SOME STABILITY RESULTS FOR PERTURBED SEMILINEAR PARABOLIC EQUATIONS

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Some stability results for perturbed semilinear parabolic equations

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ABSTRACT

We prove a nonlinear variation of constants formula for a class of semilinear parabolic equations, and use this formula to study the stability of nonlinear perturbations of such equations. As an example, we apply our results to a perturbation of a nonlinear scalar reaction diffusion equation.

Keywords and Phrases: Nonlinear parabolic equations, perturbations, stability.
1. Introduction and Preliminaries

Our goal in this paper is to prove a representation formula and some stability results for a parabolic evolutionary equation of the form

\[(1.1) \quad u' + Au = F(u) + G(t,u)\]

in a Hilbert space $X$ (we shall make precise assumptions on $A$, $F$ and $G$ in Section 2).

Our point of view consists in regarding (1.1) as a perturbation of the semilinear autonomous equation

\[(1.2) \quad u' + Au = F(u).\]

If (1.2) is linear ($F=0$), then there is a standard relationship between (1.1) and (1.2), given by the variation of constants formula

\[(1.3) \quad u(t) = T(t)u_0 + \int_0^t T(t-s) G(s,u(s)) \, ds,\]

where $\{T(t), t \geq 0\}$ is the linear semigroup generated by $-A$. In fact, (1.3) provides the standard definition of "mild solutions" of (1.1) (see Pazy [8]).

In the general case, one can still obtain a useful representation of solutions of the perturbed equation (1.1) by using a \textit{nonlinear} variation of constants formula, in which $T(t-s)$ is replaced by a different kernel which takes into account the nonlinearity $F$. A formula of this type (in infinite dimension) seems to have been first proved by Alekseev [1], and was later applied by Brauer [2],[3], Hastings [5], Strauss [9] and others to questions of stability for ODEs and FDEs. A version for evolution equations with a bounded linear operator, together with an extension of the results of [2],[3], is given in Ladas and Lashmikantham [7]. We extend this formula to the case of a sectorial evolutionary equation, thus allowing for an unbounded coefficient, and for applications to PDEs with nonlinearities involving derivatives.

The organization of the paper is as follows. In Section 2 we study the differentiability properties of solutions of (1.2) in fractional spaces. In Section 3 we prove a version of the nonlinear variation of constants formula for equation (1.1); in Section 4 we generalize the results of [2],[3],[7], to this infinite-dimensional context. Finally, in Section 5 we provide an application of these results to a nonlinear scalar reaction-diffusion
equation. We believe that this method of stability analysis is a useful alternative to the use of Lyapunov functions, particularly when dealing with perturbed nonlinear systems.

2. Regularity results

Let $X$ be a real Hilbert space with norm $|.|$ and inner product $(.,.)$, and consider the following evolutionary equation in $X$:

\begin{align*}
(2.1) \quad u' + Au &= F(u) + G(t,u) \\
    u(0) &= u_0,
\end{align*}

together with

\begin{align*}
(2.2) \quad u' + Au &= F(u) \\
    u(0) &= u_0.
\end{align*}

We shall make the following assumptions regarding (2.1)-(2.2):

(I) $A$ is a self-adjoint, positive, densely defined operator on $X$, with compact resolvent (we could also work with sectorial operators, but we prefer a more concrete framework). Then $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$. Furthermore, the powers $\{A^\alpha, \alpha \in \mathbb{R}\}$ can be defined, and $X^\alpha = D(A^\alpha)$ becomes a Hilbert space under the inner product $(u,v)_\alpha = (A^\alpha u,A^\alpha v)$. The corresponding norm is $|u|_\alpha = (u,u)_\alpha^{1/2}$. We will also introduce the notation $V = X^{1/2}$ and $\| . \| = |u|_{1/2}$, since this particular space will occur frequently.

(II) We assume that $F: V \to X$ is locally Lipschitz continuous and $G: \mathbb{R}^+ \times V \to X$ is locally Lipschitz continuous in $u$ and locally Hölder continuous in $t$ with exponent $\theta$, $0 < \theta \leq 1$. Notice that these conditions ensure the local existence and uniqueness of continuously differentiable solutions for (2.1) and (2.2) for any initial condition $u_0 \in V$. We shall assume throughout the paper that $\|u\|$ is bounded for all time, i.e., all solutions of (2.1) and (2.2) with initial condition $u_0 \in V$ belong to $BC((0,\infty); V)$. This is a convenient sufficient condition for local solutions to be global. For details, we refer to [8], Thm. 6.3.1.

(III) Finally, let us assume that $F$ is Fréchet differentiable, and $DF$ is a continuous mapping from $V$ to $L(V;X)$. 
We shall first study the properties of solutions of the "homogeneous" equation (2.2). The following result makes precise the differentiability properties of these solutions.

**Lemma 2.1** Assume that (I) and (II) hold, and let \( \alpha \) and \( \gamma \) satisfy \( 0 < \alpha \leq 1/2 \) and \( 1/2 < \gamma \leq 1 \). Then the mild solution \( u(\cdot) \) of (2.2) satisfies \( u \in C((0,\infty); X^\gamma) \cap C^1((0,\infty); X^\alpha) \). Moreover, the following estimates hold:

\[
(2.3) \quad \| u(t) \|_\gamma \leq C_1 (u_0) \cdot t \cdot (\gamma - 1/2) e^{-\delta t} + C_2(u_0), \quad t > 0,
\]

\[
(2.4) \quad \| \frac{du}{dt} \|_\alpha \leq C_3(u_0) \cdot t \cdot (\alpha + 1/2), \quad t > 0
\]

where the \( C_i \) are constants depending on \( u_0 \in V \).

**Proof.** We will need the following well-known inequality for the semigroup \( \{ T(t) \} \) (see [8], Thm. 2.6.13):

\[
(2.5) \quad \| A^\beta T(t) \|_{L(X)} \leq M_\beta t \cdot \beta e^{-\delta t} \quad \text{for} \quad t > 0,
\]

where \( \beta \geq 0 \) is given and \( M_\beta \) is a constant depending on \( \beta \), while \( \delta > 0 \) is an absolute constant.

By using the formula for a mild solution, estimate (2.5), and the fact that \( u_0 \in V \), we have

\[
\| u(t; u_0) \|_\gamma \leq \| T(t) u_0 \|_\gamma \left( 0 \right) + \int_0^t \| T(t - s) F(u(s)) \|_\gamma \, ds
\]

\[\leq M_\gamma^{-1/2} t \cdot (\gamma - 1/2) e^{-\delta t} \| u_0 \| + \int_0^t M_\gamma (t - s) \cdot \gamma e^{-\delta(t - s)} \| F(u(s)) \| \, ds\]

where \( u(t; u_0) \) denotes the solution with initial condition \( u_0 \). Since \( F: V \to X \) is locally Lipschitz continuous, and (II) is valid, we have \( \| F(u(s)) \| \leq C_4(u_0), \quad s \geq 0 \), where \( C_4 \) is a constant, therefore we obtain

\[
\| u(t; u_0) \|_\gamma \leq M_\gamma^{-1/2} \| u_0 \| \cdot t \cdot (\gamma - 1/2) e^{-\delta t} + M_\gamma C_4(u_0) \delta \gamma^{-1} \Gamma (1 - \gamma), \quad t > 0,
\]

hence (2.3) holds.
One can obtain a similar estimate for the increment \( u(t; u_0) - u(t; u_0) \) by using the Lipschitz continuity of \( F \) and the Henry-Gronwall inequality. This shows that \( u : (0, \infty) \to X \) is continuous. The fact that \( u \in C^1((0, \infty); X^\alpha) \) and the bound (2.4) then follow from Thm. 3.5.2 in [6].

**Lemma 2.2** Assume that conditions (I), (II) and (III) are satisfied. Then the mapping taking \( u_0 \) into \( u(t; u_0) \) is continuously differentiable for all \( t \geq 0 \). Let us introduce the notation

\[
H(t, a) = \frac{\partial u(t; a)}{\partial a} \in L(V), \ t \geq 0.
\]

Then \( H(t, a) \) is the mild solution of the operator equation

\[
(2.6) \quad \frac{dH}{dt} + AH = DF(u(t; a)) H
\]

\[
H(0) = I, \text{ the identity operator.}
\]

The operator \( H \) can be extended, for \( t > 0 \), to a continuous mapping from \((0, \infty) \times X^\alpha \) to \( L(X^\alpha, V) \) with \( 0 < \alpha < 1/2 \). Moreover, for \( t > 0 \) we have

\[
(2.7) \quad |H(t, a)|_{L(X^\alpha; V)} \leq M_{1/2 - \alpha} E^\alpha(C_\alpha(a)t) t^{-(1/2 - \alpha)}
\]

where \( C_\alpha(a) \) is a constant depending on \( a \),

\[
E^\alpha(t) = \sum_{m=0}^{\infty} b_m t^{m\alpha}
\]

and the coefficients are given by \( b_0 = 1, b_{m+1} = b_m \Gamma(m\alpha + \alpha + 1/2)/\Gamma(m\alpha + \alpha + 1) \).

**Proof.** The first part of this lemma, concerning the differentiability with respect to \( u_0 \) is a statement described in Thm. 3.4.4 of [6]. For completeness, we prefer to give a direct proof. By using the mild solution formulas for (2.2) and (2.6), we obtain

\[
(2.8) \quad u(t; a) = T(t) a + \int_0^t T(t - s) F(u(s; a)) \, ds
\]
(2.9) \[ \mathcal{H}(t,a) = T(t) + \int_0^t T(t - s) \mathcal{D}(u(s; a)) H(s,a) \, ds \quad \text{for } 0 \leq t < \infty. \]

For any \( \xi \in V \), we have the following estimate

(2.10) \[ \| u(t; a + \xi) - u(t; a) - H(t,a)\xi \| \]

\[ \leq \int_0^t \| T(t - s) [\mathcal{F}(u(s; a + \xi)) - \mathcal{F}(u(s; a)) - \mathcal{D}(u(s; a)) H(s,a)\xi] \| \, ds \]

\[ \leq \int_0^t M_{1/2} (t - s)^{-1/2} \int_0^1 \mathcal{D}(u(s; a) + \lambda \mathcal{A}(u(s; a, \xi)) \mathcal{A}(u(s; a, \xi)) \, d\lambda - \mathcal{D}(u(s; a)) H(s,a)\xi \| ds \]

(here \( \mathcal{A}(s; a, \xi) = u(s; a + \xi) - u(s; a) \))

\[ \leq \int_0^t M_{1/2} (t - s)^{-1/2} \rho(a) \| \mathcal{A}(s; a, \xi) - H(s,a)\xi \| \, ds \]

\[ + \int_0^t M_{1/2} (t - s)^{-1/2} \int_0^1 \| \mathcal{D}(u(s; a) + \lambda \mathcal{A}(u(s; a, \xi)) \mathcal{A}(u(s; a, \xi)) - \mathcal{D}(u(s; a)) \|_{L(V,X)} d\lambda. \]

\[ \| H(s,a) \|_{L(V)} \| \xi \| \, ds, \]

where \( \rho(a) = \sup \{ \| \mathcal{D}((1 - \lambda)u(s; a) + \lambda u(s; a + \xi)) \|_{L(V,X)} : s \geq 0, \| \xi \| \leq 1, 0 \leq \lambda \leq 1 \}. \]

By (2.9), we have

(2.11) \[ \| \mathcal{H}(t,a) \|_{L(V)} \leq 1 + \int_0^t M_{1/2} (t - s)^{-1/2} \rho_1(a) \| H(s,a) \|_{L(V)} \, ds \quad \text{for } t \geq 0, \]

where \( \rho_1(a) = \sup \{ \| \mathcal{D}(u(s; a)) \|_{L(V,X)} : s \geq 0 \}. \)

The Henry-Gronwall inequality (see [6]), implies

(2.12) \[ \| \mathcal{H}(t,a) \|_{L(V)} \leq E_{1/2} \left[ (M_{1/2} \rho_1(a))^2 t \right], \quad t \geq 0. \]
where

\begin{equation}
E_\theta(z) = \sum_{n=0}^{\infty} \frac{2^n \theta}{\Gamma(n \theta + 1)}.
\end{equation}

Locally, for \( t \in [0, t^*] \), with \( t^* \) relatively fixed, we have

\begin{equation}
\| H(t,a) \|_{L(V)} \leq K_H (t^*,a) = \text{const}.
\end{equation}

As \( \xi \to 0 \) in \( V \), by the continuous dependence of the solution on the initial data (see [6], Lemma 3.4.3), and the continuity and boundedness of \( DF(u) \), we obtain

\begin{equation}
\lim_{\xi \to 0} \int_0^1 \| DF(u(s \pm \xi) + \lambda \Delta u(s \pm \xi)) - DF(u(s ; a)\|_{L(V)} d\lambda) \| = 0 \quad \text{as} \quad \xi \to 0
\end{equation}

for \( s \in [0, t^*] \).

By inserting (2.14) and (2.15) into (2.10), we obtain

\begin{equation}
\| \Delta u(s ; a, \xi) - H(s,a)\xi \| \leq \begin{aligned}
& \theta \| D \| + \int_0^t M_{1/2}(t - s)^{-1/2} \rho(a) \| \Delta u(s ; a, \xi) - H(s,a)\xi \| ds \\
& \leq \begin{cases}
1 + \sum_{n=1}^{\infty} \frac{(M_{1/2} \rho(a) \Gamma(1/2)v(t^*)^n)}{\Gamma(n+3/2)} o(\| \xi \|) \text{ as } \xi \to 0, \text{ for } 0 \leq t \leq t^*. 
\end{cases}
\end{aligned}
\end{equation}

By applying the Henry-Gronwall inequality once again, we finally get

\begin{equation}
\| \Delta u(t ; a, \xi) - H(t,a)\xi \| \leq \begin{aligned}
& \theta \| D \| + \int_0^t M_{1/2}(t - s)^{-1/2} \rho(a) \| \Delta u(s ; a, \xi) - H(s,a)\xi \| ds \\
& \leq \begin{cases}
1 + \sum_{n=1}^{\infty} \frac{(M_{1/2} \rho(a) \Gamma(1/2)v(t^*)^n)}{\Gamma(n+3/2)} o(\| \xi \|) \text{ as } \xi \to 0, \text{ for } 0 \leq t \leq t^*. 
\end{cases}
\end{aligned}
\end{equation}

This shows that \( \frac{\partial u(t ; a)}{\partial a} = H(t,a) \) is the mild solution of (2.9), and is strongly continuous for \( t \geq 0 \).

For the second part and (2.7), we examine equation (2.8) for \( t > 0 \) by regarding both sides as mappings from \( X^\alpha \to V \), with \( 0 < \alpha < 1/2 \). Then the following estimate holds for \( t > 0 \):

\begin{equation}
\| H(t,a) \|_{L(X^\alpha, V)} \leq 
\end{equation}
\[ \leq \| T(t) \|_{L(X^\alpha, v)} + \int_0^t \| T(t - s) \|_{DF(u(s) ; a)} H(s, a) \|_{L(X^\alpha, v)} \, ds \]

\[ \leq M_{1/2} \cdot t^{(1/2 - \alpha)} + \int_0^t M_{1/2}(t - s) \cdot s^{1/2} \rho_1(a) \| H(s, a) \|_{L(X^\alpha, v)} \, ds \]

Let \( g(t) = t^{1/2 - \alpha} \| H(s, a) \|_{L(X^\alpha, v)} \). Then \( g(t) \) satisfies the differential inequality

\[(2.19) \quad g(t) \leq M_{1/2} \cdot t^{(1/2 - \alpha)} + \int_0^t M_{1/2}(t - s) \cdot s^{1/2} \rho_1(a) \cdot g(s) \, ds.\]

This inequality is of the type considered in [6], p. 189, and it follows that

\[(2.20) \quad g(t) \leq M_{1/2} \cdot d \sum_{m=0}^{\infty} b_m [M_{1/2} \rho_1(a) \Gamma(1/2)t^\alpha]^m \text{ for all } t > 0,\]

where the \( b_m \) are the coefficients defined in the statement of the Lemma. This gives the required bound and ends the proof.

3. Nonlinear Variation of Constants Formula

In this section we consider evolution equation (2.1), which we recall:

\[(3.1) \quad u' + Au = F(u) + G(t, u), \quad u(0) = a \in V.\]

We shall work under assumptions (I), (II) and (III) of Section 2.

**Theorem 3.1** The mild solution \( u(.) \) of (3.1) satisfies the nonlinear variation of constants formula:

\[(3.2) \quad u(t) = w(t, a) + \int_0^t H(t - s, u(s)) G(s, u(s)) \, ds, \quad t \geq 0,\]

where \( H(t, a) \) is defined by

\[ H(t, a) = \frac{\partial w(t ; a)}{\partial a} \]
and \( w(t; a) \) is the mild solution of the unperturbed equation (2.2) with \( w(0; a) = a \).

**Proof.** Consider the function \( s \to w(t - s; u(s)) \) as a mapping from \([0, t] \) into \( V \). We will show that its total derivative with respect to \( s \) exists and is expressed by the following formula:

\[
\frac{d}{ds} w(t - s; u(s)) = -\frac{\partial w}{\partial t} (t - s; u(s)) + H(t - s, u(s)) \frac{du(s)}{ds}, \quad \text{for } 0 < s < t, \ t \text{ given.}
\]

To prove this, let us introduce the auxiliary functions

\[
\Delta_h w(t - s; u(s)) = \frac{1}{h} \left[ w(t - (s + h); u(s + h) - w(t - s; u(s)) \right], \ h \neq 0,
\]

and

\[
\Delta u_h(s) = u(s + h) - u(s).
\]

Then we have

\[
\Delta_h w(t - s; u(s)) = -\frac{1}{h} \left[ w(t - s; u(s)) - w(t - h - s; u(s)) \right] +
\]

\[
\int_0^1 \frac{\partial w}{\partial a} (t - (s + h); u(s) + \lambda \Delta u_h(s)) d\lambda \cdot \frac{1}{h} \int_0^1 \frac{du}{ds}(s + \mu h) d\mu,
\]

where we have used the mean-value theorem in Banach spaces. By Lemma 2.1, we have, for \( 0 \leq s < t \),

\[
\lim \left\{ -\frac{1}{h} \left[ w(t - s; u(s)) - w(t - h - s; u(s)) \right] \right\} = -\frac{\partial w}{\partial t} (t - s; u(s)) \text{ in } V \text{ as } h \to 0.
\]

By [6], Thm. 3.5.2, and the regularity assumption on \( G \), we see that for any \( \alpha \) such that \( 0 < \alpha \leq 1/2 \), the mild solution \( u \) of (3.1) has the property that:

\[
t \to \| \frac{du(t)}{dt} \| \in X^\alpha \text{ is locally Hölder continuous and}
\]

\[
\| \frac{du}{dt} \|_{\alpha} \leq C(t_1) t^{-\alpha + 1/2}, \text{ for } 0 < t \leq t_1,
\]
where the constant $C(t_1)$ depends on $t_1 > 0$. Thus it follows that

$$\lim_{h \to 0} \int_0^1 \frac{du}{ds}(s + \mu h) \, d\mu = \frac{du(s)}{ds}$$

(3.7) \quad h \to 0

in $X^\alpha$, for $0 < s \leq t$. On the other hand, by Lemma 2.2, we have

$$\lim_{h \to 0} \int_0^1 \frac{\partial w}{\partial a}(t - (s + h) ; u(s) + \lambda \Delta u_h(s)) \, d\lambda = \frac{\partial w}{\partial a}(t - s ; u(s))$$

in $L(X^\alpha, V)$, for $0 \leq s < t$.

By combining (3.5), (3.7), and (3.8) and substituting into (3.4), we obtain (3.3).

Next, by the uniqueness of the solutions, we have

(3.9) \quad w(t - s ; u(s)) = w(t - h - s ; w(h ; u(s))) for $0 < s < t$ and $0 < h < t - s$.

By taking the derivative in $V$ with respect to $h$ at $h = 0$, and using Lemma 2.1, we get

$$0 = -\frac{\partial w}{\partial t}(t - s ; u(s)) + \frac{\partial w}{\partial a}(t - s ; u(s)) \frac{dw}{dt}(0 ; u(s)), 0 < s < t,$$

where $u(s) \in D(A)$ and

(3.10) \quad $\frac{dw}{dt}(0 ; u(s)) = -A(u(s) + F(u(s)))$, $0 < s < t$.

It follows that

$$\frac{d}{ds} w(t - s ; u(s)) = H(t - s ; u(s)) \left[ \frac{du(s)}{ds} + A(u(s) - F(u(s)) \right]$$

$$= H(t - s ; u(s)) G(s,u(s)), 0 < s < t.$$

Finally, since (2.9) implies that $H(t,a)$ is a continuous mapping from $[0,\infty) \times V \to L(V)$, and $u(.)$ and $g(.,u(.))$ are continuous in $V$, it follows that $w(t - s ; u(s))$ is an absolutely continuous function from $[0,t]$ into $V$. Therefore, we have
\begin{equation}
(3.13) \int_0^t \frac{d}{ds} w(t - s ; u(s)) \, ds = w(0 ; u(t)) - w(t ; u(0)) = u(t) - w(t ; a).
\end{equation}

The conclusion of the theorem now follows by combining (3.12) and (3.13).

4. Stability results

In this section we will once again consider the perturbed nonlinear equation (2.1), where \( A, F \) and \( G \) satisfy assumptions (I)-(II)-(III) introduced in Section 2. As we shall see, the nonlinear variation of constants formula is a useful tool for the study of the stability properties of (2.1), and has the advantage of avoiding the use of Lyapunov functions. In addition to the above hypotheses, we will assume the following:

Standing Hypothesis

The operator \( H(t,a) \) introduced in the previous section satisfies

\begin{equation}
(*) \quad \| H(t,a) \|_{L(V)} \leq \exp \left( \int_0^t \alpha(s) \, ds \right) \text{ for all } t \geq 0, \text{ where } \alpha \text{ is a continuous bounded function.}
\end{equation}

We assume that (4.1) holds uniformly with respect to \( a \), for \( a \) varying in an open convex subset \( D \) of \( V \), to be specified in each case.

We shall also consider the following hypothesis

\begin{equation}
(**) \quad \mu = \lim\sup_{t \to \infty} \frac{1}{t} \int_0^t \alpha(s) \, ds < 0 \text{ for all } a \in D.
\end{equation}

As we shall see, one can give simple sufficient conditions for (*) and (**) to hold in terms of the linearization at a single point.

We shall need the following preliminary result.

**Lemma 4.1** Assume that (I)-(II)-(III) and (*) are satisfied, and let \( a, b \in D \). As before, let \( w(t ; a) \) and \( w(t ; b) \) be the solutions of (2.2) with initial conditions \( a \) and \( b \), respectively. Then the following inequality holds

\begin{equation}
(4.1) \quad \| w(t ; b) - w(t ; a) \| \leq \| b - a \| \exp \left( \int_0^t \alpha(s) \, ds \right).
\end{equation}

**Proof.** Let \( \phi(\lambda) = a + \lambda(b - a) \), \( \lambda \in [0,1] \). Then we can write

\begin{equation}
(4.2) \quad \frac{\partial}{\partial \lambda} [w(t,\phi(\lambda))] = H(t,\phi(\lambda)) \phi'(\lambda) = H(t,\phi(\lambda)) (b - a) \text{ and integration gives}
\end{equation}
\[(4.3) \quad w(t; b) - w(t; a) = \left[ \int_0^1 H(t, \phi(\lambda)) \, d\lambda \right] (b - a).\]

Then (4.1) follows by taking the norm in V and using (*)

The following theorem generalizes the results of [2], [3], and [7].

**Theorem 4.2** Assume that the function \( \alpha(s) \) introduced above satisfies (*) and (**) in a neighborhood \( D \) of the origin, and assume that \( F(0) = 0 \).

Then:

(i) The zero solution of (2.2) is asymptotically stable.

(ii) If, in addition, \( G(t, u) = o(\| u \|) \) uniformly in \( t \), then the zero solution of (2.1) is also asymptotically stable.

**Proof.** (i) This follows from (4.1) by setting \( a = 0 \). (ii) Let us first estimate

\[(4.4) \quad \| u(t; b) - w(t; a) \| \leq \| w(t; b) - w(t; a) \| + \| u(t; b) - w(t; b) \|.\]

The first term on the right can be bounded by (4.1), and the second term can be bounded by using the nonlinear variation of constants representation (3.2). If we put together these estimates, we obtain

\[(4.5) \quad \| u(t; b) - w(t; a) \| \leq \| b - a \| \| \exp(\int_0^t \alpha(s) \, ds) + \int_0^t \exp(\int_0^r \alpha(s) \, ds) \| G(r, u(r, b)) \| \, dr \]

By taking \( a = 0 \), we obtain

\[(4.6) \quad \| u(t; b) \| \leq \| b \| \| \exp(\int_0^t \alpha(s) \, ds) + \int_0^t \exp(\int_0^r \alpha(s) \, ds) \| G(r, u(r, b)) \| \, dr \]

Since \( G(t, u) = o(\| u \|) \) uniformly in \( t \), we have \( \| G(t, u) \| \leq \varepsilon \| u \| \) if \( \| u \| \leq \delta \). Substituting this into (4.6), we arrive at

\[(4.7) \quad \| u(t; b) \| \leq \| b \| \| \exp(\int_0^t \alpha(s) \, ds) + \varepsilon \int_0^t \exp(\int_0^r \alpha(s) \, ds) \| u(r, b) \| \| \, dr \]

and the Gronwall inequality gives
\[(4.8) \quad \|u(t; b)\| \leq \|b\| e^{\varepsilon t} \exp \left( \int_0^t \alpha(s) \, ds \right).\]

Finally, condition (**) implies that for sufficiently small \(\varepsilon\) and \(\|b\|\), the right-hand side of (4.8) tends to 0 as \(t \to \infty\), which proves (ii).

The nonlinear variation of constants formula also yields results on the asymptotic equivalence of equations (2.1) and (2.2). As an example, we shall prove the following theorem.

**Theorem 4.3** Assume that:

(i) \(\|H(t,a)\|_{L(V)}\) is uniformly bounded for all \(t \geq 0\) and \(a\) in \(V\).
(ii) for every strongly measurable function \(v\) such that \(v(t) \in V\) for all \(t \geq 0\), one has
\[
\int_0^\infty G(s,v(s)) \, ds < \infty.
\]

Then with every solution \(u(t)\) of (2.1) we can associate a solution \(w(t)\) of (2.2) such that

\[
\lim_{t \to \infty} [u(t) - w(t)] = 0.
\]

**Proof.** Let us define \(w(t)\) by

\[
w(t) = u(t) + \int_0^t H(t-s,u(s)) \, G(s,u(s)) \, ds
\]

where \(H\) is the function in (3.2). By assumption, there exists a constant \(K\) such that \(\|H(t,a)\|_{L(V)}\) \(\leq K\), therefore

\[
\|u(t) - w(t)\|_V \leq K \int_t^\infty G(s,u(s)) \, ds \|_{L(V)}
\]

Our assumption (ii) ensures that the integral on the right-hand side tends to 0 as \(t \to \infty\), which proves the result.

**Remark 4.4** By using the above methods, one can also prove more sophisticated results on exponential and uniform asymptotic stability (see e.g. [3], [4]).
In order to apply these theorems to specific problems, one needs sufficient conditions for the exponential bound (**) to hold. In order to obtain such a condition, let us notice that the variational equation (2.6), whose fundamental solution is \( H(t : a) \), is of the form

\[
(4.9) \quad v' + (A - B(t ; a))v = 0,
\]

where \( B(t ; a) = DF(u(t ; a)) \). All the eigenvalues of the operator - \( A \) are real and negative, so the unperturbed equation (without \( B(t ; a) \)) is stable. We need a criterion that will guarantee the existence of an exponential bound for \( H \) on a subset \( D \) of \( V \) when the time-dependent perturbation \( B(t ; a) \) is added. The following result shows that such a bound exists in a sufficiently small neighborhood of the origin.

**Lemma 4.5** Assume that the operator \( A - B(t ; 0) \) is positive definite, uniformly in \( t \), with smallest eigenvalue bounded below by \( \beta > 0 \). Then there exists a neighborhood \( D \) of the origin such that (*) and (**) hold for all solutions with initial conditions in \( D \). More specifically, given \( T > 0 \), there exist constants \( \rho > 0 \) and \( M > 0 \) such that for all solutions \( v(t ; b) \) with \( \| b \| \leq \frac{\rho}{M} \) one has

\[
(4.10) \quad \| v(t ; b) \| \leq M \exp(-\beta t) \| b \| \text{ for all } t \geq T.
\]

**Proof.** This result is a particular case of [6], Thm. 5.1.1, to which we refer for a proof.

5. **An example: perturbation of a scalar reaction-diffusion equation**

In this section we shall apply the previous results to the following nonlinear boundary-value problem:

\[
(5.1) \quad u_t - u_{xx} = u - bu^3 + g(t,u,u_x), \quad t \geq 0, \quad x \in I = [0,1], \quad b > 0,
\]

\[
u(0) = u(1) = 0
\]

\[
u(x,0) = u_0(x).
\]

Let us introduce the Hilbert space \( X = L^2(I) \), and the linear operator \( A \) by \( Au = -d^2u/dx^2 \) with domain \( D(A) = H^2(I) \cap H_0^1(I) \). Then it is easy to check that \( A \) satisfies assumption (I) in Section 2. Let, as before, \( V = D(A^{1/2}) \) and notice that \( \| u \| = \| u_x \| \). The function \( F: V \rightarrow X \) defined by \( F(u) = u - bu^3 \) is locally Lipschitz continuous. We shall assume that \( G: \mathbb{R}^+ \times V \rightarrow X \) defined by \( G(t,u) = g(t,u,u_x) \) is locally Lipschitz continuous in \( u \) and locally Hölder.
continuous in t. We shall assume also that G is such that all solutions of (5.1) with initial condition in \( V \) exist for all time in \( V \).

As a first step, we prove some estimates regarding the solutions of the unperturbed equation.

**Lemma 5.1** Consider the unperturbed problem

\[
(5.2) \quad u_t - u_{xx} = u - bu^3, \quad t \geq 0, \quad x \in I = [0,1] \\
u(0) = u(1) = 0 \\
u(x,0) = u_0(x).
\]
i.e., the equation in \( X \),

\[
(5.3) \quad u' + Au = F(u), \quad u(0) = u_0.
\]

Then, for any initial condition \( u_0 \in V \), the corresponding solution \( u(.) \) exists for all time and there is an absolute constant \( R \) (independent of \( u_0 \)) such that \( u(.) \) satisfies:

\[
\lim \sup \|u(t)\| \leq R \quad \text{as} \quad t \to \infty.
\]

**Proof.** By taking the inner product of (5.2) with \( -u_{xx} \) and integrating by parts, we obtain

\[
(5.4) \quad \frac{1}{2} \frac{d}{dt} \|u_x\|^2 + \|u_{xx}\|^2 \leq \|u_x\|^2 - 3b \int_I u^2 u_x^2 \, dx
\]

We first notice that by the Poincaré inequality

\[
(5.5) \quad 3b \int_I u^2 u_x^2 \, dx \geq \frac{3b}{4} \int_I u^4 \, dx
\]

We also have, by the Schwarz and Young inequalities,

\[
(5.6) \quad \|u\| \leq (\int_I u^4 \, dx)^{1/2} \leq \frac{1}{6b} + \frac{3b}{2} \int_I u^4 \, dx.
\]

It follows that

\[
(5.7) \quad 3b \int_I u^2 u_x^2 \, dx \geq \frac{1}{2} \|u\|^2 - \frac{1}{3b}.
\]

By substituting this into (5.4), we obtain the estimate

\[
(5.8) \quad \frac{1}{2} \frac{d}{dt} \|u_x\|^2 + \|u_{xx}\|^2 \leq \|u_x\|^2 - \frac{1}{2} \|u\|^2 + \frac{1}{3b}.
\]

By the interpolation inequality we have
(5.9) \[ |u_x|^2 \leq |u| \|u_{xx}\| \leq \frac{1}{2} |u|^2 + \frac{1}{2} |u_{xx}|^2. \]

Substitution into (5.8), together with the Poincaré inequality \[ |u_x|^2 \leq |u_{xx}|^2 \] gives

(5.10) \[ \frac{d}{dt} |u_x|^2 + |u_x|^2 \leq \frac{2}{3b}, \] and the Gronwall inequality implies that

(5.11) \[ |u_x(t)|^2 \leq e^{-t} |u_x(0)|^2 + \frac{2}{3b} (1 - e^{-t}), \] hence

(5.12) \[ \limsup_{t \to \infty} \|u(t)\|^2 \leq \frac{2}{3b}. \]

This ends the proof of the theorem.

With this result in hand, we shall verify that the hypotheses (*) and (**) of Section 4 are satisfied by the fundamental solution \(H(t; a)\) of the variational equation of Eq. (5.3). The variational equation corresponding to the solution \(u = u(t; a)\) is

(5.14) \[ v_t - v_{xx} = v - 3b(u(t; a))^2 v \]

Since \(u(t; 0) = 0\), the linearization around zero reduces to \(v_t - v_{xx} = v\), and it is easily checked that the linear operator \(L\) defined on \(V\) by \(Lv = -v_{xx} - v\) is positive definite. Hence Lemma 4.5 and Theorem 4.2 can be applied, and the following holds:

**Theorem 5.2.** Assume that \(G(t,u)\) satisfies the condition \(G(t,u) = o(\|u\|)\) as \(\|u\| \to 0\), uniformly in \(t\). Then the zero solution of (2.1) is (exponentially) asymptotically stable.

**References**


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