Omega)}.
\]
This yields the conclusion in Lemma 5.1. \( \square \)

In order to study properties of equation (4.12), it is convenient to reformulate this equation within an evolution framework. To accomplish this, we shall introduce the following linear operators:
\[
A : L_2(\Omega) \rightarrow L_2(\Omega) \\
Au \equiv \Delta^2 u; \ u \in \mathcal{D}(A) \equiv \{ u \in H^4(\Omega); \Delta u + (1 - \mu)B_1 u = 0, \ \frac{\partial}{\partial \nu} \Delta u + (1 - \mu)B_2 u - u = 0 \text{ on } \Gamma \}. \tag{5.16}
\]
\[
A_N : L_2(\Omega) \rightarrow L_2(\Omega) \\
A_N u \equiv -\Delta u; \ u \in \mathcal{D}(A_N) \equiv \{ u \in H^2(\Omega); \frac{\partial}{\partial \nu} u_t = 0 \}. \tag{5.17}
\]
It is well known that both \( A \) and \( A_N \) are positive, self-adjoint operators on \( L_2(\Omega) \).

Define \( G_i : H^{-1}(\Gamma) \rightarrow L_2(\Omega), \ i = 1, 2 \), by
\[
G_{ig} \equiv v_i \iff \Delta^2 v_i = 0 \text{ in } \Omega \tag{5.18}
\]
and
\[
\Delta v_1 + (1 - \mu)B_1 v_1 = g; \ \frac{\partial}{\partial \nu} \Delta v_1 + (1 - \mu)B_2 v_1 - v_1 = 0 \text{ on } \Gamma, \tag{5.19}
\]
\[
\Delta v_2 + (1 - \mu)B_1 v_2 = 0; \ \frac{\partial}{\partial \nu} \Delta v_2 + (1 - \mu)B_2 v_2 - v_2 = g \text{ on } \Gamma. \tag{5.20}
\]

From elliptic theory, it follows that
\[
G_1 \in \mathcal{L}(L_2(\Gamma); H^{5/2}(\Omega)) \subset \mathcal{L}(L_2(\Gamma); \mathcal{D}(A^{1/2}), \tag{5.21}
\]
\[
G_2 \in \mathcal{L}(H^{-1}(\Gamma); H^{5/2}(\Omega)) \subset \mathcal{L}(H^{-1}(\Gamma); \mathcal{D}(A^{1/2}). \tag{5.22}
\]
Define \( H \equiv H^2(\Omega) \times H^1(\Omega) \), where \( H \) is endowed with the topology
\[
||u||_H^2 \equiv ||A^{1/2} u||_{L_2(\Omega)}^2 + ||(1 + \gamma^2 A_N)^{1/2} u||_{L_2(\Omega)}^2. \tag{5.23}
\]

\( A(t) : H \rightarrow H \) is defined by:
\[
A(t) \left( \begin{array}{c} u \\ v \end{array} \right) = \left\{ (1 + \gamma^2 A_N)^{-1} A[u - G_1 f_1(t, \cdot) \frac{\partial}{\partial \nu} v - G_2 g(t, \cdot) v - G_2 \frac{\partial}{\partial \tau} f_2(t, \cdot) \frac{\partial}{\partial \tau} v], \right\} \tag{5.24}
\]
where

\[\mathcal{D}(A(t)) = \{(u, v) \in H^2(\Omega) \times H^2(\Omega) : G_1 f_1(t) \frac{\partial}{\partial \nu} v + G_2 g(t) v + G_2 \frac{\partial}{\partial \tau} f_2(t) \frac{\partial}{\partial \tau} v \in \mathcal{D}(A^{3/4}) \} \]

Note that \( \mathcal{D}(A(t)) \) depends on \( t \). It can be easily verified that \( \forall t \geq 0, A(t) \) is dissipative in \( H \) equipped with the inner product as in (5.23).

With the above notation, equation (5.1) can be rewritten as an evolution
\[
\left\{ \begin{array}{l}
\frac{d}{dt} Z(t, s) = A(t) Z(t, s) \\
Z(t, s) = (z_0, z_1) \in H.
\end{array} \right. \tag{5.25}
\]

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In order to represent the solution to (4.12) via an evolution operator, we introduce the operator \( B : H^{-1}(\Gamma) \to (\mathcal{D}(A(t)))' \) (where duality is understood with respect to the \( H \) inner product),

\[
B h \equiv \begin{pmatrix} 0 \\ (1 + \gamma^2 A_N)^{-1} A G_1 h \end{pmatrix}.
\] (5.26)

Note that the operator \( B \) is unbounded on \( H \equiv H^2(\Omega \times H^1(\Omega)). \) However, it is bounded in a weaker topology of \( \mathcal{D}(A(t))' \). Indeed, let \( \phi \in \mathcal{D}(A(t)) \). Then

\[
(Bh, \phi)_H = ((1 + \gamma^2 A_N)^{-1} A G_1 h, \phi)_H = (AG_1 h, \phi)_L^2(\Gamma) = (AG_1 h, \phi)_L^2(\Gamma) 
\leq C ||h||_{H^{-1}(\Gamma)} ||\phi||_{\mathcal{D}(A(t))} \leq C ||h||_{H^{-1}(\Gamma)} ||\phi||_{\mathcal{D}(A(t))},
\]

where we have used (5.22). The above result allows us to define a family of operators,

\[
L_s : L_2(s, T; H^{-1}(\Gamma)) \to C(s, T; \mathcal{D}(A(t))'),
\] (5.27)
as follows,

\[
(L_s h)(t) \equiv \int_s^t Z(t, \tau) B h(\tau) d\tau, \quad t \geq s \geq 0.
\] (5.28)

We shall prove that, although \( B \) is unbounded in \( H \), the operator \( L_s \) is bounded.

**Lemma 5.2** For any \( T > 0 \),

\[
||L_s h||_{L_\infty(s, T; H)} \leq C ||h||_{L_2(s, T; H^{-1}(\Gamma))}.
\]

The constant \( C \) does not depend on \( T \).

**Proof:** Define \( \bar{y}(t) \equiv (L_s g)(t) \). It can be easily shown that \( \bar{y}(t) = (y(t), y_t(t)) \), where \( y(t) \) is a solution (in \( \mathcal{D}'(Q_x) \)) of the following second order equation

\[
\begin{align*}
\frac{\partial}{\partial t} y_{tt} - \gamma^2 \Delta y_{tt} + \Delta^2 y &= 0 &\text{in } Q_x \equiv \Omega \times (s, T) \\
y(t) = y_t(t) &= 0 &\text{in } \Omega \\
\Delta y + (1 - \mu) B_1 y &= -f_1(t, \cdot, \frac{\partial}{\partial n} y_t) &\text{on } \Sigma_x \equiv \Gamma \times (s, T) \\
\frac{\partial}{\partial \sigma} \Delta y + (1 - \mu) B_2 y &= y - g(t, \cdot) y_t - \frac{\partial}{\partial \sigma} \{f_2(t, \cdot, \frac{\partial}{\partial n} y_t) \} &\text{on } \Sigma_x.
\end{align*}
\] (5.29)

Multiplying equation (5.29) by \( y_t \) and integrating by parts yields

\[
\frac{d}{dt} \left\{ ||y(t)||^2_{L^2(\Omega)} + \gamma^2 ||\nabla y(t)||^2_{L^2(\Omega)} + a(y(t), y(t)) \right\} + 2 \int_\Gamma \left\{ f_1(t) (\frac{\partial}{\partial n} y_t) + g(t) y_t^2 + f_2(t) (\frac{\partial}{\partial \sigma} y_t) \right\} d\Gamma = 2 \int_\Gamma h y_t d\Gamma.
\] (5.30)

(This procedure can be rigorously justified by first taking \( h \) smooth and passing through to the limit. See [5].) Integrating from \( s \) to \( t \), \( s \leq t < T \), yields

\[
\begin{align*}
||y(t)||^2_{L^2(\Omega)} + \gamma^2 ||\nabla y(t)||^2_{L^2(\Omega)} + a(y(t), y(t)) + 2 \int_s^t \int_\Gamma \left\{ f_1(t) (\frac{\partial}{\partial n} y_t) + g(t) y_t^2 + f_2(t) (\frac{\partial}{\partial \sigma} y_t) \right\} d\Gamma d t 
&\leq 2 \int_s^t \int_\Gamma h y_t d\Gamma d t.
\end{align*}
\] (5.31)

Hence,

\[
||y(t)||^2_{H^1(\Omega)} + ||y(t)||^2_{H^2(\Omega)} \leq C \int_s^t ||y||^2_{H^{-1}(\Gamma)} d t + \epsilon \int_s^t ||y||^2_{H^{-1}(\Gamma)} d t + \epsilon \int_s^t ||y||^2_{H^1(\Gamma)} d t,
\] (5.32)

where we have used conditions (4.15)-(4.17). Taking \( \epsilon \) sufficiently small in (5.32) yields

\[
||y(t)||^2_{H^1(\Omega)} + ||y(t)||^2_{H^2(\Omega)} \leq \int_s^t ||\nabla y(t)||^2_{L^2(\Omega)} d t + C \int_s^t ||y||^2_{H^{-1}(\Gamma)} d t,
\] (5.33)

which implies

\[
||L_s h||_{L_\infty(s, T; H^{-1}(\Gamma))} \leq C ||h||_{L_2(s, T; H^{-1}(\Gamma))}.
\]
Lemma 5.3  For any $0 < \omega < \min(w_0, w_1)$,

$$\sup_{t \geq 0} \|e^{\omega t}(L_0 h)(t)\|_H \leq C \sup_{t \geq 0} \|e^{\omega t}h(t)\|_{H^{-1}(\Gamma)}. \quad (5.34)$$

Proof: We write

$$e^{\omega t}(L_0 h)(t) = \int_{t_0}^t e^{\omega (t-\tau)} Z(t, \tau) Be^{\omega \tau} h(\tau) d\tau = \sum_{l=0}^{n} \int_{t_0}^{(l+1)t_0} \int_{t_0}^{(l+1)t_0} e^{\omega (t-\tau)} Z(t, \tau) Be^{\omega \tau} h(\tau) d\tau. \quad (5.34)$$

where $(n+1)t_0 = t$ and $t_0$ is fixed. By using the evolution property,

$$\int_{t_0}^{(l+1)t_0} e^{\omega (t-\tau)} Z(t, \tau) Be^{\omega \tau} h(\tau) d\tau = e^{\omega (t-(l+1)t_0)} Z(t, (l+1)t_0) \int_{t_0}^{(l+1)t_0} e^{\omega (l+1)t_0-\tau)} Z((l+1)t_0, \tau) Be^{\omega \tau} h(\tau) d\tau. \quad (5.34)$$

From Lemma 5.1,

$$\| \int_{t_0}^{(l+1)t_0} e^{\omega (t-\tau)} Z(t, \tau) Be^{\omega \tau} h(\tau) d\tau \|_H \leq e^{\omega (t-(l+1)t_0)} e^{\omega (l+1)t_0-\tau)} \|Z((l+1)t_0, \tau) Be^{\omega \tau} h(\tau) d\tau \|_H \leq C_1 e^{\omega (t-\omega_1) t_0} \|L_{t_0}(he^{\omega})((l+1)t_0)\|_H, \quad (5.35)$$

where the second inequality follows from Lemma 5.2 and we denoted

$$(L_0 h)(t) \equiv \int_{t}^{t} e^{\omega (t-\tau)} Z(t, \tau) Bh(\tau) d\tau. \quad (5.36)$$

Notice that from Lemma 5.1 and Lemma 5.2, it follows that

$$\|(L_0 h)(t)\|_H \leq e^{C \omega t/ \omega_1} \sup_{s \leq \tau \leq t} \|(L_0 h)(\tau)\|_H \leq C \|h\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))}. \quad (5.37)$$

Indeed, let $y(t) \equiv (L_0 h)(t)$. Then

$$y(t) = \int_{t}^{t} Z(t, \tau) Bh(\tau) d\tau + \omega_2 \int_{t}^{t} Z(t, \tau) y(\tau) d\tau. \quad (5.38)$$

Hence, for $t \leq T$, applying Lemma 5.1 and Lemma 5.2,

$$\|y(t)\|_H \leq \|L_0 h\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} + \omega_2 \int_{t}^{t} \|Z(t, \tau)\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} \|y(\tau)\|_H d\tau \leq C \|h\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} + \omega_2 \int_{t}^{t} C_1 e^{\omega (t-\tau)} \|y(\tau)\|_H d\tau \leq C \|h\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} + \omega_2 \int_{t}^{t} C_1 e^{\omega (t-\tau)} \|y(\tau)\|_H d\tau. \quad (5.39)$$

Applying Gronwall’s Lemma yields

$$\|y(t)\|_H \leq C \|h\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} e^{C_1 \omega t/ \omega_1}, \quad t \leq T,$$

which proves (5.37). Combining (5.35) and (5.37) gives

$$\| \int_{t_0}^{(l+1)t_0} e^{\omega (t-\tau)} Z(t, \tau) Be^{\omega \tau} h(\tau) d\tau \|_H \leq C e^{\omega (t-\omega_1) t_0} \|h e^{\omega}((l+1)t_0)\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))}. \quad (5.38)$$

Recalling (5.34),

$$\sup_{t \geq 0} \|e^{\omega t}(L_0 h)(t)\|_H \leq C \sum_{l=0}^{n} e^{(\omega_2-\omega_1)(t-(l+1)t_0)} \|h e^{\omega}((l+1)t_0)\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} \|h e^{\omega}((l+1)t_0)\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} \|h e^{\omega}((l+1)t_0)\|_{L_2(\mathbb{R}, \Gamma; H^{-1}(\Gamma))} \sum_{l=0}^{n} e^{(\omega_2-\omega_1)(n-l)t_0}. \quad (5.39)$$
Since $\omega_2 < \min \{\omega_0, \omega_1\}$, \(\sum_{i=0}^{n} \frac{e^{(\omega_2 - \omega_1)(n-i)t_0}}{i!} < \infty\) and
\[
\|e^{w_2t} \|_{L^2(0, \infty; H^{-1}(\Gamma))} \leq C \sup_{t \geq 0} \|e^{\omega_0(t)}\|_{H^{-1}(\Gamma)},
\]
(5.40)
which concludes the proof of the lemma. □

Now we are in a position to complete the proof of Theorem 4.2. The solution to (4.12) can be written via an evolution operator as
\[
\begin{pmatrix}
  u \\
  u_t
\end{pmatrix}
(t) = Z(t, 0)
\begin{pmatrix}
  u_0 \\
  u_1
\end{pmatrix}
+ \int_0^t Z(t, \tau) Bl(\tau)d\tau + \int_0^t Z(t, \tau)
\begin{pmatrix}
  0 \\
  K(\tau)
\end{pmatrix}
\]d\tau.
(5.41)

From Lemma 5.1 and (4.13),
\[
\|Z(t, 0)
\begin{pmatrix}
  u_0 \\
  u_1
\end{pmatrix}
+ \int_0^t Z(t, \tau)
\begin{pmatrix}
  0 \\
  K(\tau)
\end{pmatrix}
\]d\tau\|_H \leq C_1 e^{-\omega_1 t}\|\|(u_0, u_1)\|_H + C_1 C_0 \int_0^t e^{-\omega_1(t-\tau)}e^{-\omega_0 \tau}d\tau
\leq Ce^{-\omega t}\|\|(u_0, u_1)\|_H + C_0,
(5.42)
where $w < \min \{\omega_0, \omega_1\}$.

As for the second term on the right-hand side of (5.3), we apply Lemma 5.3 to find
\[
e^{\omega_2 t}\|\int_0^t Z(t, \tau) Bl(\tau)d\tau\|_H \leq e^{\omega_2 t}\|\|L_0(t)\|_H
\leq C \sup_{t \geq 0} \|e^{\omega_0 t}L(t)\|_{H^{-1}(\Gamma)} \leq CC_0.
(5.43)

This proves the conclusion of Theorem 4.2 with $\omega < \min \{\omega_0, \omega_1\}$. □

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