BLOW-UP BEHAVIOR FOR SEMILINEAR HEAT EQUATIONS: MULTI-DIMENSIONAL CASE

By

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Blow-Up Behavior for Semilinear Heat Equations:
Multi-Dimensional Case

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Abstract

This paper concerns with the Cauchy problem:

\[
\begin{align*}
  u_t - \Delta u &= F(u), \quad (x, t) \in \mathbb{R}^N \times (0, T) \\
  u(x, 0) &= u_0(x)
\end{align*}
\]

where \(u_0(x)\) is continuous, nonnegative and bounded, and \(F(u) = u^p\) with \(p > 1\) or \(F(u) = e^u\). Assume that \(u\) blows up at \(x = 0\) and \(t = T\). In case \(F(u) = u^p\), let \(w(y, s) = (T - t)^{1/(p-1)}u(y(T - t)^{1/2}, t), \quad s = -\log(T - t)\). We study the large time behavior of \(w(y, s)\). In the radial case, we prove: if \(w(y, s) \neq \beta \beta^\beta \quad (\beta = (p - 1)^{-1})\), then either \(w(y, s) = \beta \beta^\beta - (2p)^{-1}\beta \beta^\beta N^{2(N-1)/2} \pi^{N/4} H(y)s^{-1} + o(1/s)\) where \(H(y) = (2N)^{-1}|y|^2 - 1\) or there exists an \(m \geq 3, k_m > 1\), constants \(C_i\) (not all zero) and polynomials \(H_{m,i}\) of degree \(m\), such that \(w(y, s) = \beta \beta^\beta - e^{(1-m/2)s}\sum_{i=1}^{k_m} C_i H_{m,i}(y) = o(e^{(1-m/2)s})\). The above convergence takes place in \(C^2_{\text{loc}}\) as well as in some weighted \(L^2\) space. For the nonradial solutions, we also obtain some results in the case \(N = 2\). Similar results also hold in the case \(F(u) = e^u\).

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§1 Introduction

This paper is concerned with nonnegative blowing up solutions of the initial value problem:

\[
u_t = \Delta u + F(u) \quad \text{in} \quad \mathbb{R}^N \times (0, T) \tag{1.1}
\]
\[ u(x,0) = u_0(x) \quad x \in \mathbb{R}^N \]  \hspace{1cm} (1.2)

where \( u_0(x) \) is continuous, nonnegative and bounded, and
\[ F(u) = u^p \quad \text{with} \quad p > 1, \quad \text{or} \quad F(u) = e^u. \] \hspace{1cm} (1.3)

It is well-known that for suitably chosen initial data, the solution of (1.1)-(1.2) blows up in finite time. For instance, if \( F(u) = u^p \) with \( 1 < p \leq 1 + 2/N \), for any nontrivial solution \( u \) of (1.1)-(1.2), there exists a finite time \( T \) such that
\[ \limsup_{t \to T} \sup_{x \in \mathbb{R}^N} u(x,t) = +\infty. \] \hspace{1cm} (1.4)

(see, e.g. [6].) We then call \( T \) the blow-up time. If for some \( x_0 \in \mathbb{R}^N \), there exists a sequence \( \{(x_n,t_n)\} \), such that \( x_n \to x_0, \quad t_n \to T \), and
\[ \lim_{n \to \infty} u(x_n,t_n) = +\infty, \]
we call \( x_0 \) a blow-up point. There are many papers concerning the number of blow-up points, their location and the behavior of \( u \) near a blow-up point, we refer to [4] for a review of recent results.

Of particular interest is the study of asymptotic behavior of solutions as \( t \) approaches the blow-up time. In this direction, the pioneering work is Giga and Kohn [9], followed by [10] and [11]. They used the well-known change of variables: for any \( a \in \mathbb{R}^N \),
\[ w_a(y,s) = (T-t)^{(p-1)/p}u(x,t), \quad y = (x-a)(T-t)^{-1/2}, \quad s = -\log(T-t). \] \hspace{1cm} (1.5)

One can check that:
\[ \frac{\partial}{\partial s} w_a = \Delta w - \frac{y}{2} \cdot \nabla w_a - \frac{1}{p-1} w_a + w_a^p. \] \hspace{1cm} (1.6)

The question of studying the blow-up behavior of \( u \) near a blow-up point is thus transformed to the studying the large time behavior of \( w \). By using the “energy” methods, Giga and Kohn were able to prove that, if \( p < (N+2)/(N-2) \), then
\[ w_a(y,s) \longrightarrow \beta^\theta \quad \text{or} \quad 0, \quad \beta = 1/(p-1), \] \hspace{1cm} (1.7)
uniformly on the sets \( |y| \leq R \) with \( R > 0 \) (if \( N \) is 1 or 2, then the conclusion is true for any \( 1 < p < \infty \)). Moreover, if \( w_a \to 0 \), then \( a \) is not a blow-up point. Similar results have been established even for \( p > (N+2)/(N+2) \), (see Theorem 2.1 below).

From now on, we shall always assume 0 is a blow-up point and \( a = 0 \); therefore, we shall suppress the subscript \( a \). By (1.7),
\[ \lim_{t \to \infty} (T-t)^{(p-1)/p}u(\xi(T-t)^{1/2},t) = \beta^\theta \]
uniformly for $|\xi| < R$ with $R > 0$. This establishes the behavior of $u$ in any space-time parabolas. It is natural to ask what happens beyond these parabolas. For example, what is the asymptotic shape of the curve where $(T - t)^{1/(p-1)}u$ is constant?

In the one-dimensional case, Galaktionov and Posachkov [7] used a formal argument to derive the ansatz

$$u(x, t) \sim (T - t)^{-1/(p-1)}[(p - 1) + \frac{(p - 1)^2}{4p(T - t)\log(T - t)}]^{-1/(p-1)}.$$  

The counterpart of $w$ is

$$w(y, s) = \beta^y[1 + \frac{(p - 1)y^2}{4ps}]^{-1/(p-1)} + O(s^{-1})$$  
(1.8)

as $s \to \infty$. Moreover, at $y = 0$,

$$w(0, s) = \beta^y[1 + \frac{1}{2ps}] + o\left(\frac{1}{s}\right).$$  
(1.9)

M. Herrero and J. Valázques [12] and S. Fillipas [3] have independently given rigorous proofs of (1.9) under some conditions on the initial value.

In this paper, we shall extend (1.9) and related results to radial solutions of (1.1)-(1.2) for any dimension $N$. As to nonradial solutions, we also obtain some results for the case $N = 2$. Before we state our theorems, let us introduce some notations. For $1 \leq q < \infty$, and any positive integer $k$, define

$$L^q_k(R^N) = \{ f \in L^q_{\text{loc}}(R^N) : \int_{R^N} |f(x)|^q \rho(x) dx < \infty \},$$

$$H^k_{\rho}(R^N) = \{ f \in L^2_k(R^N) : \text{for any } j \in [0, k], f^{(j)} \in L^2_{\rho}(R^N) \}$$

where $\rho(x) \triangleq \exp(-|x|^2/4)$. From now on, the symbols $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ will denote the norm and the inner product in $L^2_k(R^N)$ respectively, and the symbol $C$ will represent a positive constant, not necessarily the same at each occurrence. Now we can state our main results.

**Theorem 1.1.** Assume that $w$ is a bounded radial solution of (1.6) with the property that $\|w(\cdot, s) - \beta^y\| \to 0$ as $s \to \infty$. Then one of the following cases must occur: either

$$w(\cdot, s) \equiv \beta^y,$$  
(1.10)

or for $H(y) \triangleq (2N)^{-1}|y|^2 - 1$,

$$w(\cdot, s) = \beta^y - Np\beta^y \frac{1}{2s} H(y) + o\left(\frac{1}{s}\right),$$  
(1.11)

or there exists an $m \geq 3$, some constants $C_i$ (not all zero) and polynomials $H_{m,i}$ of degree $m$, such that

$$w(\cdot, s) = \beta^y + \sum_{i=1}^{k_m} C_i e^{(1-m/2)s} H_{m,i}(y) + o(e^{(1-m/2)s}).$$  
(1.12)
where convergence takes place in $H^1_\rho$ as well as in $C^2_{\text{loc}}$.

If (1.11) happens, then we have the following theorem:

**Theorem 1.2.** Let $u$ be a solution of (1.1)-(1.2) with the following properties:

(i) $F(u) = u^p$, $1 < p < \infty$.

(ii) $u_0(x)$ is a radial function and monotone decreasing in $|x|$.

(iii) $w$ is defined by (1.5) and (1.11) takes place.

If $p \geq (N + 2)/(N - 2)$, we add the assumption that $\Delta u_0 + u_0^p \geq 0$. Then

$$\lim_{t \to T}(T - t)^{1/(p-1)}u(\xi((T - t)|\log(T - t)|)^{1/2}, t) = \beta^\theta[1 + \frac{p - 1}{4p} |\xi|^2]^{-1/(p-1)}$$

uniformly on compact sets $|\xi| \leq R$ with $R > 0$.

**Remark 1.3.** Theorems 1.1 and 1.2 in the one-dimensional case were first proved in [12]. Actually, in that case, they showed that $u_0(x)$ is radial and has a single maximum at 0 implies that (1.11) takes place. In the multi-dimensional case, we are unable to prove this fact.

Concerning the non-radial case, we have the following theorem.

**Theorem 1.4.** Let $N = 2$ and $w$ be a bounded solution of (1.6) with the property that $\|w(\cdot, s) - \beta^\theta\| \to 0$ as $s \to \infty$. Then one of the following cases must occur: either

$$w(\cdot, s) \equiv \beta^\theta,$$

or

$$0 < c \leq s\|w(\cdot, s) - \beta^\theta\|_{H^1_\rho} \leq C < \infty, \quad (1.13)$$

or there exists an $m \geq 3$, some constants $C_i$ (not all zero) and polynomials $H_{m,i}$ of degree $m$, such that

$$w(\cdot, s) = \beta^\theta + e^{(1-m/2)s} \sum_{i=1}^{k_m} C_i H_{m,i}(\cdot) = O(e^{(1-m/2)s}) \quad \text{as} \quad s \to \infty$$

the convergence takes place in $H^1_\rho$ as well as $C^2_{\text{loc}}$.

**Corollary 1.5.** Assume (1.13). Then for some positive constant $C$,

$$(T - t)^{1/(p-1)}u(\xi((T - t)|\log(T - t)|)^{1/2}, t) \geq \beta^\theta(1 + C|\xi|^2)^{-1/(p-1)} \quad \text{as} \quad t \to T \quad (1.14)$$

uniformly on sets $|\xi| \leq R$ with $R > 0$.

As to the case $F(u) = e^u$, we have parallel results. Let

$$w(y, s) = u(x, t) + \log(T - t), \quad x = y(T - t)^{1/2}, \quad s = -\log(T - t). \quad (1.15)$$

Then

$$w_s = \Delta w - \frac{y}{2} \cdot \nabla w + e^w - 1. \quad (1.16)$$
Theorem 1.6. Let \( w \) be a bounded solution of (1.16) with the property that \( \|w(\cdot, s)\| \to 0 \) as \( s \to \infty \). Then we have the following alternatives, either

\[
w(\cdot, s) \equiv 0 \quad \text{for any} \quad s > 0,
\]
or

\[
w(\cdot, s) + N2^{(N-1)/2} \pi^{N/4} (2N)^{-1}|y|^2 - 1)s^{-1} = o(1/s),
\]

or there exists \( m \geq 3 \), some constants \( C_i \) (not all zero) and polynomials \( H_{m,i} \) of degree \( m \), such that:

\[
w(\cdot, s) = \sum_{i=1}^{k_m} C_i e^{(1-m/2)s} H_{m,i} + o(e^{(1-m/2)s}) \quad \text{as} \quad s \to \infty
\]

where convergence takes place in \( H^1_\rho \) as well as in \( C^2_{\text{loc}} \).

Theorem 1.7. Let \( u \) be a solution of (1.1)-(1.2) with the following properties:

(i) \( F(u) = e^u \).

(ii) \( u_0(x) \) is radial and monotone decreasing in \( |x| \). Moreover, \( \Delta u_0 + e^{u_0} \geq 0 \).

(iii) \( w \) is defined by (1.15) and (1.17) takes place.

Then

\[
\lim_{|\xi| \to 0} \{u(\xi((T - t)|\log(T - t)|)^{1/2}, t) + \log(T - t)\} = -\log(1 + |\xi|^2/4)
\]

uniformly on compact subset \( |\xi| \leq R \) with \( R > 0 \).

Theorem 1.8. Let \( N = 2 \) and \( w \) be a bounded solution of (1.16) with the property that \( \|w(\cdot, s)\| \to 0 \) as \( s \to \infty \). Then we have the following alternatives: either

\[
w \equiv 0
\]
or

\[
0 < c \leq s\|w(\cdot, s)\| \leq C < \infty,
\]

or there exists an \( m \geq 3 \), some constants \( C_i \) (not all zero), such that

\[
w(\cdot, s) = \sum_{i=1}^{k_m} C_i e^{(1-m/2)s} H_{m,i} + o(e^{(1-m/2)s}) \quad \text{as} \quad s \to \infty.
\]

Corollary 1.9. Assume that (1.18) happens. Then for some \( C > 0 \),

\[
u(\xi((T - t)|\log(T - t)|)^{1/2}, t) + \log(T - t) \geq -\log(1 + C|\xi|^2).
\]

We shall only give the proofs for the case \( F(u) = u^p \), since the proofs for the exponential case are the same with minor differences: we only need to change the function \( f(v) \) in (2.3) to the function \( \hat{f}(v) = e^v - 1 - v \).
The rest of the paper is organized as follows. In §2, we shall review some known results. Then in §3 we shall prove some results for the general case (i.e. not necessarily radial solutions). Finally, Theorem 1.1 and Theorem 1.4 are established in §4 and §5 respectively. Theorem 1.2 then can be proved by adapting the argument in [12].

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§2. Preliminaries

We shall begin with the review of known results. Assume that \( u \) is a solution of (1.1) and (1.2) which blows up at 0 and \( t = T \).

**Theorem 2.1.** (i) Let \( F(u) = u^p \) with \( 1 < p < \infty \) if \( N = 1 \) or \( 2 \), \( p < (N + 2)/(N - 2) \) if \( N \geq 3 \). Then

\[
\lim_{t \to T} u(x(T - t)^{1/2}, t)(T - t)^{(p-1)/(p)} = \beta^p
\]

uniformly on sets \( |x| \leq R \) with \( R > 0 \) (see [9], [10], [11]).

(ii) Let \( F(u) = u^p \) and \( u_0(x) \) be radial and \( \Delta u_0 + u_0^p \geq 0 \). Then the conclusion of (i) is true for any \( 1 < p < \infty \) (see [1]).

(iii) Let \( F(u) = e^u \), \( N \leq 2 \), and \( \Delta u_0 + e^{u_0} \geq 0 \). Then

\[
\lim_{t \to T} u(x(T - t)^{1/2}, t) + \log(T - t) = 0
\]

uniformly on sets \( |x| \leq R \) with \( R > 0 \) (see [14]).

(iv) Let \( F(u) = e^u \) and \( u_0(x) \) be radial and \( \Delta u_0 + e^{u_0} \geq 0 \). Then the conclusion of (iii) is true for any \( N \) (see [1]).

Note that under the assumptions of Theorem 2.1, the function \( u \) defined in (1.5) or in (1.15) is bounded.

Let \( L v = \rho^{-1} \nabla (\rho \nabla v) + v \) with \( \rho = e^{-|v|^2/4} \). Then \( L \) is a self-adjoint operator on \( L^2_\rho \). The eigenvalues of \( L \) are

\[
\lambda = 1 - \frac{m}{2}, \quad m = 0, 1, 2, \ldots
\]

and the corresponding eigenfunctions are

for \( \lambda_0 = 1 \), \( h_0 \)

for \( \lambda_1 = \frac{1}{2} \), \( h_1(y_i), \; i = 1, 2, \ldots, N \) 

\( (N \text{ distinct eigenfunctions}) \)

for \( \lambda_2 = 0 \), \( h_2(y_i), \; i = 1, 2, \ldots, N \)

\[
h_1(y_i)h_2(y_j), \quad i \neq j, \; i, j = 1, 2, \ldots, N
\]
\[ (N + \binom{N}{2}) \text{ distinct eigenfunctions} \]

......

where \( h_k(y) = H_k(y/2) \) and \( H_k(x) \) is the standard \( k \)-th Hermite polynomial. These eigenfunctions form an orthogonal basis of \( L^2_p \). For a proof, see [3].

We shall write \( H_{m,i}, \quad i = 1, \ldots, k_m \) for the eigenfunctions corresponding to the eigenvalue \( 1 - m/2 \) with the property \( \|H_{m,i}\| = 1 \). Hence the set \( \{H_{m,i}, \quad m = 0, 1, \ldots, i = 1, \ldots, k_m\} \) forms an orthonormal basis of \( L^2_p \).

Let \( w(y,s) = (T - t)^{1/(p-1)}u(x,t) \), where \( y, s \) were defined in (1.5), and let \( v(y,s) = w(y,s) - \beta^\theta \). By Theorem 2.1, \( v \to 0 \) as \( s \to \infty \) uniformly on sets \( |y| \leq R \). By the dominated convergence theorem, this implies \( v \to 0 \) in \( L^2_p \). Moreover, \( v \) solves

\[ v_s = \Delta v - \frac{y}{2} \cdot \nabla v + v + f(v) \quad (2.3) \]

where \( f(v) = (\beta^\theta + v)^p - (p - 1)^{-p/(p-1)} - p(p - 1)^{-1}v \). So \( f(s) = O(s^2) \) as \( s \to 0 \). We can expand \( v \) in terms of Hermite polynomials:

\[ v(y,s) = \sum_{m=0}^{\infty} \sum_{i=1}^{k_m} a_{m,i}(s) H_{m,i}(y) \]

and one can check that

\[ \frac{d}{ds} a_{m,i} = (1 - \frac{m}{2})a_{m,i} + < f(v), H_{m,i}> \quad (2.4) \]

Since \( u \) is nonnegative and \( w \) is bounded, we have \( |v(y,s)| \leq M \) and \( v(y,s) \geq -\beta^\theta \) for \( (y,s) \in R^N \times (0, \infty) \). Consequently, the hypotheses of the following theorem are satisfied by our function \( v \).

**Theorem 2.2.** Let \( v \) be a solution of (2.4) with the following properties:

(i) \( v(y,s) \) exists for all time \( s \) and \( v(y,s) \to 0 \) as \( s \to \infty \) uniformly on \( |y| \leq R \).

(ii) \( |v(y,s)| \leq M \) for all \( (y,s) \in R^N \times (0, \infty) \).

(iii) \( |v(y,s)| \geq -\beta^\theta \) for all \( (y,s) \in R^N \times (0, \infty) \).

Then either \( \|v(\cdot, s)\| \to 0 \) exponentially fast, or

\[ \sum_{m \neq 2} \sum_{i=1}^{k_m} a_{m,i}^2(s) = o\left( \sum_{i=1}^{k_2} a_{2,i}^2(s) \right) \quad (2.5) \]

as \( s \to \infty \).

This theorem was proved by S. Fillipas in [3]. He also proved the following theorem.

**Theorem 2.3.** If \( \|v(\cdot, s)\| \) does not decay exponentially, then

\[ \frac{d}{ds} a_{2,i} = \frac{p}{2\beta^\theta} < \left( \sum_{i=1}^{k_2} a_{2,i} H_{2,i} \right)^2, H_{2,j} > + o\left( \sum_{i=1}^{k_2} a_{2,i}^2 \right). \quad (2.6) \]
If \( \psi(s) \triangleq \|v(\cdot, s)\| \) decays exponentially, then we shall see in \( \S 3 \) that either \( v \equiv 0 \) or (1.12) happens. If \( \psi(s) \) does not decay exponentially, one of the following three cases must necessarily occur: either

\[
\limsup_{s \to \infty} s\psi(s) = \infty, \tag{2.7}
\]

or there exist positive constants \( C \) and \( c \) such that

\[
0 < c \leq \liminf_{s \to \infty} (s\psi(s)) \leq \limsup_{s \to \infty} (s\psi(s)) \leq C < \infty, \tag{2.8}
\]

or

\[
\liminf_{s \to \infty} (s\psi(s)) = 0 \quad \text{and} \quad \limsup_{s \to \infty} (e^{\varepsilon s} \psi(s)) = \infty \tag{2.9}
\]

for any \( \varepsilon > 0 \). We intend to show that the cases (2.8) and (2.10) do not occur. To this end, we need the following result (see [2]).

**Lemma 2.4.** Let \( \psi(s) \) be a nonnegative function such that \( \psi \in \mathcal{C}^0([0, \infty)) \), \( \lim_{s \to \infty} \psi(s) = 0 \) and \( \limsup_{s \to \infty} e^{\varepsilon s} \psi(s) = \infty \) for any \( \varepsilon > 0 \) (resp. \( \limsup_{s \to \infty} s\psi(s) = \infty \)). Then there exists a function \( \eta(s) \in \mathcal{C}^\infty([0, \infty)) \) such that

(i) \( \eta > 0, \ \eta' < 0, \ \lim_{s \to \infty} \eta(s) = 0 \).

(ii) \( 0 < \limsup_{s \to \infty} \psi(s)/\eta(s) < \infty \).

(iii) \( \lim_{s \to \infty} \varepsilon s\psi(s) = \infty \) for any \( \varepsilon > 0 \) (resp. \( \lim_{s \to \infty} s\eta(s) = \infty \)).

(iv) \( \eta'/\eta \) and \( \eta''/\eta \) belong to \( L^1(0, \infty) \).

(v) \( \lim_{s \to \infty} \eta'(s)/\eta(s) = \lim_{s \to \infty} \eta''(s)/\eta(s) = 0 \).

Following [12], we define

\[
\tilde{v}(y, s) = \frac{v(y, s)}{\mu(s)} \tag{2.10}
\]

where

\[
\mu(s) = \begin{cases} 
\eta(s) & \text{if (2.7) or (2.9) holds, where } \eta(s) \\
1/s & \text{if (2.8) holds.}
\end{cases} \tag{2.11}
\]

Expand \( \tilde{v} \) in terms of Hermite polynomials:

\[
\tilde{v} = \sum_{m=0}^{\infty} \sum_{i=1}^{km} b_{m,i} H_{m,i} \tag{2.12}
\]

where

\[
b_{m,i}(s) = \frac{a_{m,i}(s)}{\mu(s)}. \tag{2.13}
\]

By studying the large time behavior of \( \tilde{v} \), we shall be able to exclude (2.9) and in some special cases, (2.7) as well; see sections \( \S 3, \S 4 \) and \( \S 5 \).

We shall also need the following results.
Lemma 2.5. Assume that $v$ solves (2.4) and $|v| \leq M < \infty$. Then for any $r > 1$, $q > 1$, and $L > 0$, there exists $s^*_0 = s^*_0(q,r)$ and $C = C(r,q,L)$, such that

$$\|v(\cdot, s + s^*)\|_{r,\rho} \leq C\|v(\cdot, s)\|_{q,\rho}$$

for any $s > 0$ and any $s^* \in [s^*_0, s^*_0 + L]$.

Lemma 2.6. Let $\phi$ be a solution of the initial value problem:

$$\phi_s = \Delta \phi - \frac{y}{2} \cdot \nabla \phi + \frac{m}{2} \phi + h(y, s),$$

$$\phi(y, 0) = \phi_0(y),$$

where $\phi \in L^2_\rho$ and $h(y, s) \in L^2_{\text{loc}}((0, \infty) : L^2(R^N))$. There exists a positive constant $C$, such that for any $T > 0$, there holds

$$\|\phi(\cdot, s)\|^2 + s\|\nabla \phi(\cdot, s)\|^2 + \int_0^T s^2 \|\Delta \phi(\cdot, s)\|^2 ds + \int_0^T s^2 \|\phi_s(\cdot, s)\|^2 ds$$

$$\leq C(T + T^2 + e^{2mT})(\|\phi_0\|^2 + \int_0^T \|h(\cdot, s)\|^2 ds).$$

Lemmas 2.5 and 2.6 were proved for the one dimensional case in [12], but the arguments there can be easily adapted to apply to the multi-dimensional case.

§3 The General Case

As explained in §2, we are interested in studying the large time behavior of the function $\tilde{v} = v/\mu$ which satisfies

$$\tilde{v}_s = \Delta \tilde{v} - \frac{y}{2} \cdot \tilde{v} + \tilde{v} + \frac{\mu'}{\mu} \tilde{v} + \frac{f(\mu \tilde{v})}{\mu}. \quad (3.1)$$

Moreover, by Lemma 2.4 (ii)

$$\|\tilde{v}(\cdot, s)\| \leq M < \infty. \quad (3.2)$$

For any sequence $s_j \to \infty$, define

$$\tilde{v}_j(y, s) = \tilde{v}(y, s_j + s).$$

By standard parabolic estimates (see [13]), it follows from (3.2) that

$$|\tilde{v}_j| \leq M(R) \text{ on } B(R) \times (-R, \infty)$$

for sufficiently large $s_j$. Applying $L^p$ and Schauder estimates, we obtain

$$|\tilde{v}_j|_{C^{2+s}(B(R/2) \times (-R/2, \infty))} \leq M'(R).$$
Hence there exists a subsequence (again labeled as \( s_j \)) such that
\[
\tilde{v}_j(y, s) \rightarrow \tilde{v}_\infty(y, s) \ \text{in} \ C^2(K)
\] (3.3)
for any compact subset \( K \subset \mathbb{R}^{N+1} \). By Lemma 2.4, it follows that \( \tilde{v}_\infty \) solves
\[
z_s = \Delta z - \frac{y}{2} \cdot \nabla z + z.
\]
We wish to show that \( \tilde{v}_\infty \) is independent of \( s \). To this end, we need the following lemma.

**Lemma 3.1.** For any \( s_0 > 0 \),
\[
\int_{s_0}^{+\infty} \| \tilde{v}_s(\cdot, s) \|^2 ds < \infty.
\] (3.4)

The proof for \( N = 1 \) is due to Herrero and Velázquez [12]. For general \( N \), the proof is similar but slightly different. Since the lemma is crucial for our argument, we provide the details here.

**Proof of Lemma 3.1.** Multiplying (3.1) by \( \rho \tilde{v}_s \), and integrating over \( \mathbb{R}^N \times [s_0, s_1] \), we get
\[
\int_{s_0}^{s_1} \int_{\mathbb{R}^N} \rho \tilde{v}_s^2 dx ds = \frac{1}{2} \int_{\mathbb{R}^N} \rho |\nabla \tilde{v}(\cdot, s_1)|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \rho |\nabla \tilde{v}(\cdot, s_0)|^2
\]
\[
+ \int_{\mathbb{R}^N} \rho \tilde{v}(\cdot, s_1) - \int_{\mathbb{R}^N} \rho \tilde{v}(\cdot, s_0) - \frac{1}{2} \int_{s_0}^{s_1} \int_{\mathbb{R}^N} \rho \frac{\mu'}{\mu} \tilde{v}_s \tilde{v} + \int_{s_0}^{s_1} \int_{\mathbb{R}^N} \frac{f(\mu \tilde{v})}{\mu} \tilde{v}_s \rho
\]
\[
\leq C - \frac{1}{2} \int_{s_0}^{s_1} \int_{\mathbb{R}^N} \rho \frac{\mu'}{\mu} \tilde{v}_s \rho + \int_{s_0}^{s_1} \int_{\mathbb{R}^N} \frac{f(\mu \tilde{v})}{\mu} \tilde{v} \rho \ \text{(by (3.2))}
\]
\[
\triangleq C + J_1 + J_2
\]
where \( C \) depends on \( s_0 \), but independent of \( s_1 \). By integration by parts,
\[
J_1 = -\frac{\mu'(s_1)}{2\mu(s_0)} \int_{\mathbb{R}^N} \rho \tilde{v}^2(y, s_1) dy + \frac{\mu'(s_0)}{2\mu(s_1)} \int_{\mathbb{R}^N} \rho \tilde{v}^2(y, s_0) dy
\]
\[
+ \frac{1}{2} \int_{s_0}^{s_1} \left( \frac{\mu'}{\mu} \right)' \left( \int_{\mathbb{R}^N} \rho \tilde{v}^2 ds \right).
\]

Hence \( J_1 \) is bounded as \( s_1 \rightarrow \infty \) by (3.2) and Lemma 2.4. To estimate \( J_2 \), introduce
\[
G(y, s) \triangleq \frac{1}{\mu} [\frac{(\beta^\mu + \mu \tilde{v})^{1+p}}{(1+p)\mu} - (p-1)^{-p/(p-1)} \tilde{v} - \frac{p \mu \tilde{v}^2}{2(p-1)} - \frac{(p-1)^{-(p+1)/(p-1)}}{(p+1)\mu}],
\]
\[
g(y, s) \triangleq \frac{\mu'}{\mu^2} [-(\frac{2(\beta^\mu + \mu \tilde{v})^{p+1}}{p+1}) + (\beta^\mu + \mu \tilde{v})^p \mu \tilde{v}
\]
\[
+ (p-1)^{-p/(p-1)} \mu \tilde{v} + \frac{2(p-1)^{-(p+1)/(p-1)}}{p+1}].
\]
One readily verifies that
\[
\frac{dG(y, s)}{ds} = \frac{f(\mu \tilde{v})}{\mu} \tilde{v}_s + g(y, s),
\]
and
\[
|G(y, s)| \leq C|v|^2 \quad (v = \mu \tilde{v}),
\]
\[
|g(y, s)| \leq -C \frac{\mu'}{\mu^3} |v|^3.
\]
Therefore, by (3.2) and Lemma 2.4 again, we have for some $L > 0$,
\[
|J_2| = \left| \int_{RN} \rho G(y, s_1)dy - \int_{RN} \rho G(y, s_0)dy - \int_{s_0}^{s_1} \int_{RN} g(y, s)dyds \right|
\leq C \left( 1 - \int_{s_0}^{s_1} \frac{\mu'}{\mu^3} (s) \int_{RN} \rho(y)|v|^3(y, s)dyds \right)
\leq C \left( 1 - \int_{s_0}^{s_1} \frac{\mu'}{\mu^3} (s) |v|^3 (y, s) \right) \int_{RN} \rho(y)dy ds^{3/2} ds
\leq C \left( 1 - \int_{s_0}^{s_1} \frac{\mu'}{\mu^3} (s) \mu^3 (s - L) ds \right)
\leq C \left( 1 - \int_{s_0}^{s_1} \mu'(s) ds \right) = C \left( 1 + \mu(s_0) - \mu(s_1) \right)
\]
where we used the fact $\mu(s - L)/\mu(s) \leq M$ for any $s > 0$ which follows easily from Lemma 2.4. The proof is complete.

We return to the study of $\tilde{v}_j$. By Lemma 3.1,
\[
\int_{-R}^{R} \int_{B(R)} \rho |\tilde{v}_j| dy ds = \int_{-R}^{R} \int_{B(R)} |\tilde{v}_j|^2 \rho dy ds \rightarrow 0
\]
as $j \rightarrow \infty$. Hence
\[
\int_{-R}^{R} \int_{B(R)} |\tilde{v}_{j,s}|^2 dy ds = 0.
\]
It follows that $\tilde{v}_\infty$ is independent of $s$. We have thus proved

**Lemma 3.2.** For any sequence $s_j \rightarrow \infty$, there exists a subsequence (again labeled as $s_j$), such that
\[
\tilde{v}_j(y, s) \rightarrow \tilde{v}_\infty(y) \quad \text{in } C^2(K)
\]
for any compact subset $K \subset R^{N+1}$. Furthermore, $\tilde{v} \in L^2_\rho$ and it solves
\[
\Delta v - \frac{y}{2} \cdot \nabla v + v = 0.
\]

**Remark 3.3.** If the solution $\tilde{v}_\infty$ of (3.5) were unique, we would have $\tilde{v}(y, s) \rightarrow \tilde{v}_\infty(y)$ as $s \rightarrow \infty$. Unfortunately, $\tilde{v}_\infty$ may not be unique since there exists a family of solutions of (3.5), i.e. $\sum_{i=1}^{k_2} \gamma_i H_{2,i}(y)$ for any $\gamma_i \in R^l, i = 1, \ldots, N$. This forces us to use another approach by examining the dynamics of the coefficients $a_{2,i}$; see sections §4 and §5.
Next we want to show that
\[ \tilde{v}_j(y, s) \to \tilde{v}_\infty(y) \quad \text{in } L^2_\rho. \quad (3.6) \]

For this purpose, introduce
\[ L^2_\rho(R^N \times (0, 1)) = \{ g(y, s) : \int_0^1 \int_{R^N} g^2(y, s) \rho(y) dy ds < \infty \}. \]

**Lemma 3.4.** The set \( K \triangleq \{ \tilde{v}_j(y, s) \} \) is precompact in \( L^2_\rho(R^N \times (0, 1)) \).

**Proof.** By (3.2), \( K \) is bounded in \( L^2_\rho(R^N \times (0, 1)) \). By Lemma 2.5, for any \( R > 0 \),
\begin{align*}
\int_0^1 \int_{|y| > R} \tilde{v}_j^2(y, s) \rho(y) dy ds & \leq \left( \int_{|y| > R} \rho(y) dy \right)^{1/2} \left( \int_0^1 \int_{|y| > R} \tilde{v}^4(s_j + s, y) \rho(y) dy ds \right)^{1/2} \\
& \leq C \left( \int_{|y| > R} \rho(y) dy \right)^{1/2} \left( \int_0^1 \int_{R^N} \tilde{v}^2(s_j - L + s, y) \rho(y) dy ds \right)^{1/2} \\
& \leq C \left( \int_{|y| > R} \rho(y) dy \right)^{1/2} \to 0 \quad \text{as } R \to \infty \quad (3.7)
\end{align*}

uniformly in \( j \). Next, we multiply (3.1) by \( \rho \tilde{v} \), then integrate over \( R^N \times (0, 1) \) to get (denote \( \mu_j(s) = \mu(s_j + s) \))
\begin{align*}
\int_0^1 \int_{R^N} \rho |\nabla \tilde{v}_j|^2 & \leq \frac{1}{2} \int_{R^N} \rho \tilde{v}_j^2(y, 0) dy \\
& \quad + \int_0^1 \int_{R^N} \rho \tilde{v}_j^2 - \int_0^1 \frac{\mu_j'}{\mu_j} \int_{R^N} \rho \tilde{v}_j^2 + \int_0^1 \int_{R^N} \frac{f(\mu_j \tilde{v}_j)}{\mu_j} \rho \tilde{v}_j \\
& \leq C(1 + \int_0^1 \frac{|\mu_j'|}{\mu_j} + \int_0^1 \mu_j \int_{R^N} \rho |\tilde{v}_j|^3) \\
& \leq C \left[ 1 + \int_0^1 \mu_j \left( \int_{R^N} \rho \tilde{v}_j^2(y, s - L) dy \right)^{3/2} \right] \quad \text{(by Lemma 2.5)} \\
& \leq C \quad (3.8)
\end{align*}

where we have used the fact that \( f(s) \leq Cs^2 \). Using the mean value theorem and noting (3.4) and (3.8), we obtain
\[ \int_0^1 \int_{R^N} \rho \tilde{v}_j(s + t, y + h) - \tilde{v}_j(y, s)^2 dy ds \to 0 \quad (3.9) \]
as \( t \to 0, |h| \to 0 \). The conclusion of the lemma follows from (3.7) and (3.9) by standard results from real analysis.

By Lemma 3.4, for any sequence \( s_j \to \infty \), there exists a subsequence (again labeled as \( s_j \)), such that
\[ \tilde{v}_j(y, s) \to \tilde{v}_\infty'(y, s) \quad \text{in } L^2_\rho(R^N \times (0, 1)). \]
On the other hand, by Lemma 3.2, there exists another subsequence (again labeled as $s_j$) such that
\[ \tilde{v}_j(y, s) \longrightarrow \tilde{v}_\infty(y) \]
uniformly on compact subsets of $\mathbb{R}^N$. Consequently,
\[ \tilde{v}'_\infty(y, s) = \tilde{v}_\infty(y), \]
i.e.
\[ \tilde{v}_j(y, s) \longrightarrow \tilde{v}_\infty(y) \quad \text{in} \quad L^2_p(\mathbb{R}^N \times (0, 1)). \tag{3.10} \]

We want to show further that (3.6) holds. Since $L^2_p$ is a Hilbert space, we only need to check that
\[ \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2_j(y, 0) dy = \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2_j(y, s_j) dy \longrightarrow \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2_\infty(y) dy. \tag{3.11} \]

Note that (3.10) implies
\[ \int_0^1 \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2_j(y, s) dy \longrightarrow \int_{\mathbb{R}^N} \tilde{v}^2_\infty(y) \rho(y) dy \quad \text{as} \quad j \rightarrow \infty. \tag{3.12} \]

Hence the proof of (3.11) reduces to the following estimate:
\[
\left| \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2(y, s_j) - \int_0^1 \int_{\mathbb{R}^N} \rho(y) \tilde{v}_j(y, s) dy ds \right|
\leq \int_0^1 \int_{\mathbb{R}^N} \rho(y) |\tilde{v}^2_j(y, s) - \tilde{v}^2_j(y, 0)| dy ds
= \int_0^1 \int_{\mathbb{R}^N} |\int_0^1 2\tilde{v}_j \tilde{v}_{j,s}(y, ts) s dt| dy ds
\leq 2 \int_0^1 s ds \left( \int_0^1 \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2_j(y, ts) dy dt \right)^{1/2} \left( \int_0^1 \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2_{j,s}(y, ts) dy dt \right)^{1/2}
\leq C \int_0^1 s \left( \int_0^1 \int_{\mathbb{R}^N} \rho(y) \tilde{v}^2_j(y, s_j + ts) dy dt \right)^{1/2} ds \longrightarrow 0 \quad \text{as} \quad j \rightarrow \infty
\]

by (3.4). To summarize, we have established:

**Lemma 3.5.** For any sequence $s_j \rightarrow \infty$, there exists a subsequence (again labeled as $s_j$) such that
\[ \tilde{v}(y, s_j + s) \longrightarrow \tilde{v}_\infty(y) \quad \text{in} \quad L^2_p(\mathbb{R}^N) \]
and in $C^2(K)$ for any compact subset $K \subset \mathbb{R}^N$. Moreover, $\tilde{v}_\infty$ solves (3.5) and therefore belongs to the eigenspace corresponding to the eigenvalue $\lambda_2 = 0$. In particular,
\[ \tilde{v}_\infty = \sum_{i=1}^{k_2} \gamma_i H_{2,i} \]
for some constants $\gamma_i$, $i = 1, \ldots, N$. 

13
Now we are ready to study the large time behavior of $v(y,s)$. We begin with the case that $\|v(\cdot,s)\|$ decays exponentially, i.e.

$$\psi(s) \leq Ce^{-\epsilon s}$$  \hspace{1cm} (3.13)$$

for some $\epsilon > 0$.

**Theorem 3.6.** Assume that (3.13) holds. Then either $\psi \equiv 0$ or there exists an $m \geq 3$ and constants $C_i, \ i = 1, \ldots, k_m$, not all zero, such that

$$\|v(\cdot,s) - \sum_{i=1}^{k_m} C_i e^{1-m/2)s} H_{m,i}\| = o(e^{(1-m/2)s}).$$

The proof given in [12] for one-dimensional case works here with some trivial changes; hence, the details are omitted.

If $\psi(s)$ does not decay exponentially, then we have known that there exist three possibilities (2.7)-(2.9).

**Theorem 3.7.** (2.9) can not occur.

**Proof.** Let $V = \sum_{i=1}^{k_2} a^2_{2,i}$. We need to distinguish two types of Hermite polynomials of degree 2:

$$H_{2,i} = c_i \left( \frac{1}{2} y_i^2 - 1 \right), \ i = 1, \ldots, N,$$

$$H_{2,i} = c_2 y_j y_l, \ i = N + 1, \ldots, k_2, \ j \neq l, 1 \leq j, l \leq N.$$

For any fixed $i \leq N$, assume that $H_{2,j(i)} = c_2 y_{j(i)} y_1, \ldots, H_{2,j(i)+N-1} = c_2 y_{j(i)} y_N$. Using Theorem 2.3, we can easily check that

$$\frac{d}{ds} a_{2,i} = \nu_1 a^2_{m,i} + \sum_{l=j(i)}^{j(i)+N-1} \nu_{2,l} a^2_{2,l} + h_i, \ i = 1, \ldots, N$$  \hspace{1cm} (3.14)$$

$$\frac{d}{ds} a_{2,i} = \sum_{j \neq l} \nu_{j,l} a_{2,j} a_{2,l} + h_i \ i = N + 1, \ldots, k_2$$  \hspace{1cm} (3.15)$$

where $h_i = o(\sum_{i=1}^{k_2} a^2_{2,i})$, $\nu_1 = p(2\beta \alpha)^{-1} < H^2_{2,1}, H_{2,1} >, \nu_{2,l}$ and $\nu_{j,l}$ are computable constants. Hence

$$\frac{d}{ds} V = 2 \sum_{i=1}^{k_2} \frac{d}{ds} a_{2,i} \geq -C(\sum_{i=1}^{k_2} a^2_{2,i})^{3/2} + h$$

$$= -CV^{3/2} + h$$  \hspace{1cm} (3.16)$$

where $h = o(V^{3/2})$. Integrating (3.16), we get

$$-\frac{1}{V^{1/2}(s)} + \frac{1}{V^{1/2}(s_0)} \geq -C(s-s_0) + \int_{s_0}^{s} h(s)ds$$

14
or
\[- \frac{1}{sv^{1/2}(s)} + \frac{1}{sv^{1/2}(s_0)} \geq -C(1 - \frac{s}{s_0}) + \frac{1}{s} \int_{s_0}^{s} \tilde{h}(s) ds\] (3.17)

where \(\tilde{h}(s) \to 0\) as \(s \to \infty\).

If (2.9) happens, there exists a sequence \(s_j \to \infty\) such that \(s_j \psi(s_j) \to 0\). Since \(V(s) \leq \psi(s)\), we also have that \(s_j V(s_j) \to 0\). Setting \(s = s_j\) in (3.17) and letting \(j \to \infty\), we obtain a contradiction. The proof is complete.

\[\textbf{§4 The Radial Case}\]

Throughout this section, we shall assume that \(u(x, t)\) is a radial solution of (1.1)-(1.2); hence, \(w(y, s)\) and \(v(y, s)\) are all radial functions. We have shown that (2.9) cannot occur in §3. If \(v(y, s)\) is a radial function, we shall show also that (2.7) cannot happen.

By Lemma 3.5, for any sequence \(s_j \to \infty\), there exists a subsequence (again labeled as \(s_j\)), such that:
\[\tilde{v}(y, s_j) \to \tilde{v}_\infty(y) = \sum_{i=1}^{k_2} \gamma_i H_{2,i}(y).\] (4.1)

As we did in §3, we divide \(H_{2,i}\) into two sets: \(H_{2,i} = c_1(2^{-1}y_i^2 - 1), \quad i = 1, \ldots, N; \quad H_{2,i} = c_2 y_j y_i, \quad i = N + 1, \ldots, k_2, \quad j \neq l, \quad 1 \leq j, l \leq N\). Since \(\tilde{v}(y, s_j)\) is a radial function, so is \(\tilde{v}_\infty(y)\). It follows that
\[\gamma_1 = \cdots = \gamma_N, \quad \gamma_{N+1} = \cdots = \gamma_{k_2} = 0.\] (4.2)

**Lemma 4.1.** There holds
\[b_{2,i}(s) = \frac{a_{2,i}(s)}{\mu(s)} \to 0 \quad \text{as} \quad s \to \infty, \quad i = N + 1, \ldots, k_2.\]

**Proof.** Suppose that \(b_{2,i_0}(s_j) \geq c > 0\), for some \(s_j \to \infty, \quad N + 1 \leq i_0 \leq k_2\). Then there exists a subsequence (again labeled as \(s_j\)), such that \(\tilde{v}(y, s_j) \to \tilde{v}_\infty(y) = \gamma \sum_{i=1}^{N} H_{2,i}(y)\) by (4.2). Hence \(< \tilde{v}(y, s_j), H_{2,i_0} > \to < \tilde{v}_\infty, H_{2,i_0} >= 0\), i.e. \(b_{2,i_0}(s_j) \to 0\), a contradiction.

Next we shall remove the possibility (2.7).

**Lemma 4.2.** The case (2.7) cannot occur.

**Proof.** Assume that (2.7) is the case, i.e. \(\limsup s \psi(s) = \infty\). By Lemma 2.4 (ii), there exists a sequence \(s_j \to \infty\), such that
\[||\tilde{v}(\cdot, s_j)|| = \psi(s_j)/\mu(s_j) \geq c > 0.\] (4.3)

On the other hand, we have \(\lim_{s \to \infty} s \mu(s) = \infty\) because of Lemma 2.4 (iii). Therefore,
\[\lim_{j \to \infty} s_j \psi(s_j) = \infty.\]

15
By Theorem 2.2, we further get
\[
\lim_{j \to \infty} s_j \left( \sum_{i=1}^{k_2} a_{2,i}^2(s_j) \right)^{1/2} = \infty. \tag{4.4}
\]

For this sequence \( s_j \to \infty \), we can find a subsequence (again labeled as \( s_j \)) such that
\[
\tilde{v}(y, s_j) \to \tilde{v}_\infty(y) = \gamma \sum_{i=1}^{N} H_{2,i}(y) \text{ in } \mathbb{L}^2_{\mu}(\mathbb{R}^N). \tag{4.5}
\]

By (4.3), \( \gamma \neq 0 \). Furthermore, it follows from (4.5) that
\[
b_{2,i}(s_j) = \frac{a_{2,i}(s_j)}{\mu(s_j)} \to \gamma \neq 0 \quad i = 1, \ldots, N, \tag{4.6}
\]
\[
b_{2,i}(s_j) = \frac{a_{2,i}(s_j)}{\mu(s_j)} \to 0 \quad i = N + 1, \ldots, k_2. \tag{4.7}
\]

In particular,
\[
\frac{\sum_{i=N+1}^{k_2} a_{2,i}^2(s_j)}{\sum_{i=1}^{N} a_{2,i}^2(s_j)} \to 0 \quad \text{as } j \to \infty. \tag{4.8}
\]

By (4.4) and (4.8),
\[
s_j \left( \sum_{i=1}^{N} a_{2,i}^2(s_j) \right)^{1/2} \to +\infty. \tag{4.9}
\]

It follows from (4.6) and (4.9) that
\[
s_j \left| \sum_{i=1}^{N} a_{2,i}(s_j) \right| \to \infty \quad \text{as } j \to \infty. \tag{4.10}
\]

We now take up an approach already used in Theorem 3.7. Let \( V = \sum_{i=1}^{N} a_{2,i} \). By (3.14),
\[
\frac{dV}{ds} = \nu_1 \sum_{i=1}^{N} a_{2,i}^2 + \sum_{i>N} \nu_i a_{2,i}^2 + h
\geq \epsilon \left( \sum_{i=1}^{N} a_{2,i}^2 \right)^2 + h = \epsilon V^2 + h \tag{4.11}
\]
for some \( \epsilon > 0 \), where \( h = o(\sum_{i=1}^{N} a_{2,i}^2) = o(V^2) \). Integrating (4.11) yields
\[
-V(s) + \frac{1}{V(s_0)} \geq \epsilon(s - s_0) + \int_{s_0}^{s} g(t)dt \tag{4.12}
\]
where \( g(s) \to 0 \) as \( s \to \infty \). Setting \( s = s_j \) in (4.12) and letting \( j \to \infty \), we get a contradiction because of (4.10).
Combining Theorem 3.7 with Lemma 4.2, we conclude that if $\psi(s)$ does not decay exponentially, only (2.8) is possible. Next, we want to obtain the exact behavior of $a_{2,i}(s)$ as $s \to \infty$, $i = 1, \ldots, N$. We begin with the following lemma.

**Lemma 4.3.** Assume (2.8). There exists a $\delta > 0$, $L > 0$, such that

$$\delta \leq \frac{a_{2,i}(s)}{\sum_{i=1}^{N} a_{2,i}^2(s)} \leq 1 \quad \text{for any } s \in (L, \infty), \quad i = 1, \ldots, N.$$ 

**Proof.** Suppose for some $s_j \to \infty$, $1 \leq i_0 \leq N$, we have

$$\frac{a_{2,i_0}^2(s_j)}{\sum_{i=1}^{N} a_{2,i}^2(s_j)} \to 0 \quad \text{as } j \to \infty.$$ 

From (2.8), it follows that $s_j a_{2,i_0}(s_j) \to 0$ as $j \to \infty$. On the other hand, by Lemma 3.5, there exists a subsequence (again labeled as $s_j$), such that $\tilde{v}(y, s_j) \to \tilde{v}_\infty(y)$. We have $\tilde{v}_\infty \neq 0$ because of (2.8). Therefore,

$$\tilde{v}(y, s_j) = s_j v(y, s_j) \to \gamma \sum_{i=1}^{N} H_{2,i}(y) \quad \text{in } L^2_\rho,$$ 

This, in turn, implies

$$s_j a_{2,i_0}(s_j) \to \gamma \neq 0,$$ 

a contradiction.

Using Lemma 4.1 and Lemma 4.3, we can rewrite (3.14) as

$$\frac{d}{ds} a_{2,i} = \nu_1 a_{2,i}^2 + o(a_{2,i}^2), \quad i = 1, \ldots, N. \quad (4.13)$$

Integrating (4.13), we find that

$$- \frac{1}{a_{2,i}(s)} + \frac{1}{a_{2,i}(s_0)} = \nu_1 (s - s_0) + \int_{s_0}^{s} h_i(t)dt, \quad i = 1, \ldots, N$$

where $h_i(s) \to 0$ as $s \to \infty$. Therefore,

$$\lim_{s \to \infty} s a_{2,i}(s) = - \frac{1}{\nu_1} = - \frac{\beta}{p} 2^{(N-1)/2} \pi^{N/4}, \quad i = 1, \ldots, N.$$ 

We conclude that

$$v(y, s) = \sum_{i=1}^{N} a_{2,i}(s) H_{2,i}(y) + \sum_{i+N+1}^{k_1} a_{2,i}(s) H_{2,i}(y) + \sum_{m \neq 2}^{k_m} \sum_{i=1}^{N} a_{2,i} H_{2,i}(y)$$

$$= - \frac{\beta}{p} 2^{(N-1)/2} \pi^{N/4} \sum_{i=1}^{N} H_{2,i}(y) + o\left(\frac{1}{s}\right)$$

$$= - \frac{N \beta}{ps} 2^{(N-1)/2} \pi^{N/4} \left(\frac{1}{2N} |y|^2 - 1\right) + o\left(\frac{1}{s}\right) \quad (4.14)$$

17
where the convergence takes place in $L^2_{\rho}$. From Lemma 3.5, it follows that the convergence also takes place in $C^2_{\text{loc}}$.

To finish the proof of Theorem 1.1, we need to extend the convergence in (4.14) to $H^1_{\rho}$, which is required in the proof of Theorem 1.2. We begin with a lemma.

**Lemma 4.4.** Suppose $H_m$, $H_n$ are two different Hermite polynomials. Then

$$<\frac{\partial}{\partial y_j} H_m, \frac{\partial}{\partial y_j} H_n> = 0,$$

(4.15)

$$\|\frac{\partial}{\partial y_j} H_m\| = \left( \frac{m}{2} \right)^{1/2} \text{ or } 0.$$  

(4.16)

**Proof.** Without loss of generality, assume that $H_m = h_{m_1}(y_1) \cdots h_{m_i}(y_i)$, $m_1 + \cdots + m_i = m$, and $y_j = y_1$. Note that $h_{m_i}$ satisfies

$$h''_{m_i} - \frac{y}{2} h'_{m_i} + \frac{m_i}{2} = 0.$$  

(4.17)

We compute

$$\|\frac{\partial}{\partial y_1} H_m\|^2 = \int_{R^N} \rho \frac{\partial}{\partial y_1} (h_{m_1} \cdots h_{m_i}) \frac{\partial}{\partial y_1} H_m$$

$$= -\int_{R^N} \frac{\partial}{\partial y_1} (\rho \frac{\partial}{\partial y_1} h_{m_1} \cdots h_{m_i}) H_m$$

$$= \int_{R^N} \rho \frac{y_1}{2} h_{m_1} \cdots h_{m_i} H_m - \int_{R^N} \rho (\frac{\partial^2}{\partial y_1^2} h_{m_i}) \cdots h_{m_i} H_m \quad \text{(by (4.17))}$$

$$= \frac{m_1}{2} \int_{R^N} \rho H''_{m_i} = \frac{m_1}{2} \quad \text{(by (4.17)).}$$

Similarly, one can check (4.15).

Set

$$v - \sum_{i=1}^{N} a_{2,i} H_{2,i} = \sum_{i=N+1}^{k_2} a_{2,i} H_{2,i} + \sum_{m=0}^{k_m} \sum_{i=1}^{k_m} a_{m,i} H_{m,i} = \sum_{m=3}^{k} a_{2,i} H_{2,i},$$

$$\Delta = I_1 + I_2 + I_3.$$  

(4.18)

We proceed to estimate various terms as follows:

$$\|I_1\|_{H^1_{\rho}} = \sum_{i=1}^{k_2} |a_{2,i}| \|\nabla H_{2,i}\|$$

$$\leq C \sum_{i=1}^{k_2} |a_{2,i}(s)| = o(\frac{1}{s}).$$
$I_2$ can be dealt with similarly. To estimate $I_3$, a different approach is required. For any fixed $R > 0$, we can write
\[
v(y, s) = S_L(R)v(y, s - R) + \int_{s-R}^{s} S_L(s - \tau)f(v(\cdot, \tau))d\tau \tag{4.19}
\]
where $S_L$ is the semigroup generated by $L$ on $L^2_{\rho}$. Since $f(v) = \sum_{m=0}^{\infty} \sum_{i=1}^{k_m} < f(v), H_{2,i}> H_{2,i}$, we have
\[
v'(s) = \sum_{m=0}^{2} \sum_{i=1}^{k_m} a_{m,i}(s - R)e^{(1-m/2)R} + \int_{s-R}^{s} e^{(1-m/2)(s-\tau)} < f(v), H_{m,i}> d\tau H_{m,i} \]
\[
+ \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} a_{m,i}(s - R)e^{(1-m/2)R}H_{m,i} + \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} H_{m,i} \int_{s-R}^{s} e^{(1-m/2)(s-\tau)} < f(v), H_{m,i}> d\tau \]
\[
\triangleq J_1 + J_2 + J_3. \tag{4.20}
\]
Comparing (4.18) and (4.20), we find that
\[
I_3 = J_2 + J_3.
\]
Let us estimate the $H^1_{\rho}$ norm of $J_2$ and $J_3$. By lemma 4.4,
\[
\|J_2\|_{H^1_{\rho}} \leq \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} a_{m,i}(s - R)e^{(1-m/2)R}(1 + \frac{m}{2})^{1/2}.
\]
We can easily check that
\[
k_m \leq Cm^{N-1}. \tag{4.21}
\]
Consequently,
\[
\|J_2\|_{H^1_{\rho}} \leq \left(\sum_{m=3}^{\infty} \sum_{i=1}^{k_m} (1 + \frac{m}{2})e^{(1-m/2)R}\right)^{1/2} \left(\sum_{m=3}^{\infty} \sum_{i=1}^{k_m} a_{m,i}^2(s - R)\right)^{1/2}
\]
\[
\leq C\left(\sum_{m=3}^{\infty} m^Ne^{(1-m/2)R}\right)^{1/2}\|v(\cdot, s - R) - \sum_{i=1}^{k_2} a_{2,i}(s - R)H_{2,i}\|.
\]
But
\[
\sum_{m=3}^{\infty} m^Ne^{(1-m/2)R} = e^{-R}\sum_{j=0}^{\infty} (j + 3)^Ne^{-jR} \leq C < \infty,
\]
we conclude that
\[
\|J_2\|_{H^1_{\rho}} \leq C\|v(\cdot, s - R) - \sum_{i=1}^{k_2} a_{2,i}(s - R)\| = o\left(\frac{1}{s}\right).
\]

The estimate of $J_3$ is more involved. First we note that $f(s) \leq Cs^2$; hence, $\|f(v(\cdot, s))\| \leq Cs^{-2}$. We shall next estimate the "$\| \cdot \|$" norm of $J_3$:
\[
\|J_3\| \leq \int_{s-R}^{s} \left\| \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} e^{(1-m/2)(s-\tau)} < f(v), H_{2,i}> d\tau H_{m,i}\|\]
\[
\triangleq \int_{s-R}^{s} g(\tau)d\tau.
\]
$g(s)$ can be estimated as follows:

$$g(s) = \left[ \int_{\mathbb{R}^N} \rho \left( \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} e^{(1-m/2)(1-m/2)(s-\tau)} < f(v), H_{m,i} > H_{m,i} \right)^2 dy \right]^{1/2}$$

$$= \left( \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} e^{2(1-m/2)(s-\tau)} < f(v), H_{m,i} >^2 \right)^{1/2} \int_{\mathbb{R}^N} \rho H_{m,i}^2 \right)^{1/2} \quad \text{(recall } s - \tau > 0)$$

$$\leq \left( \sum_{m=3}^{\infty} \sum_{i=1}^{k_m} < f(v), H_{2,i} >^2 \right)^{1/2}$$

$$\leq \|f(v)\| \leq C \frac{1}{s^2}.$$ 

It follows that

$$\| J_3 \| \leq \frac{C}{s^2}. \quad (4.22)$$

We have thus proved that $\| J_3(s) \| = o(1/s)$. Observe that $J_3$ solves

$$z_s = \Delta z - \frac{y}{2} + z + (f(v) - \sum_{m=0}^{2} \sum_{i=1}^{k_m} < f(v), H_{m,i} > H_{m,i})$$

$$\Delta \equiv Lz + h(y, s)$$

where $\|h(\cdot, s)\| \leq \|f(v)\| \leq C/s^2$. Since we can write

$$J_3(\cdot, s) = S_L J_3(\cdot, s - R) + \int_{s-R}^{s} S_L(s - R) h(\cdot, \tau) d\tau$$

for any fixed $R > 0$, the conclusion that $\| J_3(s) \|_{H^1_{\rho}} = o(1/s)$ follows from Lemma 2.6 and (4.22).

We summarize:

**Theorem 4.6.** If $\psi(s)$ does not decay exponentially, then (4.14) holds, where convergence takes place in $H^1_{\rho}$ as well as $C^2_{\text{loc}}$.

Theorem 1.2 now follows from Theorem 3.6 and Theorem 4.6. Having established Theorem 1.1, we can adapt the argument in [12] to prove Theorem 1.2. We only need to notice that if $S(t)$ is the semigroup associated to the heat operator, and if $v(y, s) = \sum_m \sum_i a_{m,i}(s) H_{m,i}(y)$, then $S(t)v(y, s) = \sum_m \sum_i a_{m,i}(s)(1 - t)^{m/2} H_{m,i}(y/(1 - t)^{1/2})$. The details are omitted.

§5 The Case $N = 2$

In the case $N = 2$, if we can exclude (2.7), then Theorem 1.4 will be proved.
Let us write \( H_{2,1} = c_1(2^{-1}y_1^2 - 1), \ H_{2,2} = c_1(2^{-1}y_2^2 - 1), \ H_{2,3} = c_2 y_1 y_2. \) In \( \S 4, \) we proved that
\[
(\sum_{i=1}^{N} a_{2,i})' \geq \epsilon (\sum_{i=1}^{k_2} a_{2,i}^2)(1 + o(1)). \tag{5.1}
\]
Although we assumed solutions were radial in \( \S 4, \) the proof there applies to the general case without any changes. In the case \( N = 2, \) (5.1) reads
\[
(\sum_{i=1}^{2} a_{2,i})' \geq \epsilon (\sum_{i=1}^{3} a_{2,i}^2)(1 + o(1)) > 0. \tag{5.2}
\]
Since \( a_{2,1} + a_{2,2} \to 0 \) as \( s \to \infty, \) we see that
\[
a_{2,1} + a_{2,2} < 0. \tag{5.3}
\]
If (2.7) happens, as we showed before, there exists \( s_j \to \infty, \) such that
\[
\tilde{v}(y, s_j) \to v_{\infty}(y) = \sum_{i=1}^{3} \gamma_i H_{2,i} \tag{5.4}
\]
and
\[
s_j \psi(s_j) \to \infty. \tag{5.5}
\]
If we can show there exists another sequence \( \tau_j \) such that
\[
\tau_j |a_{2,1}^{\prime}(\tau_j) + a_{2,2}(\tau_j)| \to \infty, \tag{5.6}
\]
then we will get a contradiction to (5.2) as before. Hence the proof of the fact that (2.7) cannot occur reduces to the construction of \( \tau_j. \) To this end, we need the following lemma.

**Lemma 5.1.** There holds
\[
|a_{2,1} + a_{2,2}^\prime(s)| > |a_{2,3}|(s) \tag{5.7}
\]
for \( s \) sufficiently large.

**Proof.** The \( a_{2,i} \)'s satisfy
\[
\frac{d}{ds} a_{2,1} = \nu_{1} a_{2,1} + \frac{1}{2} \nu_{1} a_{2,3}^2 + h_1, \tag{5.8}
\]
\[
\frac{d}{ds} a_{2,2} = \nu_{1} a_{2,2} + \frac{1}{2} \nu_{1} a_{2,3}^2 + h_2, \tag{5.9}
\]
\[
\frac{d}{ds} a_{2,3} = \nu_{1}(a_{2,1} + a_{2,2}) a_{2,3} + h_3 \tag{5.10}
\]
where \( h_i = o(a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2), \ i = 1, 2, 3. \) Multiplying (5.10) by \( \text{sgn} a_{2,3}, \) then adding (5.8) and (5.9) to it, we get
\[
(a_{2,1} + a_{2,2} + |a_{2,3}|)' \geq \nu_{1}(a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2) - \nu_{1}(a_{2,1} + a_{2,2}) a_{2,3} + \sum_{i=1}^{3} h_i \geq \delta(a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2) + \sum_{i=1}^{3} h_i > 0
\]
for $s$ sufficiently large. Since $a_{2,1} + a_{2,2} + |a_{2,3}| \to 0$ as $s \to \infty$, we see that

$$a_{2,1} + a_{2,2} + |a_{2,3}| < 0$$

and (5.7) follows.

By Lemma 5.1 and (5.5), we have

$$s_j(a_{2,1}^2(s_j) + a_{2,2}^2(s_j))^{1/2} \to \infty.$$  \hspace{1cm} (5.12)

Let us begin the proof of the existence of $\tau_j \to \infty$ such that (5.6) holds. Three cases can occur:

(i) $\gamma_1 \gamma_2 = 0$. Then we let $\tau_j = s_j$ and (5.6) follows from (5.12).

(ii) $\gamma_1 \gamma_2 > 0$. In this case, $a_{2,1}(s_j)$ and $a_{2,2}(s_j)$ have the same sign when $j$ is sufficiently large. We can again let $\tau_j = s_j$ and (5.6) follows from (5.12).

(iii) $\gamma_1 \gamma_2 < 0$. Without loss of generality, assume that $\gamma_1 > 0$ and $\gamma_2 < 0$. Then $a_{2,1}(s_j) > 0$ and $a_{2,2}(s_j) < 0$ for $j$ sufficiently large. From (5.4) and (5.12), it follows that

$$s_j a_{2,1}(s_j) \to \infty,$$  \hspace{1cm} (5.13)

$$s_j |a_{2,2}(s_j)| \to \infty$$  \hspace{1cm} (5.14)

as $j \to \infty$. Using the fact that $a_{2,1}(s_j)/\mu(s_j) \to \gamma_1$, $a_{2,2}(s_j)/\mu(s_j) \to \gamma_2$, we obtain that

$$\frac{|a_{2,1}(s_j)|}{(a_{2,1}^2(s_j) + a_{2,2}^2(s_j))^{1/2}} \geq c > 0.$$  

Consequently,

$$a_{2,1}'(s_j) > 0 \quad \text{(recall (5.8))}$$

for $j$ sufficiently large. Since $a_{2,1}(s) \to 0$ as $s \to \infty$, there exists an $s_0 > s_j$ such that $a_{2,1}'(s_0) = 0$. Let

$$\tau_j = \inf \{s > s_j : a_{2,1}'(s) = 0\}.$$  

From the fact that $a_{2,1}'(\tau_j) = 0$ and (5.8), it follows that

$$a_{2,1}^2(\tau_j) + a_{2,3}^2(\tau_j) = o(a_{2,1}^2(\tau_j) + a_{2,2}^2(\tau_j)) = o(a_{2,2}^2(\tau_j)).$$  \hspace{1cm} (5.15)

On the other hand, since $a_{2,1}'(s) \geq 0$ on $(s_j, \tau_j)$, we have

$$a_{2,1}(\tau_j) \geq a_{2,1}(s_j) > 0.$$  \hspace{1cm} (5.16)

Combining (5.15) with (5.16), we conclude that

$$\tau_j a_{2,1}(\tau_j) + a_{2,2}(\tau_j) \geq \tau_j a_{2,1}(\tau_j) \geq s_j a_{2,1}(s_j) \to \infty \quad \text{as} \quad j \to \infty$$

by (5.13).
We summarize

**Theorem 5.2.** In the case $N = 2$, (2.7) can not occur.

Theorem 1.4 now follows from Theorem 3.6 anf Theorem 5.2.

Having established Theorem 1.4, we can argue as in [12] to prove Corollary 1.5. Indeed, all we need to prove (1.14) in that argument is

$$
\sum_{i=1}^{k_2} a_{2,i}(s) H_{2,i}(\frac{x}{(T-t)^{1/2}}) \geq \frac{C}{\log(T-t)} + o(\frac{1}{\log(T-t)}).
$$

(5.17)

and (5.17) follows from (1.13).

**References**


<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>637</td>
<td>Xinfu Chen</td>
<td>Generation and propagation of the interface for reaction–diffusion equations</td>
</tr>
<tr>
<td>638</td>
<td>Philip Korman</td>
<td>Dynamics of the Lotka–Volterra systems with diffusion</td>
</tr>
<tr>
<td>639</td>
<td>Harlan W. Stech</td>
<td>Generic Hopf bifurcation in a class of integro-differential equations</td>
</tr>
<tr>
<td>640</td>
<td>Stephane Laederich</td>
<td>Periodic solutions of non linear differential difference equations</td>
</tr>
<tr>
<td>641</td>
<td>Peter J. Olver</td>
<td>Canonical Forms and Integrability of BiHamiltonian Systems</td>
</tr>
<tr>
<td>642</td>
<td>S.A. van Gils, M.P. Krupa and W.F. Langford</td>
<td>Hopf bifurcation with nonsemisimple 1:1 Resonance</td>
</tr>
<tr>
<td>643</td>
<td>R.D. James and D. Kinderlehrer</td>
<td>Frustration in ferromagnetic materials</td>
</tr>
<tr>
<td>644</td>
<td>Carlos Rocha</td>
<td>Properties of the attractor of a scalar parabolic P.D.E.</td>
</tr>
<tr>
<td>645</td>
<td>Debra Lewis</td>
<td>Lagrangian block diagonalization</td>
</tr>
<tr>
<td>646</td>
<td>Richard C. Churchill and David L. Rod</td>
<td>On the determination of Ziglin monodromy groups</td>
</tr>
<tr>
<td>647</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>A nonlocal diffusion equation arising in terminally attached polymer chains</td>
</tr>
<tr>
<td>648</td>
<td>Peter Gritzmann and Victor Klee</td>
<td>Inner and outer j- Radii of convex bodies in finite-dimensional normed spaces</td>
</tr>
<tr>
<td>649</td>
<td>P. Szmolyan</td>
<td>Analysis of a singularly perturbed traveling wave problem</td>
</tr>
<tr>
<td>650</td>
<td>Stanley Reiter and Carl P. Simon</td>
<td>Decentralized dynamic processes for finding equilibrium</td>
</tr>
<tr>
<td>651</td>
<td>Fernando Reitich</td>
<td>Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions</td>
</tr>
<tr>
<td>652</td>
<td>Russell A. Johnson</td>
<td>Cantor spectrum for the quasi-periodic Schrödinger equation</td>
</tr>
<tr>
<td>653</td>
<td>Wenxiang Liu</td>
<td>Singular solutions for a convection diffusion equation with absorption</td>
</tr>
<tr>
<td>654</td>
<td>Deborah Brandon and William J. Hrusa</td>
<td>Global existence of smooth shearing motions of a nonlinear viscoelastic fluid</td>
</tr>
<tr>
<td>655</td>
<td>James F. Reineck</td>
<td>The connection matrix in Morse–Smale flows II</td>
</tr>
<tr>
<td>656</td>
<td>Claude Baesens, John Guckenheimer, Seunghwan Kim and Robert Mackay</td>
<td>Simple resonance regions of torus diffeomorphisms</td>
</tr>
<tr>
<td>657</td>
<td>Willard Miller, Jr.</td>
<td>Lecture notes in radar/sonar: Topics in Harmonic analysis with applications to radar and sonar</td>
</tr>
<tr>
<td>658</td>
<td>Calvin H. Wilcox</td>
<td>Lecture notes in radar/sonar: Sonar and Radar Echo Structure</td>
</tr>
<tr>
<td>659</td>
<td>Richard E. Blahut</td>
<td>Lecture notes in radar/sonar: Theory of remote surveillance algorithms</td>
</tr>
<tr>
<td>660</td>
<td>D.V. Anosov</td>
<td>Hilbert’s 21st problem (according to Bolibruch)</td>
</tr>
<tr>
<td>661</td>
<td>Stephane Laederich</td>
<td>Ray–Singer torsion for complex manifolds and the adiabatic limit</td>
</tr>
<tr>
<td>662</td>
<td>Geneviève Raugel and George R. Sell</td>
<td>Navier-Stokes equations in thin 3d domains: Global regularity of solutions I</td>
</tr>
<tr>
<td>663</td>
<td>Emanuel Parzen</td>
<td>Time series, statistics, and information</td>
</tr>
<tr>
<td>664</td>
<td>Andrew Majda and Kevin Lamb</td>
<td>Simplified equations for low Mach number combustion with strong heat release</td>
</tr>
<tr>
<td>665</td>
<td>Ju. S. Il'yashenko</td>
<td>Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation</td>
</tr>
<tr>
<td>666</td>
<td>James F. Reineck</td>
<td>Continuation to gradient flows</td>
</tr>
<tr>
<td>667</td>
<td>Mohamed Sami Elbialy</td>
<td>Simultaneous binary collisions in the collinear N–body problem</td>
</tr>
<tr>
<td>668</td>
<td>John A. Jacquez and Carl P. Simon</td>
<td>Aids: The epidemiological significance of two different mean rates of partner-change</td>
</tr>
<tr>
<td>669</td>
<td>Carl P. Simon and John A. Jacquez</td>
<td>Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations</td>
</tr>
<tr>
<td>670</td>
<td>Matthew Stafford</td>
<td>Markov partitions for expanding maps of the circle</td>
</tr>
<tr>
<td>671</td>
<td>Ciprian Foias and Edriss S. Titi</td>
<td>Determining nodes, finite difference schemes and inertial manifolds</td>
</tr>
<tr>
<td>672</td>
<td>M.W. Smiley</td>
<td>Global attractors and approximate inertial manifolds for abstract dissipative equations</td>
</tr>
<tr>
<td>673</td>
<td>M.W. Smiley</td>
<td>On the existence of smooth breathers for nonlinear wave equations</td>
</tr>
<tr>
<td>674</td>
<td>Hitay Özbay and Janos Turi</td>
<td>Robust stabilization of systems governed by singular integro-differential equations</td>
</tr>
<tr>
<td>675</td>
<td>Mary Silber and Edgar Knobloch</td>
<td>Hopf bifurcation on a square lattice</td>
</tr>
<tr>
<td>676</td>
<td>Christophe Golé</td>
<td>Ghost circles for twist maps</td>
</tr>
<tr>
<td>677</td>
<td>Christophe Golé</td>
<td>Ghost tori for monotone maps</td>
</tr>
<tr>
<td>678</td>
<td>Christophe Golé</td>
<td>Monotone maps of $T^n \times R^n$ and their periodic orbits</td>
</tr>
<tr>
<td>679</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Hypergeometric expansions of Heun polynomials</td>
</tr>
<tr>
<td>680</td>
<td>Victor A. Pliss and George R. Sell</td>
<td>Perturbations of attractors of differential equations</td>
</tr>
<tr>
<td>681</td>
<td>Avner Friedman and Peter Knabner</td>
<td>A transport model with micro- and macro-structure</td>
</tr>
<tr>
<td>682</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>A note on group contractions and radar ambiguity functions</td>
</tr>
<tr>
<td>683</td>
<td>George R. Sell</td>
<td>References on dynamical systems</td>
</tr>
<tr>
<td>684</td>
<td>Shui-Nee Chow, Kening Lu and George R. Sell</td>
<td>Smoothness of inertial manifolds</td>
</tr>
<tr>
<td>685</td>
<td>Shui-Nee Chow, Xiao-Biao Lin and Kening Lu</td>
<td>Smooth invariant foliations in infinite dimensional spaces</td>
</tr>
</tbody>
</table>