PROJECTION FINITE ELEMENT METHODS FOR SEMICONDUCTOR DEVICE EQUATIONS

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Abstract. In this paper a class of nonstandard finite element methods, which we call projection finite element methods, is introduced to numerically solve the stationary drift-diffusion semiconductor device equations in two and three space dimensions. The methods are based on the use of nonconforming finite elements and the projection of coefficients into finite element spaces, produce symmetric and positive definite systems of algebraic equations, allow to design optimal order multigrid methods for the solution of the linear systems, and yield error estimates of high order. Numerical results are presented to show the performance of the methods.

1. Introduction. The stationary drift-diffusion semiconductor device equations are described by the coupled system of nonlinear partial differential equations [12], [14]:

\begin{align}
\frac{\lambda^2}{\Delta \psi} = C(x) - n + p, & \quad x \in \Omega, \\
\text{div}(\nabla n - n \nabla \psi) = R(\psi, n, p), & \quad x \in \Omega, \\
\text{div}(\nabla p + p \nabla \psi) = R(\psi, n, p), & \quad x \in \Omega,
\end{align}

where $\lambda$ is the normed Debye length, $\psi$ is the (scaled) potential, $n$ and $p$ are the (scaled) electron and hole concentrations, $C$ is the doping profile, $R$ is the carrier recombination-generation rate, and $\Omega$ is the device in $\mathbb{R}^2$ or $\mathbb{R}^3$. Introducing the change of variables [3], [5]

\begin{align}
n = u e^\psi, \quad p = v e^{-\psi},
\end{align}

the system (1.1) can be written as

\begin{align}
\frac{\lambda^2}{\Delta \psi} = C(x) - u e^\psi + v e^{-\psi}, & \quad x \in \Omega, \\
\text{div}(e^{\psi} \nabla u) = R(\psi, u, v), & \quad x \in \Omega, \\
\text{div}(e^{-\psi} \nabla v) = R(\psi, u, v), & \quad x \in \Omega.
\end{align}

Then, having used iteration procedures and Gummel's method [10], the nonlinear system (1.3) can be decoupled and linearized so that linear equations of the following form have to be solved at each iteration step:

\begin{align}
-\text{div}(a(x) \nabla \phi) = f(x), & \quad x \in \Omega.
\end{align}

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It is well known that standard finite element or finite difference methods are not effective choices for equation (1.4a) mainly due to the fact that the potential \( \psi \) might be fairly big in applications and thus \( a(x) = e^{-\psi(x)} \) (resp., \( e^{\psi(x)} \)) could be a considerable source of problems in computations. Recently, some mixed finite element methods for approximating the solution of (1.4a) have been introduced [3], [4], [5]. However, these methods are restricted to two dimensions and to triangulations having acute angles only. Also, the mixed formulations given in [3], [4] are difficult to handle and are in general expensive from a computational point of view. Moreover, error bounds for these methods are unsatisfactory in practice.

In this paper a class of nonstandard finite element methods, which we call projection finite element methods, is introduced to numerically solve equation (1.4a). The methods are based on the use of nonconforming finite elements and on the projection of coefficients into finite element spaces and are not restricted to triangulations having acute angles only. It is shown that these methods are essentially equivalent to some mixed finite element methods and thus the main features of the mixed methods are preserved here. Furthermore, it is proven that the methods under consideration produce symmetric and positive definite finite element systems and allow us to develop simple and optimal order multigrid algorithms for the solution of the linear systems, and that error estimates of high order can be obtained. Finally, the methods can be easily extended to three dimensions.

In the next section projection finite element methods on rectangular elements in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) are defined and analyzed. Then, in §3, multigrid methods for the solution of the linear systems produced by the projection methods are developed. In §4, the corresponding triangular projection methods are introduced. Finally, in §5, numerical results are presented to test the performance of these methods.

We shall consider the boundary condition

\[
\begin{align*}
(1.4b) & \quad \phi = \phi_D, \quad x \in \partial\Omega_D, \\
(1.4c) & \quad \partial\phi/\partial\nu = 0, \quad x \in \partial\Omega_N,
\end{align*}
\]

where \( \partial\Omega = \partial\Omega_D \cup \partial\Omega_N \) and \( \partial\Omega_D \cap \partial\Omega_N = \emptyset \).

2. Rectangular projection finite element methods. We first consider \( \Omega \) to be a planar polygonal domain. Let \( \{T_k\}_{k \geq 1} \) be a regular sequence of partitions of \( \Omega \) into rectangles oriented along the coordinate axes and having maximum diameter \( h_k \) [9]. For each \( k \), the intersections of the Dirichlet and Neumann segments are vertices of rectangles
only. Associated with each $T_k$, we introduce the spaces

$$W_k = \{ v : v|_T \in P_0(T), \forall T \in T_k \},$$

$$N_k = \{ v : v|_T = a_T^{\frac{1}{2}} + a_T^2 x + a_T^3 y + a_T^4 (x^2 - y^2), \quad a_T^i \in \mathbb{R}, \forall T \in T_k;$$

$v$ is continuous at the midpoints of interior edges and

vanishes at the midpoints of boundary edges in $\partial \Omega_D$,}

$$B_k = \left\{ \varphi : \varphi|_T = \gamma_T \left( 4 - 12 \left( \frac{(x - x_T)^2}{h_{kT_x}^2} + \frac{(y - y_T)^2}{h_{kT_y}^2} \right) \right), \quad \gamma_T \in \mathbb{R}, \forall T \in T_k \right\},$$

$$M_k = N_k \oplus B_k,$$

where $(x_T, y_T)$ is the center of $T$ and $h_{kT_x}$ and $h_{kT_y}$ are the $x$-length and $y$-length of $T$, respectively. Namely, on each element, $B_k$ is the set of $P_2$-bubble functions and $M_k$ is thus the usual nonconforming space augmented with the $P_2$-bubbles. On $T = [-1,1]^2$, the $P_2$-bubble is $4 - 3(x^2 + y^2)$, which vanishes at the two quadratic Gauss points on each edge.

We now introduce our projection finite element method for approximating the solution of (1.4):

Find $\phi_k \in M_k + \phi_D$ such that

$$\sum_{T \in T_k} (\alpha_k^{-1} \nabla \phi_k, \nabla v)_T = (P_k f, v), \quad \forall v \in M_k,$$

where $P_k$ denotes the $L^2$-projection onto $W_k$ and $\alpha_k = P_k a^{-1}$. Notice the differences between (2.1) and the standard Galerkin method. First, on the left-hand side of (2.1), $\alpha_k^{-1}$ appears in place of $a$. That is, we take the harmonic average $\alpha_k^{-1}$ of the coefficient $a(x)$ instead of the coefficient itself so that the coefficients of system (2.1) are of reasonable size. This is particularly useful when $a$ is of the form $e^{-\psi}$ or $e^{\psi}$ since $\psi$ might be fairly big, as mentioned before. Secondly, on the right-hand side of (2.1), $P_k f$ appears in place of $f$. This enables us to compute directly the electric field and the current $\sigma = -a \nabla \phi$ by using a very simple formula, as shown in Theorem 2 below. The formula below for calculating the field and current variables are very important in practice since these variables are the ones with which one is primarily concerned.

The following result can be found in [6], [7], [8].

**Theorem 1.** Problem (2.1) has a unique solution $\phi_k$ in $M_k$. Moreover, there is a constant $C$ independent of $k$ such that

$$\left( \sum_{T \in T_k} \| \nabla \phi - \nabla \phi_k \|_T^2 \right)^{1/2} \leq C h_k (\| \phi \|_2 + \| a \|_1),$$

(2.2a)

$$\| \phi - \phi_k \| \leq C h_k^2 (\| a \|_1 + \| f \|_1),$$

(2.2b)
where $\| \cdot \|$ and $\| \cdot \|_m$ represent the norms of $L^2(\Omega)$ and $H^m(\Omega)$, respectively, for $m = 1, 2$.

We remark that $\phi_k$ approximates $\phi$ with a higher order of accuracy than the usual numerical solution produced by the lowest-order mixed finite element methods [3], [4]. It is also straightforward to see that problem (2.1) produces a symmetric and positive definite system of algebraic equations if the coefficient $a$ is strictly positive.

We now prove a theorem concerning the calculation of the approximate electric field and current. This theorem shows a relationship between the method under consideration and the lowest-order rectangular mixed method; the numerical field $\sigma_k = -\alpha_k^{-1} \nabla \phi_k$ is the quantity produced by the mixed method [6], [7], [8].

**Theorem 2.** $\sigma_k$ at a point $(x, y) \in T \in T_k$ is evaluated by the formula

\[
\sigma_k(x, y) = -\alpha_k^{-1} \nabla z_k(x, y) + \frac{P_k f|_T}{h_{kT}^2 + h_{kTz}^2} (h_{kTz}^2 (y - y_T), h_{kTy}^2 (x - x_T)),
\]

where $z_k \in N_k + \phi_D$ is the solution of

\[
\sum_{T \in T_k} (\alpha_k^{-1} \nabla z_k, \nabla v)_T = (P_k f, v), \quad \forall v \in N_k.
\]

Moreover, $\sigma_k$ has continuous normal components at the interelement boundaries.

The theorem implies that $\sigma_k$ can be computed from the solution of the standard non-conforming Galerkin method modified in a virtually cost-free manner. Namely, one simply adds to a standard program the mean of the right-hand side function on each $T$. Also, the strong continuity property for $\sigma_k$ stated in the theorem is important in applications [3], [4], [5].

**Proof:** By the definition of $M_k$, let $\phi_k = z_k + \xi_k$ with $\xi_k \in B_k$; then, it follows from the definition of $\sigma_k$ that

\[
\sigma_k = -\alpha_k^{-1} (\nabla z_k + \nabla \xi_k).
\]

Now, by the orthogonality of $N_k$ and $B_k$, the definition of $M_k$, (2.1), and (2.4), we see that $\xi_k$ satisfies the equation

\[
(\alpha_k^{-1} \nabla \xi_k, \nabla \varphi)_T = (P_k f, \varphi)_T, \quad \forall \varphi \in B(T), \ T \in T_k,
\]

where $B(T) = B_k|_T$, so that

\[
\Delta \xi_k = -\alpha_k P_k f, \quad \text{on each} \quad T.
\]

Hence, the desired result (2.3) follows from the definition of $B_k$ and a few simple calculations.

The continuity of $\sigma_k$ can be easily seen from formula (2.3) and the definition of $N_k$, as shown in [11]; we omit the details. This completes the proof. $\blacksquare$
We shall now extend the results above to three space dimensions. For this, let \( \Omega \) be a polygonal domain in \( \mathbb{R}^3 \) and let \( T_k \) be now a decomposition of \( \Omega \) into rectangular parallelepipeds having maximum diameter \( h_k \) and oriented along the coordinate axes. In the present case, the form of problem (2.1) formally remains the same with the following new definitions of \( N_k \) and \( B_k \):

\[
N_k = \{ v : v|_T = a_T^1 + a_T^2 x + a_T^3 y + a_T^4 z + a_T^5 (x^2 - y^2) + a_T^6 (x^2 - z^2), \\
a_T^i \in \mathbb{R}, \forall T \in T_k; \ v \text{ is continuous at the centers of interior faces and vanishes at the centers of boundary faces in } \partial \Omega_D \},
\]

\[
B_k = \{ \varphi : \varphi|_T = \gamma_T \left( 5 - 12 \left( \frac{(x - x_T)^2}{h^2_{kT_x}} + \frac{(y - y_T)^2}{h^2_{kT_y}} + \frac{(z - z_T)^2}{h^2_{kT_z}} \right) \right), \\
\gamma_T \in \mathbb{R}, \forall T \in T_k \}.
\]

The \( P_2 \)-bubble on \( T = [-1, 1]^3 \) is \( 5 - 3(x^2 + y^2 + z^2) \), which is equal to zero at the four tensor product quadratic Gauss points on each face. Furthermore, the results in Theorems 1 and 2 are still true; the formula (2.3) is accordingly modified as follows:

\[
\sigma_k = -\alpha_k^{-1} \nabla z_k + P_k f|_T \left( \frac{1}{h^2_{kT_x}} + \frac{1}{h^2_{kT_y}} + \frac{1}{h^2_{kT_z}} \right)^{-1} \left( \frac{x - x_T}{h^2_{kT_x}}, \frac{y - y_T}{h^2_{kT_y}}, \frac{z - z_T}{h^2_{kT_z}} \right), \\
(x, y, z) \in T \in T_k.
\]

3. The multigrid algorithm. In this section we design a multigrid algorithm for the projection method (2.1) and then for the numerical field \( \sigma_k \) introduced in Theorem 2 as a by-product. We consider the two-dimensional version of (2.1); the extension to the three-dimensional case is trivial from the discussion above.

We need to assume a structure of our family of partitions \( \{T_k\}_{k \geq 1} \). Let \( T_1 \) be given and let \( T_{k+1} \) be constructed by connecting the midpoints of the edges of the rectangles in \( T_k \).

For each \( k \), define

\[
a_k(v, w) = \sum_{T \in T_k} (\alpha_k^{-1} \nabla v, \nabla w)_T , \ \forall v, w \in M_k,
\]

and let \( A_k : M_k \to M_k \) be defined by

\[
a_k(v, w) = (A_k v, w)_{T_k} , \ \forall v, w \in M_k.
\]

Since \( a_k(\cdot, \cdot) \) is symmetric positive definite on \( M_k \), the operator \( A_k \) is symmetric positive definite with respect to \((\cdot, \cdot)_{T_k}\) and standard inverse estimates [9] yield that

\[
(3.1) \quad \text{spectral radius of } A_k \leq Q h_k^{-2},
\]
where $Q$ is a constant independent of $k$.

Note that, since $M_{k-1} \nsubseteq M_k$, the spaces $M_k$ are nonnested. It is known that natural injection operators do not work for nonnested finite element spaces. Hence, we need to introduce intergrid transfer operators. Following [2], [8], we define the coarse-to-fine intergrid transfer operators $I_{k-1}^k : M_{k-1} \rightarrow M_k$ such that $I_{k-1}^k : N_{k-1} \rightarrow N_k$ and $I_{k-1}^k : B_{k-1} \rightarrow B_k$.

If $v \in N_{k-1}$ and $i$ is a midpoint of an edge $e$ of a rectangle in $T_k$, then, $I_{k-1}^k v \in N_k$ is given by

$$
(I_{k-1}^k v)(i) = \begin{cases} 
0 & \text{if } e \subset \partial \Omega_D, \\
v(i) & \text{if } e \subset \partial \Omega_N \text{ or } e \not\subset T \text{ for any } T \in T_{k-1}, \\
\frac{1}{2}(v|_{T_1(i)} + v|_{T_2(i)}) & \text{if } e \subset T_1 \cap T_2 \text{ for some } T_1, T_2 \in T_{k-1}.
\end{cases}
$$

For $\varphi \in B_{k-1}$, $I_{k-1}^k \varphi \in B_k$ is simply determined by

$$(I_{k-1}^k \varphi, 1)_T = \frac{1}{4} (\varphi, 1)_T,$$

where $T \in T_k$ is one of the four rectangles obtained from subdividing $\hat{T} \in T_{k-1}$.

The fine-to-coarse intergrid transfer operator $I_{k-1}^k : M_k \rightarrow M_{k-1}$ is then defined as usual [2], [8]:

$$(I_{k-1}^k v, w)_{T_{k-1}} = (v, I_{k-1}^k w)_{T_k}, \quad \forall v \in M_k, \ w \in M_{k-1}.$$

We are now ready to define our multigrid algorithm for problem (2.1) or the equivalent linear system

$$(3.2) \quad A_k \phi_k = f_k,$$

where $f_k \in M_k$ and $(f_k, v)_{T_k} = (P_k f, v), \ \forall v \in M_k$. For $k = 1, 2, \cdots$, approximate solutions $\hat{\phi}_k \in M_k$ to problem (3.2) are obtained as follows.

(3.3a) For $k = 1$, $\hat{\phi}_k$ is obtained by a direct method.

(3.3b) For $k \geq 2$, $\hat{\phi}_k$ are obtained recursively by

(i) $\hat{\phi}_0 = I_{k-1}^k \hat{\phi}_{k-1},$

(ii) $\hat{\phi}_l = \text{MG}(k, \phi_{k-1}^l, f_k), \ 1 \leq l \leq r;$

(iii) $\hat{\phi}_r = \phi_r^k.$

Here $r$ is a positive integer independent of $k$ and the $k$th level iteration with initial guess $\phi_{l-1}^k$ yields $\text{MG}(k, \phi_{l-1}^k, f_k)$ as an approximate solution to problem (3.2) by means of the following smoothing and correction steps.

(3.3c) (Smoothing step) The approximation $g_j \in M_k$, $j = 1, 2, \cdots, m$, is defined recursively from the initial guess $g_0 = \phi_{l-1}^k$ by the equations

$$
g_j - g_{j-1} = Q^{-1} h_k^2 (f_k - A_k g_{j-1}), \quad j = 1, \cdots, m,$$

6
where \( m \) is the number of smoothing steps and \( Q \) is defined by (3.1).

(3.3d) (Correction step) \( MG(k, \phi_{k-1}^0, f_k) = g_m + I_{k-1}^k q_p \) where \( q_j \in M_{k-1} \) \( (j = 0, \cdots, p, \ p = 2 \ or \ 3) \) is defined recursively by

\[
q_0 = 0, \\
q_j = MG(k - 1, q_{j-1}, \hat{f}_k), \ j = 1, \cdots, p,
\]

where \( \hat{f}_k = I_{k-1}^k (f_k - A_k g_m) \).

We now consider the multigrid approximation \( \hat{\sigma}_k \) to \( \sigma_k \). From the definition of \( \sigma_k \), it is defined as

(3.4) \[
\hat{\sigma}_k = -\alpha_k^{-1} \nabla \hat{\phi}.
\]

In order to preserve the continuity property of \( \sigma_k \) stated in Theorem 2, we introduce the average \( \Lambda_k \hat{\sigma}_k \) of \( \hat{\sigma}_k \) as in [2], [8]. Let \( e \) be an edge of \( T \) in \( T_k \) and \( n_e \) be a unit normal of \( e \). If \( e \in \partial \Omega \), then

\[
(\Lambda_k \hat{\sigma}_k \cdot n_e)|_e = (\hat{\sigma}_k|_T \cdot n_e)|_e; \text{ if } e \text{ is the common edge of } T_1 \text{ and } T \text{ in } T_k,
\]

then,

\[
(\Lambda_k \hat{\sigma}_k \cdot n_e)|_e = ((\hat{\sigma}_k|_T \cdot n_e)|_e + (\hat{\sigma}_k|_{T_1} \cdot n_e)|_e)/2.
\]

The next theorem shows the convergence of the multigrid method (3.3).

**Theorem 3.** Let \( \hat{\phi}_k \) and \( \hat{\sigma}_k \) be defined by (3.3) and (3.4), respectively. Then, if \( m \) and \( r \) in (3.3) are large enough, there is a constant \( C \) independent of \( k \) such that

\[
\| \sigma_k - \Lambda_k \hat{\sigma}_k \| \leq C h_k \| f \|,
\]

\[
\| \sigma - \Lambda_k \hat{\sigma}_k \| \leq C h_k (\| f \| + \| a \|_1),
\]

\[
\| \phi_k - \hat{\phi}_k \| + \left( \sum_{T \in T_k} \| \nabla (\phi_k - \hat{\phi}_k) \|_T^2 \right)^{1/2} \leq C h_k \| f \|.
\]

Moreover, if \( f \in H^1(\Omega) \),

\[
\| \phi_k - \hat{\phi}_k \| \leq C h_k^2 \| f \|_1,
\]

\[
\| \phi - \hat{\phi}_k \| \leq C h_k^2 (\| f \|_1 + \| a \|_1).
\]

The proof can be found in [8]. The requirement in Theorem 3 on the largeness of \( m \) ensures that the \( k \)th level iteration in (3.3) is a contraction for \( k = 1, 2, \cdots \). Let \( m_k = \text{dim}(M_k) \); it can be seen that the total work for obtaining \( \hat{\phi}_k \) is \( O(m_k) \) [1], [8]. Thus, the cost for computing \( \hat{\sigma}_k \) is also \( O(m_k) \).
4. Triangular projection finite element methods. In this section we shall describe the triangular analogue of the projection finite element method on rectangles introduced in §2, which may deserve the name "projection" better. For a planar domain \( \Omega \), let \( T_k \) be a regular partition of \( \Omega \) into triangles of diameter not greater than \( h_k \). For each \( T \) in \( T_k \), let \((\lambda_1, \lambda_2, \lambda_3)\) represent the barycentric coordinates of a point of \( T \). On the triangle \( T \), we define the \( P_2 \)-bubble

\[
\xi_T(x) = 2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2),
\]

which vanishes at the two Gaussian quadrature points of each side of \( T \) and is equal to unity at the barycenter of \( T \). Then, we introduce the spaces

\[
N_k = \{ v : v|_T \in P_1(T), \forall T \in T_k; \ v \text{ is continuous at the midpoints of interior edges and vanishes at the midpoints of boundary edges in } \partial \Omega_D \},
\]

\[
B_k = \{ \varphi : \varphi|_T = \gamma_T \xi_T(x), \gamma_T \in \mathbb{R}, \forall T \in T_k \}.
\]

The definition of \( M_k \) formally remains the same as before. We shall also need the following space:

\[
V_k = \{ \tau : \tau|_T = (a_T + b_T x, c_T + b_T y), (x, y) \in T, \ a_T, b_T, c_T \in \mathbb{R}, \forall T \in T_k \},
\]

which is the Raviart-Thomas space [13]. We are now in a position to formulate the triangular projection finite element method:

Find \( \phi_k \in M_k + \phi_D \) such that

\[
(4.1) \quad \sum_T (\alpha_k^{-1} P_{V_k}(\nabla \phi_k), \nabla v)_T = (P_k f, v), \forall v \in M_k,
\]

where \( P_{V_k} \) indicates the standard \( L^2 \)-projection onto \( V_k \) and \( P_k \) is defined as in §2. The projections introduced in system (4.1) have similar meanings to those in (2.1). The point we should here stress is the introduction of the projection operator \( P_{V_k} \). The technique of introducing \( P_{V_k} \) allows us to derive an analogous formula to (2.3) for the calculation of the approximate electric field and current in the present case, as stated below.

**Theorem 4.** Problem (4.1) has a unique solution \( \phi_k \) in \( M_k \). Moreover, \( \sigma_k = -\alpha_k^{-1} P_{V_k}(\nabla \phi_k) \) at a point \((x, y) \in T \in T_k \) is computed by the formula

\[
\sigma_k(x, y) = -\alpha_k^{-1} \nabla z_k(x, y) + P_k f|_T(x - x_T, y - y_T)/2,
\]

where \((x_T, y_T)\) is the center of gravity of the triangle \( T \) and \( z_k \in N_k + \phi_D \) is the solution of problem (2.4) with \( N_k \) defined above, and \( \sigma_k \) has continuous normal components at the interelement boundaries.

Again, \( \sigma_k \) can be obtained from the solution of the \( P_1 \)-nonconforming finite element method, as shown here. Moreover, the error estimates (2.2) in Theorem 1 hold in the present case. Finally, if \( T_1 \) is given and each \( T_{k+1} \) is a regular refinement of \( T_k \) into four times as many elements, then a multigrid algorithm similar to (3.3) can be developed for (4.1) and Theorem 3 remains valid.

The extension of the triangular projection method to three space dimensions can be carried out in the same manner as in §2.
5. **Numerical result.** In this section a numerical result for solving (1.1a) is presented by using the rectangular projection method (2.1). A uniform mesh of $96 \times 32$ points is used over the domain $\Omega = (0.0, 0.6) \times (0.0, 0.2)$ (see Figure 1). The Dirichlet boundary segments are of the form

\[
\partial \Omega_D = \{(x, y) : 0 < x < 0.1, \ y = 0.2\} \cup \{(x, y) : 0.2 < x < 0.4, \ y = 0.2\} \\
\cup \{(x, y) : 0.5 < x < 0.6, \ y = 0.2\},
\]

and all the other parts of the boundary are the Neumann segments. In applications, the three parts above of $\partial \Omega_D$ represent the source, gate, and drain contacts, respectively. The boundary datum is given by

\[
\phi_D = \begin{cases} 
0.198, & (x, y) \in (0, 0.1) \times \{y = 0.2\}, \\
-0.63, & (x, y) \in (0.2, 0.4) \times \{y = 0.2\}, \\
2.198, & (x, y) \in (0.5, 0.6) \times \{y = 0.2\}.
\end{cases}
\]

In (1.1a), the parameters are chosen as follows:

\[
C/\lambda^2 = \begin{cases} 
30, & (x, y) \in [0, 0.1] \times [0.15, 0.2] \cup [0.5, 0.6] \times [0.15, 0.2], \\
10, & \text{elsewhere},
\end{cases}
\]

and $n = p = 0$. Figure 2 very well demonstrates the layer structure of the potential $\phi$. The peaks of the electric field in Figures 3 and 4 are due to its singularities around the intersections of the Dirichlet and Neumann segments.

**REFERENCES**


*Keywords.* semiconductor device, projection finite element method, numerical result, multigrid method

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FIG. 1. The uniform mesh.

FIG. 2. The potential $\phi$. 
FIG. 3. The horizontal electric field $\sigma_1$.

FIG. 4. The vertical electric field $\sigma_2$. 
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