INVISCID LIMITS AND REGULARITY ESTIMATES FOR THE
SOLUTIONS OF THE 2-D DISSIPATIVE
QUASI-GEOSTROPHIC EQUATIONS

By

Jiahong Wu

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455
Inviscid Limits and Regularity Estimates for the Solutions of the 2-D Dissipative Quasi-geostrophic Equations

Jiahong Wu
School of Mathematics
The Institute for Advanced Study
Princeton, NJ 08540

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Abstract

We discuss two important topics of turbulence theory: inviscid limit and decay of Fourier spectrum for the 2-D dissipative quasi-geostrophic (QGS) equations. In the first part we consider inviscid limits for both smooth and weak solutions of the 2-D dissipative QGS equations and prove that the classical solutions with smooth initial data tend to the solutions of the corresponding non-dissipative equations as the dissipative coefficient tends to zero. Here the convergence is in the strong $L^2$ sense and we give the optimal convergence rate. For the weak solutions of the dissipative QGS equations with $L^2$ initial data, we obtain weak $L^2$ inviscid limit result. In the second part we use the methods of Foias-Temam [8] and Doering-Titi [7] developed for the Navier-Stokes equations to establish exponential decay of spatial Fourier spectrum for the solutions of the dissipative QGS equations, but we treat general norms and our method of estimating the nonlinear terms are different.
1 Introduction

We consider the 2-D surface quasi-geostrophic (QGS) equations

\[ \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \quad 0 \leq \alpha \leq 1 \]

where \( \kappa > 0 \) in the case of the dissipative equations and \( \kappa = 0 \) for the non-dissipative equations. Here the velocity \( u = (u_1, u_2) \) is determined from \( \theta \) by a stream function \( \psi \):

\[ (u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right) \]

where \( \psi \) satisfies

\[ (-\Delta)^{\frac{1}{2}} \psi = -\theta \]

These equations have been under intensive investigations because of mathematical importance and potential applications in meteorology and oceanography ([3],[9],[10]). As pointed out in [3], the 2-D dissipative and non-dissipative QGS equations are strikingly analogous to the 3-D Navier-Stokes and the Euler equations. Inviscid limit results for the Navier-Stokes equations with smooth and rough initial data have been established ([1],[2], [4],[5],[6]). Naturally the inviscid limit problem for the solutions of the dissipative QGS equations rises and so far we have seen no work in this direction.

We know from [3] that the QGS equations (defined on \( \mathbb{R}^2 \) or \( \mathbb{T}^2 \)) with smooth initial data admit unique classical solutions for short times and the quantity

\[ \int_0^T \| \nabla \theta(\cdot, s) \|_{L^\infty} ds \]

is responsible for possible singularity formation. The control of this quantity is also important in our proof of the inviscid limit results for smooth solutions.

Both dissipative and non-dissipative QGS equations on \( \mathbb{T}^2 \) with \( L^2 \) initial data have weak solutions in the distribution sense ([11]). Inviscid limits for weak solutions are in general hard to obtain and in this case we only obtain a weak \( L^2 \) result without a rate. We believe that explicit rates can be given in certain negative Sobolev norms.
The second part is devoted to regularity estimates for the dissipative QGS equations. Although we use the ideas of Foias-Temam [8] and Doering-Titi [7] developed for the Navier-Stokes equations, our methods of estimating the non-linear terms is significantly different from theirs because of the special structure of the QGS equations. We treat the norms $\|e^{it\Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}$ (see notations in Section 3) and a special consequence of our estimates for $\|e^{it\Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}$ is that as long as $\|\Lambda^\beta \theta\|_{L^2}^2$ remains bounded, the Fourier spectrum decays exponentially at high wave numbers. In a similar fashion as Doering and Titi [7] argue for the Navier-Stokes equations, these exponential decay estimates can be used to obtain bounds on small length scales defined through the exponential decay rate.

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2 Inviscid Limits

We consider the 2-D dissipative ($\kappa > 0$) and the non-dissipative ($\kappa = 0$) quasi-geostrophic equations on $\mathbb{R}^2$ or $\mathbb{T}^2$:

$$ \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa \Lambda^{2\alpha} \theta = 0, \quad 0 \leq \alpha \leq 1 $$

where $\Lambda = (-\Delta)^{1/2}$ is the Riesz potential operator and the velocity $u$ is defined from the stream function $\psi = -\Lambda^{-1} \theta$ by

$$ u = (u_1, u_2) = \left( -\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right) $$

We use $\mathbb{D}^2$ to denote either $\mathbb{R}^2$ or $\mathbb{T}^2$. We first consider the smooth initial data case: $\theta_0 \in H^k(\mathbb{D}^2)(k \geq 3)$. As shown by Constantin, Majda and Tabak [3], the QGS equations with smooth initial data have local (in time) smooth solutions and the Beale-Kato-Majda type blowup conditions have been obtained. More precisely,

**Proposition 2.1** If the initial data $\theta|_{t=0} = \theta_0 \in H^k(\mathbb{D}^2)$ for some $k \geq 3$, then both the QGS and the dissipative QGS equations have a unique smooth solution for a small time interval, respectively. Furthermore, the solution $\theta^{QG}$ of the QGS equations satisfies

$$ \int_0^t \| \nabla \theta^{QG}(\cdot, s) \|_{L^\infty} ds < \infty, $$

$$ \int_0^t \| \theta^{QG}(\cdot, s) \|_{L^2}^2 ds < \infty $$

for any $t$ belonging to the existence interval $[0, T^*)$.

This proposition admits the possibility of finite-time singularity formation and consequently, the inviscid limit results for the smooth solutions are valid only for the time period before the possible breakdown. We need further estimates on the solutions.
Proposition 2.2 Let $\theta^Q$ and $\theta^{DQ}$ be the smooth solutions of the QGS and the dissipative QGS equations with the same initial data $\theta_0 \in H^k(D^2)(k \geq 3)$. Respectively, $u^Q$ and $u^{DQ}$ are the corresponding velocities and $\psi^Q$ and $\psi^{DQ}$ are the stream functions. Then for any $t$ in the maximal time interval $[0, T^*)$ (when the smooth solutions exist),

1. The solution $\theta^Q$ of the QGS equation satisfies:

$$\int_{D^2} G(\theta^Q(x,t))dx = \int_{D^2} G(\theta_0)dx,$$

where $G$ is a continuous function with $G(0) = 0$. Especially,

$$\|\theta^Q(\cdot,t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad 1 \leq p \leq \infty,$$

furthermore,

$$\|u^Q(\cdot,t)\|_{L^2} = \|\theta^Q(\cdot,t)\|_{L^2} = \|\theta_0\|_{L^2},$$

$$\|u^Q(\cdot,t)\|_{L^q} \leq C_q\|\theta^Q(\cdot,t)\|_{L^q}, \quad 1 < q < \infty,$$

where $C_q$ is a constant depending on $q$.

2. The solution $\theta^{DQ}$ of the dissipative QGS equation obeys

$$\|\theta^{DQ}(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad 1 \leq p \leq \infty,$$

$$\|u^{DQ}(\cdot,t)\|_{L^q} \leq C_q\|\theta^{DQ}(\cdot,t)\|_{L^q}, \quad 1 < q < \infty.$$

Proof. The proof of this proposition is classical. The $L^p$ estimates are the classical inequality for the Calderon-Zygmund singular integrals while the $L^p$ bounds come from energy estimates. We omit details.

We now state the inviscid limit theorem for smooth solutions.

Theorem 2.3 Let $\theta^Q$ and $\theta^{DQ}$ be the smooth solutions of the QGS equations and the dissipative QGS equations with the same initial data $\theta_0 \in H^k(D^2)(k \geq 3)$. If $[0, T^*)$ is the maximal time interval of smooth existence, then for any $t < T^*$,

$$\|\theta^Q(\cdot,t) - \theta^{DQ}(\cdot,t)\|_{L^2(D^2)} \leq C \kappa,$$

where $C$ is a constant depending on $\theta_0$ and $T^*$ only.
The convergence rate $O(\kappa)$ is optimal. In the proof we only treat the case $\mathbb{D}^2 = \mathbb{R}^2$ and the case $\mathbb{D}^2 = \mathbb{T}^2$ is easier.

**Proof of Theorem 2.3.** Consider the difference 

$$\theta(x, t) = \theta^{DQG}(x, t) - \theta^{QG}(x, t)$$

between the solutions of the QGS and the dissipative QGS equations and let $u(x, t)$ be the corresponding velocity difference. This difference $\theta$ satisfies

$$\frac{\partial \theta}{\partial t} + u^{DQG} \cdot \nabla \theta + u \cdot \nabla \theta^{QG} + \kappa \Lambda^{2\alpha}(\theta + \theta^{QG}) = 0$$

where $\Lambda^{2\alpha} = (-\Delta)^\alpha$. Multiplying by $\theta$ and integrating in space

$$\frac{1}{2} \frac{d}{dt} \int \theta^2 \, dx + \kappa \int (\Lambda^{2\alpha} \theta) \theta \, dx = I + II + III$$

where

$$I = \int (u^{DQG} \cdot \nabla \theta) \theta \, dx, \quad II = \int (u \cdot \nabla \theta^{QG}) \theta \, dx$$

$$III = \kappa \int (\Lambda^{2\alpha} \theta^{QG}) \theta \, dx$$

We estimate these three terms and start with the first one. Clearly the estimates in Proposition 2.2 guarantee that $I$ is integrable. We show that it is actually zero. Let $\chi$ be a smooth cut-off function: $\chi(x) = 1$ if $|x| < 1$ and $\chi(x) = 0$ if $|x| > 2$ and $\chi_r = \chi(r^2)$ for $r > 0$. Using the Dominated Convergence Theorem and the divergence theorem,

$$I = \lim_{r \to \infty} \int (u^{DQG} \cdot \nabla \theta) \theta \chi_r \, dx = - \lim_{r \to \infty} \frac{1}{2r} \int \chi' \cdot u^{DQG} \theta^2 \, dx$$

since the last integral is bounded,

$$| \int \chi' \cdot u^{DQG} \theta^2 \, dx | \leq \int |u^{DQG}| \theta^2 \, dx \leq \| \theta \|^2_{L^4} \leq 4 \| \theta_0 \|_{L^4} \| \theta_0 \|^2_{L^4}$$

we obtain $I = 0$. $II$ and $III$ can be estimated by using Proposition 2.2,

$$|II| \leq \| \nabla \theta^{QG} \|_{L^\infty} \| u \|_{L^2} \| \theta \|_{L^2} = \| \nabla \theta^{QG} \|_{L^\infty} \| \theta \|^2_{L^2}$$

6
\[ |III| \leq \frac{\kappa^2}{2} \int (\Lambda^{2\alpha} \theta^{QG})^2 dx + \frac{1}{2} \int \theta^2 dx \]

Collecting these estimates,

\[ \frac{d}{dt} \int \theta^2 dx + \kappa \| \Lambda^\alpha \theta \|_{L^2}^2 dx \leq \mathcal{P}(t) \| \theta \|_{L^2}^2 + \kappa^2 \| \theta^{QG} \|_{2\alpha}^2 \]

where

\[ \mathcal{P}(t) = 2 \| \nabla \theta^{QG}(\cdot, t) \|_{L^\infty} + 1 \]

By Gronwall’s inequality,

\[ \| \theta \|_{L^2}^2 \leq e^{\int_0^t \mathcal{P}(s) ds} \| \theta_0 \|_{L^2}^2 + \kappa^2 \int_0^t e^{\int_s^t \mathcal{P}(\tau) d\tau} \| \theta^{QG} \|_{2\alpha}^2 d\tau \]

Noting that \( \theta_0 = 0 \) and using the result of Proposition 2.1, especially,

\[ \int_0^t \| \nabla \theta^{QG}(\cdot, s) \|_{L^\infty} ds < \infty, \quad \int_0^t \| \theta^{QG}(\cdot, s) \|_{L^2}^2 ds < \infty \]

we obtain

\[ \| \theta \|_{L^2} \leq C\kappa, \]

which completes the proof of Theorem 2.3.

We now turn to weak solutions of these equations corresponding to \( L^2 \) initial data. We restrict ourselves to the periodic domain \( T^2 = [0, L] \times [0, L] \). We quote the result of Resnick [11] on the existence of weak solutions.

**Proposition 2.4** Let \( \theta_0 \in L^2(T^2) \) and \( T > 0 \) be arbitrarily fixed. Then there exist weak solutions \( \theta^{QG} \in L^\infty([0, T]; L^2(T^2)) \) and \( \theta^{DQG} \in L^\infty([0, T]; L^2(T^2)) \cap L^2([0, T]; H^\alpha(T^2)) \) of the QGS and the dissipative QGS equations, respectively. That is, for each test function \( \phi \in C^\infty(T^2) \),

\[ \int \theta^{QG} \phi dx - \int \theta_0 \phi dx - \int_0^T \int_{T^2} \theta^{QG} (u^{QG} \cdot \nabla \phi) dx dt = 0, \]

\[ \int \theta^{DQG} \phi dx - \int \theta_0 \phi dx - \int_0^T \int_{T^2} \theta^{DQG} (u^{DQG} \cdot \nabla \phi) dx dt = 0, \]

where \( u^{QG} \) and \( u^{DQG} \) are the velocities corresponding to \( \theta^{QG} \) and \( \theta^{DQG} \).
These weak solutions are constructed by using classical Galerkin approximations. The weak $L^2$ inviscid limit result is an easy consequence of this construction method.

**Theorem 2.5** Let $\theta_0 \in L^2(\mathbb{T}^2)$ and $\theta^{QG}$ and $\theta^{DQG}$ be the weak solutions of the QGS and the dissipative QGS equations with the same initial data $\theta_0$. Then for any arbitrarily fixed $T > 0$ and any $\phi \in L^2(\mathbb{T}^2)$,

$$\limsup_{\kappa \to 0} \langle \theta^{DQG}(\cdot, t) - \theta^{QG}(\cdot, t), \phi \rangle = 0, \quad \text{for any } t \leq T,$$

(1)

**Proof** Consider the $n$-th Galerkin approximation $\{\theta_n^{QG}\}$ and $\{\theta_n^{DQG}\}$, which are in the space $S_n$ spanned by the Fourier modes $e^{imx}$ with $0 < |m| \leq n$ and satisfy

$$\frac{\partial \theta_n}{\partial t} + P_n(u_n \cdot \nabla \theta_n) + \kappa \Lambda^{2a} \theta_n = 0,$$

$$\theta_n|_{t=0} = P_n \theta_0,$$

where $P_n$ is the orthogonal projection from $L^2$ onto $S_n$ and $\kappa = 0$ in the case of $\theta^{QG}$. As we know from [11], for some subsequences

$$\theta_n^{DQG} \rightharpoonup \theta^{DQG}, \quad \theta_n^{QG} \rightharpoonup \theta^{QG} \quad \text{weakly in } L^2(\mathbb{T}^2),$$

So we have for any $\epsilon > 0$ by taking large $n$,

$$|\langle \theta^{DQG}(\cdot, t) - \theta^{QG}(\cdot, t), \phi \rangle| \leq \epsilon + |\langle \theta_n^{DQG} - \theta_n^{QG}, \phi \rangle|$$

$$\leq \epsilon + \|\phi\|_{L^2} \|	heta_n^{DQG} - \theta_n^{QG}\|_{L^2} \leq \epsilon + C_n \kappa$$

(2)

which implies (1). Here we’ve applied the inviscid limit result for smooth solutions to $\theta_n^{DQG} - \theta_n^{QG}$.

**Remark 2.6** Since the constants $C_n$ in the inequality (2) depends on $n$, we obtain no convergence rate. An explicit rate may exist in negative Sobolev norms.
3 Regularity Estimates

We consider the 2-D dissipative QGS equations with smooth initial data on the torus $T^2 = [0, L] \times [0, L]$, which admits a unique classical local solution. Let $\theta$ be this solution. We will estimate the quantity

$$\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}^2$$

where the operators $\Lambda^\beta$ and $e^{\lambda \Lambda^\alpha}$ are defined through the Fourier transform

$$\Lambda^\beta f = L^{-2} \sum_k e^{i k x} |k|^{\beta} \hat{f}(k)$$

$$e^{\lambda \Lambda^\alpha} f = L^{-2} \sum_k e^{i k x + \lambda |k|^\alpha} \hat{f}(k)$$

with $\hat{f}(k)$ being the k-th Fourier mode of $f$,

$$\hat{f}(k) = \frac{1}{V} \int_{T^2} e^{-i k \cdot x} f(x) dx.$$  

It is easy to see from these notations that $e^{\lambda \Lambda^\alpha}$ commutes with $\Lambda^\beta$ and partial derivatives for periodic boundary conditions considered here.

We obtain bounds for the quantity $\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}^2$, which lead to the exponential decay of the Fourier spectrum of $\theta$. The precise estimates are

**Theorem 3.1** Consider the 2-D dissipative QGS equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa \Lambda^{2\alpha} \theta = 0, \quad \kappa > 0, \quad \frac{1}{2} < \alpha \leq 1$$  \hspace{1cm} (3)

on the 2-D torus $T^2 = [0, L] \times [0, L]$. Let the initial data $\theta_0 \in H^k(T^2)$ with mean zero and $\theta$ be the unique smooth solution. We take $\beta$:

$$\beta > 0, \quad \beta + 2\alpha > 2.$$  

Then $\theta$ satisfies for any $\gamma > 0$

$$\|e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta\|_{L^2}^2 \leq \frac{e^{2 \gamma t^2 \sum_k \|\Lambda^{2\alpha} \theta_0\|_{L^2}^2}}{1 - C(\|\Lambda^\beta \theta_0\|_{L^2}^2)^{N-1} \kappa^{-1} (\gamma^2 - 1) \left( e^{\frac{2\gamma}{\kappa}} - 1 \right)^{N-1}}.$$  \hspace{1cm} (4)
which is finite for \( t \in [0, t^*] \),

\[
t^* = \frac{\kappa}{2(N-1)\gamma^2} \log \left( 1 + \frac{\kappa^{M-1}\gamma^2}{C\|\Lambda^\beta \theta_0\|_{L^2}^{2(N-1)}} \right)
\]

Here \( C \) is a constant and \( M,N \) are given by

\[
M = \frac{\alpha + \sigma}{\alpha - \sigma}, \quad N = 1 + \frac{\alpha}{\alpha - \sigma}, \quad \sigma = \begin{cases} 1 - \alpha, & \text{if } \beta \geq 1 \\ 2 - \beta - \alpha, & \text{if } \beta \leq 1 \end{cases}
\]

A special consequence of this theorem is that each Fourier mode amplitude can be individually controlled. In fact, a rough estimate gives

\[
e^{2\gamma t |k|^\alpha |k|^{2\beta}} |\theta(k,t)|^2 \leq \sum_k e^{2\gamma t |k|^\alpha |k|^{2\beta}} |\theta(k,t)|^2 = L^2 \|e^{\gamma t \Lambda^\alpha \Lambda^\beta \theta}\|_{L^2}^2
\]

Thus the \( k \)-th mode is bounded by

\[
|\theta(k,t)|^2 \leq \frac{L^2}{|k|^{2\beta}} e^{\frac{2\gamma t}{2} - 2\gamma t |k|^\alpha \|\Lambda^\beta \theta_0\|_{L^2}^2} \left( 1 - C(\|\Lambda^\beta \theta_0\|_{L^2}^2)^{N-1} \kappa^{-(M-1)} \gamma^{-2} \left( e^{\frac{2(N-1)\gamma^2}{2} - 1} \right) \right) \frac{1}{N-1}
\]

for \( t \in [0, t^*] \).

Doering and Titi [7] establish the exponential decay of power spectrum for the flow field of the 3-D Navier-Stokes equations. Similar analysis based on these bounds can be made to conclude that if \( \|\Lambda^\beta \theta\|_{L^2} \) is bounded uniformly in time, then after a transient time of length \( t^*/2 \) the Fourier spectrum of \( \theta \) decays exponentially at high wave numbers. Furthermore, the associated decay length can be defined and estimated in terms of the dissipation rates.

The main difficulty in proving the estimate (4) is how to bound the non-linear term properly. We need the inequalities for the Calderon-Zygmund type singular integrals. We'll also use the following lemma concerning the operator \( \Lambda^s \), which is proved in [11],[12].

**Lemma 3.2** For \( s > 0 \) and \( 1 < r < p \leq \infty \),

\[
\|\Lambda^t(uv)\|_{L^r} \leq C(\|u\|_{L^p}\|\Lambda^s v\|_{L^q} + \|v\|_{L^p}\|\Lambda^s u\|_{L^q})
\]

where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) and \( C \) is a constant.
\[ |(\nu, \gamma)\theta|_{\nu, \gamma, 1, \nu, \gamma} \sum_{\gamma} \cdot |(\nu, \gamma)\eta|_{\nu, \gamma, 1, \nu, \gamma} \sum_{\gamma} \tau - T \]

So this term is bounded by

\[ \omega |\nu, \gamma| + \omega |\gamma| \geq \omega |\nu, \gamma| + \gamma | = \omega |\gamma| \]

for \( 1 \geq \omega \geq 0 \)

\[ \sum_{\gamma = \nu + \gamma} (\nu, \gamma)\theta(\gamma)\theta_{\nu, \gamma, 1, \nu, \gamma} \sum_{\gamma - T} = (\theta \cdot n)_{\nu, \gamma, 1, \nu, \gamma} \]

can be better seen from the Fourier transform. To obtain further estimates, we break up the kernel term \((\theta \cdot n)_{\nu, \gamma, 1, \nu, \gamma}\) as

\[ \theta_{\nu, \gamma, 1, \nu, \gamma} = \theta \quad n_{\nu, \gamma, 1, \nu, \gamma} = n \]

notations are Riesz transforms. For brevity, we use the following

\[ x_p (\theta \cdot n)_{\nu, \gamma, 1, \nu, \gamma} (V^{-1} \phi, V^{-1} \psi) = H \]

where

\[ x_p (\theta \cdot n)_{\nu, \gamma, 1, \nu, \gamma} (V^{-1} \phi, V^{-1} \psi) \int - = II \]

Now we deal with the second term. Since the operators commute,

\[ \varepsilon ||\theta_{\nu, \gamma, 1, \nu, \gamma}||_{\nu -} = x_p (\theta_{\nu, \gamma, 1, \nu, \gamma}) (\theta_{\nu, \gamma, 1, \nu, \gamma}) \int \nu - = III \]

\[ x_p ((\theta \Delta \cdot \eta)_{\nu, \gamma, 1, \nu, \gamma}) (\theta_{\nu, \gamma, 1, \nu, \gamma}) \int - = II \]

\[ x_p (\theta_{\nu, \gamma, 1, \nu, \gamma}) (\theta_{\nu, \gamma, 1, \nu, \gamma}) \int \nu = I \]

where

\[ III + II + I = x_p \left( \theta_{\nu, \gamma, 1, \nu, \gamma} + \theta_{\nu, \gamma, 1, \nu, \gamma} \right) (\theta_{\nu, \gamma, 1, \nu, \gamma}) \int \frac{2p}{P} + \frac{1}{I} \]

Using the equation (3.3),

Proof of Theorem 3.1.
Now using Hölder’s inequality and the boundedness of Riesz transform,

\[ |II| \leq C\|\tilde{\theta}\|_{H^{\sigma+\beta}}\|\Lambda^{\beta+1-\alpha}(\tilde{u} \cdot \tilde{\theta})\|_{L^2} \]

where \( C \) is a constant.

Now we take \( q = 2 \) if \( \beta > 1 \) and \( q = \frac{2}{\beta} \) if \( \beta \leq 1 \). Choose \( p \) such that

\[ \frac{2}{p} + \frac{2}{q} = 1 \]

and \( \sigma = 2 - \frac{2}{q} - \alpha \). The condition that \( \beta + 2\alpha > 2 \) implies that \( 0 < \sigma < \alpha \).

We apply Lemma 2.2 to obtain

\[ |II| \leq C\|\tilde{\theta}\|_{H^{\sigma+\beta}} \left( \|\tilde{u}\|_{L^p}\|\Lambda^{\beta+1-\alpha}\tilde{\theta}\|_{L^q} + \|\tilde{\theta}\|_{L^p}\|\Lambda^{\beta+1-\alpha}\tilde{u}\|_{L^q} \right) = II_1 + II_2 \]

Using Sobolev imbeddings

\[ H^\beta \hookrightarrow H^{\frac{2}{q}} \hookrightarrow L^p, \quad H^{\beta+\sigma} \hookrightarrow L^p_{\beta+1-\sigma} \]

and the Gagliardo-Nirenberg interpolation (since \( \sigma < \alpha \)):

\[ \beta + \sigma = \frac{\alpha}{\alpha + \beta} (\beta + \alpha) + \left( 1 - \frac{\alpha}{\alpha + \beta} \right) \beta \]

we have

\[ II_1 \leq C\|\tilde{\theta}\|_{H^{\sigma+\beta}}\|\tilde{u}\|_{H^\beta}\|\tilde{\theta}\|_{H^{\beta+\sigma}} \leq C\|\tilde{\theta}\|_{H^\beta}\|\tilde{\theta}\|_{H^{\beta+\sigma}} \]

\[ \leq C\|\tilde{\theta}\|_{H^{\sigma+\beta}}^{1+\frac{\sigma}{\alpha}}\|\tilde{\theta}\|_{H^\beta}^{2-\frac{\sigma}{\alpha}} \]

where \( C \) is constant depending on \( \alpha \) and \( \beta \). By Young’s inequality

\[ II_1 \leq \frac{\kappa}{8}\|\tilde{\theta}\|_{H^{\sigma+\beta}}^2 + \frac{C}{\kappa M} \left( \|\tilde{\theta}\|_{H^\beta}^2 \right)^N \]

where \( N = \frac{2\alpha - \sigma}{\alpha - \sigma} \) and \( M = \frac{\alpha + \sigma}{\alpha - \sigma} \).

A similar estimate results in the same bound for \( II_2 \).

Collecting the estimates for \( I, II \) and \( III \) and reintroducing \( \tilde{\theta} = e^{\gamma t A} \theta \),

\[ \frac{d}{dt}\|e^{\gamma t A^\beta} \theta\|^2_{L^2} \leq -\kappa\|e^{\gamma t A^\beta} \Lambda^{\beta+\alpha} \theta\|^2_{L^2} \]

12
\[ + \frac{2\gamma^2}{\kappa} \| e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta \|_{L^2}^2 + \frac{C}{\kappa M} \| e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta \|_{L^2}^{2N} \]

Now we let

\[ Z(t) = \| e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta \|_{L^2}^2 \]

the differential inequality becomes

\[ \frac{dZ}{dt} \leq \frac{2\gamma^2}{\kappa} Z + \frac{C}{\kappa M} Z^N \]

An elementary algebra results

\[ \frac{dY}{dt} \leq \frac{Ce^{2(N-1)\frac{\gamma^2 t}{\kappa}}}{\kappa M} Y^N \]

where \( Y = e^{-\frac{2\gamma^2 t}{\kappa}} Z \). After a simple calculation,

\[ Y \leq \frac{Y_0}{\left( 1 - CY_0^{N-1}e^{-(M-1)\gamma^2 \left( e^{\frac{2(N-1)\gamma^2 t}{\kappa}} - 1 \right)} \right)^{\frac{1}{N-1}}} \]

Reintroducing \( Z(t) = \| e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta \|_{L^2}^2 \) and noting that \( Z(0) = \| \Lambda^\beta \theta_0 \|_{L^2}^2 \) that

\[ \| e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta \|_{L^2}^2 \leq \frac{\frac{e^{2\gamma^2 t}}{\kappa}}{\left( 1 - C\left( \| \Lambda^\beta \theta_0 \|_{L^2}^2 \right)^{N-1}e^{-(M-1)\gamma^2 \left( e^{\frac{2(N-1)\gamma^2 t}{\kappa}} - 1 \right)} \right)^{\frac{1}{N-1}}} \]

This means that \( \| e^{\gamma t \Lambda^\alpha} \Lambda^\beta \theta \|_{L^2}^2 \) is finite on the interval \([0, t^*)\), where

\[ t^* = \frac{\kappa}{2(N-1)\gamma^2} \log \left( 1 + \frac{\kappa^{M-1} \gamma^2}{C \| \Lambda^\beta \theta_0 \|_{L^2}^{2(N-1)}} \right) \]

The smaller the initial decay rate \( \| \Lambda^\beta \theta_0 \|_{L^2} \) is, the larger \( t^* \). Similarly, the larger the parameter \( \gamma \), the shorter \( t^* \).
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<td>1348</td>
<td>A.V. Fursikov</td>
<td>Certain optimal control problems for Navier-Stokes system with distributed control function</td>
</tr>
<tr>
<td>1349</td>
<td>F. Gesztesy, R. Nowell &amp; W. Pötz</td>
<td>One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics</td>
</tr>
<tr>
<td>1350</td>
<td>F. Gesztesy &amp; H. Holden</td>
<td>On trace formulas for Schrödinger-type operators</td>
</tr>
<tr>
<td>1351</td>
<td>X. Chen</td>
<td>Global asymptotic limit of solutions of the Cahn-Hilliard equation</td>
</tr>
<tr>
<td>1352</td>
<td>X. Chen</td>
<td>Lorenz equations, Part I: Existence and nonexistence of homoclinic orbits</td>
</tr>
<tr>
<td>1353</td>
<td>X. Chen</td>
<td>Lorenz equations Part II: “Randomly” rotated homoclinic orbits and chaotic trajectories</td>
</tr>
<tr>
<td>1354</td>
<td>X. Chen</td>
<td>Lorenz equations, Part III: Existence of hyperbolic sets</td>
</tr>
<tr>
<td>1356</td>
<td>C. Liu</td>
<td>The Helmholtz equation on Lipschitz domains</td>
</tr>
<tr>
<td>1357</td>
<td>G. Avalos &amp; I. Lasiecka</td>
<td>Exponential stability of a thermoelastic system without mechanical dissipation</td>
</tr>
<tr>
<td>1358</td>
<td>R. Lipton</td>
<td>Heat conduction in fine scale mixtures with interfacial contact resistance</td>
</tr>
<tr>
<td>1359</td>
<td>V. Odisharia &amp; J. Peradze</td>
<td>Solvability of a nonlinear problem of Kirchhoff shell</td>
</tr>
<tr>
<td>1360</td>
<td>P.J. Olver, G. Sapiro &amp; A. Tannenbaum</td>
<td>Affine invariant edge maps and active contours</td>
</tr>
<tr>
<td>1361</td>
<td>R.D. James</td>
<td>Hysteresis in phase transformations</td>
</tr>
<tr>
<td>1362</td>
<td>A. Sei &amp; W. Symes</td>
<td>A note on consistency and adjointness for numerical schemes</td>
</tr>
<tr>
<td>1363</td>
<td>A. Friedman &amp; B. Hu</td>
<td>Head-media interaction in magnetic recording</td>
</tr>
<tr>
<td>1364</td>
<td>A. Friedman &amp; J.J.L. Veldzquez</td>
<td>Time-dependent coating flows in a strip, part I: The linearized problem</td>
</tr>
<tr>
<td>1365</td>
<td>X. Ren &amp; M. Winter</td>
<td>Young measures in a nonlocal phase transition problem</td>
</tr>
<tr>
<td>1366</td>
<td>K. Bhattacharya &amp; R.V. Kohn</td>
<td>Elastic energy minimization and the recoverable strains of polycrystalline shape-memory materials</td>
</tr>
<tr>
<td>1367</td>
<td>G.A. Chechkin</td>
<td>Operator pencil and homogenization in the problem of vibration of fluid in a vessel with a fine net on the surface</td>
</tr>
<tr>
<td>1368</td>
<td>M. Carme Calderer &amp; B. Mukherjee</td>
<td>On Poiseuille flow of liquid crystals</td>
</tr>
<tr>
<td>1369</td>
<td>M.A. Pinsky &amp; M.E. Taylor</td>
<td>Pointwise Fourier inversion: A wave equation approach</td>
</tr>
<tr>
<td>1370</td>
<td>D. Brandon &amp; R.C. Rogers</td>
<td>Order parameter models of elastic bars and precursor oscillations</td>
</tr>
<tr>
<td>1371</td>
<td>H.A. Levine &amp; B.D. Sleeman</td>
<td>A system of reaction diffusion equations arising in the theory of reinforced random walks</td>
</tr>
<tr>
<td>1372</td>
<td>B. Cockburn &amp; P.-A. Gremaud</td>
<td>A priori error estimates for numerical methods for scalar conservation laws. Part II: Flux-splitting monotone schemes on irregular Cartesian grids</td>
</tr>
<tr>
<td>1373</td>
<td>B. Li &amp; M. Luskin</td>
<td>Finite element analysis of microstructure for the cubic to tetragonal transformation</td>
</tr>
<tr>
<td>1374</td>
<td>M. Luskin</td>
<td>On the computation of crystalline microstructure</td>
</tr>
<tr>
<td>1375</td>
<td>J.P. Matos</td>
<td>On gradient young measures supported on a point and a well</td>
</tr>
<tr>
<td>1376</td>
<td>M. Nitsche</td>
<td>Scaling properties of vortex ring formation at a circular tube opening</td>
</tr>
<tr>
<td>1377</td>
<td>J.L. Bona &amp; Y.A. Li</td>
<td>Decay and analyticity of solitary waves</td>
</tr>
<tr>
<td>1378</td>
<td>V. Isakov</td>
<td>On uniqueness in a lateral cauchy problem with multiple characteristics</td>
</tr>
<tr>
<td>1379</td>
<td>M.A. Kouritzin</td>
<td>Averaging for fundamental solutions of parabolic equations</td>
</tr>
<tr>
<td>1380</td>
<td>T. Aktosun, M. Klaus &amp; C. van der Mee</td>
<td>Integral equation methods for the inverse problem with discontinuous wavespeed</td>
</tr>
<tr>
<td>1381</td>
<td>P. Morin &amp; R.D. Spies</td>
<td>Convergent spectral approximations for the thermomechanical processes in shape memory allows</td>
</tr>
<tr>
<td>1382</td>
<td>D.N. Arnold &amp; X. Liu</td>
<td>Interior estimates for a low order finite element method for the Reissner-Mindlin plate model</td>
</tr>
<tr>
<td>1383</td>
<td>D.N. Arnold &amp; R.S. Falk</td>
<td>Analysis of a linear-linear finite element for the Reissner-Mindlin plate model</td>
</tr>
<tr>
<td>1384</td>
<td>D.N. Arnold, R.S. Falk &amp; R. Winther</td>
<td>Preconditioning in H(div) and applications</td>
</tr>
<tr>
<td>1385</td>
<td>M. Lavrentiev</td>
<td>Nonlinear parabolic problems possessing solutions with unbounded gradients</td>
</tr>
<tr>
<td>1386</td>
<td>O.P. Bruno &amp; P. Laurence</td>
<td>Existence of three-dimensional toroidal MHD equilibria with nonconstant pressure</td>
</tr>
<tr>
<td>1387</td>
<td>O.P. Bruno, F. Reitich, &amp; P.H. Leo</td>
<td>The overall elastic energy of polycrystallin martensitic solids</td>
</tr>
<tr>
<td>1388</td>
<td>M. Fila &amp; H.A. Levine</td>
<td>On critical exponents for a semilinear parabolic system coupled in an equation and a boundary condition</td>
</tr>
<tr>
<td>1390</td>
<td>J.M. Berg &amp; H.G. Kwatny</td>
<td>Unfolding the zero structure of a linear control system</td>
</tr>
<tr>
<td>1391</td>
<td>A. Sei</td>
<td>High order finite-difference approximations of the wave equation with absorbing boundary conditions: A stability analysis</td>
</tr>
<tr>
<td>1392</td>
<td>A.V. Coward &amp; Y.Y. Renardy</td>
<td>Small amplitude oscillatory forcing on two-layer plane channel flow</td>
</tr>
<tr>
<td>1393</td>
<td>V.A. Pliss &amp; G.R. Sell</td>
<td>Approximation dynamics and the stability of invariant sets</td>
</tr>
</tbody>
</table>
1394 J.G. Cao & P. Roblin, A new computational model for heterojunction resonant tunneling diode
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1443 M.A. Kouritzin, Stochastic processes and perturbation problems defined by parabolic equations with a small parameter
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