SELF–ORGANIZED CRITICALITY: ANALYSIS AND SIMULATION OF A 1D SANDPILE

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Abstract. To study the self–organization of systems, their approach towards a critical state, and the statistical properties at criticality, so–called mathematical sandpiles have been suggested. In this paper we analyze elementary properties of a slope–based one–dimensional model, for which one boundary is an abyss, the other is a wall.

Our analysis is based on properties of the Markov matrix. Some numerical results for sandpiles with small lattice sizes are also included.

Key words. random evolution, discrete time dynamical system, Markov matrix, self–organized criticality, sandpile

AMS(MOS) subject classifications. 60J10, 82C20, 82C27.

1. Introduction. The paradigm of self–organized criticality was introduced by Bak, Tang, and Wiesenfeld [2] about a decade ago and has attracted considerable interest, mostly in the physics community. Presumably, the concept can be used to explain — or at least describe — statistical features of a wide variety of open systems with many components. Applications of the concept have been attempted in many areas, ranging from geology to biology and economics. In [1] Bak gives a popular account with references to the scientific literature.

The mathematical models — so–called sandpiles — can be viewed as discrete–time dynamical systems, whose law of evolution contains a simple random element and simple local deterministics rules. Complexity arises since the system has a large number of components.

The evolution takes place in a finite but large set $\mathcal{S}_L$ of stable sandpile configurations. Here L denotes the number of sites of a one–dimensional lattice. After a grain of sand is dropped at a randomly chosen site $1 \leq r \leq L$, a simple toppling rule is applied until a new stable configuration is reached. (In many cases, the drop will not lead to any instability, and then no topplings occur.) Thus, the random evolution can be described by L operators $E_r : \mathcal{S}_L \to \mathcal{S}_L$, where r denotes the dropping site.

In numerical simulations, one observes that the sandpiles evolve towards a (non–unique) critical state. Basically, such a state is reached, when the average slope of the pile becomes critical. At this stage, on average, as many grains of sand fall into the abyss as are dropped onto the pile.

Though this behavior is quite plausible, the difficulties of analyzing even the simplest slope–based models are formidable, and a rigorous anal-

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ysis of the statistical features, which can be observed in numerical simulations, are currently out of reach.

In this paper, we first give a formal description of the so–called LL–model of [3] in Section 2. Section 3 describes elementary properties of the dynamics, which are expressed in terms of the Markov matrix. The set $\mathcal{S}_L$ of all stable configurations can be decomposed as

$$\mathcal{S}_L = \mathcal{T}_L \cup \mathcal{R}_L, \quad \mathcal{T}_L \cap \mathcal{R}_L = \emptyset$$

where $\mathcal{T}_L$ and $\mathcal{R}_L$ are the sets of transient and recurrent states, respectively. The evolution operators E_r map $\mathcal{R}_L$ into itself, which implies that the Markov matrix P is reducible,

$$P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} .$$

Here P_{22} corresponds to the random evolution in the set of all recurrent states. For the spectral radius of P_{11} we will prove that

$$1 - \frac{1}{L} \le \rho(P_{11}) < 1 ,$$

and conjecture equality $1 - \frac{1}{L} = \rho(P_{11})$. The spectral radius $\rho(P_{11})$ is related to the expected time needed to evolve from a transient into a recurrent state.

The long term statistical properties of the evolution depend on the properties of P_{22}. We will show that P_{22} is an irreducible matrix, which is cyclic of index 2, i.e., 1 and -1 are the only eigenvalues of P_{22} of modulus $\rho(P_{22}) = 1$. A further result, proved in Section 4, concerns the Frobenius eigenvector ψ of P_{22}^T. We normalize $\sum_j \psi_j = 1$, so that ψ is the unique stationary probability vector of the evolution in $\mathcal{R}_L$. Our result implies the estimate

$$0 < \psi_j \le 2^{-L} \quad \text{for all} \quad j ,$$

which says, in particular, that each single recurrent state occurs with very small probability for large L. (This is, of course, quite common for problems in statistical mechanics.)

With a similar technique, we prove in Section 5 a rigorous upper bound for the expected value of the average slope. Section 6 shows some results of numerical simulations.

In Appendix A we show how the number N_L of all stable states can be computed recursively. In Appendix B results for $L = 3$ are summarized, and in Appendix C we give a description of the evolution in terms of slopes and trapping sites. As suggested in [3], we expect that this formulation will speed up computations for large L. These formulae are included here since the corresponding formulae in [3] contain misprints.

This paper owes much to the ideas presented in [3]. Our main intend is to lay a mathematically rigorous foundation for further developments. So far, we have not been able, however, to rigorously analyze the more advanced scaling issues, which are addressed in [3].

2. Formal Description of the Dynamics. We begin with a precise formulation of the LL–model of [3]. Consider a one–dimensional lattice with $L+2$ sites $0, 1, \dots, L, L+1$, and let u_s denote the "number of grains of sand" at site s. Thus, the integer vector[1]

$$u = (u_0, u_1, \dots, u_L, u_{L+1}) \in N^{L+2}$$

describes the *configuration (or site)* of a one–dimensional sandpile. The sites 0 and $L+1$ are used to formulate *boundary conditions*, which we take as

$$u_0 = 0, \quad u_{L+1} = u_L \ , \tag{2.1}$$

corresponding to an abyss at the left boundary 0 and a wall at the right boundary $L+1$. With

$$\mathcal{A}_L = \{u \in N^{L+2} : \ u_0 = 0, \ u_{L+1} = u_L\}$$

we denote the (infinite) set of all states satisfying the boundary conditions (2.1).

If $u \in \mathcal{A}_L$, then the site s is called *stable* for u if $u_s \le u_{s-1} + 2$, and *unstable* otherwise. (Here $1 \le s \le L$.) We call u *stable*, if all its sites are stable, and denote the set of all stable states with $\mathcal{S}_L$,

$$\mathcal{S}_L = \{u \in \mathcal{A}_L : \ u_s \le u_{s-1} + 2, \ s = 1, \dots, L\} \ .$$

Clearly, if $u \in \mathcal{S}_L$ then

$$0 \le u_s \le 2s \quad \text{for} \quad s = 1, \dots, L \ ,$$

and therefore the set $\mathcal{S}_L$ is finite. The number $\#\mathcal{S}_L$ of its elements grows fast with L. Table 2.1 shows the number of stable states $\#\mathcal{S}_L$ and of recurrent stable states $\#\mathcal{R}_L$ (see Appendix A) for $1 \le L \le 16$.

The (discrete time) dynamics, which we want to study, takes place in the set of stable states $\mathcal{S}_L$. If $u \in \mathcal{S}_L$ is given, the *evolution step*

$$u \to u'$$

can be described by two half–time steps, which are roughly as follows: First, one grain of sand is added to a randomly chosen site. Then, if the new state is unstable, a simple *toppling rule* is applied until a stable state

[1] With N we denote the set of nonnegative integers.

TABLE 2.1
The number of stable and of recurrent states

L	$\#\mathcal{S}_L$	$\#\mathcal{R}_L$	Percentage
1	3	2	66.67%
2	12	5	41.67%
3	55	14	25.45%
4	273	42	15.38%
5	1,428	132	9.24%
6	7,752	429	5.53%
7	43,263	1,430	3.31%
8	246,675	4,862	1.97%
9	1,430,715	16,796	1.17%
10	8,414,640	58,786	0.70%
11	50,067,108	208,012	0.42%
12	300,830,572	742,900	0.25%
13	1,822,766,520	2,674,440	0.15%
14	11,124,755,664	9,694,845	0.09%
15	68,328,754,959	35,357,670	0.05%
16	422,030,545,335	129,644,790	0.03%

u' is reached. The toppling rule says that two grains of sand topple from any unstable site s to site $s-1$.

We now formalize this description and first define the *toppling operator*

$$T : \mathcal{A}_L \to \mathcal{A}_L$$

as follows. If $v \in \mathcal{A}_L$ is stable, then $Tv = v$, i.e., T is the identity on the set of stable states $\mathcal{S}_L$. If v is unstable, then from each unstable site s of v two grains of sand topple to site $s-1$. (To be unambiguous, toppling takes place from the unstable sites of v *only*. If a site s is stable for v but becomes unstable through toppling from sites $s \pm 1$, no toppling takes place from site s in the step $v \to Tv$.) After the topplings, the boundary conditions are updated. We give pseudocode of a procedure that overwrites v with Tv:

```
for s = 1 : L
    if s is unstable for v
        v_s = v_s - 2
        v_{s-1} = v_{s-1} + 2
    end
end
v_0 = 0;   v_{L+1} = v_L
```

The following property of T is easily shown:

LEMMA 2.1. *For any $v \in \mathcal{A}_L$ there exists $k \in N$ with $T^k v \in \mathcal{S}_L$.*

Proof: Consider the functional

$$\sigma(v) = \sum_{s=1}^{L} s v_s , \quad v \in \mathcal{A}_L \ .$$

Clearly,

$$\sigma(Tv) \le \sigma(v) - 2 \qquad \text{if } v \text{ is unstable.}$$

Therefore, $T^k v$ is stable for $k \ge \frac{1}{2}\sigma(v)$. □

Let $1 \le r \le L$. To have a short notation for the dropping of one grain of sand at site r, we define the operator

$$R_r : \mathcal{A}_L \to \mathcal{A}_L$$

by the equations

$$\begin{aligned} (R_r u)_r &= u_r + 1 \ ; \\ (R_r u)_s &= u_s \quad \text{for} \quad s = 0, \ldots, L, \ s \ne r \ ; \\ (R_r u)_{L+1} &= (R_r u)_L \ . \end{aligned}$$

The two half–steps of the evolution $u \to u'$ can now be described as follows.

1. Choose a site $1 \le r \le L$ at random, giving each site equal probability $1/L$, and obtain $v = R_r u$.
2. If v is stable let $u' = v$. Otherwise, apply the toppling operator T repeatedly until a stable state $u' = T^k v$ is reached.

We denote the result of the evolution step by

$$u' = E_r u = T^k R_r u \tag{2.2}$$

and note that the exponent $k = k(u, r)$ is a function of the randomly chosen site r and of the initial state $u \in \mathcal{S}_L$. To summarize, for each site $1 \le r \le L$ we have defined an operator

$$E_r : \mathcal{S}_L \to \mathcal{S}_L$$

mapping any given stable state u to the stable state $u' = T^k R_r u$. If $u \in \mathcal{S}_L$ is an initial state and $1 \le r_n \le L$ is a sequence of randomly chosen dropping sites, we obtain the sequence of stable configurations

$$u^n = E_{r_n} \cdots E_{r_1} u$$

of sandpiles.

3. Elementary Properties of the Dynamics. We write the set $\mathcal{S}_L$ of all stable states as the disjoint union of all *transient states* and of all *recurrent states*,

$$\mathcal{S}_L = \mathcal{T}_L \cup \mathcal{R}_L, \quad \mathcal{T}_L \cap \mathcal{R}_L = \emptyset \ .$$

To begin with, we *define* the sets $\mathcal{R}_L$ and $\mathcal{T}_L$ by

$$\mathcal{R}_L = \{u \in \mathcal{S}_L : \ u_s \geq s, \ 1 \leq s \leq L\}, \quad \mathcal{T}_L = \mathcal{S}_L \setminus \mathcal{R}_L$$

and will justify the terminology *transient* and *recurrent* below.

We first show that the set $\mathcal{R}_L$ is closed under evolution.

LEMMA 3.1. *If $u \in \mathcal{R}_L$ and $1 \leq r \leq L$, then $E_r u \in \mathcal{R}_L$.*

Proof: Consider any $v \in \mathcal{A}_L$ with

$$v_s \geq s, \quad s = 1, \ldots, L \ ,$$

and let $w = Tv$. We show that

$$w_s \geq s, \quad s = 1, \ldots, L \ . \tag{3.1}$$

Indeed, if s is stable for v, then $w_s \geq v_s \geq s$. If s is unstable for v, then

$$v_s \geq v_{s-1} + 3 \geq s + 2 \ ,$$

and therefore $w_s \geq v_s - 2 \geq s$. This proves that $w = Tv$ satisfies (3.1). The lemma now follows from the representation $E_r u = T^k R_r u$. □

Next we consider any $u \in \mathcal{T}_L$, a state we have called transient, and show that evolution into $\mathcal{R}_L$ occurs if the dropping sites are properly chosen,

$$E_{r_n} \cdots E_{r_1} u \in \mathcal{R}_L \ . \tag{3.2}$$

LEMMA 3.2. *For any $u \in \mathcal{T}_L$ there exist finitely many dropping sites r_j such that (3.2) holds. Moreover, the r_j can be chosen such that no topplings occur in the transition from u to $E_{r_n} \cdots E_{r_1} u$.*

Proof: Define

$$\delta_s = u_s - s, \quad 0 \leq s \leq L \ ,$$

and choose a site r with

$$\delta_r = \min_s \delta_s \ .$$

Since $u \in \mathcal{T}_L$ we have

$$\delta_r < 0, \quad r \geq 1 \ .$$

Now consider $v = R_r u$, i.e., drop one grain at the site r. We claim that $v = R_r u$ is stable. Indeed, we have to test stability only for site r, where $v_r = u_r + 1$. From

$$\delta_{r-1} = u_{r-1} - (r-1) \geq u_r - r = \delta_r$$

we conclude

$$u_r \leq u_{r-1} + 1 \ ,$$

and therefore,

$$v_r = u_r + 1 \leq u_{r-1} + 2 = v_{r-1} + 2 \ .$$

This proves that $v = R_r u$ is stable, and consequently $E_r u = R_r u$. The result of the lemma follows by repeated application of suitable dropping operators $R_{r_1}, R_{r_2}, \ldots$. □

Our next result says that the set of recurrent states $\mathcal{R}_L$ does not contain any nontrivial subset which is closed under evolution.

LEMMA 3.3. *For any two recurrent states $u, v \in \mathcal{R}_L$ there are finitely many dropping sites r_j with*

$$v = E_{r_n} \cdots E_{r_1} u \ .$$

Proof: 1) First assume that u and v are two different states in $\mathcal{R}_L$ with

$$u_s \leq v_s, \quad 1 \leq s \leq L \ .$$

Let r denote the *smallest* site with

$$u_r < v_r \ ,$$

and set $w = R_r u$. We show that w is stable. Indeed,

$$\begin{aligned} w_r &= u_r + 1 \\ &\leq v_r \\ &\leq v_{r-1} + 2 \\ &= u_{r-1} + 2 \\ &= w_{r-1} + 2 \end{aligned}$$

The inequalities $w_s \leq w_{s-1} + 2$ for $s \neq r$ follow from the stability of u. Since $w = R_r u$ is stable, we have

$$w = E_r u = R_r u \ .$$

Using induction in $m = \sum_s (v_s - u_s)$ we obtain that

$$v = E_{r_n} \cdots E_{r_1} u = R_{r_n} \cdots R_{r_1} u$$

if the dropping sites r_j are suitably chosen.

2) For the second part of the proof, let us denote the *lowest state* in $\mathcal{R}_L$ by U, i.e.,

$$U_s = s, \quad 1 \le s \le L \ .$$

First consider any state $u \in \mathcal{R}_L, u \ne U$, and let r denote the *smallest* site with

$$u_r > r \ .$$

Then the state $w = R_r u$ is unstable since

$$w_r = u_r + 1 > r + 1 \quad \text{and} \quad w_{r-1} = u_{r-1} = r - 1 \ .$$

To compute

$$u' = E_r u = T^k R_r u = T^k w, \quad k = k(u,r) \ge 1,$$

the toppling operator T has to be applied a number of times to w before a stable state u' is reached. Using the equations

$$w_s = s \quad \text{for} \quad s < r, \quad w_r > r + 1 \ ,$$

it is easy to see that at least two grains of sand fall into the abyss in the step

$$w \to T^k w = u' \ .$$

Therefore,

$$\sum_{s=1}^{L} u'_s < \sum_{s=1}^{L} u_s, \quad u' = E_r u.$$

A simple induction argument shows the existence of sites r_j with

$$U = E_{r_n} \cdots E_{r_1} u \ ,$$

where U is the lowest state in $\mathcal{R}_L$. Combining this with the first part of the proof, the lemma follows. □

4. Properties of the Markov Matrices.

4.1. Definition of the Markov Matrix P. Let $N = N_L = \#\mathcal{S}_L$ denote the number of stable states corresponding to the lattice $0, 1, \ldots, L, L+1$; see Table 1 for the values of $N_1, N_2, \ldots, N_{16}$. If we enumerate the states in $\mathcal{S}_L$, $u^1, \ldots, u^N$, we can associate an $N \times N$ matrix $P = (p_{ij})$ to the evolution. By *definition*, p_{ij} is the probability that the state $u = u^i$ goes over into the state $u' = u^j$ in one time step. Thus, the so–called *Markov matrix* P can be computed as follows:

```
P = 0
for i = 1 : N
    for r = 1 : L
        determine j with E_r u^i = u^j
        p_ij = p_ij + 1/L
    end
end
```

It is obvious that

$$0 \le p_{ij} \le 1, \quad \sum_{i=1}^{N} p_{ij} = 1 \ .$$

Thus, by definition, P *is a row–stochastic matrix.* With $\bar{e}$ we denote the column vector

$$\bar{e} = (1, 1, \ldots, 1)^T \tag{4.1}$$

and express the identity $\sum_j p_{ij} = 1$ in matrix form by

$$P\bar{e} = \bar{e} \ . \tag{4.2}$$

Remark: In the case under consideration here, *every entry p_{ij} of P is either 0 or $1/L$.* In other words, if $r, r' \in \{1, 2, \ldots, L\}$ and $r \neq r'$, then $E_r u \neq E_{r'} u$ for every $u \in \mathcal{S}_L$. This can be seen as follows. With every $u \in \mathcal{S}_L$ associate the vector

$$\mu = \mu(u) = (\mu_1, \ldots, \mu_L), \quad \mu_s = u_s \ mod \ 2 \ ,$$

i.e., $\mu_s = 0$ if u_s is even and $\mu_s = 1$ if u_s is odd. Then the vector corresponding to $E_r u$ is

$$\mu(E_r u) = (\nu_1, \ldots, \nu_L)$$

where

$$\nu_s = \mu_s \quad \text{for} \quad s \neq r, \quad \text{and} \quad \nu_r \neq \mu_r \ ,$$

because toppling does not change the μ–vector. Therefore,

$$\mu(E_r u) \neq \mu(E_{r'} u) \quad \text{if} \quad r \neq r' \ ,$$

and $E_r u \neq E_{r'} u$ follows. (The mapping $u \to \mu(u)$ will also be used below to obtain information about the stationary probability vector associated with the P.)

4.2. Example. Consider the case $L = 2$ with $N_2 = 12$ stable states as an example. We order the states in $\mathcal{S}_2$ as follows. (For simplicity, the boundary values $u_0^i = 0, u_3^i = u_2^i$ are not displayed.)

i	u^i
1	00
2	10
3	20
4	01
5	11
6	21
7	02
8	22
9	13
10	24
11	12
12	23

The first seven states lie in $\mathcal{T}_L$, the remaining five in $\mathcal{R}_L$. The states in $\mathcal{T}_L$ are ordered so that the first states end with 0, the next states end with 1, etc. The states in $\mathcal{R}_L$ are ordered so that the first three states are *even* ($u_1^i + u_2^i$ is even), the remaining two states are *odd*. One obtains the Markov matrix

$$P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} \tag{4.3}$$

with

$$P_{11} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_{12} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P_{22} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

For later reference we note that $P_{22}^T \psi = \psi$ where ψ is the probability vector

$$\psi = \frac{1}{8}(1, 2, 1, 2, 2)^T .$$

4.3. The Spectral Radii $\rho(P_{11})$, $\rho(P_{22})$. The block structure (4.3) of P is valid for any L, as follows directly from the fact that $\mathcal{R}_L$ is closed under evolution; see Lemma 3.1.

LEMMA 4.1. *Let $M = M_L = \#\mathcal{R}_L$ and $N = N_L = \#\mathcal{S}_L$ denote the number of recurrent states and stable states, respectively. We order the states $u^1, \ldots, u^N$ in $\mathcal{S}_L$ so that $u^1, \ldots, u^{N-M}$ are transient, whereas $u^{N+1-M}, \ldots, u^N$ are recurrent. Then the Markov matrix P has the block structure (4.3), where P_{11} has size $(N-M)\times(N-M)$ and P_{22} has size $M \times M$.* □

Recall that a square matrix A is called *reducible* if there is a permutation matrix Q that puts A into the form

$$Q^T A Q = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} ,$$

and *irreducible* otherwise. (Here the diagonal blocks A_{11} and A_{22} are non–empty and square.) Clearly, P *is reducible*, and from Lemma 3.3 one obtains that P_{22} is irreducible.

LEMMA 4.2. *The matrix P_{22} corresponding to the evolution in the set of recurrent states $\mathcal{R}_L$ is irreducible.* □

Next we consider the spectral radii of P_{11} and P_{22}. We use the notation

$$\rho(A) = \max\{|\lambda| : \ \lambda \in \sigma(A)\}$$

for the *spectral radius* of A, where $\sigma(A)$ is the set of eigenvalues of A. Furthermore, the *matrix norm* corresponding to the (vector) maximum norm in $\mathbb{R}^m$ is

$$|A|_\infty = \max_{1\le i\le m} \sum_{j=1}^m |a_{ij}|, \quad A \in \mathbb{R}^{m\times m} .$$

THEOREM 4.3. *The diagonal blocks P_{11} and P_{22} of the Markov matrix P satisfy*

$$\rho(P_{11}) < 1, \quad \rho(P_{22}) = 1 .$$

Proof: 1) Since P_{22} is non–negative and irreducible, the equation $\rho(P_{22}) = 1$ follows from $P_{22}\bar{e} = \bar{e}$ by Frobenius' theorem.

2) For P_{11} we have[2]

$$P_{11}\bar{e} \le \bar{e} \ , \tag{4.4}$$

thus

$$\rho(P_{11}) \le |P_{11}|_\infty \le 1 .$$

[2] If $x, y \in \mathbb{R}^m$ we write $x \le y$ if $x_i \le y_i$ for all $i = 1, \ldots, m$. Similarly, we write $x < y$ if $x_i < y_i$ for all $i = 1, \ldots, m$.

Assuming that $\rho(P_{11}) = 1$, one obtains from Frobenius' theorem that the number 1 is an eigenvalue of P_{11},

$$P_{11}\psi = \psi, \quad \psi \neq 0 \ . \tag{4.5}$$

We will lead this to a contradiction. First, we can apply the non–negative matrix P_{11} repeatedly to the inequality (4.4) and obtain

$$P_{11}^k \bar{e} \leq P_{11}^{k-1}\bar{e} \leq \cdots \leq \bar{e} \ , \quad k = 1, 2, \ldots$$

Fix $1 \leq i \leq N - M$. By Lemma 3.2, there are indices $i = j_0, j_1, \ldots, j_n$ with

$$p_{j_k, j_{k+1}} > 0 \quad \text{for} \quad k = 0, \ldots, n-1 \ ;$$

$$1 \leq j_k \leq N - M \quad for \quad k = 0, \ldots, n-1 \ ; \quad N - M < j_n \leq N \ .$$

Since $j_n > N - M$ and $p_{j_{n-1}, j_n} > 0$ it follows that

$$\sum_{\nu=1}^{N-M} p_{j_{n-1}, \nu} < 1 \ ,$$

or in matrix form,

$$(P_{11}\bar{e})_{j_{n-1}} < 1 \ .$$

Using this strict inequality and the strict positivity $p_{j_{n-2}, j_{n-1}} > 0$, it follows that

$$(P_{11}^2 \bar{e})_{j_{n-2}} = \sum_{\nu=1}^{N-M} p_{j_{n-2}, \nu}(P_{11}\bar{e})_\nu < 1 \ .$$

Inductively, one obtains

$$(P_{11}^n \bar{e})_i < 1 \ . \tag{4.6}$$

To summarize, for any $1 \leq i \leq N - M$, there is a positive integer $n = n(i)$ with (4.6). Now let $\bar{n} = \max_i n(i)$. Using the inequalities $P_{11}^k \bar{e} \leq \bar{e}$ and the strict inequalities (4.6) we have derived the strict (vector) inequality

$$(I + P_{11} + \cdots + P_{11}^{\bar{n}})\bar{e} < (\bar{n} + 1)\bar{e} \ .$$

This implies the norm estimate

$$|\sum_{k=0}^{\bar{n}} P_{11}^k|_\infty < \bar{n} + 1 \ . \tag{4.7}$$

However, the equation $P_{11}\psi = \psi$ (see (4.5)) yields

$$(\sum_{k=0}^{\bar{n}} P_{11}^k)\psi = (\bar{n}+1)\psi \ .$$

Thus, $\bar{n}+1$ is an eigenvalue of the matrix $\sum_{k=0}^{\bar{n}} P_{11}^k$, which contradicts (4.7). Therefore, the assumption $\rho(P_{11}) = 1$ leads to a contradiction, and the estimate $\rho(P_{11}) < 1$ is proved. □

Remark: We have computed $\rho(P_{11})$ for $L = 1,2,3,4,5$. For these values of L we have obtained that $\rho(P_{11}) = (L-1)/L$ and conjecture that this formula holds in general. However, we can only prove the following partial result.

LEMMA 4.4. *Let P_{11} denote the block of the Markov matrix specified above. Then we have*

$$\frac{L-1}{L} \le \rho(P_{11}) < 1 \ .$$

Proof: The lower bound for $\rho(P_{11})$ is a consequence of the following three observations:

1. If $u \in \mathcal{T}_L$ satisfies $u_L \ge 1$, then $(E_r u)_L \ge 1$ for all $1 \le r \le L$.
2. If $u_L = 0$ and $1 \le r < L$, then $(E_r u)_L = 0$.
3. If $u_L = 0$ then $(E_L u)_L = 1$.

Now order the states in $\mathcal{T}_L$ so that the first states satisfy $u_L = 0$ and the remaining states satisfy $u_L \ge 1$. Then, because of the first observation, P_{11} has the block form

$$P_{11} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \ ,$$

where A has size $N_0 \times N_0$, if there are N_0 states u in $\mathcal{T}_L$ with $u_L = 0$. Also, because of the second and third observation, the row sum of each row of A is $(L-1)/L$. This implies

$$A\bar{e} = \frac{L-1}{L}\bar{e} \ ,$$

and the result follows. □

Remark: In Lemma 3.2 and, therefore, in the proof of Theorem 4.3 one can choose $n \le \frac{1}{2}L(L+1)$. Furthermore, the non–zero entries of P all equal $\frac{1}{L}$. For these reasons, the method of proof of Theorem 4.3 can be used to obtain a nontrivial upper bound

$$\rho(P_{11}) \le 1 - \varepsilon_L, \quad \varepsilon_L > 0 \ .$$

In this way, one does not obtain the desired upper bound $\rho(P_{11}) \le (L-1)/L$, however.

4.4. The Stationary Probability Vector for P. Recall that a vector

$$\pi = (\pi_1, \dots, \pi_N)^T$$

is called a *probability vector* if

$$0 \le \pi_i \le 1, \quad \sum_{i=1}^{N} \pi_i = 1 \ .$$

If the system is in the state u^i at time $t = n$, then, with probability p_{ij}, it is in the state u^j at time $t' = n + 1$. Therefore, if π is the probability distribution of the states u^i at time $t = n$, then the system is in the state u^j at time $t' = n + 1$ with probability

$$\pi'_j = \sum_{i=1}^{N} \pi_i p_{ij} \ .$$

Thus, the vector

$$\pi' = P^T \pi$$

gives the probability distribution at time $t' = n+1$. A probability vector π is called *stationary* for P (or, more precisely, for the Markov process with Markov matrix P), if

$$P^T \pi = \pi \ .$$

In general, by the Perron–Frobenius theorem, any row–stochastic matrix has a stationary probability vector. We will show that this vector is unique for the matrices P under consideration here.

THEOREM 4.5. *The Markov matrix P associated with the LL–sandpile evolution has exactly one stationary probability vector π. This vector has the form*

$$\pi = (0, \dots, 0, \psi_1, \dots, \psi_M)^T, \quad \psi_i > 0 \ , \tag{4.8}$$

with zero probabilities corresponding to the states in $\mathcal{T}_L$ and positive probabilities ψ_i corresponding to the states in $\mathcal{R}_L$.

Proof: Since P_{22} is non–negative, irreducible and since $\rho(P_{22}) = 1$, there exists a unique vector $\psi \in \mathbb{R}^M$ with

$$P_{22}^T \psi = \psi, \quad \sum_{i=1}^{M} \psi_i = 1 \ .$$

This vector is positive, $\psi > 0$. If we define π by (4.8), then we obtain

$$P^T \pi = \pi$$

using the block structure (4.3) of P. Since $\rho(P_{11}) < 1$ and since $\lambda_1 = 1$ is a simple eigenvalue of P_{22}, $\lambda_1 = 1$ is also a simple eigenvalue of P. This implies uniqueness of π. □

4.5. P_{22} is Cyclic of Index Two. In this subsection we will focus on the dynamics within the set $\mathcal{R}_L$ of recurrent states. The matrix P_{22} is the corresponding Markov matrix. Recall the concept of a cyclic non–negative irreducible matrix.

Definition: Let $A \in \mathbb{R}^{m\times m}$ denote an irreducible, non–negative matrix with exactly h distinct eigenvalues λ_j of modulus $\rho(A)$,

$$|\lambda_j| = \rho(A), \quad j = 1, \dots, h \ .$$

If $h = 1$ then A is called *primitive*. If $h \geq 2$ then A is called *cyclic* (or *imprimitive*) of *index* h.

If A is an irreducible, non–negative matrix, which is cyclic of index $h \geq 2$, then the characteristic polynomial of A has the form (see [4], for example)

$$\det(zI - A) = z^k(z^h - \rho^h)(z^h - \delta_2\rho^h)\cdots(z^h - \delta_r\rho^h) \ .$$

Here $\rho = \rho(A) > 0$ and $|\delta_i| < 1$ for $2 \leq i \leq r$, if $r \geq 2$. In particular, the spectrum $\sigma(A)$, regarded as a subset of the complex plane, goes over into itself under a rotation by the angle $2\pi/h$ about the origin. All eigenvalues λ_j of A with maximal absolute value $|\lambda_j| = \rho(A)$ are simple roots of $\det(zI - A)$.

We show that the matrix P_{22} is cyclic of index 2.

THEOREM 4.6. *For $L \geq 2$, the matrix P_{22}, which is the Markov matrix of the evolution in the set of recurrent states $\mathcal{R}_L$, is cyclic of index 2.*

Proof: Call $u \in \mathcal{R}_L$ *even* if $\sum_{s=1}^{L} u_s$ is even, and *odd* otherwise. Then

$$\mathcal{R}_L = \mathcal{R}_L^{even} \cup \mathcal{R}_L^{odd}, \quad \mathcal{R}_L^{even} \cap \mathcal{R}_L^{odd} = \emptyset \ ,$$

where $\mathcal{R}_L^{even}$ is the set of all even states in $\mathcal{R}_L$ and $\mathcal{R}_L^{odd}$ the set of all odd states in $\mathcal{R}_L$. Both these sets are non–empty for $L \geq 2$. Clearly, if u is even, then

$$u' = E_r u = T^k R_r u$$

is odd and vice versa. Therefore, if we order the states $u^1, \dots, u^M$ in $\mathcal{R}_L$ so that the first M^e states are even and the remaining states are odd, then P_{22} has the form

$$P_{22} = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \ . \tag{4.9}$$

If $M^o = M - M^e$ denotes the number of odd states, then A_1 has size $M^e \times M^o$ and A_2 has size $M^o \times M^e$. Now consider

$$P_{22}\begin{pmatrix} \bar{e} \\ -\bar{e} \end{pmatrix} = \begin{pmatrix} -A_1\bar{e} \\ A_2\bar{e} \end{pmatrix} = -\begin{pmatrix} \bar{e} \\ -\bar{e} \end{pmatrix} \ .$$

This shows that -1 is an eigenvalue of P_{22}, and consequently P_{22} is cyclic of index $h \geq 2$.

Next consider

$$P_{22}^2 = \begin{pmatrix} A_1 A_2 & 0 \\ 0 & A_2 A_1 \end{pmatrix} .$$

We show that the square matrices $A_1 A_2$ *and* $A_2 A_1$ *are irreducible and primitive.* If this is shown, then it follows that $\lambda_1 = 1$ is the *only* eigenvalue of P_{22}^2 with modulus one. (The eigenvalue $\lambda_1 = 1$ of P_{22}^2 has geometric and algebraic multiplicity two.) Therefore, the only eigenvalues of P_{22} with modulus one are 1 and -1, proving that P_{22} is cyclic of index 2.

It remains to show that $A_1 A_2$ and $A_2 A_1$ are irreducible and primitive. For definiteness, consider $A_1 A_2$. (The arguments for $A_2 A_1$ are similar.) To show that $A_1 A_2$ is irreducible, consider two arbitrary even states $u, v \in \mathcal{R}_L^{even}$. By Lemma 3.3 there exist dropping sites $r_1, \ldots, r_n$ with

$$v = E_{r_n} \cdots E_{r_1} u .$$

It follows that n is even, and we can group two successive operators $E_{r_2} E_{r_1}$ etc. together and write

$$v = (E_{r_n} E_{r_{n-1}}) \cdots (E_{r_2} E_{r_1}) u . \tag{4.10}$$

All intermediate states

$$(E_{r_j} E_{r_{j-1}}) \cdots (E_{r_2} E_{r_1}) u$$

lie in $\mathcal{R}_L^{even}$. The matrix $A_1 A_2$ is the Markov matrix of the random evolution with time steps

$$u \in \mathcal{R}_L^{even} \quad \rightarrow \quad E_r E_{r'} u \in \mathcal{R}_L^{even} .$$

Since any two given states $u, v \in \mathcal{R}_L^{even}$ can be connected as in (4.10), it follows that $A_1 A_2$ is irreducible.

We now show that $A_1 A_2$ has at least one positive diagonal element. To this end, consider the two states

$$U = (1, 2, 3, 4, \ldots, L)$$

and

$$V = (1, 3, 3, 4, \ldots, L) ,$$

which differ only for $s = 2$, where $V_2 = 3, U_2 = 2$. (We do not display the boundary values.) Clearly, both states lie in $\mathcal{R}_L$, and either U or V is even. It is easy to see that

$$E_1^2 U = U \quad \text{and} \quad E_1^2 V = V .$$

Therefore, if $U = u^i$ is even or $V = u^i$ is even, the diagonal element $(A_1A_2)_{ii}$ is positive.

Since A_1A_2 is a non–negative irreducible matrix with at least one positive diagonal element, there is a power $(A_1A_2)^q, q \geq 1$, for which all matrix entries are positive.(See, for example, [4].) By Theorem 2.5 of [4], this implies that A_1A_2 is primitive. □

4.6. Evolution of Probabilities for P_{22}. Let us first summarize some results for P_{22}, which follow from our previous considerations. As in the proof of Theorem 4.6, we order the states $u^1, \ldots, u^M$ in $\mathcal{R}_L$ so that the first M^e states are even and the remaining states are odd. Then we have

$$P_{22} = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} , \quad P_{22}^2 = \begin{pmatrix} A_1A_2 & 0 \\ 0 & A_2A_1 \end{pmatrix} .$$

We set

$$(P_{22}^T)^2 =: \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} . \tag{4.11}$$

Here $A = (A_1A_2)^T$ has size $M^e \times M^e$ and $B = (A_2A_1)^T$ has size $M^o \times M^o$. Both matrices, A and B, are non–negative, irreducible, and primitive. They also satisfy

$$A^T\bar{e} = \bar{e}, \quad B^T\bar{e} = \bar{e} .$$

The matrix P_{22} has a unique stationary probability vector, which we denote by ψ,

$$P_{22}^T\psi = \psi, \quad \sum_{i=1}^{M} \psi_i = 1, \quad \psi > 0 .$$

Let ψ^0 denote a chosen probability distribution for the states in $\mathcal{R}_L$ at time $t = 0$, and let

$$\psi^n = (P_{22}^T)^n\psi^0$$

denote the corresponding probability distribution at time $t = n$. It is suggestive to believe that $\psi^n \to \psi$ as $n \to \infty$. However, since -1 is an eigenvalue of P_{22}, the sequence ψ^n does not converge, in general. Only the two subsequences

$$\psi^{2n} \quad \text{and} \quad \psi^{2n+1}$$

converge. To describe their limits, we introduce the *Frobenius' eigenvectors* f and g of A and B, respectively. They are the unique vectors with

$$Af = f, \quad \sum_i f_i = 1, \quad f > 0 ;$$

$$Bg = g, \quad \sum_i g_i = 1, \quad g > 0 \ .$$

LEMMA 4.7. *Let ψ^n, f, and g be the quantities described above. Set*

$$\alpha = \sum_{i=1}^{M^e} \psi_i^0 \ .$$

(I.e., α is the probability that the system is in an even state at time $t = 0$.) Then we have

$$\psi^{2n} \rightarrow \begin{pmatrix} \alpha f \\ (1-\alpha) g \end{pmatrix} \ , \quad \psi^{2n+1} \rightarrow \begin{pmatrix} (1-\alpha) f \\ \alpha g \end{pmatrix}$$

as $n \rightarrow \infty$.

Proof: We partition ψ^n into its even and odd parts,

$$\psi^n = \begin{pmatrix} (\psi^n)^e \\ (\psi^n)^o \end{pmatrix}$$

and obtain

$$(\psi^{2n})^e = A^n (\psi^0)^e, \quad (\psi^{2n})^o = B^n (\psi^0)^o \ ,$$

$$(\psi^{2n+1})^e = A^n (\psi^1)^e, \quad (\psi^{2n+1})^o = B^n (\psi^1)^o \ .$$

The properties of A and B imply convergence

$$(\psi^{2n})^e \rightarrow \alpha f, \quad (\psi^{2n})^o \rightarrow \beta g \ ,$$

$$(\psi^{2n+1})^e \rightarrow \gamma f, \quad (\psi^{2n+1})^o \rightarrow \delta g \ ,$$

where

$$\alpha = \bar{e}^T (\psi^0)^e, \quad \beta = \bar{e}^T (\psi^0)^o \ ,$$

$$\gamma = \bar{e}^T (\psi^1)^e, \quad \delta = \bar{e}^T (\psi^1)^o \ .$$

Clearly, $\beta = 1 - \alpha, \delta = 1 - \gamma$. Furthermore,

$$(\psi^1)^o = A_2^T (\psi^0)^e \ ,$$

and therefore $\delta = \alpha$. This proves the lemma. □

Corollary: *If ψ^n and ψ are defined as above, then we have*

$$\frac{1}{2}(\psi^n + \psi^{n+1}) \rightarrow \psi \quad as \quad n \rightarrow \infty \ .$$

Thus, in order to have a convergent probability distribution for $n \to \infty$*, it suffices to average over two consecutive time steps.*

Since $f > 0, g > 0$, there is a positive number $\kappa = \kappa_L$, depending only on the lattice size L, with

$$f > 2\kappa \bar{e}, \quad g > 2\kappa \bar{e} \ .$$

Also, in Lemma 4.7 we have $\alpha \geq \frac{1}{2}$ or $1 - \alpha \geq \frac{1}{2}$. Therefore, we conclude from Lemma 4.7 that

$$\psi_j^{2n} > \kappa \quad \text{or} \quad \psi_j^{2n+1} > \kappa \ ,$$

where $1 \leq j \leq M$ is arbitrary and n is sufficiently large. Thus we have proved the following result.

LEMMA 4.8. *Let* $\psi^n = (P_{22}^T)^n \psi^0$ *denote the sequence of probability distributions of the states in* $\mathcal{R}_L$ *with initial distribution* ψ^0*. There is a positive number* κ_L*, depending only on the lattice size* L*, with the following property: For any* $n_1 \in N$ *and any* $j \in \{1, \ldots, M\}$*, there is an integer* $n = n(j, n_1)$ *with*

$$\psi_j^n > \kappa_L > 0 \quad \textit{and} \quad n \geq n_1 \ .$$

The Frobenius eigenvectors ψ, f, and g of P_{22}^T, A, and B (see (4.11)) are related in simple ways. We show the following result.

LEMMA 4.9. *Let* ψ *and* $\bar{\psi}$ *denote the eigenvectors of* P_{22}^T *to the eigenvalues 1 and -1, respectively. We normalize* $\sum_i \psi_i = 1$*. Then we have*

$$\psi = \frac{1}{2} \begin{pmatrix} f \\ g \end{pmatrix}, \quad \bar{\psi} = \gamma \begin{pmatrix} f \\ -g \end{pmatrix}, \quad \gamma \in I\!R \ .$$

Furthermore,

$$g = A_1^T f, \quad f = A_2^T g \ . \tag{4.12}$$

Proof: We first show (4.12) and recall

$$A_2^T A_1^T f = f, \quad A_1^T A_2^T g = g \ .$$

The first equation implies $A_1^T A_2^T A_1^T f = A_1^T f$. Since the eigenvalue 1 of $B = A_1^T A_2^T$ is simple and since $\bar{e}^T A_1^T f = 1 = \bar{e}^T g$ holds, the equation $A_1^T f = g$ follows. The second equation in (4.12) is shown in the same way.

To show the relations between $\psi, \bar{\psi}, f$, and g, we set

$$\bar{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

and note that $span\{\psi, \bar{\psi}\} = span\{\bar{f}, \bar{g}\}$. (Both spaces agree with the eigenspace of $(P_{22}^T)^2$ to the eigenvalue 1.) It follows that $\psi = \alpha \bar{f} + (1-\alpha)\bar{g}$. We compute

$$\psi = P_{22}^T \psi = P_{22}^T(\alpha \bar{f} + (1-\alpha)\bar{g}) = \begin{pmatrix} (1-\alpha)A_2^T g \\ \alpha A_1^T f \end{pmatrix} = \begin{pmatrix} (1-\alpha) f \\ \alpha g \end{pmatrix} ,$$

which yields $\alpha f = (1-\alpha) f$. Clearly, this implies $\alpha = \frac{1}{2}$.

Next we consider $\bar{\psi}$ and first note that

$$P_{22}^T \bar{\psi} = -\bar{\psi} \quad \text{and} \quad \bar{e}^T P_{22}^T = \bar{e}^T$$

implies $\bar{e}^T \bar{\psi} = 0$. Since

$$\bar{\psi} = \alpha \bar{f} + \beta \bar{g}$$

and $\bar{e}^T \bar{f} = \bar{e}^T \bar{g} = 1$ we obtain $\beta = -\alpha$. This proves that $\bar{f} - \bar{g}$ is an eigenvector of P_{22}^T to the eigenvalue -1. □

4.7. Note on Terminology "Recurrent" and "Transient". Consider again the random dynamics

$$u \in \mathcal{S}_L \to E_r u \in \mathcal{S}_L$$

in the set $\mathcal{S}_L = \mathcal{T}_L \cup \mathcal{R}_L$ of all stable states. As above, we order the states $u^1, \cdots, u^N$ in $\mathcal{S}_L$ so that $u^1, \ldots, u^{N-M} \in \mathcal{T}_L$ and $u^{N+1-M}, \ldots, u^N \in \mathcal{R}_L$. The Markov matrix P has the block structure (4.3). The following theorem is an immediate consequence of our results.

THEOREM 4.10. *Let $\pi^0 = (\pi_1^0, \ldots, \pi_N^0)^T$ denote any initial probability distribution of the states in $\mathcal{S}_L$, and let $\pi^n = (P^T)^n \pi^0$ denote the resulting probability distribution at time $t = n$. Then we have*

$$\lim_{n\to\infty} \pi_j^n = 0 \quad \text{if} \quad u^j \in \mathcal{T}_L \; ; \tag{4.13}$$

$$\limsup_{n\to\infty} \pi_j^n > \kappa_L > 0 \quad \text{if} \quad u^j \in \mathcal{R}_L \; . \tag{4.14}$$

The number $\kappa_L > 0$ depends only on L.

Proof: We write π^n in the partitioned form

$$\pi^n = \begin{pmatrix} \phi^n \\ \psi^n \end{pmatrix} , \quad \phi^n \in \mathbb{R}^{N-M}, \quad \psi^n \in \mathbb{R}^M \; .$$

Clearly,

$$\phi^n = (P_{11}^T)^n \phi^0 \; ,$$

and we have proved that $\rho(P_{11}) < 1$. Therefore, for any $\varepsilon > 0$ there is a constant $C(L, \varepsilon)$, independent of π^0 and of n, with

$$\sum_{j=1}^{N-M} \phi_j^n \le C(L,\varepsilon)(\rho(P_{11}) + \varepsilon)^n \; .$$

If $\varepsilon > 0$ is small enough, the left side of the above inequality tends to zero as $n \to \infty$. This proves (4.13).

Furthermore, we obtain

$$\sum_{j=1}^{M} \psi_j^{n_0} \geq \frac{1}{2}$$

if n_0 is large enough. Also,

$$\psi^{n+n_0} \geq (P_{22}^T)^n \psi^{n_0} \quad \text{for} \quad n \geq 0 \ .$$

We can apply the previous lemma to the sequence $(P_{22}^T)^n \psi^{n_0}, n = 1, 2, \ldots$ and obtain (4.14). □

Recall that π_j^n is the probability that the system is in the state u^j at time $t = n$, under the assumption that π^0 is a given probability distribution at time $t = 0$. The theorem says that the probability of the occurrence of a state u^j at time $t = n$ tends to zero as $n \to \infty$ if and only if $u^j \in \mathcal{T}_L$. If $u^j \in \mathcal{R}_L$, then the probability of the occurrence of u^j for arbitrarily large times remains bounded away from zero. *In this probabilistic sense, the states in $\mathcal{T}_L$ are transient, whereas the states in $\mathcal{R}_L$ are recurrent.*

The number $\kappa_L > 0$ gives a lower bound on the probability of the reoccurrence of the transient states. It would be interesting to know how κ_L behaves for increasing L.

4.8. An Odd/Even Shadow Dynamics. Recall that $\mathcal{R}_L$ is the set of M recurrent states and that P_{22} is the Markov matrix of the random evolution

$$u \in \mathcal{R}_L \to E_r u \in \mathcal{R}_L \ .$$

The matrix P_{22} has a unique stationary probability vector ψ,

$$P_{22}^T \psi = \psi, \quad \sum_i \psi_i = 1, \quad \psi > 0 \ .$$

The considerations in this subsection will give some limited information about ψ.

Let

$$\mathcal{K}_L = \{0, 1\}^L$$

denote the set of all vectors

$$\mu = (\mu_1, \ldots, \mu_L)$$

whose components μ_s are either 0 or 1. With every vector $u \in \mathcal{R}_L$ we associate the following vector in $\mathcal{K}_L$,

$$Qu = \mu \ ,$$

where $\mu_s = u_s \ mod \ 2$, i.e., the vector $\mu = Qu$ keeps track of the parity of the components u_s. The starting point of our considerations is the simple observation that toppling does not change the parity of any component of u.

For $1 \le r \le L$ define the operator

$$\bar{E}_r : \mathcal{K}_L \to \mathcal{K}_L$$

by

$$(\bar{E}_r \mu)_s = \mu_s \quad \text{if} \quad s \ne r, \quad (\bar{E}_r \mu)_r \ne \mu_r \ .$$

The fact that toppling does not affect Qu implies the equation

$$QE_r u = \bar{E}_r Qu, \quad u \in \mathcal{R}_L, \quad 1 \le r \le L \ . \tag{4.15}$$

If we assume, as before, that every dropping site $1 \le r \le L$ is chosen with equal probability, we obtain a rather simple random evolution in $\mathcal{K}_L$,

$$\mu \in \mathcal{K}_L \to \bar{E}_r \mu \in \mathcal{K}_L \ . \tag{4.16}$$

Because of (4.15), the simple evolution (4.16) can be considered as a shadow of the more complex evolution in $\mathcal{R}_L$. There are 2^L elements in $\mathcal{K}_L$. We denote the Markov matrix corresponding to the evolution (4.16) by $\bar{P}$. Thus, $\bar{P}$ has size $2^L \times 2^L$, and we use the notation $\bar{P}_{\mu\nu}$ to denote the probability of the one–time–step transition $\mu \to \nu$. Clearly,

$$\bar{P}_{\mu\nu} = \begin{cases} \frac{1}{L} & \text{if } \mu \text{ and } \nu \text{ differ in exactly one position} \\ 0 & \text{otherwise} \end{cases}$$

It follows that $\bar{P}$ is non–negative, irreducible, row–stochastic, and *symmetric.* (The two transitions

$$\mu \to \nu \quad \text{and} \quad \nu \to \mu$$

always have the same probability.) For this reason, *the vector $2^{-L}\bar{e}$ is the unique stationary probability vector for $\bar{P}$.* This has the following implication for stationary probability distribution of the dynamics in $\mathcal{R}_L$.

THEOREM 4.11. *Let ψ denote the stationary probability vector of P_{22}. For each $\mu \in \mathcal{K}_L$ we have*

$$\sum_{Qu^j = \mu} \psi_j = 2^{-L} \ . \tag{4.17}$$

Here the sum is taken over all indices $j \in \{1, \ldots, M\}$ with $Qu^j = \mu$.

Proof: Denote the entries of P_{22} by $p_{ij}, 1 \le i, j \le M$. Then we have

$$\sum_{i=1}^{M} \psi_i p_{ij} = \psi_j, \quad 1 \le j \le M \ .$$

Summation over all j with $Qu^j = \nu$ yields

$$\sum_{i=1}^{M} \sum_{Qu^j=\nu} \psi_i p_{ij} = \sum_{Qu^j=\nu} \psi_j =: \bar{\psi}_\nu \ ,$$

and therefore,

$$\sum_{\mu \in \mathcal{K}_L} \sum_{Qu^i=\mu} \sum_{Qu^j=\nu} \psi_i p_{ij} = \bar{\psi}_\nu \ . \tag{4.18}$$

We now evaluate the sum

$$\sum_{Qu^j=\nu} p_{ij} \ ,$$

which is the probability that u^i goes over into a state u^j with $Qu^j = \nu$ in one time step. We distinguish between two cases. In the first case, assume the row index i satisfies $Qu^i =: \mu$ where μ and ν differ in exactly one position. If the dropping index r agrees with this position, then $Q(E_r u^i) = \nu$, and if the dropping index r does not agree with this position, then $Q(E_r u^i) \neq \nu$. Therefore,

$$\sum_{Qu^j=\nu} p_{ij} = \frac{1}{L} \quad \text{if } Qu^i \text{ differs from } \nu \text{ in exactly one position.}$$

(In fact, the sum has exactly one non–zero entry.) In the second case, where $Qu^i = \mu$ does not differ from ν in exactly one position, a one–step transition $u^i \to u^j$ is impossible if $Qu^j = \nu$. Therefore,

$$\sum_{Qu^j=\nu} p_{ij} = 0 \quad \text{if } Qu^i \text{ does not differ from } \nu \text{ in exactly one position.}$$

In other words,

$$\sum_{Qu^j=\nu} p_{ij} = \bar{P}_{\mu\nu} \quad \text{if} \quad Qu^i = \mu \ .$$

Thus we obtain from (4.18),

$$\bar{\psi}_\nu = \sum_{\mu \in \mathcal{K}_L} \Big(\sum_{Qu^i=\mu} \psi_i \Big) \bar{P}_{\mu\nu} = \sum_{\mu \in \mathcal{K}_L} \bar{\psi}_\mu \bar{P}_{\mu\nu} \ . \tag{4.19}$$

This says that the vector $(\bar{\psi}_\mu)_{\mu \in \mathcal{K}_L}$ is a stationary probability vector for $\bar{P}$. However, since the vector $2^{-L}\bar{e}$ is the *only* stationary probability vector of $\bar{P}$, equation (4.17) follows. □

4.9. Example. Let $L = 2$. The even and odd states in $\mathcal{R}_L$ are as follows (we do not display the boundary values),

$$\begin{aligned} \mathcal{R}_L^{even} &= \{(22),(13),(24)\}, \quad M^e = 3 \ , \\ \mathcal{R}_L^{odd} &= \{(12),(23)\}, \quad M^o = 2 \ . \end{aligned}$$

The corresponding matrix P_{22} reads

$$P_{22} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} .$$

Note that (4.9) implies

$$P_{22}^2 = \begin{pmatrix} A_1 A_2 & 0 \\ 0 & A_2 A_1 \end{pmatrix} .$$

In the case $L = 2$ one obtains

$$A_1 A_2 = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad A_2 A_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

The rank 1 matrix $A_1 A_2$ has the eigenvalue zero of geometric multiplicity 2, and the simple eigenvalue one. The matrix $A_2 A_1$ has the simple eigenvalues zero and one. Since we already know that P_{22} has -1 as an eigenvalue, it follows that the eigenvalues of P_{22} are 0, 1, and -1. The eigenvalue zero has geometric multiplicity 3.

The Frobenius eigenvectors of $A = (A_1 A_2)^T$ and $B = (A_2 A_1)^T$ are

$$f = \frac{1}{4}(1,2,1)^T \quad \text{and} \quad g = \frac{1}{2}(1,1)^T$$

respectively.

5. The Average Slope. A quantity of interest is the average slope of the evolving sandpiles. For a single state $u \in \mathcal{R}_L$ the average slope is simply u_L/L. As in the previous section, let ψ, f, and g denote the Frobenius eigenvectors of P_{22}^T, A, and B, respectively. (See (4.11).) It will be convenient to index these vectors by their corresponding states. We discuss the following quantities

$$\sigma_L = \frac{1}{L} \sum_{u \in \mathcal{R}_L} \psi_u u_L \ , \tag{5.1}$$

$$\sigma_L^{even} = \frac{1}{L} \sum_{u \in \mathcal{R}_L^{even}} f_u u_L \ , \tag{5.2}$$

$$\sigma_L^{odd} = \frac{1}{L} \sum_{u \in \mathcal{R}_L^{odd}} g_u u_L \ . \tag{5.3}$$

Since

$$\psi = \frac{1}{2}\begin{pmatrix} f \\ g \end{pmatrix}$$

we obtain that

$$\sigma_L = \frac{1}{2}(\sigma_L^{even} + \sigma_L^{odd}) \ .$$

Example: In case $L = 2$ we have

$$\begin{aligned} \psi &= \frac{1}{8}(1,2,1,2,2)^T \\ f &= \frac{1}{4}(1,2,1)^T \\ g &= \frac{1}{2}(1,1)^T \end{aligned}$$

for the states

$$(22),(13),(24);(12),(23) \ .$$

One obtains

$$\sigma_2 = \frac{11}{8}, \quad \sigma_2^{even} = \frac{3}{2}, \quad \sigma_2^{odd} = \frac{5}{4} \ .$$

It would be interesting to know how the average slopes σ_L etc. behave for large L. In [3] it is claimed that

$$\sigma_L \approx \frac{3}{2} - kL^{-1/3} \tag{5.4}$$

for large L, but the arguments are not rigorous. The above relation is confirmed numerically, however [3]. In this paper we give a rigorous proof of the estimates

$$\sigma_L \le \frac{3}{2}, \quad \sigma_L^{even} \le \frac{3}{2}, \quad \sigma_L^{odd} \le \frac{3}{2} \ .$$

Our proof is based on a generalization of Theorem 4.11, which we show next.

5.1. A General Result on Stationary Probabilities. THEOREM 5.1. *Let $\mathcal{M}, \bar{\mathcal{M}}, \Lambda$ denote finite non–empty sets, and let $Q : \mathcal{M} \to \bar{\mathcal{M}}$ denote a map. Assume that for each $\lambda \in \Lambda$ maps*

$$E_\lambda : \mathcal{M} \to \mathcal{M}, \quad \bar{E}_\lambda : \bar{\mathcal{M}} \to \bar{\mathcal{M}}$$

are given so that

$$\bar{E}_\lambda Qu = QE_\lambda u \ , \quad \forall u \in \mathcal{M}, \quad \forall \lambda \in \Lambda \ .$$

Assuming that each $\lambda \in \Lambda$ is chosen with equal probability, the families of maps E_λ and $\bar{E}_\lambda$ determine random evolutions in $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively. The corresponding Markov matrices are denoted by

$$P = (p_{uv}) \quad \textit{and} \quad \bar{P} = (\bar{p}_{\mu\nu}) \ .$$

Let (ψ_u) denote a stationary probability vector for P, and assume that $\bar{P}$ has the unique stationary probability vector $(\bar{\psi}_\mu)$. Then we have

$$\sum_{Qu=\mu} \psi_u = \bar{\psi}_\mu \ , \quad \forall \mu \in \bar{\mathcal{M}} \ .$$

Proof: 1) If Λ has L elements, then

$$\bar{p}_{\mu\nu} = \frac{1}{L} \#\{\lambda \ : \ \bar{E}_\lambda \mu = \nu\} \ . \tag{5.5}$$

Fix $\nu \in \bar{\mathcal{M}}$ and $u \in \mathcal{M}$ and consider the sum

$$\sum_{Qv=\nu} p_{uv} =: sum$$

taken over all $v \in \mathcal{M}$ with $Qv = \nu$. Clearly, sum is the probability that u goes over into the set $\{v : Qv = \nu\}$ in one time step. Therefore,

$$sum = \frac{1}{L} \#\{\lambda \ : \ QE_\lambda u = \nu\} \ . \tag{5.6}$$

Set $Qu =: \mu$. Then, by assumption,

$$QE_\lambda u = \bar{E}_\lambda Qu = \bar{E}_\lambda \mu \ ,$$

and consequently the λ values occurring in the sets in (5.5) and (5.6) agree. This proves the equation

$$\sum_{Qv=\nu} p_{uv} = \bar{p}_{\mu\nu} \quad \text{if} \quad Qu = \mu \ . \tag{5.7}$$

2) Summing the identities

$$\sum_{u \in \mathcal{M}} \psi_u p_{uv} = \psi_v, \quad v \in \mathcal{M} \ ,$$

over all v with $Qv = \nu$, we obtain

$$\sum_u \sum_{Qv=\nu} \psi_u p_{uv} = \sum_{Qv=\nu} \psi_v =: h_\nu \ .$$

Now write the sum over $u \in \mathcal{M}$ as a double sum,

$$\sum_{\mu \in \bar{\mathcal{M}}} \sum_{Qu=\mu} \sum_{Qv=\nu} \psi_u p_{uv} = h_\nu \ .$$

Using (5.7) we find that

$$\sum_{\mu \in \bar{\mathcal{M}}} \Big(\sum_{Qu=\mu} \psi_u \Big) \bar{p}_{\mu\nu} = h_\nu \ ,$$

i.e.,

$$\sum_{\mu} h_\mu \bar{p}_{\mu\nu} = h_\nu \ .$$

This shows that h is a stationary probability vector for $\bar{P}$. The final result follows from our assumption that $\bar{\psi}$ is the *unique* stationary probability vector for $\bar{P}$. □

5.2. Evolution in $\mathcal{R}_L$. We first apply the previous theorem to the evolution in $\mathcal{R}_L$. For $u \in \mathcal{R}_L$ define the *slope vector* $v = Su$ by

$$v_s = u_s - u_{s-1}, \quad 1 \le s \le L \ ,$$

and note

$$u_L = \sum_{s=1}^{L} v_s \ .$$

Since u is stable, we have $v \le 2\bar{e}$. The vector $v = Su$ has a representation

$$v = \bar{e} + \mu - 2g \ , \tag{5.8}$$

where

$$\mu \in \mathcal{K}_L = \{0,1\}^L \quad \text{and} \quad g \in N^L$$

are uniquely determined. (Thus, the components of g are non–negative integers.) Now consider an evolution step

$$u \to u'' = E_r u = T^k R_r u = T^k u'$$

with intermediate step

$$u' = R_r u = u + e^r \ .$$

Here

$$e^r = (0, \ldots, 0, 1, 0, \ldots, 0), \quad e^r_r = 1 \ .$$

Let $v' = Su', v'' = Su''$ denote the corresponding slopes with decompositions

$$v' = \bar{e} + \mu' - 2g', \quad v'' = \bar{e} + \mu'' - 2g'' \ .$$

We have

$$v' = Su' = S(u + e^r) = v + Se^r$$

with

$$Se^r = e^r - e^{r+1} \quad \text{for} \quad r < L, \quad Se^L = e^L \ .$$

The equation $v' = v + Se^r$ implies

$$\mu' - 2g' = \mu - 2g + Se^r \ .$$

Therefore, the vectors μ and μ' differ exactly in positions r and $r+1$ if $r < L$ and differ exactly in position L if $r = L$. We define the corresponding operator

$$\bar{E}_r : \mathcal{K}_L \to \mathcal{K}_L$$

by

$$\begin{array}{rcll} (\bar{E}_r \mu)_s & = & \mu_s & \text{if} \quad s \notin \{r, r+1\} \ , \\ (\bar{E}_r \mu)_s & \neq & \mu_s & \text{if} \quad s \in \{r, r+1\} \ . \end{array}$$

Our derivation shows that $\mu' = \bar{E}_r \mu$.

Since toppling changes each site by an even integer, we have

$$u'' = u' + 2h, \quad h \in \mathbb{Z}^L \ .$$

Therefore,

$$v'' = Su'' = v' + 2Sh \ ,$$

which implies that $\mu'' = \mu'$. Defining the operator

$$Q : \mathcal{R}_L \to \mathcal{K}_L$$

by the assignment $Qu = \mu$, our derivation shows that

$$QE_r u = \bar{E}_r Q u, \quad u \in \mathcal{R}_L, \quad 1 \le r \le L \ .$$

Next we consider the Markov matrix $\bar{P} = (\bar{p}_{\mu\nu})$ corresponding to the random evolution determined by

$$\bar{E}_r : \mathcal{K}_L \to \mathcal{K}_L, \quad 1 \le r \le L \ .$$

LEMMA 5.2. *The matrix $\bar{P}$ is non–negative, row–stochastic, irreducible, and symmetric.*

Proof: Clearly, $\bar{P}$ is non–negative and $\bar{P}\bar{e} = \bar{e}$. Now let $\mu, \nu \in \mathcal{K}_L$ be arbitrary. Then there exists $1 \le r \le L$ with $\bar{E}_r \mu = \nu$ if and only if either μ and ν differ in exactly two positions $r, r+1$ with $r < L$ or μ and ν differ exactly in position L. For such pairs μ, ν we have $\bar{p}_{\mu\nu} = \frac{1}{L}$, and $\bar{p}_{\mu\nu} = 0$ otherwise. This shows that $\bar{P}$ is symmetric.

To show that $\bar{P}$ is irreducible, we let $\mu, \nu \in \mathcal{K}_L$ be arbitrary with $\mu \ne \nu$. Set

$$\mu^{(2)} = \begin{cases} \bar{E}_1 \mu & \text{if } \mu_1 \ne \nu_1 \\ \mu & \text{if } \mu_1 = \nu_1 \end{cases}$$

Then we have $\mu_1^{(2)} = \nu_1$ and set

$$\mu^{(3)} = \begin{cases} \bar{E}_2 \mu^{(2)} & \text{if } \mu_2^{(2)} \ne \nu_2 \\ \mu^{(2)} & \text{if } \mu_2^{(2)} = \nu_2 \end{cases}$$

etc. It follows that there exist $r_1, \ldots, r_n$ with

$$\nu = \bar{E}_{r_n} \cdots \bar{E}_{r_1} \mu \ ,$$

proving that $\bar{P}$ is irreducible. □

Because of the previous lemma, the vector $\bar{\psi} = 2^{-L}\bar{e}$ is the unique stationary probability vector for $\bar{P}$, and Theorem 5.1 yields

$$\sum_{Qu=\mu} \psi_u = 2^{-L}, \quad \mu \in \mathcal{K}_L \ .$$

Now consider

$$\sigma_L = \frac{1}{L} \sum_{\mu \in \mathcal{K}_L} \sum_{Qu=\mu} \psi_u u_L \ .$$

If

$$v = Su = \bar{e} + \mu - 2g, \quad \mu = Qu \ ,$$

is the slope vector of u, then

$$u_L = \sum_{s=1}^{L} v_s \le L + \sum_{s=1}^{L} \mu_s$$

since $g \ge 0$. One obtains

$$\begin{aligned} \sigma_L &\le 1 + \frac{1}{L} \sum_{\mu \in \mathcal{K}_L} \sum_{Qu=\mu} \psi_u \sum_{s=1}^{L} \mu_s \\ &= 1 + \frac{2^{-L}}{L} \sum_{\mu \in \mathcal{K}_L} \sum_{s=1}^{L} \mu_s \ . \end{aligned}$$

It is easy to show that

$$\sum_{\mu \in \mathcal{K}_L} \sum_{s=1}^{L} \mu_s = 2^L \, \frac{L}{2} \; . \tag{5.9}$$

(Group the 2^L elements of $\mathcal{K}_L$ into pairs $\mu, \bar{\mu}$ with $\mu + \bar{\mu} = \bar{e}$.) Combining (5.9) with the above estimate of σ_L, we obtain

$$\sigma_L \leq \frac{3}{2} \; .$$

5.3. Evolution in $\mathcal{R}_L^{even}$ and in $\mathcal{R}_L^{odd}$. A random evolution step in $\mathcal{R}_L^{even}$ has the form

$$u \to E_{r'} E_r u, \quad 1 \leq r, r' \leq L \; ,$$

and Theorem 5.1 can be applied with

$$\Lambda = \{(r, r') \; : \; 1 \leq r, r' \leq L\} \; .$$

The operator $Q : \mathcal{R}_L^{even} \to \mathcal{K}_L$ is defined as in the previous subsection, but is restricted to $\mathcal{R}_L^{even}$. Furthermore,

$$E_\lambda = E_{r'} E_r, \quad \bar{E}_\lambda = \bar{E}_{r'} \bar{E}_r \quad \text{for} \quad \lambda = (r, r') \; .$$

The Markov matrix of the family $\bar{E}_\lambda$ is

$$\bar{P}^{even} = \bar{P}^2 \; ,$$

where $\bar{P}$ is the Markov matrix of $\bar{E}_r$ as defined in the previous subsection. Therefore, $2^{-L}\bar{e}$ is the unique stationary probability vector for $\bar{P}^{even}$. The remaining arguments are the same as in the previous subsection, showing that

$$\sigma_L^{even} \leq \frac{3}{2} \; .$$

The proof of

$$\sigma_L^{odd} \leq \frac{3}{2}$$

is similar.

6. Simulation. First two figures in Fig. 6.1 show two configurations of sandpiles with $L = 25$. These were reached during an evolution starting with the flat sandpile $u = 0$ after $n = 100$ and $n = 1,000$ time steps, respectively. The state in the first figure is not critical, whereas state in the second figure shows a critical state. The rest two figures in Fig. 6.1 show the number of topplings that occurred for two different simulations, both starting at $u = 0$ and using $L = 25$. Clearly, these two figures differ in their details, as must be expected for a random evolution, but appear to share some *statistical features*, at least in a non–technical sense.

In Fig. 6.2 we show the evolution of the average slope $\frac{1}{L}u_L^{(n)}$ as a function of time n. We always started at $u^{(0)} = 0$ and made runes for $L = 20, 40, 80, 160$. Roughly speaking, the quantity $\frac{1}{L}u_L^{(n)}$ first increases linearly as a function of n. At $n = cL^2$ with $c \approx 1.3$, the increase stops and then $\frac{1}{L}u_L^{(n)}$ fluctuates about σ_L.

For $L = 160$ it took approximately $40,000$ evolution steps until criticality was reached, which required about $60Mflops$ of computation.

A. The Number of Stable States. We denote $\#\mathcal{S}_L$ by N_L and let $N_{L,j}$ denote the number of stable states $u \in \mathcal{S}_L$ with $u_L = j$, $j = 0, 1, \ldots, 2L$.

For each stable state $u \in \mathcal{S}_{L-1}$ with $u_{L-1} = j$, we may append 0, 1, 2, ..., j+2 at site L to obtain $j + 3$ stable states of $\mathcal{S}_L$. Therefore, we have the following formula:

$$N_L = \sum_{j=0}^{2(L-1)} (j+3) N_{L-1,j} \tag{A1}$$

The numbers $N_{i,j}$ for $i = 1, 2$ are as follows:

$$N_{1,0} = N_{1,1} = N_{1,2} = 1 \ , \tag{A2}$$

$$N_{2,0} = N_{2,1} = N_{2,2} = N_1 = 3, \quad N_{2,3} = 2, \quad N_{2,4} = 1 \ . \tag{A3}$$

We will now derive formulae, which will allow to compute $N_{i,j}$ for $0 \le j \le 2i$ (and then N_i) from $N_{i-1,j}$ and N_{i-1} and N_{i-2}.

Consider the projection

$$p : \mathcal{S}_i \to \mathcal{S}_{i-1}$$

where, for each $u \in \mathcal{S}_i$, $p(u)$ is the truncated state obtained from u by dropping the last component. Clearly, p establishes a 1-1 correspondence between the set of stable states $u \in \mathcal{S}_i$ with $u_i = 0$ (or 1 or 2) and the set $\mathcal{S}_{i-1}$. This implies that

$$N_{i,0} = N_{i,1} = N_{i,2} = N_{i-1} \ . \tag{A4}$$

Also, p establishes a 1-1 correspondence between the set of stable states $u \in \mathcal{S}_i$ with $u_i = 3$, and the set of stable states $v \in \mathcal{S}_{i-1}$ with $v_{i-1} \neq 0$; therefore,

$$N_{i,3} = N_{i-1} - N_{i-1,0} = N_{i-1} - N_{i-2} \ . \tag{A5}$$

Similarly,

$$N_{i,4} = N_{i-1} - N_{i-1,0} - N_{i-1,1} = N_{i-1} - 2N_{i-2} \ , \qquad (A6)$$

$$N_{i,5} = N_{i-1} - N_{i-1,0} - N_{i-1,1} - N_{i-1,2} = N_{i-1} - 3N_{i-2} \ . \qquad (A7)$$

Now consider $N_{i,j}$ for $j = 6, 7, \ldots, 2i-1$. For each stable state $u \in \mathcal{S}_i$ with $u_i = j-1$, we may redefine $u_i = j$ without changing stability unless $u_{i-1} = j-3$. Therefore,

$$N_{i,j} = N_{i,j-1} - N_{i-1,j-3} \ . \qquad (A8)$$

Finally, there is only one stable state in $\mathcal{S}_i$ with $u_i = 2i$, thus

$$N_{i,2i} = 1 \ . \qquad (A9)$$

B. Example When $L = 3$. When $L = 3$, there are 55 stable states, 41 of which are transient and 14 of which are recurrent. There are 7 even and 7 odd recurrent states, namely:

$$\begin{aligned} \mathcal{R}_L^{even} &= \{(123),(233),(224),(134),(244),(235),(246)\}, \\ \mathcal{R}_L^{odd} &= \{(223),(133),(243),(124),(234),(135),(245)\}. \end{aligned}$$

The matrix P_{22} corresponding to the evolution in the set of recurrent states $\mathcal{R}_L$ is

$$P_{22} = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$$

with

$$A_1 = \frac{1}{3}\begin{pmatrix} 1&1&0&1&0&0&0 \\ 0&1&1&0&1&0&0 \\ 0&0&1&1&1&0&0 \\ 0&0&0&1&1&1&0 \\ 0&0&0&1&1&0&1 \\ 0&0&0&0&1&1&1 \\ 0&0&0&1&1&0&1 \end{pmatrix}, \quad A_2 = \frac{1}{3}\begin{pmatrix} 1&1&1&0&0&0&0 \\ 1&1&0&1&0&0&0 \\ 1&1&0&0&1&0&0 \\ 1&0&1&1&0&0&0 \\ 0&0&0&1&1&1&0 \\ 1&0&0&1&0&1&0 \\ 1&0&0&0&0&1&1 \end{pmatrix}$$

Hence,

$$(P_{22}^T)^2 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with

$$A = \frac{1}{9}\begin{pmatrix} 3&2&2&2&2&2&2 \\ 2&2&1&0&0&0&0 \\ 2&0&1&1&1&0&1 \\ 2&2&2&3&2&2&2 \\ 0&2&2&1&1&1&1 \\ 0&1&1&2&2&3&2 \\ 0&0&0&0&1&1&1 \end{pmatrix}, \quad B = \frac{1}{9}\begin{pmatrix} 1&1&1&1&0&1&1 \\ 2&2&2&1&0&1&1 \\ 2&1&1&1&0&0&0 \\ 2&2&2&3&2&2&2 \\ 2&2&2&2&3&2&2 \\ 0&1&0&1&2&2&1 \\ 0&0&1&0&2&1&2 \end{pmatrix}$$

The stationary probability vector of the matrix P_{22} is

$$\begin{array}{rllllllll} \psi = (& 1/8, & 11/252, & 1/18, & 1/8, & 48/911, & 41/504, & 19/1134, & \\ & 1/24, & 85/1512, & 25/756, & 1/8, & 1/8, & 13/189, & 19/378 &)^T \end{array}$$

The matrices A and B have the same eigenvalues, namely

$$1,\ 2/9,\ 1/9,\ 1/9,\ 1/9,\ 0,\ 0$$

The two Frobenius eigenvectors are

$$\begin{array}{rllllllll} f = (& 1/4, & 11/126, & 1/9, & 1/4, & 96/911, & 41/252, & 19/567 &)^T \\ g = (& 1/12, & 85/756, & 25/378, & 1/4, & 1/4, & 26/189, & 19/189 &)^T \end{array}$$

One obtains,

$$\sigma_3^{even} = \frac{2207}{1701} \approx 1.2975 \ , \quad \sigma_3^{odd} = \frac{167}{126} \approx 1.3254 \ .$$

Starting from the flat sandpile $u = (000)$, it takes about 11 time steps before a recurrent state is reached.

According to our theory, the probability distributions at even/odd time steps approach

$$\psi^{2n} \rightarrow \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad \psi^{2n+1} \rightarrow \begin{pmatrix} 0 \\ g \end{pmatrix}$$

as $n \rightarrow \infty$. Therefore, when one averages in time over even n and odd n, the average slope should approach the two quantities σ_3^{even} and σ_3^{odd}, respectively. Indeed, we have computed these time– averages of average slopes over even/odd time steps for the interval between $n_0 = 3L^2 + 1 = 28$ and $n_1 = 1027$. Our results were 1.2968 and 1.3243, respectively. This is in good agreement with the values given above based on the Frobenius eigenvectors.

C. The Evolution in Terms of Slopes and Trapping Sites. Let $u = (u_j)_{0 \le j \le L+1}$ denote a stable state which satisfies the boundary conditions

$$u_0 = 0, \quad u_{L+1} = u_L \ .$$

Its slope vector $v = (v_j)_{1 \le j \le L+1}$ is defined by

$$v_j = u_j - u_{j-1}, \quad 1 \le j \le L+1 \ ,$$

and we set

$$\epsilon_j = v_j - 1, \quad 1 \le j \le L+1 \ .$$

Note that $\epsilon_{L+1} = -1$ and $\epsilon_j \le 1, \forall j$.

A site j is called a *trapping site* if $\epsilon_j < 0$. Clearly, from $\epsilon = (\epsilon_j)$ we can recover u, and therefore the evolution can be formulated in terms of ϵ. Let ϵ be given, and let $1 \le j \le L$ denote the dropping site. We use the notation $\epsilon' = \tilde{E}_j \epsilon$ for the next ϵ–vector. Then ϵ' can be obtained as follows:

Case 1, $\epsilon_j \le 0$: In this case no avalanche is generated. We have

$$\begin{aligned} \epsilon'_j &= \epsilon_j + 1 \;, \\ \epsilon'_{j+1} &= \epsilon_{j+1} - 1 \;, \\ \epsilon'_{L+1} &= -1 \;. \end{aligned}$$

Case 2, $\epsilon_j = 1$ and $\epsilon_{j+1} \le 0$: In this case an avalanche occurs, but there is no back avalanche. Let k denote the *nearest trapping site to the left* of j. (I.e., k is the largest integer with $1 \le k < j$ and $\epsilon_k < 0$; if no such k exists, set $k = 0$.) We have

Case 2a, $k = 0$:

$$\begin{aligned} \epsilon'_j &= \epsilon_j - 1 = 0 \;, \\ \epsilon'_{j+1} &= \epsilon_{j+1} + 1 \;, \\ \epsilon'_{L+1} &= -1 \;. \end{aligned}$$

Case 2b, $k > 0$:

$$\begin{aligned} \epsilon'_k &= \epsilon_k + 2 \;, \\ \epsilon'_{k+1} &= \epsilon_{k+1} - 2 \;, \\ \epsilon'_j &= \epsilon_j - 1 \;, \\ \epsilon'_{j+1} &= \epsilon_{j+1} + 1 \;, \\ \epsilon'_{L+1} &= -1 \;. \end{aligned}$$

Remark: If $k + 1 = j$, then the two settings

$$\begin{aligned} \epsilon'_{k+1} &= \epsilon_{k+1} - 2 \;, \\ \epsilon'_j &= \epsilon_j - 1 \;, \end{aligned}$$

have to be replaced by

$$\epsilon'_j = \epsilon_j - 3 \;.$$

Case 3, $\epsilon_j = 1$ and $\epsilon_{j+1} = 1$: In this case, there is a back avalanche. Let $k < j$ be determined as in Case 2, and let m denote the *nearest trapping site to the right* of j. (I.e., m is the smallest integer with $m > j + 1$ and $\epsilon_m < 0$; such an m always exists since $\epsilon_{L+1} = -1$.)

Case 3a, $k = 0$:

$$\begin{aligned} \epsilon'_j &= \epsilon_j - 1 = 0 \;, \\ \epsilon'_{j+1} &= \epsilon_{j+1} - 1 = 0 \;, \\ \epsilon'_m &= \epsilon_m + 2 \;, \\ \epsilon'_{L+1} &= -1 \;. \end{aligned}$$

Case 3b, $k > 0$:

$$\begin{aligned}
\epsilon'_k &= \epsilon_k + 2 \ , \\
\epsilon'_m &= \epsilon_m + 2 \ , \\
\epsilon'_j &= \epsilon_j - 1 \ , \\
\epsilon'_{j+1} &= \epsilon_{j+1} - 1 \ , \\
\epsilon'_{k+m-j} &= \epsilon_{k+m-j} - 2 \ , \\
\epsilon'_{L+1} &= -1 \ .
\end{aligned}$$

Remark: If $k + m - j = j$, then the two settings

$$\begin{aligned}
\epsilon'_{k+m-j} &= \epsilon_{k+m-j} - 2 \ , \\
\epsilon'_j &= \epsilon_j - 1 \ ,
\end{aligned}$$

have to be replaced by

$$\epsilon'_j = \epsilon_j - 3 \ .$$

Similarly, if $k + m - j = j + 1$, then the two settings

$$\begin{aligned}
\epsilon'_{k+m-j} &= \epsilon_{k+m-j} - 2 \ , \\
\epsilon'_{j+1} &= \epsilon_{j+1} - 1 \ ,
\end{aligned}$$

have to be replaced by

$$\epsilon'_{j+1} = \epsilon_{j+1} - 3 \ .$$

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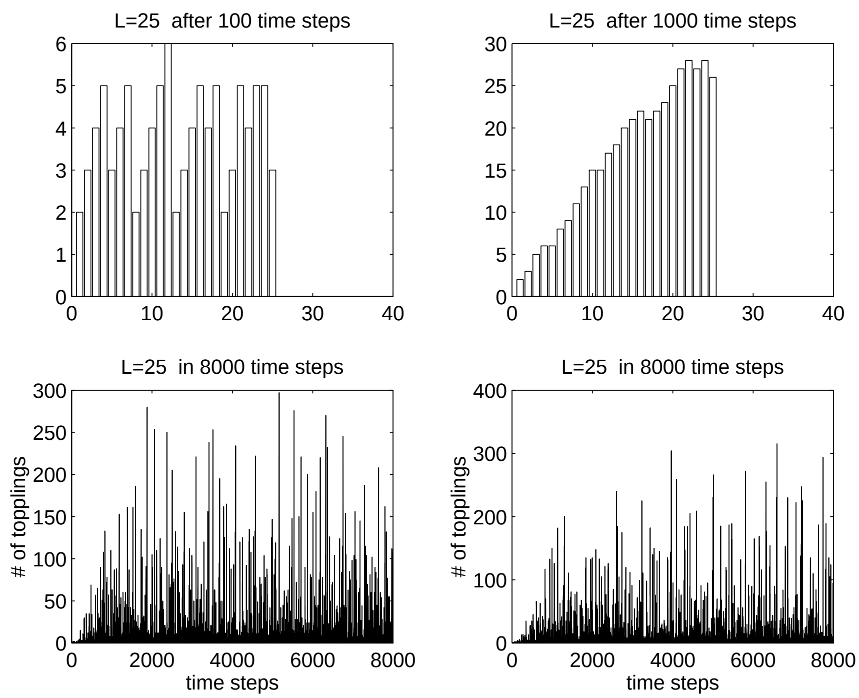

FIG. 6.1. *Configurations and size of avalanches*

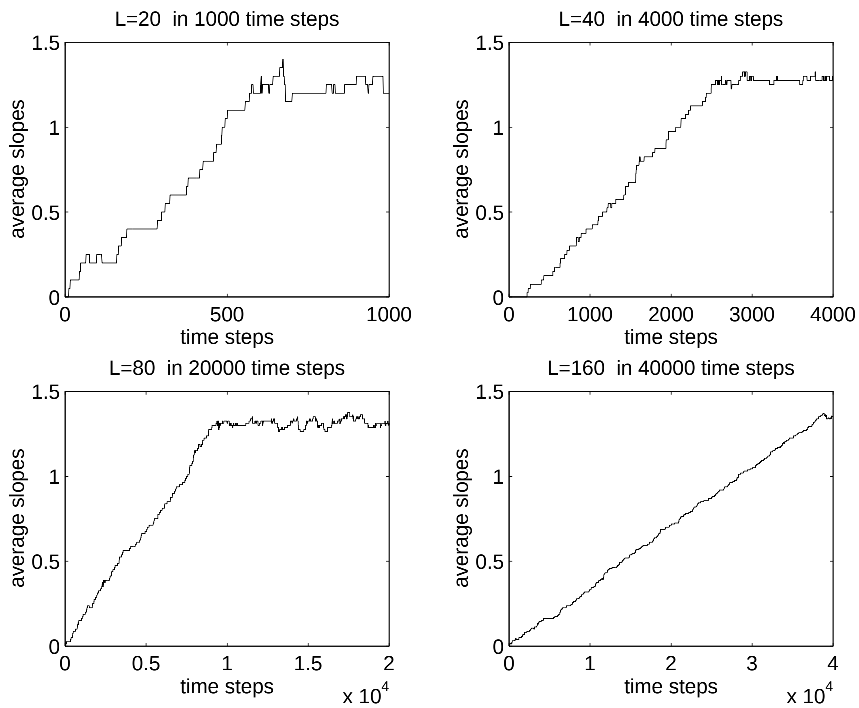

FIG. 6.2. *Average slopes*