VISCOSITY DEPENDENT BEHAVIOUR
OF VISCOELASTIC POROUS MEDIA

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VISCOSITY DEPENDENT BEHAVIOUR OF VISCOELASTIC POROUS MEDIA

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Abstract. In this paper we study the convergence process for the homogenization problem of a mixture formed by a viscous incompressible fluid and a linear viscoelastic solid. Depending on the order of magnitude of the viscosity coefficient, different homogenized behaviours can be obtained. By using a compactness result of Nguetseng [6] we are able to obtain in an unitary way the convergence proofs for a viscosity having a magnitude of the form $O(\varepsilon^{2\gamma})$ for $\gamma \geq 0$, recovering thus results obtained in [3] and [11] for $\gamma = 0$ and $\gamma = 1$, together with the transition behaviour $0 < \gamma < 1$, and the behaviour for $\gamma > 1$.

1. Introduction.

The motion of a mixture formed by a viscous incompressible fluid and a linear viscoelastic solid is considered, in the general framework of homogenization. The solid and the fluid part of the mixture are supposed to be periodically distributed with a small period.

Generally, different behaviours can be obtained for various orders of magnitude of the coefficients. Thus, the limit displacement and velocity may or may not depend on the local variables. In the case of the mixture of an elastic solid and a viscous fluid, having viscosity $O(1)$, it was shown in [1] that the homogenized behaviour is given by an elastic solid with non-instantaneous memory. Analogous results as in [1] can be obtained if the solid is viscoelastic [11].

A totally different behaviour is obtained in the case of a slightly viscous fluid. This problem, for a fluid having a viscosity of order $O(\varepsilon^2)$, was studied in [3] (see also [4]) by using matched asymptotic expansions. The homogenized behaviour is described by a mixture of a viscoelastic solid with memory and a fluid that satisfies a Darcy law with memory.

In this paper we study the convergence process for a viscosity of the form $\mu = O(\varepsilon^{2\gamma})$, for $\gamma \geq 0$, by using the compactness result of Nguetseng [6]. We are thus able to recover the results of [11] for $\gamma = 0$, to prove the convergence for the asymptotics in [3] for $\gamma = 1$, and also to obtain the transition behaviour for $0 < \gamma < 1$, and the behaviour for $\gamma > 1$.

We obtain that there is a critical value for $\gamma = 1$ that separates two different global behaviours. Thus for values $\gamma < 1$ the homogenized behaviour is given by a viscoelastic law

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with non-instantaneous memory, whereas for $\gamma \geq 1$ the homogenized behaviour is described by a mixture of a viscoelastic solid, with memory, and an ideal fluid that satisfies a Darcy law with memory.

The problem is formulated in section 2. and the formal asymptotic results are obtained in section 3.. The last two sections are devoted to proving the convergence results.

2. A mixture of a linear viscoelastic solid and a viscous fluid.

Let us suppose that we have a periodic mixture of a linear viscoelastic solid with a viscous fluid, the solid part being connected.

If the open, bounded domain occupied by the mixture, in the reference configuration is denoted by $\Omega$, $\Omega \subset \mathbb{R}^3$, and the solid (resp. fluid part) is denoted by $\Omega_{\varepsilon}$ (resp. $\Omega_{f\varepsilon}$) we consider them periodic in the following sense:

\begin{align}
\bar{\Omega} &= \bar{\Omega}_{\varepsilon} \cup \bar{\Omega}_{f\varepsilon} \\
\Omega_{\varepsilon} &= \{ x \in \Omega / x = \varepsilon(n + y), y \in Y_s \cup (\partial Y_s \cap \partial Y), n \in \mathbb{Z}^3 \} \\
\Omega_{f\varepsilon} &= \{ x \in \Omega / x = \varepsilon(n + y), y \in Y_f \cup (\partial Y_f \cap \partial Y), n \in \mathbb{Z}^3 \}
\end{align}

where:

\begin{align}
Y &= \left( -\frac{1}{2}, \frac{1}{2} \right)^3, \quad Y_s, Y_f \subset Y, \quad \bar{Y} = \bar{Y}_s \cup \bar{Y}_f, \quad \Gamma = \bar{Y}_s \cap \bar{Y}_f
\end{align}

with $Y_s, Y_f$ open, connected, disjoint and $\Gamma$ smooth.

Then the behaviour of the mixture is given by:

\begin{align}
\begin{cases}
\rho^s \frac{\partial^2 u^s}{\partial t^2} - \frac{\partial \sigma^s_{ii}}{\partial x_i} = f_i & \text{in } \Omega_{\varepsilon} \times (0, T) \\
\rho^f \frac{\partial v^f}{\partial t} - \frac{\partial \sigma^f_{ii}}{\partial x_i} = f_i & \text{in } \Omega_{f\varepsilon} \times (0, T) \\
\text{div } v^\varepsilon = 0 & \text{in } \Omega_{f\varepsilon} \times (0, T) \\
\sigma^s_{ij} = a_{ijkh} e_{kh}(u^\varepsilon) + b_{ijkh}^s e_{kh}(v^\varepsilon) \\
\sigma^f_{ij} = -p^\varepsilon \delta_{ij} + 2\mu(\varepsilon) e_{ij}(v^\varepsilon) \\
v^\varepsilon = \frac{\partial u^\varepsilon}{\partial t}
\end{cases}
\end{align}

where $u, v, \sigma, f, p, \mu$ denote the displacement vector, the velocity, the stress tensor, the body forces, the pressure and respectively the viscosity.

We suppose that $a_{ijkh}, b_{ijkh}^s$ are real numbers that satisfy:

\begin{align}
a_{ijkh} &= a_{jikh} = a_{khij}, \quad b_{ijkh}^s = b_{jikh}^s = b_{khiij}^s \\
a_{ijkh} \xi_{ij} \xi_{kh} \geq \alpha \xi_{ij} \xi_{ij}, \quad b_{ijkh}^s \xi_{ij} \xi_{kh} \geq \beta \xi_{ij} \xi_{ij}
\end{align}

for all $\xi_{ij} \in \mathbb{R}, \xi_{ij} = \xi_{ji}$, where $\alpha, \beta > 0$. 

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We will also suppose that the viscosity \( \mu(\varepsilon) \) is of the form

\[
\mu(\varepsilon) = \varepsilon^{2\gamma} \mu, \quad \text{with } \mu > 0 \text{ and } \gamma \geq 0
\]  

The natural transmission conditions on the solid-fluid boundary are the continuity of the displacement and of the normal stress:

\[
[u^\varepsilon] = 0, \quad [\sigma^\varepsilon_{ij} n_j] = 0 \quad \text{on } \Gamma_\varepsilon
\]

where \( \Gamma_\varepsilon = \partial \Omega_{se} \cap \partial \Omega_{f\varepsilon} \) and \( n = (n_1, n_2, n_3) \) denotes the unitary normal, exterior to \( \Omega_{se} \).

The initial and boundary conditions are:

\[
u^\varepsilon(0) = \frac{\partial u^\varepsilon}{\partial t}(0) = 0, \quad u^\varepsilon = 0 \text{ on } \partial \Omega
\]

In order to give the variational formulation we introduce the following notations:

\[
w^\varepsilon(x) = w \left( \frac{x}{\varepsilon} \right), \text{ for any } Y\text{-periodic function}
\]

\[
a_{ijkh}(y) = \begin{cases} 0 & y \in Y_f \\ a_{ijkh} & y \in Y_s \end{cases}, \quad b^f_{ijkh}(y) = \begin{cases} b^f_{ijkh} & y \in Y_f \\ 0 & y \in Y_s \end{cases},
\]

\[
b^s_{ijkh}(y) = \begin{cases} 0 & y \in Y_f \\ b^s_{ijkh} & y \in Y_s \end{cases},
\]

\[
\rho(y) = \begin{cases} \rho^f & y \in Y_f \\ \rho^s & y \in Y_s \end{cases}, \quad e_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
\]

where \( b^f_{ijkh} = \mu(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}) \).

The variational formulation of the problem is:

\[
\text{find } u^\varepsilon, p^\varepsilon, \text{ functions of } t \text{ with values in } H^1_0(\Omega), L^2(\Omega_{f\varepsilon}) \text{ respectively, such that}
\]

\[
\int_\Omega \rho^s \frac{\partial^2 u^\varepsilon}{\partial t^2} w_i + \int_\Omega a^{\varepsilon}_{ijkh} e_{kh}(u^\varepsilon) e_{ij}(w) + \int_\Omega (b^{\varepsilon}_{ijkh} + \varepsilon^{2\gamma} b^{\varepsilon f}_{ijkh}) e_{kh} \left( \frac{\partial u^\varepsilon}{\partial t} \right) e_{ij}(w) -
\]

\[
- \int_{\partial \Omega_{f\varepsilon}} p^\varepsilon \text{ div } w = \int_{\Omega_{f\varepsilon}} f w, \quad \text{for all } w \in H^1_0(\Omega)
\]

\[
\text{div } \frac{\partial u^\varepsilon}{\partial t} = 0 \quad \text{in } \Omega_{f\varepsilon}
\]
The existence and uniqueness of a solution can be obtained either by semigroup results (as in [9]), or by reducing the problem, by means of the Laplace transform, to an elliptic one:

\[
\begin{align*}
\text{find } & \hat{u}^\varepsilon \in H_0^1(\Omega), \hat{p}^\varepsilon \in L^2(\Omega_{f,\varepsilon}), \text{div } \hat{u}^\varepsilon = 0 \text{ in } \Omega_{f,\varepsilon}, \\
\lambda^2 & \int_\Omega \hat{u}^\varepsilon w + \int_\Omega (a_{ij}^\varepsilon + \lambda b_{ij}^\varepsilon) e_{kh}(\hat{u}^\varepsilon) e_{ij}(w) - \int_{\Omega_{f,\varepsilon}} \hat{p}^\varepsilon \text{div } w = \int_\Omega \hat{f} w \\
& \text{for all } w \in H_0^1(\Omega), \quad Re\lambda > \lambda_0 > 0
\end{align*}
\]

where \( b_{ij}^\varepsilon = b_{ij}^{e,\varepsilon} + \varepsilon^2 \gamma b_{ij}^{f,\varepsilon} \). The coercivity of the left side on the space \( V_\varepsilon = H_0^1(\Omega) \cap \{ w/ \text{div } w = 0 \text{ in } \Omega_{f,\varepsilon} \} \) ensures, by the Lax-Milgram lemma, the existence and uniqueness of \( \hat{u}^\varepsilon \) and the existence and uniqueness of \( \hat{p}^\varepsilon \in L^2(\Omega_{f,\varepsilon})/R \) results from the characterization of the space \( V_{\varepsilon,1} \) (see [10]).

3. Asymptotic expansions. Due to the fact that the first order term in the expansion of the displacement in the solid part does not depend on the local variable \( y \), whereas it does in the fluid [8], we search for two-scale asymptotic expansions in the form:

\[
\begin{align*}
& u^\varepsilon(x,t) = u^0(x,t) + u^r(x,y,t) + \varepsilon u^1(x,y,t) + \ldots \\
& p^\varepsilon(x,t) = p^0(x,t) + \varepsilon p^1(x,y,t) + \ldots
\end{align*}
\]

where \( y = \frac{x}{\varepsilon} \) and \( u^r, u^1, p^1, \ldots \) are \( Y \)-periodic in \( y \). The first term in the expansion of the velocity is dependent on the local variable and is split into two parts \( u^0 \) and \( u^r \) where \( u^r \) represents the relative displacement of the fluid with respect to the solid and thus \( u^r(x,y,t) = 0 \) for \( y \in Y_s \).

The balance of mass (2.15) yields that:

\[
\begin{align*}
\text{div}_y v^r &= 0 \\
\text{div}_x (v^0 + v^r) + \text{div}_y v^1 &= 0
\end{align*}
\]

(3.2)

By taking in (2.14) test functions of the form:

\[
w = \omega^\varepsilon(x)\phi(x), \quad \text{with } \omega \in V_Y, \quad \phi \in \mathcal{D}(\Omega)
\]

(3.3)

where \( V_Y = \{ w | w \in H^1(Y), w = 0 \text{ in } Y_s, \text{div } w = 0 \text{ in } Y_f \} \), we obtain the relative motion, when \( \varepsilon \to 0 \), given by:

i) if \( 0 \leq \gamma < 1 \)

\[
\int_{Y_f} \mu \frac{\partial v_i^r}{\partial y_j} \frac{\partial \omega_i}{\partial y_j} = 0, \quad \text{for all } \omega \in V_Y
\]

so that \( v^r = 0 \).
ii) if $\gamma = 1$

$$
\int_{Y_f} \rho^f \frac{\partial(v^0 + v^r)}{\partial t} \omega + \int_{Y_f} b_{ijkh}^f e_{khy}(v^r) e_{i,jy}(\omega) - \int_{Y_f} p^0 \text{div}_x \omega = \int_Y f \omega, \text{ for all } \omega \in V_Y.
$$

(3.4)

If we denote by $H_Y$ the completion of $V_Y$ for the norm associated with the scalar product

$$(u, w)_{H_Y} = \int_{Y_f} u_i w_i,$$

we obtain for $v^r$ the evolution problem:

$$(3.5) \quad \rho^f \frac{\partial v^r}{\partial t} + \mu A_1 v^r = \left( f_i - \frac{\partial p^0}{\partial x_i} - \rho^f \frac{\partial v^0_i}{\partial t} \right) \phi^i$$

$$v^r(0) = 0$$

where $A_1$ is the selfadjoint operator associated by the representation theorem with the form $(v, w)_{V_Y}$ and the vectors $\phi^i$ are defined by:

$$(3.6) \quad \int_{Y_f} \omega_i = (\phi^i, \omega)_{H_Y} \quad \text{for all } \omega \in H_Y$$

Than $v^r$ can be expressed in the form:

$$(3.7) \quad v^r = \frac{1}{\rho^f} \int_0^t e^{-\frac{t-s}{\tau} \mu A_1(t-s)} \phi^i \left( f_i - \frac{\partial p^0}{\partial x_i} - \rho^f \frac{\partial v^0_i}{\partial t} \right) (s) ds.$$

iii) if $\gamma > 1$

$$(3.8) \quad \rho^f \frac{\partial v^r}{\partial t} = \left( f_i - \frac{\partial p^0}{\partial x_i} - \rho^f \frac{\partial v^0_i}{\partial t} \right) \phi^i$$

and $v^r$ has the form:

$$(3.9) \quad v^r = \frac{1}{\rho} \int_0^t \phi^i \left( f_i - \frac{\partial p^0}{\partial x_i} - \rho^f \frac{\partial v^0_i}{\partial t} \right) (s) ds$$
For obtaining the form of the $\varepsilon$ term in the expansion of the displacement, we take in (2.14) test functions of the form:

\begin{equation}
(3.10) \quad w = \varepsilon \omega^\varepsilon(x) \phi(x), \quad \text{with } \omega \in V \gamma, \quad \phi \in D(\Omega).
\end{equation}

We thus obtain by matching the powers of $\varepsilon$:

\begin{equation}
(3.11) \quad \int_{\Omega} a_{ij\varepsilon h}(e_{kk\varepsilon} u^0 + e_{kk\varepsilon} u^1)(e_{ij\varepsilon} \omega)^\varepsilon \phi + \\
+ \int_{\Omega} (b_{ij\varepsilon k} + \varepsilon^2 b_{ij\varepsilon k}^\varepsilon)(e_{kk\varepsilon} \left( \frac{\partial u^0}{\partial t} \right) + e_{kk\varepsilon} \left( \frac{\partial u^1}{\partial t} \right))(e_{ij\varepsilon} \omega)^\varepsilon \phi + \\
+ \varepsilon^{2\gamma-1} \int_{\Omega^f} b_{ij\varepsilon k} e_{kk\varepsilon} v^\gamma(e_{ij\varepsilon} \omega)^\varepsilon \phi - \int_{\Omega^f} p^0(\text{div}_\gamma \omega)^\varepsilon \phi + O(\varepsilon) = 0
\end{equation}

where $e_{ij\varepsilon} \omega^\varepsilon = \varepsilon^{-1}(e_{ij\varepsilon} \omega)^\varepsilon$, in the sense of the extension (2.11).

The third integral, that contains the relative velocity, will always vanish for $\varepsilon \to 0$, because for $\gamma < 1$, $v^\gamma$ vanishes, and for $\gamma \geq 1$ the term is of order $O(\varepsilon)$.

The form of $u^1$, in the expansion of the displacement, will be different depending if $\gamma = 0$ or $\gamma > 0$.

i) if $\gamma = 0$ we obtain for $\varepsilon \to 0$:

\begin{equation}
(3.12) \quad \int_{\tilde{V}} \left( a_{ij\varepsilon h} + b_{ij\varepsilon k} \frac{\partial}{\partial t} \right) \left( \frac{\partial u_k}{\partial x_h} \right) \frac{\partial \omega_i}{\partial y_j} + \int_{\tilde{V}} \left( a_{ij\varepsilon k} + b_{ij\varepsilon k} \frac{\partial}{\partial t} \right) \left( \frac{\partial u_k}{\partial y_h} \right) \frac{\partial \omega_i}{\partial y_j} - \int_{\tilde{V}} p^0 \text{div}_\gamma \omega = 0
\end{equation}

where $b_{ij\varepsilon k} = b_{ij\varepsilon k}^f + b_{ij\varepsilon k}^s$

If $\tilde{V}$ is the subspace of $H^1(\gamma)$ containing functions with zero mean value, having the scalar product:

\begin{equation}
(3.13) \quad (u, v)_{\tilde{V}} = \int_{\tilde{V}} b_{ij\varepsilon k} \frac{\partial u_i}{\partial y_j} \frac{\partial v_k}{\partial y_h}
\end{equation}

and we define $A_2 \in \mathcal{L}(\tilde{V}, \tilde{V})$, $m^{kk} \in \tilde{V}$, $n^{kk} \in \tilde{V}$, $\psi \in \tilde{V}$, for all $u, v \in \tilde{V}$ by the following:

\begin{equation}
(3.14) \quad (A_2 u, v) = \int_{\tilde{V}} a_{ij\varepsilon k} \frac{\partial u_i}{\partial y_j} \frac{\partial v_k}{\partial y_h}
\end{equation}

\begin{equation}
(3.15) \quad (m^{kk} v, v) = \int_{\tilde{V}} a_{ij\varepsilon k} \frac{\partial v_i}{\partial y_j}
\end{equation}

\begin{equation}
(3.16) \quad (n^{kk} v, v) = \int_{\tilde{V}} b_{ij\varepsilon k} \frac{\partial v_i}{\partial y_j}
\end{equation}

\begin{equation}
(3.17) \quad (\psi, v) = \int_{\tilde{V}} \text{div} v
\end{equation}
then the problem (3.12) can be expressed in the form:

\[
\frac{\partial u^1}{\partial t} + A_2 u^1 = - \left( m^{kh} + n^{kh} \frac{\partial}{\partial t} \right) \frac{\partial u_k^0}{\partial x_h} - p^0 \psi \\
u^1(0) = 0
\]

and therefore:

\[
u^1 = -n^{kh} \frac{\partial u_k^0}{\partial x_h} - \int_0^t e^{-A_2(t-s)} \left( A_2 n^{kh} - m^{kh} \right) \frac{\partial u_k^0}{\partial x_h} - p^0 \psi \right) (s) ds
\]

ii) if \( 0 < \gamma \), from (3.11), for \( \varepsilon \to 0 \) we get:

\[
\int_{Y_\varepsilon} \left( a_{ijkh} + b_{ijkh}^* \frac{\partial}{\partial t} \right) \frac{\partial u_k^0}{\partial x_h} \frac{\partial \omega_i}{\partial y_j} + \\
+ \int_{Y_\varepsilon} \left( a_{ijkh} + b_{ijkh}^* \frac{\partial}{\partial t} \right) \frac{\partial u_k^1}{\partial x_h} \frac{\partial \omega_i}{\partial y_j} - \int_{Y^0} p^0 \text{div}_y \omega = 0
\]

and therefore \( u^1 \) can be put in a form analogous to (3.19), where the integrals in (3.14)–(3.16) are on \( Y_\varepsilon \).

The homogenized stress tensor can be obtained from (2.5)_4 and (2.5)_5. Thus:

i. if \( \gamma = 0 \)

\[
\sigma_{ij}^0 = \int_{Y_\varepsilon} \left( a_{ijkh} + b_{ijkh}^* \frac{\partial}{\partial t} \right) \left( \frac{\partial u_k^0}{\partial x_h} + \frac{\partial u_k^1}{\partial y_h} \right)
\]

ii. if \( \gamma > 0 \)

\[
\sigma_{ij}^0 = \int_{Y_\varepsilon} \left( a_{ijkh}^* + b_{ijkh}^* \frac{\partial}{\partial t} \right) \left( \frac{\partial u_k^0}{\partial x_h} + \frac{\partial u_k^1}{\partial y_h} \right)
\]

By using the expression (3.19), for \( u^1 \), and the analogous form for the case \( \gamma > 0 \), the homogenized stress is obtained in the form:

\[
\sigma_{ij}^0 = \alpha_{ijkh}(u^0) + \beta_{ijkh}(\frac{\partial u^0}{\partial t}) - \alpha_{ij}p^0 + \\
+ \int_0^t g_{ijkh}(t-s)e_{kh}(u^0)(s)ds + \int_0^t g_{ij}(t-s)p^0(s)ds
\]
where:

\[
\alpha_{ikh} = \left( a_{ikh} - a_{ijm} \frac{\partial}{\partial y_m} n_i^{kh} + b_{ijm} \frac{\partial}{\partial y_m} r_k^{kh} \right) -
\]

\[
\beta_{ikh} = \left( b_{ikh} - b_{ijm} \frac{\partial}{\partial y_m} n_i^{kh} \right) -
\]

\[
\alpha_{ij} = \left( b_{ikh} \frac{\partial \psi_k}{\partial y_h} \right) -
\]

\[
g_{ikh}(s) = a_{ijm} \frac{\partial}{\partial y_m} (e^{-A_2 \phi} r_i^{kh}) + b_{ijm} \frac{\partial}{\partial y_m} (e^{-A_2 \phi} r_i^{kh}) -
\]

\[
g_{ij}(s) = a_{ijm} \frac{\partial}{\partial y_m} (e^{-A_2 \phi}) + b_{ijm} \frac{\partial}{\partial y_m} (e^{-A_2 \phi}) -
\]

\[
r^{kh} = A_2 n^{kh} - m^{kh}
\]

where \( \sim \) denotes the average on \( Y \):

\[
\bar{z} = \frac{1}{|Y|} \int_Y dy
\]

The constitutive equation (3.23) contains an elastic term, a viscoelastic term with instantaneous memory, and two long memory terms that decay exponentially.

The macroscopic equation can be obtained from the balance of momentum when \( \varepsilon \to 0 \)

\[
\int_{\Omega} \left( \frac{\partial^2 u_i^0}{\partial t^2} + n \rho_f \frac{\partial^2 \bar{u}_i}{\partial t^2} \right) w_i + \int_{\Omega} \sigma_{ij} \frac{\partial w_i}{\partial x_j} - n \int_{\Omega} \rho^{0} \text{div} w =
\]

\[
= \int_{\Omega} f_i w_i, \text{ for all } w \in H_0^1(\Omega)
\]

(3.30)

Thus the balance of momentum is:

i. if \( 0 \leq \gamma < 1 \)

\[
\tilde{\rho} \frac{\partial^2 u_i^0}{\partial t^2} - \frac{\partial \sigma_{ij}^T}{\partial x_j} = f_i
\]

(3.31)

ii. if \( 1 \leq \gamma \)

\[
\tilde{\rho} \frac{\partial^2 u_i^0}{\partial t^2} + n \rho_f \frac{\partial \bar{u}_i}{\partial t} - \frac{\partial \sigma_{ij}^T}{\partial x_j} = f_i
\]

(3.32)

where the total (effective) stress is given by:

\[
\sigma_{ij}^T = \sigma_{ij}^0 - n \rho^0 \delta_{ij}
\]

(3.33)
Thus the homogenized behaviour is given by:

i. a viscoelastic solid with non-instantaneous memory that satisfies the balance of momentum (3.31), the constitutive equation (3.23), if $0 \leq \gamma < 1$

ii. a mixture formed by a viscoelastic solid with non-instantaneous memory that satisfies the balance of momentum (3.32), the constitutive equation (3.23), and an ideal fluid that satisfies a Darcy law with memory (3.7) if $\gamma = 1$ or (3.9) if $1 < \gamma$ and the balance of mass, obtained from (3.2):

\[(3.34.) \quad \text{div}_x \tilde{\sigma}^\varepsilon = -n \text{div}_x v^0 + \int \text{div}_y v^1_{\tilde{Y}}\]

4. Estimates.

We will first prove some estimates for the solutions of (2.16).

The norm in $L^2(D)$ will be denoted by $|| \cdot ||_D$ and the norm in $H^1(D)$ will be denoted by $|| \cdot ||_D$ and the various constants independent on $\varepsilon$ will all be denoted by $C$.

**Lemma 4.1.** The solution $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ of (2.16) satisfies the following estimates:

\[(4.1) \quad |\tilde{u}^\varepsilon|_{\Omega} < C, \quad |\partial_{x_i} \tilde{u}^\varepsilon|_{\Omega_{\varepsilon}} < C\]

\[(4.2) \quad \varepsilon^\gamma |\partial_{x_i} \tilde{u}^\varepsilon|_{\Omega_{\varepsilon}} < C, \quad \varepsilon^\gamma ||\tilde{u}^\varepsilon||_{\Omega} < C\]

\[(4.3) \quad |\tilde{p}^\varepsilon|_{\Omega_{\varepsilon}} < C\]

**Proof.** By taking $w = \tilde{u}^\varepsilon$ in (2.16) we get:

\[(4.4) \quad \lambda^2 \int_\Omega \rho(\tilde{u}^\varepsilon)^2 + \int_\Omega (a_{ijkh}^\varepsilon + \lambda b_{ijkh}^\varepsilon) e_{ij}(\tilde{u}^\varepsilon) e_{kh}(\tilde{u}^\varepsilon) = \int_\Omega \tilde{f}\tilde{u}^\varepsilon\]

and by retaining only the first term in the left member, the first estimate in (4.1) is obtained. The second estimate of (4.1) is obtained from (2.7), (4.1), and:

\[\int_{\Omega_{\varepsilon}} a_{ijkh} e_{ij}(\tilde{u}^\varepsilon) e_{kh}(\tilde{u}^\varepsilon) \leq ||\tilde{f}||_\Omega |\tilde{u}^\varepsilon|_\Omega\]

and an extension lemma (see [2], [7]).

To prove (4.2) we get from (4.4):

\[\varepsilon^{2\gamma} \int_{\Omega_{\varepsilon}} b_{ijkh}^\varepsilon e_{ij}(\tilde{u}^\varepsilon) e_{kh}(\tilde{u}^\varepsilon) + \int_{\Omega_{\varepsilon}} b_{ijkh}^\varepsilon e_{ij}(\tilde{u}^\varepsilon) e_{kh}(\tilde{u}^\varepsilon) \leq C\]
and use the coercivity of the coefficients.

Next using the surjectivity of the operator \( \text{div} : H^1_0(\Omega) \to L^2(\Omega)/\mathbb{R} \) (see [10] chapter 1, lemma 2.4 and [2] lemma 5.1) we choose \( w \in H^1_0(\Omega) \) such that:

\[
(4.5) \quad \text{div} \, w = \tilde{p}^\varepsilon, \quad \|w\| \leq C|\tilde{p}^\varepsilon|_{\Omega_{f,\varepsilon}}
\]

we get:

\[
|\tilde{p}^\varepsilon|_{\Omega_{f,\varepsilon}}^2 \leq \lambda^2 C|\tilde{u}^\varepsilon| \, |w| + C(1 + \lambda)\|\tilde{u}^\varepsilon\| \, \|w\| + C|w|
\]

and using (4.1), (4.2) and (4.5) we obtain (4.3). \( \Box \)

5. Convergence results.

5.1. General convergence results. In the sequel we will recall a compactness result, obtained in [6], that gives information on the convergence of the product of two \( L^2 \) weakly convergent sequences, one of which being periodic.

We will denote by \( L^2_p \) and \( H^1_p \) the spaces that contain the extensions by periodicity of functions belonging to \( L^2(Y) \), respectively to \( H^1(Y) \), and by \( \mathcal{X}(\Omega) \) the space of continuous functions with compact support in \( \Omega \). More precisely:

**Lemma 5.1.** Let \( v^\varepsilon \in L^2(\Omega) \) be bounded:

\[
|v^\varepsilon|_{\Omega} \leq C
\]

Then there exists a subsequence and there exists \( v \in L^2(\Omega, L^2_p) \), such that for any \( w \in L^2_p \) and \( \phi \in \mathcal{X}(\Omega) \) we have:

\[
(5.1) \quad \int \frac{v^\varepsilon w^\varepsilon \phi}{\Omega} \to \int \frac{v(x,y)w(y)\phi(x)}{\Omega \times Y}
\]

Using the result above, and having in mind the estimates obtained in the previous section, we can prove the following:

**Theorem 5.2.** Let \( v^\varepsilon \in H^1(\Omega) \) such that:

\[
(5.2) \quad |v^\varepsilon|_{\Omega} < C, \quad \left| \frac{\partial v^\varepsilon}{\partial x_i} \right|_{\Omega_{f,\varepsilon}} < C, \quad \varepsilon \gamma \left| \frac{\partial v^\varepsilon}{\partial x_i} \right|_{\Omega_{f,\varepsilon}} < C
\]

then there exists a subsequence, such that for all \( w \in L^2_p \) and all \( \phi \in \mathcal{X}(\Omega) \):

\[
(5.3) \quad \int \frac{v^\varepsilon w^\varepsilon \phi}{\Omega} \to \int \frac{(v^0(x) + v^r(x,y))w(y)\phi(x)}{\Omega \times Y}
\]

\[
(5.4) \quad \int \frac{\partial v^\varepsilon}{\partial x_i} w^\varepsilon \phi {\Omega_{f,\varepsilon}} \to \int \frac{\left( \frac{\partial v^0}{\partial x_i}(x) + \frac{\partial v^1}{\partial y_i}(x,y) \right) w(y)\phi(x)}{\Omega \times Y_{f,\varepsilon}}
\]

\[
(5.5) \quad \varepsilon \gamma \int \frac{\partial v^\varepsilon}{\partial x_i} w^\varepsilon \phi {\Omega_{f,\varepsilon}} \to \int \frac{V_i(x,y)w(y)\phi(x)}{\Omega \times Y_{f,\varepsilon}}
\]
where \( v^r = 0 \) for \( 0 \leq \gamma < 1 \) and:

\[
V_i(x, y) = \begin{cases} \frac{\partial v^r}{\partial y_i}(x, y) & \text{for } \gamma = 1 \\ 0 & \text{for } \gamma > 1 \\ \frac{\partial u}{\partial y_i}(x, y) & \text{for } 0 < \gamma < 1 \end{cases}
\]

(5.6)

with \( v^0 \in H^1(\Omega), v^r \in L^2(\Omega, L^2_p), v^r = 0 \) a.e. in \( Y_s, v^1 \in L^2(\Omega, H^1_p), u \in L^2(\Omega, H^1_p(Y_f)) \).

Proof. From the previous lemma there exists a subsequence and \( v \in L^2(\Omega, L^2_p) \) that satisfies (5.1). We next define \( z_i^\varepsilon = \frac{\partial v^\varepsilon}{\partial x_i} \) in \( \Omega_{se}, z_i^\varepsilon = 0 \) in \( \Omega_{fe} \). It is obvious that \( (z_i^\varepsilon) \) is bounded in \( L^2 \) and by applying again the lemma, we get:

\[
\int_{\Omega} z_i^\varepsilon w^\varepsilon \phi \to \int_{\Omega \times Y} z_i^0(x, y) w(y) \phi(x)
\]

By taking test functions \( w \) that vanish on \( Y_s \) we get that \( z_i^0 = 0 \) in \( Y_f \).

We prove next that for \( y \in Y_s, v \) does not depend on \( y \). Indeed for \( w \in \mathcal{D}(Y_s) \):

\[
\varepsilon \int_{\Omega} z_i^\varepsilon w^\varepsilon \phi = - \int_{\Omega_{se}} v^\varepsilon \left( \frac{\partial w}{\partial y_i} \right)^\varepsilon \phi - \varepsilon \int_{\Omega_{se}} v^\varepsilon w^\varepsilon \frac{\partial \phi}{\partial x_i}
\]

and by passing to the limit, we obtain that \( v(x, y) = v^0(x) \), for \( y \in Y_s \). We define:

\[
(5.7) \quad v^r(x, y) = v(x, y) - v^0(x)
\]

and thus \( v^r = 0 \) in \( Y_s \) and (5.3) is obtained. It can be easily proved that \( v^0 \in H^1(\Omega) \).

In order to obtain (5.4) we pass to the limit in:

\[
\int_{\Omega_{se}} \frac{\partial v^\varepsilon}{\partial x_i} w_i^\varepsilon \phi = - \int_{\Omega_{se}} v^\varepsilon w_i^\varepsilon \frac{\partial \phi}{\partial x_i}
\]

where \( \text{div } w = 0 \). We obtain:

\[
\int_{Y_s} \left( z_i^0 - \frac{\partial v^0}{\partial x_i} \right) w_i(y) = 0
\]

Therefore there exists \( v^1 \in L^2(\Omega, H^1_p) \) so that for \( y \in Y_s \):

\[
z_i^0(x, y) = \frac{\partial v^0}{\partial x_i}(x) + \frac{\partial v^1}{\partial y_i}(x, y)
\]
which proves (5.4).

For proving (5.5), we deduce using the estimate (5.2) that there exists $V \in L^2(\Omega, L^2(Y))$ such that:

$$
\varepsilon^{\gamma} \int_{\Omega_{\varepsilon}} \frac{\partial v^\varepsilon}{\partial x_i} w^\varepsilon \phi \rightarrow \int_{\Omega \times Y} V(x, y) w(y) \phi
$$

For $\gamma > 0$, using test functions $w(y) = 0$ in $Y_f$, we conclude that $V(x, y) = 0$ for $y \in Y_s$.

The various cases in (5.6) for $V$ can be obtained by passing to the limit in:

$$
\varepsilon^{\gamma} \int_{\Omega} \frac{\partial v^\varepsilon}{\partial x_i} w^\varepsilon \phi = -\varepsilon^{-1} \int_{\Omega} v^\varepsilon \left( \frac{\partial w}{\partial y_i} \right)^\varepsilon \phi - \varepsilon \int_{\Omega} v^\varepsilon w^\varepsilon \frac{\partial \phi}{\partial x_i}
$$

for suitable choices of $w$.

It can be observed that if $\gamma < 1$ by multiplying the previous relation by $\varepsilon^{1-\gamma}$, when $\varepsilon \rightarrow 0$ we get:

$$
\int_{\Omega \times Y} (v^0 + v^r) \frac{\partial w^\varepsilon}{\partial y_i} \phi = 0
$$

We get from here that in this case $v^r$ is constant with respect to $y$, and is therefore zero. \[\square\]

**Lemma 5.3.** Let $v^\varepsilon \in H^1(\Omega)$ such that:

$$
|v^\varepsilon|_\Omega < C, \quad \left| \frac{\partial v^\varepsilon}{\partial x_i} \right|_{\Omega_{\varepsilon}} < C, \quad \text{div} v^\varepsilon = 0 \text{ in } \Omega_{\varepsilon}
$$

then $v^0, v^r, v^1$ given by (5.3) and (5.4) satisfy:

$$
\text{div}_x \int_{Y_f} (v^0 + v^r) = \int_{Y_s} \text{div}_y v^1
$$

**Proof.** By using the hypothesis:

$$
\int_{\Omega} \text{div} v^\varepsilon w = \int_{\Omega_{\varepsilon}} \text{div} v^\varepsilon w \quad \text{for all } w \in \mathcal{D}(\Omega)
$$

If we integrate by parts the left side and make $\varepsilon$ tend to zero, we obtain by (5.3) and (5.4):

$$
- \int_{\Omega \times Y} (v^0_i + v^r_i) \frac{\partial w}{\partial x_i} = \int_{\Omega \times Y_s} (\text{div}_x v^0 + \text{div}_y v^1) w
$$

Integrating again by parts we obtain (5.9). \[\square\]
5.2. Balance of mass. From the incompressibility condition satisfied by the solution of (2.16) in the fluid domain, and by lemma 5.3, we get:

\[ n \text{ div}_x u^0 + \text{div}_x \hat{u}^\epsilon = \int_{Y_s} \text{div}_y u^1 \]

which represents the balance of mass for the macroscopic fields.

5.4. The local problem. For obtaining the local problem we take in the variational formulation (2.16) the test functions of the form \( w = \epsilon \omega^e \theta \) with \( \theta \in \mathcal{D}(\Omega) \), \( \omega \in H^1_p \):

\[
\epsilon \int_{\Omega} \lambda^2 \rho^e \hat{u}^e \omega^e \theta + \int_{\Omega} c_{ijkh}^e \hat{u}^e (e_{ij}(\omega))^e \theta + \\
+ \epsilon \int_{\Omega} c_{ijkh}^e \hat{u}^e \omega^e_{ij} \theta_j + \epsilon^{2\gamma} \int_{\Omega} \lambda b_{ijkh}^e \hat{u}^e (e_{ij}(\omega))^e \theta \\
+ \epsilon^{2\gamma+1} \int_{\Omega} \lambda b_{ijkh}^e \hat{u}^e \omega^e_{ij} \theta_j - \epsilon \int_{\Omega} \hat{p}^e (\text{div} \omega)^e \theta - \epsilon \int_{\Omega} \hat{p}^e \omega^e_{i} \theta_i = \epsilon \int_{\Omega} f \omega^e \theta
\]

where we have used that \( (e_{ij}(\omega))^e = \epsilon e_{ij}(\omega^e) \) and we have denoted \( c_{ijkh}^e = a_{ijkh} + \lambda b_{ijkh}^e \).

It is easy to observe that excepting the second, the sixth and eventually the fourth term all the terms in (5.11) tend to zero. Indeed using theorem 5.2 we get to this conclusion by choosing \( w = \rho \omega \) in (5.3), \( w = c_{ijkh}^e \omega \) in (5.4), \( w = b_{ijkh}^e \omega_i \) in (5.4) for the first, third and fifth term respectively.

If \( \gamma > 0 \) we get when \( \epsilon \) tends to zero:

\[
\int_{\Omega \times Y_s} c_{ijkh}^e \partial_{x_h}^0 (x) + \partial_{x_h}^1 (x, y) \frac{\partial \omega_i}{\partial y_j} \theta - \int_{\Omega \times Y_f} p^0 (x, y) \text{div} \omega (y) \theta = 0,
\]

for all \( \theta \in \mathcal{D}(\Omega) \), \( \omega \in H^1_p (Y) \)

By choosing \( \omega \in \mathcal{D}(Y_f) \) we get that \( p^0 \) does not depend on \( y \). Next (5.12) implies that \( u^1 \) is linear with respect to the derivatives of \( u^0 \):

\[
u^1 (x, y) = - \frac{\partial u^0}{\partial x_m} (x) \chi^{lm} (y) - p^0 (x) \chi (y)
\]

with \( \chi^{lm} \) the unique solution of the local problem.

\[
\int_{Y_s} c_{ijkh}^e \frac{\partial \chi^{lm}}{\partial y_h} \frac{\partial \omega_i}{\partial y_j} = \int_{Y_s} c_{ijlm}^e \frac{\partial \omega_i}{\partial y_j}, \text{ for all } \omega \in H^1_p (Y_s)
\]
and $\chi$ is the unique solution of:

$$
(5.15) \quad \int_{Y_s} c_{ijkh}^s \frac{\partial \chi_k}{\partial y_h} \frac{\partial \omega_i}{\partial y_j} = \int_{Y_s} \text{div} \omega(y), \quad \text{for all } \omega \in H_p^1(Y_s)
$$

If $\gamma = 0$ the forth term has a non zero contribution and thus instead of (5.12) we will have:

$$
(5.16) \quad \int_{\Omega \times Y} c_{ijkh} \left( \frac{\partial u_k^0}{\partial x_h}(x) + \frac{\partial u_k^1}{\partial y_h}(x,y) \right) \frac{\partial \omega_i}{\partial y_j} \theta - \int_{\Omega \times Y_f} p^0(x,y) \text{div} \omega(y) \theta = 0
$$

for all $\theta \in \mathcal{D}(\Omega), \omega \in H_p^1(Y_f)$

where:

$$
(5.17) \quad c_{ijkh}(y) = \begin{cases} 
  c_{ijkh}^s & y \in Y_s \\
  \lambda b_{ijkh}^f & y \in Y_f 
\end{cases}
$$

Due to the contribution of the first integral on $Y_f$, $p^0$ will be in this case dependent on $y$. In this case $u^1$ will have a similar form as in (5.13), the coefficients being solutions of a local problem on all $Y$.

**5.5. Relative velocity.** In the previous section we have observed that the relative displacement $u^r$, given by theorem (5.2) is equal to zero if $0 \leq \gamma < 1$. Let us obtain the problem satisfied by the relative displacement $u^r$ for $\gamma \geq 1$, and thus justify (3.7) and (3.9).

Let us take the test function in (2.16) of the form $w = \omega^\varepsilon \theta$ with $\theta \in \mathcal{D}(\Omega), \omega \in H_p^1$, $\omega = 0$ in $Y_s$, div $\omega = 0$ in $Y_f$. We get:

$$
(5.18) \quad \lambda^2 \int_{\Omega} \rho^f u^\varepsilon \omega^\varepsilon \theta + \lambda \varepsilon^{2\gamma - 2} \int_{\Omega_f} b_{ijkh}^f e_k(e_k^\varepsilon)(e_{kh}^\varepsilon)^\varepsilon \theta + \\
+ \lambda \varepsilon^{2\gamma} \int_{\Omega_f} b_{ijkh}^f e_k(e_k^\varepsilon) \omega^\varepsilon e_i \theta - \int_{\Omega_f} \tilde{p}^\varepsilon \omega^\varepsilon \nabla \theta - \int_{\Omega} f \omega^\varepsilon \theta
$$

We can distinguish the following two cases:

i) if $\gamma = 1$, if $\varepsilon$ tends to zero, from (5.18) we obtain by using theorem 5.2:

$$
(5.19) \quad \lambda^2 \int_{\Omega \times Y_f} \rho^f (u^0 + u^r) \omega \theta + \lambda \int_{\Omega \times Y_f} b_{ijkh}^f \frac{\partial u_k^r}{\partial y_h} \frac{\partial \omega_i}{\partial y_j} \theta - \\
- \int_{\Omega \times Y_f} p^0 \omega \nabla \theta = \int_{\Omega \times Y_f} f(x) \omega(y) \theta(x)
$$

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and therefore:

\[(5.20) \quad \lambda^2 \rho f \int_{Y_f} u^r \omega + \lambda \mu \int_{Y_f} \frac{\partial u_i^r}{\partial y_j} \frac{\partial \omega_i}{\partial y_j} = \left( f_i - \lambda^2 \rho f u_i^0 - \frac{\partial p^0}{\partial x_i} \right) \int_{Y_f} \omega_i \]

for all \( \omega \in H^1_p \), \( \text{div} \omega = 0 \) in \( Y_f \), \( \omega = 0 \) in \( Y_s \).

ii) if \( \gamma > 1 \) it implies that \( 2\gamma - 2 > 0 \) and thus the second term is tending to zero. We obtain in this case

\[(5.21) \quad \lambda^2 \rho f \int_{Y_f} u^r \omega = \left( f_i - \lambda^2 \rho f u_i^0 - \frac{\partial p^0}{\partial x_i} \right) \int_{Y_f} \omega_i \]

for all \( \omega \in H^1_p \), \( \text{div} \omega = 0 \) in \( Y_f \), \( \omega \in 0 \) in \( Y_s \).

Remark. The relative velocity is equal to zero in the case \( \gamma \geq 1 \) if \( Y_f \) is strictly included in \( Y \), because the integral in the right members of both (5.20) and (5.21) is zero. Thus if the fluid part is not connected, the relative velocity is always zero.

5.6 The macroscopic equation. We want next to obtain the problem satisfied by the weak limit of \( (\hat{u}_e) \).

Let us suppose \( \gamma > 0 \) and tend to the limit in (2.16) for \( w \in \mathcal{D}(\Omega) \) we get:

\[(5.22) \quad \lambda^2 \int_{\Omega \times Y} \rho(u^0 + u^r)w + \int_{\Omega \times Y} c_{ijklh}^s \left( \frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_i^1}{\partial y_j} \right) \frac{\partial w_k}{\partial x_l} - \int_{\Omega \times Y_f} p^0 \text{div} w = \int_{\Omega} \hat{f}w. \]

If \( n \) is the porosity, \( n = |Y_f|/|Y| \), we have:

\[(5.23) \quad \int_{\Omega \times Y_f} p^0 \text{div} w = n \int_{\Omega} p^0 \text{div} w \]

we have to compute next, using (5.13), the integral:

\[(5.24) \quad \int_{Y_s} c_{ijklh}^s \left( \frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_i^1}{\partial y_j} \right) \]

which can be put, by standard computations ([5]), using (5.14) and (5.15), in the form

\[(5.25) \quad a_{lmkh}^h \frac{\partial u_p^0}{\partial x_m} - s_{lm}p^0 \]

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where the homogenized coefficients are defined by:

\[
(5.26) \quad a_{lmkh}^h = \int_{\hat{Y}_s} c_{ijpq}^s \frac{\partial (q_{p}^{kh} - \chi_{p}^{kh})}{\partial y_q} \frac{\partial (q_{i}^{lm} - \chi_{i}^{lm})}{\partial y_j}
\]

\[
(5.27) \quad s_{lm} = \int_{\hat{Y}_s} \text{div} \chi^{lm}
\]

with $\chi^{lm}$ solutions of the local problem (5.14) and $q_{i}^{lm} = y_m \delta_{ii}$

**Remark.** From (5.26) it can be easily proved ([5]) that the homogenized coefficients are symmetric and positive definite:

\[
(5.28) \quad a_{ijkh}^h = a_{jikh}^h = a_{kikh}^h, \quad a_{ijkh}^h \xi_i \xi_j \xi_k \geq \alpha^h \xi_i \xi_j \xi_k \text{ for all } \xi_i \in \mathbb{R}, \xi_{ij} = \xi_{ji}
\]

Thus the homogenized equation takes the form:

\[
(5.29) \quad \lambda^2 \int_{\Omega \times Y_f} \rho^f (u^0 + u^r)w + \lambda^2 (1 - n) \int_{\Omega} \rho^s u^0 w + \int_{\Omega} a_{ijkh}^h \frac{\partial u_{i}^{0}}{\partial x_j} \frac{\partial w_{k}}{\partial x_h} - \int_{\Omega} p^0 (s_{ij}^h + n \delta_{ij}) \frac{\partial w_{i}}{\partial x_j} = \int_{\Omega} f w, \text{ for all } w \in H^1_0(\Omega)
\]

i) if $0 < \gamma < 1$, we have proved that $u^r = 0$ and thus (5.29) takes the form:

\[
(5.30) \quad \lambda^2 \int_{\Omega} \tilde{u}^0 w + \int_{\Omega} a_{ijkh}^h \frac{\partial u_{i}^{0}}{\partial x_j} \frac{\partial w_{k}}{\partial x_h} - \int_{\Omega} p^0 (s_{ij}^h + n \delta_{ij}) \frac{\partial w_{i}}{\partial x_j} = \int_{\Omega} f w, \text{ for all } w \in H^1_0(\Omega)
\]

Using the balance of mass (5.10) with $v^r = 0$ the uniqueness of $u^0$ is an obvious consequence of (5.30) and (5.28).

ii) if $\gamma = 1$, the relative velocity satisfies (5.20). If in (5.29) we take $w = v^0, v^0 \in H^1(\Omega)$ and in (5.20) $\omega = v^r(x, \cdot), v^r \in L^2(\Omega, L^2_\mu)$ we obtain:

\[
(5.31) \quad \lambda^2 \int_{\Omega \times Y_f} \rho^f (u^0 + u^r)(v^0 + v^r) + \lambda^2 (1 - n) \int_{\Omega} \rho^s u^0 v^0 +
\]

\[
+ \int_{\Omega} a_{ijkh}^h \frac{\partial u_{i}^{0}}{\partial x_j} \frac{\partial v_{k}^{0}}{\partial x_h} + \lambda \mu \int_{\Omega \times Y_f} \frac{\partial u_{i}^{r}}{\partial y_j} \frac{\partial v_{i}^{r}}{\partial y_j} - 
\]

\[
- \int_{\Omega} p^0 \left( s_{ij}^h \frac{\partial v_{i}^{0}}{\partial x_j} + \text{div} \int_{Y_f} (v^0 + v^r) \right) = \int_{\Omega} f (v^0 + \tilde{v}^r)
\]
For proving the uniqueness of $u^0$ and $u^r$ we proceed by contradiction. If the difference of two solutions is $w^0, w^r$ we get:

$$
\lambda^2 \int_{\Omega \times Y} \rho^f (w^0 + w^r)^2 + \lambda^2 (1 - n) \int_{\Omega} \rho^g (w^0)^2 +
$$

$$
\int_{\Omega} a_{ijkl}^h \frac{\partial w_i^0}{\partial x_j} \frac{\partial w_k^0}{\partial x_l} + \lambda \mu \int_{\Omega \times Y} \frac{\partial w_i^r}{\partial y_j} \frac{\partial w_i^r}{\partial y_j} = 0
$$

(5.32)

and thus $w^0 = w^r = 0$.

iii) if $\gamma > 1$ we get the same equation as in (5.31) without the term in $\mu$, and from (5.32) we can still get uniqueness.

iv) if $\gamma = 0$ we obtain in the limit, from (2.16), taking into account that $u^r = 0$:

$$
\lambda^2 \int_{\Omega \times Y} \rho u^0 w + \int_{\Omega \times Y} c_{ijkl} \left( \frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_i^1}{\partial x_j} \right) \frac{\partial v_k}{\partial x_l} -
$$

$$
\int_{\Omega \times Y} p^0 \text{div } w = \int_{\Omega} \hat{f} w, \quad \text{for all } w \in D(\Omega)
$$

(5.33)

The second integral can be put, in the form (5.25) where the homogenized coefficients are defined by:

$$
a_{l,m}^{jk} = \int_{\Omega} c_{ijkl} \frac{\partial (q_p^{kh} - \chi_p^{kh})}{\partial y_q} \frac{\partial (q_i^{lm} - \chi_i^{lm})}{\partial y_j}
$$

(5.34)

$$
s_{lm} = \int_{\Omega} \text{div } \chi^{lm}
$$

(5.35)

where $\chi^{lm}$ are the solution of the corresponding local problem (5.16). The homogenized equations are in this case:

$$
\lambda^2 \int_{\Omega} \rho u^0 w + \int_{\Omega} a_{l,m}^{jk} \frac{\partial u^0_i}{\partial x_j} \frac{\partial w_k}{\partial x_l} - \int_{\Omega} p^0 (s_{i,j}^h + n \delta_{ij}) \frac{\partial w_i}{\partial x_j} =
$$

$$
= \int_{\Omega} \hat{f} w, \quad \text{for all } w \in H^1_0(\Omega)
$$

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