$H^\infty$ OPTIMAL CONTROLLER DESIGN FOR A CLASS OF DISTRIBUTED PARAMETER SYSTEMS

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$H^\infty$ Optimal Controller Design for a Class of Distributed Parameter Systems

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Abstract

This paper is a tutorial on a frequency domain design method for the two block $H^\infty$ optimal control of a class of SISO distributed parameter systems. Uncertainties in the system are assumed to be unmodeled dynamics represented in the form of either multiplicative, or additive, or coprime factor perturbations of a nominal infinite dimensional plant. Performance of the closed loop system is measured in terms of the energy amplification from external input signals to the outputs of interest. In this paper transfer functions of the nominal plant and the weights are used in order to derive a solution to two block $H^\infty$ control problems. The solution presented here uses results from operator theory. The $H^\infty$ optimal performance and controller in this method are computed from a finite set of linear equations.
1 Introduction

In this paper we study the problem of \(H^\infty\) optimal controller design for single input single output (SISO), linear time invariant, possibly infinite dimensional systems. These types of distributed models appear in many engineering applications such as distillation column (Morari and Zafiriou, 1989), abstract model of an aircraft (Enns et al., 1992), dynamics of an airfoil in steady or unsteady flow (Balakrishnan, 1978), (Özbay and Turi, 1991), flexible beams (Bontsema et al., 1988), (Joshi, 1989), (Lenz et al., 1991), servohydraulic material testing (Srinivasan and Shaw, 1990), heat conduction in multilayer solid structures (Vajta, 1991), viscoelastic models, (MacCamy, 1991), etc. The main reason to use feedback in the control of dynamical systems is to design against uncertainties. In a typical control system there are two kinds of uncertainties: (i) disturbances or noises, and (ii) modelling errors. The purpose of feedback control is to achieve certain performance specifications in the closed loop system despite these uncertainties.

We will see in Section 3 that one can incorporate both disturbances and unmodelled dynamics in the plant into a single design criteria using an \(H^\infty\) optimal control approach. This is one of the most important features of the \(H^\infty\) control theory which makes it fundamentally different than the classical \(H^2\) (LQG) control theory. There are many other control engineering problems such as sensitivity minimization, model matching, tracking, robust stabilization, etc., which can be put in the framework of the \(H^\infty\) control, see e.g. Francis (1987), Kwakernaak (1985), Zames (1981), for detailed discussions.

Recently there have been important developments in \(H^\infty\) control theory. For finite dimensional systems the solutions to \(H^\infty\) problems can be obtained using state space methods. In this approach computations leading to \(H^\infty\) controllers are reduced to finding solutions to algebraic Riccati equations, (Ball and Cohen, 1986), (Doyle et al., 1989), (Francis, 1987), (Glover and Doyle, 1988), etc. Therefore, in this case computations involve simple linear algebra, which are easy to implement on the computer. There are also game theoretic interpretations of these solutions, see e.g. Başar (1989) and references therein. Although it is possible to extend these results to infinite dimensional systems, in this case one has to be careful in using state space methods since more complicated semigroup theory and operator valued Riccati equations are involved. See the survey paper Curtain (1990) for the details of the state space \(H^\infty\) control problems for infinite dimensional systems.

In this paper a frequency domain (input/output) approach will be used for the \(H^\infty\) control of a class of distributed systems. In other words, instead of the state space representation, the transfer function will be taken as a model of the plant to be controlled. Usually, partial differential equations or functional differential equations are given as mathematical models for distributed parameter systems, because these are the simplest and most natural representations for many systems such as flexible beams, delay systems, heat equations, airfoil dynamics, etc. Therefore, one of the reasons to use distributed models in the controller design is that such infinite dimensional models may represent the dynamics of the physical system better than their finite dimensional approximates. On the other hand, in some cases infinite dimensional models which contain a few parameters are used for physical phenomenon which can otherwise be better explained by very high
order finite dimensional models. Thus, the economical representation of the system is another important reason why distributed models are used in practice. For example there is only one parameter, $h$, in the representation of the time delay element $e^{-hs}$. In general transfer functions of distributed parameter systems are transcendental functions in the Laplace transform variable $s$, along with a few parameters, such as time delay, stiffness or damping coefficient of a beam. Effects of these parameters on the control design can be studied easily in the frequency domain approach.

Roughly speaking $H^\infty$ control is a worst case design against the uncertainties. In most engineering applications uncertainties (disturbances and modelling errors) can be effectively modelled by low order rational weights. Using certain results from operator theory, one can show that several $H^\infty$ problems can be solved by finding the eigenvalues and eigenvectors of a "Hankel + Toeplitz" type of operator, see e.g. Flamm and Yang (1990), Verma and Jonckheere (1984), Zames and Mitter (1988) and references therein. This operator is similar to a so-called "skew Toeplitz" operator, (see e.g. Bercovici et al. (1988), Foias and Tannenbaum (1988), Foias et al. (1988), Özbay and Tannenbaum (1990), Özbay et al. (1990)), so the eigenvalues and eigenvectors of this skew Toeplitz operator also give solutions to $H^\infty$ control problems. In this paper we will present the skew Toeplitz approach. We will see that for a class of infinite dimensional plants, $H^\infty$ optimal controllers can be computed by solving a finite set of linear equations.

This result makes the $H^\infty$ control theory applicable to distributed plants directly, i.e. one can actually compute the optimal controller without going through any approximations on the infinite dimensional plant. For some special types of weights this theory has been applied to the robust control of such distributed systems as flexible beams, and abstract model of an unstable aircraft (plant with a time delay), (Lenz et al., 1991), (Enns et al., 1992). Approximations of the optimal controller are needed for efficient implementation. The skew Toeplitz method indicates also how one can obtain reduced order controllers keeping the system stable and yielding a performance close to the optimum, (Özbay and Tannenbaum, 1991), (Özbay, 1992).

The purpose of this paper is to present the skew Toeplitz theory in a unified compact form. All the previous papers published on this method contain some unavoidably complicated formulae (developed from operator theoretic results, which most control engineers are not familiar with) for several different cases of $H^\infty$ control of distributed parameter systems. We hope that this tutorial paper is going make the skew Toeplitz methods clear, and engineers and scientists working in the area of distributed parameter systems will find the techniques of this method useful.

The rest of this paper is organized as follows. In the next section we present mathematical preliminaries and some results from operator theory. Then, in Section 3 we define the two-block $H^\infty$ optimal control problems that we are going to consider. Section 4 reduces the $H^\infty$ control problems to a norm computation problem for a certain infinite rank operator, and shows how the singular values and vectors of this operator can be computed from a finite set of linear equations. A numerical example is given in Section 5. Approximation of the optimal controller is discussed in Section 6, and concluding remarks are made in Section 7. For the reader who is not familiar with Hardy spaces a brief background information is included in the Appendix.
2 Mathematical background

In order to make a precise definition of the $H^\infty$ control problems to be considered in the paper we need to present preliminary background from the theory of bounded analytic functions and the operator theory. We refer the interested reader to Foias and Frazho (1990), Garnett (1981), Halmos (1982), Hoffman (1962), and Nikolskii (1986) for complete details of the material presented in this section.

2.1 Conformal map between the unit disc and the right half plane

The examples considered in this paper are continuous time plants. However, discrete time systems could also be treated, because the method presented here is independent of the time domain we are working. The systems will be represented by their transfer functions, which are functions of the Laplace transform variable $s \in RHP$ (in the case of continuous time systems) or functions of the $Z$-transform variable $z \in D$ (for discrete time systems). Our solution to the $H^\infty$ control problems will be derived using functions defined on the unit disc ($z$-plane). This does not limit us to discrete time systems, since we can find a conformal map between the right half plane ($RHP$) and the unit disc ($D$). A simple example of such a map is

$$z = \frac{s-1}{s+1}, \quad s = \frac{1+z}{1-z},$$

where $s \in RHP$ and $z \in D$. This conformal map transforms every point in $RHP$ to a unique point in $D$ and vice versa, the imaginary axis (boundary of $RHP$) is mapped to the unit circle (boundary of $D$). In particular, for the above example the points $j\infty$ and 0 in the $s$-plane are mapped to the points 1 and $-1$ in the $z$-plane.

Any function $F \in H^\infty$ defined on $RHP$ can be represented in terms of a function $f \in H^\infty(D)$, and vice versa:

$$f(z) = F\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad F(s) = f\left(\frac{s-1}{s+1}\right).$$

The conformal map between $RHP$ and $D$ preserves all the important properties of $F(s)$ as a bounded analytic function: e.g. $f(z)$ is a bounded analytic function on $D$ and

$$\|F\|_\infty = \text{ess sup}_{\omega \in \mathbb{R}} |F(j\omega)| = \text{ess sup}_{\theta \in [0,2\pi]} |f(e^{j\theta})| = \|f\|_\infty.$$

In view of the above facts we can transform the problem data from $RHP$ to $D$. For example if $P(s)$ represents the transfer function of the plant, it can also be represented by $p(z) = P\left(\frac{1+z}{1-z}\right)$, as a function defined on the unit disc. Conversely, if the controller is given as a function of $z$, i.e. $c(z)$, then, its transfer function can be obtained from the inverse map, i.e. $C(s) = c\left(\frac{s-1}{s+1}\right).$
2.2 Inner-Outer factorizations

An important tool in the solution of $H^\infty$ control problems is inner-outer factorizations.

**Definition 2.1:** A function $m \in H^\infty(D)$ is called inner if $|m(z)| \leq 1$ for all $z \in D$ and $|m(e^{j\theta})| = 1$ almost everywhere $\theta \in [0, 2\pi]$.

Examples of inner functions include

$$m_1(z) = \frac{z - a}{1 - az}, \quad a \in (-1, 1)$$

$$m_2(z) = e^{-\frac{|h|}{1-z}}$$

$$m_3(z) = m_1(z)m_2(z)$$

$$m_4(z) = e^{\frac{\sum a_k}{1-a_k z}} \prod_{k=1}^\infty \left(\frac{z-a_k}{1-a_k z}\right), \quad |a_k| < 1, \quad \text{and} \quad \sum_{k=1}^\infty (1-|a_k|) < \infty.$$

Since inner functions have constant magnitude on the unit circle they generalize all pass transfer functions. In the above examples we see that $m_2(z)$, $m_3(z)$ and $m_4(z)$ are not defined at $z = 1$, (recall that the point $z = 1$ corresponds to $s = j\infty$ in the $s-$plane and $e^{-j\infty}$ is not defined). Also note that $a_\infty := \lim_{k \to \infty} a_k$ must lie on the unit circle, and at that point $m_4(z)$ is not well defined. Such points on the unit circle are the essential singularities of inner functions. Note that rational inner functions have no essential singularities. We will see that this is an important distinction between finite and infinite dimensional systems.

An important property of inner functions is that they don't change the norm, i.e. given any $f \in H^\infty(D)$ and any inner $m$ we have $\|mf\|_\infty = \|f\|_\infty$. Note that $m^*$ also has constant magnitude on the unit circle: $|\left( (m(e^{j\theta}))^* \right)| = |m^*(e^{-j\theta})| = 1$, therefore $m^*$ does not change the norm either. That is for any $f \in L^\infty(D)$ we have $\|m^*f\|_\infty = \|f\|_\infty$. Any function in $L^\infty(T)$ having unit magnitude almost everywhere (a.e.) on the unit circle will be called unitary.

**Definition 2.2:** A function $g \in H^\infty(D)$ is called outer if the closure of $(g \mathcal{L}_+)$ in $H^2(D)$ is the whole space $H^2(D)$, where $\mathcal{L}_+ = \{\sum_{k=0}^n a_k z^k, \ a_k \in C, \ n \geq 0\}$.

Outer functions generalize minimum phase functions: they don't have a zero inside the unit disc, but may have zeros on the unit circle. So, if $g$ is an outer function, with $\inf_{\theta} |g(e^{j\theta})| > 0$ then it is invertible in $H^\infty(D)$, i.e. there is another outer function $h \in H^\infty(D)$ such that $g(z)h(z) = 1$, for all $z \in D$.

**Theorem 2.3** Let $f$ be a function in $H^\infty(D)$ then it admits an inner-outer factorization of the form

$$f(z) = m(z)g(z),$$

where $m$ is inner and $g$ is outer. Note that $|f(e^{j\theta})| = |g(e^{j\theta})|$ a.e. $\theta \in [0, 2\pi]$ and hence $\|f\|_\infty = \|g\|_\infty$. □
An inner/outer factorization can be done by finding a spectral factor $g(e^{i\theta})$ of $|f(e^{i\theta})|^2$. Whenever $f(z)f(z^{-1})$ has a finite dimensional state space realization $g$ is finite dimensional, and it can be found by solving an algebraic Riccati equation, see for example Francis (1987). In the infinite dimensional case if one wants to use the state space approach, one has to first find a realization for $f(z)f(z^{-1})$ and solve an operator valued Riccati equation to obtain the outer factor of $f(z)$. Such an equation would be difficult to solve, let alone finding a state space realization. Without going into state space realizations one can compute the numerical values of the inner and outer factors of $f(z)$ at a given point $z$, see Flamm et al. (1991) for a detailed discussion on an “integral type” formula from Hoffman (1962), pp. 133, 70, 63. In some special cases of infinite dimensional $f(z)f(z^{-1})$ we can still obtain an inner/outer factorization without going through a state space realization or the integral type formula, see Section 3 for a delay system example, and Lenz et al. (1991) for a flexible beam system example.

2.3 The shift and compressed shift operators

The techniques of Section 4, where the $H^\infty$ control problem is solved, heavily depend on the use of shift and compressed shift operators. Basic properties of these operators are summarized below.

The second Hardy space $H^2(D)$ is the space of functions $f(z): D \to \mathbb{C}$ having analytic power series expansion

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \quad \text{with} \quad \sum_{k=0}^{\infty} |f_k|^2 < \infty.$$ 

The shift operator, denoted by $S$, can be represented by multiplication by $z$, i.e. shift $S$ acting on $f \in H^2(D)$ generates

$$g(z) := (Sf)(z) = zf(z) = 0 + f_0 z^1 + f_1 z^2 + \ldots \in H^2(D).$$

So, $S$ "shifts the coefficients to the right." The adjoint of the shift operator, denoted by $S^*$, "shifts the coefficients to the left"

$$h(z) := (S^*f)(z) = z^{-1}(f(z) - f_0) = f_1 + f_2 z + f_3 z^2 + \ldots \in H^2(D).$$

An important point to remark is that $S^*S^k$ is the identity (for any integer $k \geq 1$), however $S^kS^*k$ is not:

$$(S^*S^k f)(z) = f(z),$$

$$(S^kS^*f)(z) = f(z) - \sum_{\ell=0}^{k-1} f_\ell z^\ell = \sum_{\ell=k}^{\infty} f_\ell z^\ell.$$ 

Given any inner function $m(z)$ we can decompose $H^2(D)$ into two orthogonal subspaces, $mH^2$ and $H(m)$ where

$$mH^2 = \{mf : f \in H^2(D)\},$$
and $H(m)$ is the orthogonal complement of $mH^2$ in $H^2(D)$, i.e. $H(m) = H^2(D) \ominus mH^2$. So, any function $h \in H^2(D)$ has an orthogonal decomposition $h(z) = g(z) + m(z)f(z)$ where $f \in H^2(D)$ and $g \in H(m)$. If $g \in H(m)$ then $m^*(z^{-1})g(z)$ is of the form

$$m^*(z^{-1})g(z) = \sum_{i=1}^{\infty} \phi_{-i} z^{-i}$$  \hspace{1cm} \text{(1)}$$

where the right hand side converges on the outside of unit disc, for some coefficients $\phi_{-i} \in \mathbb{C}, i \geq 1$ such that $\sum_{i=1}^{\infty} |\phi_{-i}|^2 < \infty$. Another interesting fact about $H(m)$ is that if $m$ is a rational inner function (i.e. finite Blaschke product of the form $m(z) = z^n \prod_{k=1}^{N} \left( \frac{z-a_k}{1-\overline{a_k}z} \right), |a_k| < 1, k = 1, \ldots, N$) then $H(m)$ is finite dimensional, e.g. for $m(z)$ given above $H(m)$ is spanned by the functions $\{1, z, \ldots, z^{n-1}, \frac{1}{1-a_1z}, \ldots, \frac{1}{1-a_Nz}\}$. This observation is very useful in the solution of $H^\infty$ control problems associated with finite dimensional systems, or infinite dimensional plants with finitely many unstable modes.

Now we can define an orthogonal projection operator $P_{H(m)} : H^2(D) \to H(m)$. This operator acting on an element $h \in H^2(D)$ generates $P_{H(m)}h = g$, the $H(m)$ component of $h = g + mf$.

The compressed shift operator, denoted by $T$, associated with $H(m)$ is defined as follows: $T : H(m) \to H(m)$, and $Tg = P_{H(m)}Sg$, for $g \in H(m)$. More precisely

$$(Tg)(z) = (P_{H(m)}Sg)(z)$$
$$= P_{H(m)}zg(z)$$
$$= zg(z) - m(z)\phi_{-1}$$  \hspace{1cm} \text{(2)}$$

where $\phi_{-1} \in \mathbb{C}$ is as in (1). The adjoint of $T$ is $T^* = S^*|_{H(m)}$.

We can define multiplication operators using the shift operator. For example given any $f \in H^\infty(D)$ the operator $f(S)$ is obtained by formally replacing $z$ by $S$ in the power series expansion of $f(z)$:

$$f(S) = \sum_{k=0}^{\infty} f_k S^k \quad \text{and} \quad (f(S)g)(z) = f(z)g(z) \quad \forall \ g \in H^2(D).$$

In a similar fashion we can define the compression of the operator $f(S)$ on $H(m)$ by formally replacing $z$ by $T$ in the power series expansion of $f(z)$:

$$f(T) = \sum_{k=0}^{\infty} f_k T^k = P_{H(m)}f(S)|_{H(m)}. \hspace{1cm} \text{(3)}$$

For any $x \in H(m)$ we can compute $y = f(T)x$ as follows:

$$y = P_{H(m)}fx = fx - mP_+m^*fx.$$
Note that $m(T) = 0$, the zero operator. It is easy to see this from (3), for any $g \in H(m)$ we have $m(T)g = P_{H(m)}m(S)g = P_{H(m)}mg$ but $mg \in mH^2$, so $P_{H(m)}mg = 0$. Thus for all $g \in H(m)$ we have $m(T)g = 0$.

As we have mentioned above when $m(z)$ is rational $H(m)$ is a finite dimensional subspace of $H^2(D)$. Then, since $T$ acts from $H(m)$ to $H(m)$ it is a finite dimensional linear operator, i.e. given a basis for $H(m)$ we can express $T$ as a square matrix of size equal to the dimension of $H(m)$.

An important fact, which is used in the $H^\infty$ control of distributed parameter systems with finitely many unstable modes, is that if $m = m_1m_2$ with $m_1$ arbitrary inner and $m_2$ rational inner, then the operator $m_1(T)$ is of finite rank. One can see this easily from the following. First observe that

$$H(m) = H^2 \ominus m_1m_2H^2 = (H^2 \ominus m_1H^2) \oplus m_1(H^2 \ominus m_2H^2) = H(m_1) \oplus m_1H(m_2).$$

So, any $g \in H(m)$ can be decomposed into two orthogonal components $g = g_1 + m_1g_2$ where $g_1 \in H(m_1)$ and $g_2 \in H(m_2)$. Then for $g = g_1 + m_1g_2$ we have

$$m_1(T)g = P_{H(m)}m_1g = P_{H(m_1)}m_1g_1 + m_1P_{H(m_2)}m_1g_2 = 0 + m_1P_{H(m_2)}m_1g.$$  \hspace{1cm} (4)

Since $m_2$ is rational $H(m_2)$ is finite dimensional, and $P_{H(m_2)}$ is finite rank. Hence $m_1(T)$ is finite rank. In the $H^\infty$ control of distributed systems $m_1$ comes from the RHP zeros (may be infinitely many) and time delays, and $m_2$ comes from finitely many RHP poles of the plant to be controlled. We will see this in Section 4.2.

2.4 Commutant lifting theorem

One of the key steps of Section 4 uses the commutant lifting theorem. In order to keep the discussion at the minimum technical level we will present a special case (which is used in the 1-block $H^\infty$ control problem) of this theorem.

Theorem 2.4: Let $A : H(m) \to H(m)$ be a bounded linear operator commuting with the compressed shift, i.e. $AT = TA$. Then, there exists $y \in \hat{H}^\infty(D)$ with $y(z) = \sum_{k=0}^\infty y_kz^k$ such that

$$A = y(T) = \sum_{k=0}^\infty y_kT^k$$

and $\|A\| = \|y\|_\infty$. The function $y$ is called the minimal dilation of $A$, in the sense that any other function $\hat{y} \in H^\infty(D)$ satisfying $\hat{y}(T) = A$ has norm $\|\hat{y}\|_\infty \geq \|A\|$.  \hspace{1cm} $\Box$

Remark: This theorem is known as Sarason’s theorem, see Sarason (1967) for the proof. For the two block $H^\infty$ problem we will need a more general (multivariable) version of this theorem which is due to Sz-Nagy and Foias, see e.g. Foias and Frazho (1990) pp. 153–158, Sz-Nagy and Foias (1970) pp. 258–259.  \hspace{1cm} $\Box$
2.5 Hankel and Toeplitz operators

Hankel and Toeplitz operators play an important role in the solution of \( H^\infty \) control problems. For example the optimal \( H^\infty \) performance can be computed from the eigenvalues of a "Hankel + Toeplitz" type operator. These operators are defined as follows.

Given a function \( v \in L^\infty(T) \) with two sided Fourier series expansion

\[
v(e^{j\theta}) = \sum_{k=-\infty}^{\infty} v_k e^{jk\theta},
\]

the Hankel, \( \Gamma_v \), and Toeplitz, \( \Upsilon_v \), operators with symbol \( v \) are defined as follows; \( \Gamma_v : H^2(D) \to L^2(T) \oplus H^2(D) \), \( \Upsilon_v : H^2(D) \to H^2(D) \),

\[
\Gamma_v f = P_- v f \quad \Upsilon_v f = P_+ v f.
\]

for \( f \in H^2(D) \). In terms of Fourier coefficients \( f_k \), \( k = 0, 1, \ldots \), of \( f(z) \) we can express Hankel and Toeplitz operators as infinite matrices:

\[
(\Gamma_v f)(z) = [z^{-1} \, z^{-2} \, z^{-3} \, \ldots] \begin{bmatrix} v_{-1} & v_{-2} & v_{-3} & \cdots \\ v_{-2} & v_{-3} & v_{-4} & \cdots \\ v_{-3} & v_{-4} & v_{-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}
\]

and

\[
(\Upsilon_v f)(z) = [1 \, z \, z^2 \, \ldots] \begin{bmatrix} v_0 & v_{-1} & v_{-2} & \cdots \\ v_1 & v_0 & v_{-1} & \cdots \\ v_2 & v_1 & v_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}
\]

We can also represent the shift and the compressed shift operators in terms of certain infinite size matrices. However, we will not use the matrix representation of any operator we are dealing with. All we need is to understand the action of the above operators on elements in their domains.

3 Two-block \( H^\infty \) problems

Consider the feedback control system shown in Fig.1, where \( P \) denotes the plant to be controlled, and \( C \) represents the controller to be designed. The external signals are \( r, d, u, n \), reference, actuator disturbance, output disturbance, and measurement noise, respectively. The internal signals of interest are \( e, u, y \), measured error, command input, and the plant output, respectively. We will assume that the energy contents of the
external signals are finite, i.e. these signals belong to $L^2[0, \infty)$. We will consider linear time-invariant systems $P$ and $C$ represented by their transfer functions $P(s)$ and $C(s)$, respectively. (See below for precise assumptions on these transfer functions.) Throughout the paper we consider plants and controllers whose transfer functions can be represented as ratios of two $H^\infty$ functions, i.e. $P(s) = P_n(s)/P_d(s)$ and $C(s) = C_n(s)/C_d(s)$ for some $P_n, P_d, C_n, C_d \in H^\infty$.

**Definition 3.1:** Let $C(s) = 0$ and $v = 0$ in Fig. 1, i.e. the system is open loop and the output is $y = Pu = Pd$. Then, we say that the open loop system $P$ is stable if it is a bounded linear operator from $L^2[0, \infty)$ to $L^2[0, \infty)$; that is for every $u \in L^2[0, \infty)$ the output $y = Pu$ is in $L^2[0, \infty)$ and

$$
\sup_{u \neq 0} \frac{\|y\|_2}{\|u\|_2} < \infty. \tag{5}
$$

One can interpret this definition as follows: The plant $P$ is stable if all finite energy command signals $u$ give rise to finite energy outputs $y$, and the maximum energy amplification from $u$ to $y$ is finite.

The stability of $P$ is equivalent to having its transfer function $P(\cdot)$, in $H^\infty$. In fact the maximum energy amplification, given by (5), is the norm of this operator, which is equal to the $H^\infty$ norm of the transfer function, i.e. $P(s) = y(s)/u(s)$ and

$$
\sup_{u \neq 0} \frac{\|y\|_2}{\|u\|_2} = \text{ess sup}_{\omega \in \mathbb{R}} |P(j\omega)| = \|P\|_\infty.
$$

So, from a control system theoretic point of view $H^\infty$ can be seen as the set of transfer functions of all stable systems.

The above definition of stability is an input/output stability concept in the sense of bounded energy amplification. There are several other stability definitions in the case
of distributed parameter systems, see e.g. Callier and Desoer (1978), Willems (1971), Yamamoto (1990). In the finite dimensional systems case all these definitions lead to the “usual” definition of stability.

**Definition 3.2:** The closed loop system $(C, P)$ is stable if all transfer functions (from any external input to any internal signal) are in $H^\infty$.

This definition means that the closed loop system stability is equivalent to the following: all finite energy external inputs give rise to finite energy internal signals and the maximum energy amplification in the system is finite. It is easy to see from Fig.1 that all transfer functions can be expressed in terms of the following four functions:

$$
S(s) := (1 + P(s)C(s))^{-1},
$$

$$
T(s) := 1 - S(s) = P(s)C(s)(1 + P(s)C(s))^{-1},
$$

$$
C(s)S(s) = C(s)(1 + P(s)C(s))^{-1},
$$

$$
P(s)S(s) = P(s)(1 + P(s)C(s))^{-1}.
$$

For example, the sensitivity function $S$ is the transfer function from $v$ to $y$, or from $r$ to $e$; the complementary sensitivity function $T$ is the transfer function from $n$ to $y$, or from $r$ to $y$; $PS$ is the transfer function from $d$ to $y$; $CS$ is the transfer function from $r$ to $u$; etc. Therefore, the closed loop system is stable if and only if all four transfer functions $S, T, CS, PS$ are in $H^\infty$.

### 3.1 Stabilization

In a controller design the most important requirement is stability of the closed loop system $(C, P)$, with controller $C$ and plant $P$. If $(C, P)$ is a stable closed loop system, then we say that the controller $C$ stabilizes the plant $P$. In this section we will characterize the set of all stabilizing controllers for a given plant.

We will assume that $P$ has a factorization of the form $P(s) = N(s)/D(s)$, where $N, D \in H^\infty$ such that there exist $X, Y \in H^\infty$ satisfying the Bezout identity

$$
N(s)X(s) + D(s)Y(s) = 1.
$$

If such a factorization holds for $P$ then $N, D$ are called coprime factors of $P$. In fact existence of such a factorization is necessary for the existence of a stabilizing controller, Smith (1989b).

**Theorem 3.3** (Smith (1989b), Youla et al. (1976)): A controller $C$, which is a ratio of two $H^\infty$ functions, stabilizes the plant $P$ if and only if $C$ is in the form

$$
C(s) = \frac{X(s) + D(s)Q(s)}{Y(s) - N(s)Q(s)},
$$

where $Q \in H^\infty$ is the free parameter to be chosen according to design specifications, other than stabilization. □
We will need the following assumption on the plant.

**Assumption 1** (On the plant): Unless otherwise stated we will consider the following class of SISO, LTI, possibly infinite dimensional plants throughout the paper:

\[ P(s) = \frac{M_n(s)N_1(s)N_2(s)}{M_d(s)}, \quad s \in RHP \]

when transformed to \( z \)-plane (via a conformal map \( z = \frac{s-a}{s+a} \), with \( a > 0 \)) \( P(s) \) becomes

\[ p(z) = \frac{m_n(z)n_1(z)n_2(z)}{m_d(z)}, \quad z \in D \]

where \( m_n \) is an arbitrary (possibly infinite dimensional) inner function, \( m_d \) is a rational inner function, \( n_1 \) is possibly an infinite dimensional outer function with \( n_1^{-1} \in H^\infty(D) \), and \( n_2 \) is a rational outer function. We will assume that \( P(j\omega) \) is continuous on \( jR_e \) except at finitely many points. We will also assume that \( m_n \) has finitely many essential singularities, and that \( m_n(0) \neq 0 \neq m_d(0) \).

The key restrictions in this assumption are that \( n_2 \) and \( m_d \) are rational. Rationality of \( n_2 \) is a technical requirement, which can be relaxed if we allow improper controllers (see Lenz et al. (1991) for an example). Having \( m_d \) rational means that the plant can have only finitely many unstable modes. This assumption is crucial in our solution of the \( H^\infty \) control problems, since the complexity of computations depend on the order of \( m_d(z) \). Another restriction is that the plant does not have any poles on the imaginary axis (i.e. \( m_d \) is inner). The theory presented here does not need this assumption, but we will use it to avoid technical details involving “outer factor absorption,” see Flamm (1990). The assumption that \( m_n(0) \neq 0 \neq m_d(0) \) is without loss of generality, because we can always choose the parameter \( a \) in the conformal map in such a way that this is satisfied (i.e. \( a \) is chosen such that \( P(s) \) does not have any pole or zero at \( s = a \)).

**Example 1:** An infinite dimensional plant example obeying the above assumptions is

\[ P(s) = \frac{e^{-hs}(s - 0.05)}{(s + 1)(s + 0.1 - e^{-h_1s})}, \quad h_1 = 2\ln\left(\frac{5}{3}\right), \quad h > 0. \]

Note that the only point in the closed \( RHP \) where the term \((s + 0.1 - e^{-h_1s})\) becomes zero is \( s = 0.5 \). So, the plant has only one pole in the closed \( RHP \). We can easily check that the multiplicity of this pole is 1. Therefore, in this example we can identify the components of \( P \) as follows

\[ M_n(s) = \frac{e^{-hs}s - 0.05}{s + 0.05}, \quad M_d(s) = \frac{s - 0.5}{s + 0.5} \]
\[ N_1(s) = \frac{(s - 0.5)(s + 0.05)}{(s + 0.1 - e^{-h_1 s})(s + 0.5)} \]

\[ N_2(s) = \frac{1}{s + 1}. \]

Note that \( N_1^{-1} \in H^\infty \), in fact

\[ N_1(0.5)^{-1} = \left( \frac{1}{0.55} \right) \left( 1 + \frac{6}{5} \ln\left( \frac{5}{3} \right) \right), \quad \text{and} \quad N_1(\infty)^{-1} = 1. \]

One can use a conformal map in order to transform these functions to the \( z \)-domain.

As mentioned in Section 2.2, in general it may be difficult to perform inner/outer factorizations for an arbitrary infinite dimensional plant. We have seen in the above example that in some cases, the special factorization needed in Assumption 1 can be done by identifying the poles and zeros of the plant. Also, note that in the above example the choice of \( N_1 \) and \( N_2 \) is not unique. For example we could have chosen

\[ N_1'(s) = \frac{(s - 0.5)(s + 0.05)}{(s + 0.1 - e^{-h_1 s})(s + 1)} \]

\[ N_2'(s) = \frac{1}{s + 0.5}, \]

instead of the above \( N_1 \) and \( N_2 \), and still satisfy Assumption 1.

For an arbitrary plant whose transfer function is a ratio of two \( H^\infty \) functions, e.g. \( P = N/D \) with \( N, D \in H^\infty \), we can check if it satisfies Assumption 1 as follows. First of all we need to look at the zeros of \( D(s) \) in the right half plane, there should be finitely many, and none on the imaginary axis. Then, we can obtain the Bode magnitude plot for \( 20 \log |P(j\omega)| \), if it converges to a finite number as \( \omega \to \infty \), (i.e. if \( P \) is not strictly proper) then the assumption is satisfied, we can choose \( N_2 = 1 \). If this plot converges to \( -\infty \) as \( \omega \to \infty \), then the rate of decay has to be an integer multiple of \( -20 \) dB per decade, i.e. as \( \omega \to \infty \) the function has to "look like" the Bode magnitude plot of a rational function. This is necessary to have a rational \( N_2 \in H^\infty \) and to have \( N_1^{-1} \in H^\infty \).

We can obtain a characterization of stabilizing controllers for the plant considered in Example 1 by using Theorem 3.3 and solving the corresponding Bezout equation as shown below.

**Example 2:** Let us consider the plant given in Example 1. We can define \( N(s) = M_n(s)N_1(s)N_2(s) \) and \( D(s) = M_d(s) \). Then, the Bezout equation can be solved as follows. Note that

\[ Y(s) = \frac{1 - N(s)X(s)}{D(s)}. \]

Since \( D(s) \) has a single zero in the closed \( RHP \) (at \( s = 0.5 \)), we need to find an \( X \in H^\infty \) such that \( X(0.5) = N(0.5)^{-1} \). Therefore, we can simply choose

\[ X(s) = N(0.5)^{-1} = e^{h/2}(\frac{10}{3} + 4 \ln(\frac{5}{3})). \]
In general $X(s)$ is constructed from the interpolation conditions $X(p_i) = N(p_i)^{-1}$ where $p_i \in \text{RHP}$ are the zeros of $D(s)$ (i.e. poles of $P(s)$) with multiplicity 1. If a pole $p_i$ has multiplicity $k \geq 2$ then we also require $\left(\frac{\partial^k}{\partial s^k}\right)N(s)X(s)|_{s=p_i} = 0$ for all $i = 1, \ldots, k - 1$. There are finitely many interpolation conditions, if the plant has finitely many RHP poles. In such cases $X(s)$ can always be chosen as a rational function, in fact by Lagrange interpolation $x(z)$ can be chosen as a polynomial of degree $(\ell - 1)$, where $\ell$ is the dimension of $M_d$. On the other hand note that, when $X$ and $D$ are rational and $N$ is infinite dimensional, $Y$ is infinite dimensional.

### 3.2 Robust stability

In the above discussion we have assumed that the plant transfer function is given by $P(s)$, and we have characterized the set of all controllers stabilizing this fixed plant. However, usually $P(s)$ is a nominal representation of the actual plant, whose transfer function is denoted by $P_\Delta(s)$. The part of $P_\Delta$ which does not appear in $P$ is called the unmodeled dynamics. There are a number of ways to represent the unmodeled dynamics, e.g. multiplicative, additive, coprime factor perturbations of the nominal plant.

**Multiplicative Perturbation:**

$$P_\Delta(s) = P_m(s) = P(s) \left(1 + \Delta_m(s)\right)$$

**Additive Perturbation:**

$$P_\Delta(s) = P_a(s) = P(s) + \Delta_a(s),$$

**Coprime Factor Perturbation:**

$$P_\Delta(s) = P_{cf}(s) = \frac{N(s) + \Delta_N(s)}{D(s) + \Delta_D(s)}, \quad \Delta_{cf}(s) = [\Delta_N(s) \ \Delta_D(s)],$$

where $P(s) = N(s)/D(s)$ is the nominal plant and $\Delta(s)$ is the unmodeled dynamics. In general $\Delta(s)$ is unknown, but a frequency dependent upper bound function $W(s)$ (called the uncertainty weight) can be introduced to represent the uncertainty in the form

$$\|\Delta(j\omega)\| < |W(j\omega)| \quad \forall \ \omega \in \mathbb{R} \quad (8)$$

for example

$$|W_m(j\omega)| > |\Delta_m(j\omega)|, \quad (9)$$

$$|W_a(j\omega)| > |\Delta_a(j\omega)|, \quad (10)$$

$$|W_{cf}(j\omega)|^2 > |\Delta_N(j\omega)|^2 + |\Delta_D(j\omega)|^2 \quad (11)$$
where $W_m(s), W_a(s), W_{cf}(s)$ are known functions. The actual plant is in the form $P_\Delta$, with $\Delta$ unknown; but the nominal plant $P$ and the uncertainty weight $W$, satisfying (8), are known. We will assume that in the case of multiplicative and additive perturbations the actual plant $P_\Delta$ and the nominal plant $P$ have the same number of right half plane poles; when dealing with coprime factor perturbations we will relax this condition.

Note that $W_m$ and $W_a$ have to satisfy:

$$|W_m(j\omega)P(j\omega)| > |\Delta_a(j\omega)|$$  \hspace{1cm} (12)

and

$$\left| \frac{W_a(j\omega)}{P(j\omega)} \right| > |\Delta_m(j\omega)|.$$  \hspace{1cm} (13)

Our assumptions on the nominal plant imply that $|P(j\omega)| = |N_1(j\omega)N_2(j\omega)|$. This means that we can see $W_mN_1N_2$ as an additive uncertainty bound, and $\frac{W_a}{N_1N_2}$ as a multiplicative uncertainty bound. Therefore, if $W_a = W_mN_1N_2$, then whether we consider multiplicative or additive perturbations does not make any difference.

**Assumption 2 (On the weights):** We assume that $W_{cf}, W_{cf}^{-1}, (W_mN_2), (W_mN_2)^{-1}$, are rational functions in $H^\infty$, and the additive uncertainty weight is given by

$$W_a = W_mN_1N_2.$$  \hspace{1cm} (14)

We further assume that there exist $\omega_o$ and $K \geq 2$, such that

$$|W_m(j\omega)| > K \quad \text{for all} \quad \omega \geq \omega_o.$$  \hspace{1cm} (15)

Assumption 2 imply that $W_m$ must be improper, whenever $N_2$ is strictly proper, in which case (15) is automatically satisfied. The purpose of choosing $W_a$ in the form (14) is to make the additive uncertainty problem the same as the multiplicative uncertainty problem; though such a choice leads to an infinite dimensional $W_a$, whenever $N_1$ is infinite dimensional.

Considering the unmodeled dynamics in the plant we require that the controller $C$, which is fixed, stabilizes not only the nominal plant $P$ but also all possible plants $P_\Delta$ with $\Delta$ satisfying (8), for a given weight $W$. If a controller meets this requirement then we say that $C$ *robustly stabilizes the plant*. Necessary and sufficient conditions for a controller $C$ to robustly stabilize the plant are given by the following.

**Theorem 3.4:** Consider the classes of plants described by $P_m$, $P_a$ and $P_{cf}$, with the nominal plant $P = N/D$ (where $N, D$ coprime factors of $P$, and $P_m$, $P_a$ and $P$ have the same number of right half plane poles), and the uncertainty bounds given by (9), (10) and (11), for some weights $W_a$, $W_m$ and $W_{cf}$, respectively. Suppose that the nominal plant satisfies Assumption 1, and the weights satisfy Assumption 2. Let $C$ be a controller,
which is a ratio of two $H^\infty$ functions, such that $C(j\omega)$ is continuous on $j\mathbb{R}_+$ except at finitely points, and $C$ stabilizes the nominal plant $P$. When dealing with $P_a$ or $P_m$ we also assume that $C$ has finitely many poles in the closed $RHP$. Then, $C$ robustly stabilizes the plant if and only if

case (i): multiplicative perturbations

$$\|W_mPC(1+PC)^{-1}\|_\infty \leq 1,$$

(16)

case (ii): additive perturbations

$$\|W_aC(1+PC)^{-1}\|_\infty \leq 1,$$

(17)

case (iii): coprime factor perturbations

$$\left\|W_{cf}D^{-1}\left[\begin{array}{c} (1+PC)^{-1} \\ C(1+PC)^{-1} \end{array}\right]\right\|_\infty \leq 1.$$

(18)

Proof: For coprime factor perturbations case see Georgiou and Smith (1990) p.683, (this was first shown by Vidyasagar and Kimura (1986) for the finite dimensional case). Since $W_a$ satisfies (14) $P_m$ is the same as $P_a$, and

$$\|W_mPC(1+PC)^{-1}\|_\infty = \|W_aC(1+PC)^{-1}\|_\infty.$$

Therefore, the multiplicative perturbation case is the same as the additive perturbation case, and it is sufficient to prove the theorem for either of these cases. For the sufficiency part, the proof of Chen and Desoer (1982) goes through because $P$ and $C$ have finitely many poles in the closed $RHP$, and (15) implies that as $\omega \to \infty$ the Nyquist plot of $P(j\omega)C(j\omega)$ lies within the unit circle. This can be seen from the following:

$$PC = \frac{PC(1+PC)^{-1}}{1-PC(1+PC)^{-1}}.$$

Then, $|PC| < 1$ if

$$|PC(1+PC)^{-1}| < \frac{1}{2}.$$

This means that $|PC| < 1$ if

$$|W_mPC(1+PC)^{-1}| < \frac{|W_m|}{2}.$$
On the other hand, by Assumptions 1 and 2, $|W_m(j\omega)| > 2$ and $P(j\omega)C(j\omega)$ is continuous for all $\omega \geq \omega'$, (for some $\omega' < \infty$). Hence, we have that

$$\|W_mPC(1 + PC)^{-1}\|_{\infty} \leq 1$$

implies $|P(j\omega)C(j\omega)| < 1$ for all $\omega \geq \omega'$. Obviously, if $N_2$ is strictly proper, then so is $P$, and hence the Nyquist plot converges to the origin as $\omega \to \infty$. But this is not necessary, a weaker condition of the form (15) is sufficient to apply the Nyquist criterion.

For the necessity part we must show that if

$$\|W_mPC(1 + PC)^{-1}\|_{\infty} > 1,$$

then there exists an admissible multiplicative perturbation $\Delta_m$ destabilizing the closed loop system. Such a perturbation can be constructed exactly the same way as in the finite dimensional case (see e.g. Doyle and Stein (1981)), because of the assumptions on continuity of $P$ and $C$. □

**Special Case:** If $P$ and $\Delta_a$ (or $\Delta_m$) are stable, then in the above theorem we don’t need the assumption that $C$ has finitely many closed RHP poles. In this case the proof goes as follows: necessity part is the same because if (17) is not satisfied then one can find a destabilizing perturbation. For sufficiency we note that

$$(1 + P_aC)^{-1} = (1 + PC)^{-1} \frac{1}{1 + \Delta_aC(1 + PC)^{-1}}.$$

(19)

Since $C$ stabilizes $P$, $(1 + PC)^{-1}$ and $C(1 + PC)^{-1}$ are in $H^\infty$. Therefore, when $\Delta_a$ is in $H^\infty$, (17) implies that $(1 + \Delta_aC(1 + PC)^{-1})^{-1}$ is in $H^\infty$, and hence $(1 + P_aC)^{-1} \in H^\infty$ for all admissible $\Delta_a$. The identity (19) further implies that $C(1 + P_aC)^{-1}$ and $P_a(1 + P_aC)^{-1}$ are also in $H^\infty$. Hence the closed loop system $(C, P_a)$ is stable for all admissible $P_a$.

**Example 3:** Consider the plant given in Example 1, and let $W_m$ be given as

$$W_m(s) = (s + 0.2).$$

Note that $W_m(s)N_2(s) = \frac{s + 0.2}{s + 1}$ and $|W_m(j\omega)| \to \infty$ as $\omega \to \infty$. So, $W_m$ satisfies the conditions of Assumption 2. Let the controller $C$ be given as in Theorem 3.3, where $N, D, X, Y$ are as defined in Example 2. Then, condition (16) becomes

$$\|W_mM_nN_1N_2(X + M_dQ)\|_{\infty} \leq 1.$$  (20)

Therefore, a **robustly stabilizing controller exists** if and only if there exists a $Q \in H^\infty$ satisfying (20). Since $M_n$ is inner and $W_mN_1N_2$ is invertible in $H^\infty$, (20) can be reduced to

$$\|R + M_dQ_1\|_{\infty} \leq 1$$  (21)
where $Q_1 = (W_m N_2) N_1 Q$ (or $Q = (W_m N_2)^{-1} N_1^{-1} Q_1$), and $R = W_m N_1 N_2 X$. Hence a robustly stabilizing controller exists if and only if there exists a $Q_1 \in H^\infty$ satisfying (21). Since $M_d(0.5) = 0$ the left hand side of (21) is greater or equal to $|R(0.5)|$ for all $Q_1$. On the other hand for $Q_1 = Q_{1,\text{opt}}$, defined by

$$Q_{1,\text{opt}} = \frac{R(s) - R(0.5)}{M_d(s)} \in H^\infty,$$

the left hand side of (21) is equal to $|R(0.5)|$. Thus a robustly stabilizing controller exists if and only if $|R(0.5)| \leq 1$. Note that

$$|R(0.5)| = \frac{|W_m(0.5) N_1(0.5) N_2(0.5) X(0.5)|}{M_m(0.5)} = \frac{0.7 e^{h/2} 0.55}{0.45}.$$

We conclude that a robustly stabilizing controller exists if and only if

$$h \leq 2 \ln \left( \frac{9}{7.7} \right) \approx 0.312.$$

In other words if the delay is "too large," we cannot find a robustly stabilizing controller for this system, and the largest allowable delay is $2 \ln(\frac{9}{7.7})$.

### 3.3 Robust performance

The above discussion summarized by Theorem 3.4 gives conditions for a controller to robustly stabilize the plant. Besides stability we are interested in the performance of the closed loop system. A typical performance condition is the sensitivity reduction, which can be stated as follows. Given a desired upper bound $W_d(s)$ for the sensitivity function (we will assume that $W_d$ is rational and $W_d, W_d^{-1} \in H^\infty$) we want to find a robustly stabilizing controller $C$ such that

$$|(1 + P_\Delta(j\omega)C(j\omega))^{-1}| \leq |W_d(j\omega)| \quad \text{a.e.} \quad \omega \in \mathbb{R},$$

(22)

where $P_\Delta$ represents the actual plant, which can be any transfer function of the form $P_m, P_a, P_{cf}$, with the nominal plant $P$ and the uncertainty weights $W_m, W_a, W_{cf}$, respectively. The function $(1 + P_\Delta C)^{-1}$ is the sensitivity of the closed loop system with the actual plant $P_\Delta$ and the controller $C$. The problem (22) is called the robust performance problem. It is difficult to design a controller satisfying the necessary and sufficient conditions for the robust performance problem. On the other hand for the additive and multiplicative perturbation cases there is a simple sufficient condition, given by Theorem 3.5 below, which leads to a two block $H^\infty$ controller design.

**Theorem 3.5:** Assume that the classes of plants described by the multiplicative or the additive perturbations, as defined in Theorem 3.4. Consider a controller satisfying the
assumptions stated in Theorem 3.4. Then $C$, solves the robust performance problem if it stabilizes the nominal plant $P$ and satisfies the robust performance inequality

case (i): multiplicative perturbations

$$\left\| \begin{bmatrix} W_d^{-1}(1 + PC)^{-1} \\ W_m P C (1 + PC)^{-1} \end{bmatrix} \right\|_\infty \leq \frac{1}{\sqrt{2}}$$

(23)

case (ii): additive perturbations

$$\left\| \begin{bmatrix} W_d^{-1}(1 + PC)^{-1} \\ W_a C (1 + PC)^{-1} \end{bmatrix} \right\|_\infty \leq \frac{1}{\sqrt{2}}.$$  

(24)

**Proof:** This is a consequence of Theorem 3.4, Doyle et al. (1992), the details are given below. A necessary condition is robust stability, which is automatically satisfied if (23) or (24) holds. Also note that by Assumption 2 on the weights the problems (23) and (24) are identical. Therefore, we may consider either of these two cases. We want a stabilizing $C$ satisfying

$$|W_d^{-1}(j\omega)(1 + (P(j\omega) + \Delta_a(j\omega))C(j\omega))^{-1}| \leq 1$$

(25)

almost everywhere on the imaginary axis, and for all $\Delta_a$ satisfying our additive uncertainty assumptions. Note that (25) is equivalent to having

$$|W_d^{-1}(1 + PC)^{-1}| \leq |1 + \Delta_a C (1 + PC)^{-1}|, \quad \text{a.e.}$$

(26)

for all admissible $\Delta_a$, (we have dropped the dependence on ($j\omega$) for notational convenience). The inequality (26) is satisfied if and only if

$$|W_d^{-1}(1 + PC)^{-1}| \leq 1 - |W_a C (1 + PC)^{-1}|, \quad \text{a.e.}$$

(27)

It is easy to see that condition (27) is satisfied if the following holds

$$|W_d^{-1}(1 + PC)^{-1}|^2 + |W_a C (1 + PC)^{-1}|^2 \leq \frac{1}{2}, \quad \text{a.e.}$$

(28)

This concludes the proof. □

The above theorem gives only a *sufficient* condition for the robust performance problem. The conservatism is in the step where we go from (27) to (28).

**Remark 1:** By Assumption 1, $D$ is equal to $M_d$, which is inner, so (18) can be reduced to

$$\left\| W Cf \left[ \begin{bmatrix} (1 + PC)^{-1} \\ C(1 + PC)^{-1} \end{bmatrix} \right] \right\|_\infty \leq 1,$$

(29)
which looks similar to (24): if one chooses the weights $W_d^{-1} = W_a = W_{ef}$ then the two problems are the same. On the other hand, such a choice is possible, without violating Assumption 2 or rationality of $W_d$, only if $N_1$ is rational. This restricts the class of distributed plants which can be handled in problem (29).

### 3.4 Disturbance attenuation

Disturbance attenuation is another issue where we are faced with a control problem similar to (24) and (23): Consider the closed loop system shown in Figure 1. Let us assume that $d = r = 0$, and

$$v \in \mathcal{D}_v := \{W_n v_1 : v_1 \in L^2[0, \infty), \|v_1\|_2 \leq 1\}$$

$$n \in \mathcal{D}_n := \{W_n n_1 : n_1 \in L^2[0, \infty), \|n_1\|_2 \leq 1\},$$

where $W_v$ and $W_a$ are LTI "weights" shaping the frequency and magnitude of the output disturbance, and the measurement noise, respectively. A disturbance attenuation problem is to find a controller $C$ stabilizing the closed loop system and minimizing

$$\gamma_1(C) = \sup_{v \in \mathcal{D}_v, n \in \mathcal{D}_n} \|y\|_2.$$

This problem is equivalent to finding a stabilizing controller which minimizes

$$\gamma_1(C) = \left\| \begin{bmatrix} W_v & 0 \\ W_n & PC^{-1} \end{bmatrix} \right\|_\infty. \tag{30}$$

Similarly we can define another disturbance attenuation problem by assuming $d = r = n = 0$: find a controller $C$, stabilizing the closed loop system and minimizing

$$\gamma_2(C) = \sup_{v \in \mathcal{D}_v} \left\| \begin{bmatrix} y \\ W_u u \end{bmatrix} \right\|_2,$$

where $W_u$ is a LTI weight. It can be shown that

$$\gamma_2(C) = \left\| \begin{bmatrix} W_v & 0 \\ W_u & PC^{-1} \end{bmatrix} \right\|_\infty. \tag{31}$$

Note that the problem (30) is similar to the problem (23) and the problem (31) is similar to the problem (24), provided the weights are chosen appropriately. The major difference between the problems associated with disturbance attenuation and the problems arising in robust performance is that in Theorem 3.5 (for robust performance) we considered controllers with finitely many closed RHP poles. However, there is no such restriction in disturbance attenuation problems. Therefore, we may first solve the
disturbance attenuation problem. If the resulting controller has finitely many closed RHP poles, then this controller also solves the robust performance problem. Otherwise, we can approximate the controller by a finite dimensional one, in such a way that the approximate controller stabilizes the closed loop system with infinite dimensional plant, and satisfies the robust performance inequality (23) (or (24)). This issue will be discussed in Section 6.

We conclude that several cases of robust stabilization, robust performance and disturbance attenuation problems can be solved by finding the $H^\infty$ optimal controller $C_{opt}$ from the following two-block $H^\infty$ problem:

$$
\gamma_{opt} := \inf_{C \text{ stabilizing } P} \left\| \begin{bmatrix} W_1 (1 + PC)^{-1} \\ W_2 PC (1 + PC)^{-1} \end{bmatrix} \right\|_\infty,
$$

(32)

where $C_{opt}$ stabilizes the nominal plant $P$ and achieves the $H^\infty$ optimal performance $\gamma_{opt}$, i.e.

$$
\gamma_{opt} := \left\| \begin{bmatrix} W_1 (1 + PC_{opt})^{-1} \\ W_2 PC_{opt} (1 + PC_{opt})^{-1} \end{bmatrix} \right\|_\infty,
$$

and $W_1$ and $W_2$ are appropriate weights related to the control problems defined above. We will assume that $W_1$ is rational with $W_1, W_1^{-1} \in H^\infty$, and $W_2 = W_m$ satisfying Assumption 2. In the rest of the paper we will study the problem defined by (32) for the class of distributed parameter plants $P$ satisfying Assumption 1.

4 Solution of the $H^\infty$ control problems

In Section 4.1 we consider the simplest $H^\infty$ problem for distributed systems, namely the one block problem for stable plants, in order to give the basic idea behind the skew Toeplitz approach. For the class of plants defined in Section 3, we will show that the two block problem can be solved using the results of Section 4.1.

4.1 One-block problem for stable distributed plants

In this section we consider the one block problem defined by setting $w_2 = 0$ in (32). Although Assumption 2 is violated by this choice, the problem gets simplified considerably. To further simplify the problem we will also assume that $p \in H^\infty(D)$, and that $p = m_n n_1$, i.e. $n_2(z) = 1 = m_d$. The assumption $n_2 = 1$ means that the outer part of the plant is invertible; and it helps to avoid technical difficulties associated with the outer factor absorption problem (see Flamm (1990) for a detailed discussion of this problem). The simplifying assumptions we make in this section correspond to the sensitivity minimization problem for a plant $P(s)$ which is assumed to be stable and not strictly proper. From the control theoretic point of view this problem is not as interesting as the two block problem, but its solution can be used to find a solution to the two block problem.
defined in Section 3. We will illustrate this idea in Section 4.2. Throughout this section \( m_n \) will be denoted by \( m \).

Using Theorem 3.3, (choosing \( d(z) = 1, n(z) = p(z), x(z) = 0, y(z) = 1 \), in (6)) we can show that all stabilizing controllers are in the form

\[
c(z) = \frac{q(z)}{1 - p(z)q(z)} : q \in H^\infty (D).
\] (33)

Replacing (33) into (32) we have

\[
\gamma_{opt} = \inf_{q \in H^\infty (D)} \| w_1 (1 - pq) \|_\infty.
\] (34)

Since \( w_1 \) and \( n_1 \) are invertible in \( H^\infty (D) \) we can absorb them in the free parameter \( q \): defining \( q_1 = w_1 n_1 q \), or \( q = q_1 n_1^{-1} w_1^{-1} \), we have

\[
\gamma_{opt} = \inf_{q_1 \in H^\infty (D)} \| w_1 - m q_1 \|_\infty.
\] (35)

This is called a one-block \( H^\infty \) control problem. After finding an optimal \( q_1^{opt} \) achieving the optimal performance \( \gamma_{opt} \) we can find the optimal controller from

\[
c_{opt}(z) = \frac{q_{opt}(z)}{1 - p(z)q_{opt}(z)} = \frac{q_1^{opt}(z)n_1^{-1}(z)}{w_1(z) - m(z)q_1^{opt}(z)}.
\] (36)

The problem (35) is equivalent to the Nehari problem, and its solution is given in terms of the norm of the Hankel operator \( \Gamma_{m^*, w_1} \) whose symbol is

\[
(m^* w_1)(e^{j\theta}) = m^*(e^{-j\theta}) w_1(e^{j\theta})
\]

That is

**Theorem 4.1 (Nehari, 1957)**

\[
\gamma_{opt} = \| \Gamma_{m^*, w_1} \|.
\]

**Proof:** Since \( m^* \) is unitary on the unit circle we have

\[
\gamma_{opt} = \inf_{q_1 \in H^\infty (D)} \| m^*(w_1 - mq_1) \|_\infty
\]

\[
= \inf_{q_1 \in H^\infty (D)} \| m^* w_1 - q_1 \|_\infty.
\]

The last equality means that \( \gamma_{opt} \) is the smallest distance from the \( L^\infty (T) \) function \( m^* w_1 \) to the \( H^\infty (D) \) functions. So, the result follows from Nehari (1957). \( \Box \)
There is a strong relation between $\Gamma_{m \cdot w_1}$ and the operator $w_1(T)$, which can be defined as in (3).

**Theorem 4.2:**

$$\gamma_{opt} = \|\Gamma_{m \cdot w_1}\| = \|w_1(T)\|. $$

**Proof:** This can be deduced by combining Sarason’s theorem (Sarason, 1967) with Nehari’s theorem (Nehari, 1957), as follows. Using Theorem 4.1 we want to show that $\|\Gamma_{m \cdot w_1}\| = \|w_1(T)\|$. Note that when $f \in mH^2$ we have $m^* w_1 f \in H^2(D)$. So $\Gamma_{m \cdot w_1} f = 0$ for all $f \in mH^2$. Hence

$$\|\Gamma_{m \cdot w_1}\| = \|\Gamma_{m \cdot w_1} H(m)\|. $$

We claim that $\Gamma_{m \cdot w_1} H(m) = m^* w_1(T)$. Assume this is true for a moment, then we have, since $m^*$ is unitary,

$$\|\Gamma_{m \cdot w_1} H(m)\| = \|m^* w_1(T)\| = \|w_1(T)\| $$

and this completes the proof.

Now we prove the above claim. Take any $f \in H(m)$ and define $w_1(T)f =: g$. But

$$g = P_{H(m)} w_1 f = w_1 f - m P_{+} m^* w_1 f, \text{ so}$$

$$m^* g = m^* w_1 f - P_{+} m^* w_1 f, \text{ hence}$$

$$P_{-} m^* g = P_{-} m^* w_1 f.$$ 

Since $g = w_1(T)f \in H(m)$, we have $m^* g \in L^2(T) \oplus H^2(D)$. Thus for any $f \in H(m)$ we have

$$m^* w_1(T)f = m^* g = P_{-} m^* g = P_{-} m^* w_1 f = \Gamma_{m \cdot w_1} f$$

as claimed. $\square$

In order to compute the $H^\infty$ optimal performance level $\gamma_{opt}$ we want to find the norm of $w_1(T)$. Since the operator $w_1(T)$ is infinite rank, its norm is the largest of two quantities: the essential norm, denoted by $\|w_1(T)\|_e$, and the largest singular value. See Appendix B for a definition of the essential norm. For the operator $w_1(T)$ the essential norm can be computed as (see e.g. Foias et al. (1988), Foias and Tannenbaum (1988), Özbay and Tannenbaum (1990))

$$\|w_1(T)\|_e = \max\{|w_1(e^{i\theta})| : e^{i\theta} \text{ is an essential singularity of } m(z)\}. \quad (37)$$

Note that when $m$ has finitely many essential singularities it is trivial to compute the essential norm: we simply have to evaluate $w_1$ at finitely many points.
The identity (37) gives a lower bound for $\gamma_{opt}$. On the other hand if we choose $q_1 = 0$ in (35) we get an upper bound for $\gamma_{opt}$: $\|w_1\|_\infty$. Let us assume $\|w_1(T)\| > \|w_1(T)\|_e$ so that the norm is achieved at the largest singular value. Then, in order to compute $\gamma_{opt}$, we need to find the largest singular value of $w_1(T)$ between $\|w_1(T)\|_e$ and $\|w_1\|_\infty$.

By definition, $\rho$ is a singular value of $w_1(T)$ if and only if there exists a nonzero singular vector $y \in H(m)$. Now we are going to study the necessary and sufficient conditions for a given number $\rho \in (\|w_1(T)\|_e, \|w_1\|_\infty)$ to be a singular value, i.e. existence of a singular vector for $\rho$. For this purpose we write the singular value singular vector equation:

$$\left(\rho^2 I - w_1(T)^* w_1(T)\right) y = 0. \quad (38)$$

Recall that the weight $w_1$ we consider is rational and $w_1 \in H^\infty(D)$. So we can write $w_1(z) = b(z)/k(z)$ where $b(z)$ and $k(z)$ are polynomials with $1/k \in H^\infty(D)$. Let $n$ be the maximum of degrees of $b(z)$ and $k(z)$, i.e.

$$b(z) = b_0 + b_1 z^1 + \cdots + b_n z^n$$

$$k(z) = k_0 + k_1 z^1 + \cdots + k_n z^n$$

$k_n$ or $b_n$ is nonzero by construction. With this notation it can be seen from (3) that $w_1(T) = b(T)/k(T)^{-1}$. Now define $k(T)^{-1} y =: u$, since $1/k \in H^\infty(D)$ and $\|T\| \leq 1$, $u \in H(m)$ if and only if $y \in H(m)$. Then, as a summary of the above discussion, we have the following result.

Lemma 4.3: Assume that $\gamma_{opt} > \|w_1(T)\|_e$. Then $\gamma_{opt}$ is the largest value of $\rho$ for which there is a non-zero $u \in H(m)$ satisfying

$$\left(b(T)^* b(T) - \rho^2 k(T)^* k(T)\right) u = 0. \quad (39)$$

Proof: Immediate from the above discussion. □

Note that (39) is in the form $A_\rho u = 0$, where

$$A_\rho := b(T)^* b(T) - \rho^2 k(T)^* k(T).$$

Operators of the form $A_\rho$ are called skew Toeplitz, (Bercovici et al., 1988). Conditions on the invertibility of this skew Toeplitz operator determines the $H^\infty$ optimal performance, $\gamma_{opt}$. Note that since $b$ and $k$ are polynomials in $z$, $A_\rho$ is in the form

$$A_\rho =: \sum_{i,j=0}^n c_{ij} T^i T^j \quad \text{where} \quad c_{ij} c_{ji}^* \in \mathbb{C}. \quad (40)$$

So $A_\rho$ is a polynomial in powers of $T^*$ and $T$. As mentioned in Section 2.3 when $m$ is rational, $T$ is finite dimensional (i.e. a square matrix of finite size), and in this case
the skew Toeplitz operator \( A_\rho \) is a finite size square matrix and we can easily determine conditions on its invertibility. However, when \( m \) is an arbitrary inner function \( A_\rho \) is infinite dimensional, and this is the case studied in the rest of this paper.

Recall, from the proof of Theorem 4.2, that on \( H(m) \) we have \( w_1(T)^*w_1(T) = \Gamma_m^{*_{w_1}} \Gamma_m^{*_{w_1}} \). Therefore, the skew Toeplitz operator \( A_\rho \) can be expressed in terms of two special type of Hankel operators defined on \( H(m) \):

\[
A_\rho = \Gamma_m^{*_{w_1}} \Gamma_m^{*_{w_1}} - \rho^2 \Gamma_m^{*_{w_1}} \Gamma_m^{*_{w_1}}.
\]

The optimal controller can be found from \( q_1^{opt} \) using commutant lifting theorem as follows. It can be seen from (3) that \( w_1(T)^*T = Tw_1(T) \), i.e. the operator \( w_1(T) \) commutes with the compressed shift. Therefore, by the commutant lifting theorem (or in this special case by Sarason’s theorem) there exists a minimal dilation \( s_{opt} \in H^\infty \) such that \( s_{opt}(T) = w_1(T) \) and \( \|s_{opt}\|_\infty = \|w_1(T)\| = \gamma_{opt} \). Since \( s_{opt}(T) = w_1(T) \), \( s_{opt} \) and \( w_1 \) can differ\(^1\) only by a term \( mq \) where \( q \in H^\infty \), i.e. we must have \( s_{opt} = w_1 - mq_{q_1^{opt}} \), for some \( q_1^{opt} \in H^\infty \).

We can obtain \( s_{opt} \) (and hence \( q_1^{opt} \), and \( c_{opt} \)) from the optimal performance \( \gamma_{opt} \), i.e. the largest singular value of \( w_1(T) \), and a singular vector \( y_{opt} = k(T)u_{opt} \) corresponding to \( \gamma_{opt} \) satisfying the equivalent singular value singular vector equation (39):

\[
(b(T)^*b(T) - \gamma_{opt}^2 k(T)^*k(T)) u_{opt} = 0.
\]

**Theorem 4.4:** The minimal dilation \( s_{opt}(z) = w_1(z) - m(z)q_1^{opt}(z) \) of \( w_1(T) \) is given by

\[
s_{opt} = \frac{w_1(T)y_{opt}}{y_{opt}} = \frac{b(T)u_{opt}}{k(T)u_{opt}}.
\]

**Proof:** The result follows from Proposition 5.1 of Sarason (1967): The second equality is obvious from the definition of \( u_{opt}, y_{opt} \) and \( w_1 = b/k \). For the first equality we need to show that

\[
s_{opt}(z)y_{opt}(z) = (w_1(T)y_{opt})(z) = (s_{opt}(T)y_{opt})(z) = (P_{H(m)}s_{opt}y_{opt})(z).
\]

In other words we need to show that \( s_{opt}y_{opt} \) lies completely in \( H(m) \), i.e.

\[
\|s_{opt}y_{opt}\| = \|P_{H(m)}s_{opt}y_{opt}\|.
\]

Let us define

\[
\hat{y}_{opt} = \gamma_{opt}^{-1} w_1(T)y_{opt}.
\]

\(^1\)Note that \( m(T) = 0 \).
Then from (38) we have

\[ y_{opt} = \gamma_{opt}^{-1} w_1(T)^* \tilde{y}_{opt}, \]

and the following inequalities hold

\[
\|w_1(T)^*\| \|\tilde{y}_{opt}\| = \|w_1(T)^*\| \|\gamma_{opt}^{-1} w_1(T) y_{opt}\| = \|w_1(T) y_{opt}\| = \|s_{opt}(T) y_{opt}\|
\]
\[
\leq \|s_{opt} y_{opt}\|
\]
\[
\leq \|s_{opt}\| \|y_{opt}\| = \gamma_{opt} \|\gamma_{opt}^{-1} w_1(T)^* \tilde{y}_{opt}\|
\]
\[
\leq \|w_1(T)^*\| \|\tilde{y}_{opt}\|.
\]

Thus all above inequalities are in fact equalities, and in particular (43) holds. This completes the proof. □

We now develop a finite determinantal formula for computing \( \gamma_{opt} \) and \( s_{opt} \) (hence \( c_{opt} \)). Our starting point is Lemma 4.3, that is we want to examine the conditions under which there exists a non-zero \( u \in H(m) \) satisfying the singular value singular vector equation (39):

\[
(b(T)^* b(T) - \rho^2 k(T)^* k(T)) u = A_\rho u = 0.
\]

We are going to write the left hand side explicitly, and this will give us the necessary and sufficient conditions on \( \rho \) for the existence of a non-zero \( u \in H(m) \). Before going into details we would like to present the main idea behind the computations below. First thing to note is that the skew Toeplitz operator \( A_\rho \) is a polynomial in \( T^j \) and \( T^{*-j}, j = 1, 2, \ldots, n \). Recall from Section 2.3 that applying \( T \) to an element \( u \in H(m) \) we get

\[
(Tu)(z) = zu(z) - m(z) \phi_{-1},
\]

where \( \phi_{-1} \) comes from the expansion of \( m^* u \) which is in \( L^2(T) \ominus H^2(D) \):

\[
m^* u = \phi_{-1} z^{-1} + \phi_{-2} z^{-2} + \ldots
\]

(44) (the right hand side converges on and outside the unit circle). Again from Section 2.3 we have

\[
(T^* u)(z) = z^{-1} u(z) - z^{-1} \phi_0
\]

where \( \phi_0 \) comes from the expansion of \( u \):

\[
u(z) = \phi_0 + \phi_1 z + \phi_2 z^2 + \ldots.
\]

(45)
Since $A_\rho$ is a polynomial, applying $T^*$ and $T$, $j=1,2,\ldots,n$, to $u \in H(m)$ recursively we get a polynomial (in $z$ and $z^{-1}$ up to powers $n$) multiplying $u(z)$ and additional terms involving $\phi_0,\ldots,\phi_{n-1}$ and $\phi_{-1},\ldots,\phi_{-n}$. That is we can show that $A_\rho u$ is of the form

$$(A_\rho u)(z) = \left( b(z^{-1})b(z) - \rho^2 k(z^{-1})k(z) \right) u(z) - \left[ r_1(z) \ r_2(z) \ \ldots \ r_{2n}(z) \right] \phi,$$

where $\phi = [\phi_{-n}^* \ \ldots \ \phi_{-1}^* \ \phi_0^* \ \ldots \ \phi_{n-1}^*]^*$, and $r_j(z)$, $j = 1,\ldots,2n$, are explicitly computable functions, depending on the parameter $\rho$. Since for $\rho$ to be a singular value there has to be a non-zero $u \in H(m)$ satisfying $A_\rho u = 0$, we must have

$$u(z) = \frac{R_\rho(z)\phi}{b(z^{-1})b(z) - \rho^2 k(z^{-1})k(z)} \quad (46)$$

where $R_\rho(z)$ is a $1 \times 2n$ vector of functions and $\phi$ is a $2n \times 1$ constant vector. Since the denominator of (46) vanishes at its $2n$ roots $z_1,\ldots,z_{2n}$, we will see that for $u$ to be a non-zero element of $H(m)$ we must have a non-zero $\phi \in C^{2n}$ such that (assuming $z_j$'s are distinct)

$$R_\rho(z_j)\phi = 0, \quad \forall \ j = 1,\ldots,2n.$$ 

This gives $2n$ equations in $2n$ unknowns, and there is a non-zero solution if and only if the $2n \times 2n$ complex matrix

$$\begin{bmatrix}
R_\rho(z_1) \\
\vdots \\
R_\rho(z_{2n})
\end{bmatrix}$$

is singular. This gives a rank type (or determinantal) formula for $\rho$ to be a singular value of $w_1(T)$. In the discussion below we will derive an explicit formula for $R_\rho(z)$; then carefully studying the resulting $2n$ system of linear equations we will be able to reduce the number of equations to $n$.

We now study the action of each term of $A_\rho$ on $u \in H(m)$. Recall the definitions of Section 2.3, and note that we have

$$b(T)u = b(S)u - m(S)P_+m^*bu.$$ 

Since $b(z)$ is an $n$-th order polynomial and $m^*u$ has an expansion of the form (44) the projection $P_+m^*bu$ is a polynomial of degree $n-1$. An explicit computation shows that

$$P_+m^*bu = V_+(z)B\phi_-$$

where $V_+(z):= [1 \ z \ \ldots \ z^{n-1}]$,

$$B := \begin{bmatrix} b_n & \ldots & b_1 \\ 0 & \ddots & \vdots \\ 0 & 0 & b_n \end{bmatrix}, \quad \text{and} \quad \phi_- := \begin{bmatrix} \phi_{-n} \\ \vdots \\ \phi_{-1} \end{bmatrix}.$$
Similarly,

\[ k(T)u = k(S)u - m(S)(V_+(z)K\phi_-), \quad \text{where} \quad K := \begin{bmatrix} k_n & \cdots & k_1 \\ 0 & \ddots & \vdots \\ 0 & \cdots & 0 & k_n \end{bmatrix}. \]

Since the operator \( T^* \) is the same as \( S^* \), we can rewrite (39) as follows

\[
\begin{aligned}
(b(S)^*b(S) - \rho^2 k(S)^*k(S))u &- b(S)^*(m(z)V_+(z)B\phi_-) \\
&- \rho^2 k(S)^*(m(z)V_+(z)K\phi_-) = 0
\end{aligned}
\]

Recall the action of \( S^* \) from Section 3.3. Then, we can check that

\[
(b(S)^*(m(z)V_+(z))) (z) = V_+(z)m(z)b(z^{-1}) - V_-(z)B^*M,
\]

where, \( B^* \) is the transpose of \( B \),

\[
V_-(z) := [z^{-n} \cdots z^{-1}], \quad M := \begin{bmatrix} m_0 & 0 & 0 \\ \vdots & \ddots & 0 \\ m_{n-1} & \cdots & m_0 \end{bmatrix},
\]

and \( m(z) =: m_0 + m_1 z + m_2 z^2 + \cdots \). Now define a 2n-th order polynomial

\[
\lambda_\rho(z) := z^n(b(z^{-1})b(z) - \rho^2 k(z^{-1})k(z)).
\]

Note that \( \lambda_\rho(z) \) is of the form \( \lambda_\rho(z) = z^n(\lambda_0 + \cdots + \lambda_0 + \ldots + \lambda_n z^n) \) with \( \lambda_i = \lambda_{-i} \). Further define

\[
\Lambda := \begin{bmatrix} \lambda_{-n} & 0 & 0 \\ \vdots & \ddots & 0 \\ \lambda_{-1} & \cdots & \lambda_{-n} \end{bmatrix}, \quad \text{and} \quad \phi_+ := \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \end{bmatrix},
\]

where \( \phi_0, \ldots, \phi_{n-1} \) are as in (45). Then we have

\[
(b(S)^*b(S) - \rho^2 k(S)^*k(S))u = z^{-n}\lambda_\rho(z)u(z) - V_-(z)\Lambda\phi_+.
\]

Finally defining \( L := B^*MB - \rho^2 K^*MK \), and \( \tilde{b}(z) := z^n b(z^{-1}) \), similarly for \( \tilde{k}(z) \), we see that (39) is equivalent to

\[
\lambda_\rho(z)u(z) = R_-(z)\phi_- + R_+(z)\phi_+,
\]

where

\[
R_-(z) := V_+(z)\left(m(z)(\tilde{b}(z)B - \rho^2 \tilde{k}(z)K) - L\right),
\]

\[
R_+(z) := V_+(z)\Lambda.
\]
This is the explicit form of the entries of the matrix \( R_\rho(z) \), which gives 2\( n \) equations for \( u \) to be a non-zero element of \( H(m) \). The equations are derived as follows.

First we make the following assumption for simplicity.

**Assumption 3:** The roots of \( \lambda_\rho(z) \) are all non-zero and distinct.

This assumption holds generically and can be relaxed easily, see Foias et al. (1988). We need another assumption which also holds generically.

**Assumption 4:** If \( \zeta \) is a root of \( \lambda_\rho \) then \( m(\zeta) \neq 0 \).

Let us enumerate the roots of \( \lambda_\rho \) as \( z_1, z_2, \ldots, z_{2n} \) in such a way that first \( r \) of them are inside the closed unit disc \( \bar{D} \), and the rest are outside. Note that by symmetry if \( \zeta \) is a root of \( \lambda_\rho \) then so is \( 1/\zeta \). Therefore we can order the roots in such a way that \( z_{n+i} = 1/z_i, \ i = 1, 2, \ldots, n. \) Then, since \( u \in H(m) \), it has to be analytic in the closed unit disc \( \bar{D} \), so we must have

\[
R_-(z_i)\phi_- + R_+(z_i)\phi_+ = 0 \quad i = 1, \ldots, r. \tag{48}
\]

Also, since \( m^*u \) is analytic outside \( \bar{D} \) we must satisfy

\[
R_-(z_i)\phi_- + R_+(z_i)\phi_+ = 0 \quad i = r + 1, \ldots 2n. \tag{49}
\]

Assumption 4 is used in (49). The following result, which gives the 2\( n \)-equation type of determinantal formula, was obtained by Foias et al. (1988).

**Theorem 4.5** (Foias et al., 1988): Suppose that the Assumptions 3 and 4 hold and let \( \|w_1(T)\|_e < \rho < \|w_1\|_\infty \). Then \( \rho \) is a singular value of \( w_1(T) \) if and only if there exists a non-zero \( \begin{bmatrix} \phi_- \\ \phi_+ \end{bmatrix} \in \mathbb{C}^{2n} \) which satisfies 2\( n \) equations given by (48) and (49):

\[
\begin{bmatrix}
R_-(z_1) & R_+(z_1) \\
\vdots & \vdots \\
R_-(z_{2n}) & R_+(z_{2n})
\end{bmatrix}
\begin{bmatrix}
\phi_- \\
\phi_+
\end{bmatrix} = 0. \tag{50}
\]

**Proof:** (See also Foias and Frazho (1990), Foias et al. (1988), Foias and Tannebaum (1989), Özbay et al. (1990).) From the arguments above the necessity of (50) is obvious. For sufficiency part suppose that there exists a non-zero \( \hat{\phi} = \begin{bmatrix} \hat{\phi}_- \\ \hat{\phi}_+ \end{bmatrix} \in \mathbb{C}^{2n} \), such that

\[
\begin{bmatrix}
R_-(z_1) & R_+(z_1) \\
\vdots & \vdots \\
R_-(z_{2n}) & R_+(z_{2n})
\end{bmatrix}
\hat{\phi} = 0.
\]

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Then it is possible to find a non-zero $\hat{u} \in H(m)$ from equation (47):

$$\hat{u}(z) := \frac{R_-(z) \hat{\phi}_- + R_+(z) \hat{\phi}_+}{\lambda_\nu(z)},$$

(by (50) $\hat{u}$ is in $H(m)$). We now want to show that the vector $\phi$ defined from $\hat{u}$ is precisely $\hat{\phi}$, so that the singular value singular vector equation is consistent. That is the vector $\phi$ for $\hat{u}$, which is given by the coefficients of

$$\hat{u}(z) = \phi_0 + \phi_1 z^1 + \phi_2 z^2 + \cdots,$$

$$(m^* \hat{u})(z) = \phi_- z^{-1} + \phi_- z^{-2} + \cdots,$$

must be the same as $\hat{\phi}$. Note that $\hat{u}$ satisfies

$$\hat{u}(z) = \frac{b(S)^* (m(z) V_+(z)) B \hat{\phi}_- - \rho^2 k(S)^* (m(z) V_+(z)) K \hat{\phi}_- + V_-(z) \Lambda \hat{\phi}_+}{b(z^{-1}) b(z) - \rho^2 k(z^{-1}) k(z)}. \tag{51}$$

Multiplying $\hat{u}(z)$ by the denominator of the right hand side of the above equation, and taking the orthogonal projection onto $L^2(T) \ominus H^2(D)$ we obtain

$$V_-(z) \Lambda \hat{\phi}_+ = V_-(z) \Lambda \hat{\phi}_+.$$

Since the entries of $V_-(z)$ span an $n-$dimensional space and $\Lambda$ is nonsingular (because Assumption 3 implies that $\lambda_\nu(0) \neq 0$, and this means $\lambda_- \neq 0$ hence $\Lambda$ is invertible) we have that $\hat{\phi}_+ = \phi_+$. Also multiplying $\hat{u}$ by the denominator of (51) and $m^*$, and then taking the orthogonal projection on $H^2(D)$ we get

$$b(S)^* m(S)^* b(S) \hat{u} - \rho^2 k(S)^* m(S)^* k(S) \hat{u} = b(S)^* (V_+(z) B \hat{\phi}_-) - \rho^2 k(S)^* (V_+(z) K \hat{\phi}_-), \tag{52}$$

where we have used the properties

$$P_+ m^* b(z^{-1}) b(z) = m(S)^* b(S)^* b(S) = b(S)^* m(S)^* b(S).$$

Note also that

$$m(S)^* b(S) \hat{u} = P_+ m^* b \hat{u} = V_+(z) B \hat{\phi}_-.$$

Hence we conclude from (52) that

$$b(S)^* (V_+(z) B \hat{\phi}_-) - \rho^2 k(S)^* (V_+(z) K \hat{\phi}_-) = b(S)^* (V_+(z) B \hat{\phi}_-) - \rho^2 k(S)^* (V_+(z) K \hat{\phi}_-).$$

This can be re-written as

$$V_+(z) (\hat{B} B - \rho^2 \hat{K} K) (\phi_+ - \hat{\phi}_-) = 0,$$
where
\[ \hat{B} = \begin{bmatrix} b_0 & \cdots & b_{(n-1)} \\ 0 & \ddots & \vdots \\ 0 & 0 & b_0 \end{bmatrix}, \quad \text{and} \quad \hat{K} = \begin{bmatrix} k_0 & \cdots & k_{(n-1)} \\ 0 & \ddots & \vdots \\ 0 & 0 & k_0 \end{bmatrix}. \]

Note that the entries of \( V_+(z) \) span an \( n \)-dimensional space and \( \hat{B}B - \rho^2 \hat{K}K \) is an upper triangular matrix whose diagonal entries are \( b_0k_n - \rho^2 k_0k_n \) which is equal to \( \lambda_n \) (non-zero by Assumption 3). Thus we conclude that \( \phi_- = \hat{\phi}_- \), so \( \hat{u} \in H(m) \) defined above is a singular vector for the singular value \( \rho \).

Now we will show that \( 2n \) equations (50) can be reduced to \( n \) equations. First define \( F(z) := m(z)(\hat{b}(z)B - \rho^2 \hat{K}(z)K) \). Then (50) can be expressed as:
\[
\begin{bmatrix}
V_+(z_1)(F(z_1) - \mathcal{L}) & V_+(z_1) \Lambda \\
\vdots & \vdots \\
V_+(z_{2n})(F(z_{2n}) - \mathcal{L}) & V_+(z_{2n}) \Lambda
\end{bmatrix}
\begin{bmatrix}
\phi_- \\
\phi_+
\end{bmatrix} = 0. \tag{53}
\]

Introducing the Vandermode matrices
\[
\mathcal{V}_+ := \begin{bmatrix} V_+(z_1) \\ \vdots \\ V_+(z_n) \end{bmatrix}, \quad \mathcal{V}_- := \begin{bmatrix} V_+(z_1^{-1}) \\ \vdots \\ V_+(z_n^{-1}) \end{bmatrix}
\]

and defining
\[
\mathcal{F}_+ := \begin{bmatrix} V_+(z_1)F(z_1) \\ \vdots \\ V_+(z_n)F(z_n) \end{bmatrix}, \quad \mathcal{F}_- := \begin{bmatrix} V_+(z_1^{-1})F(z_1^{-1}) \\ \vdots \\ V_+(z_n^{-1})F(z_n^{-1}) \end{bmatrix}
\]

with \( \phi_+ := \mathcal{V}_+ \Lambda \phi_+ \), (53) becomes
\[
\begin{bmatrix}
E_{11} & I \\
E_{21} & \mathcal{V}_- \mathcal{V}_+^{-1}
\end{bmatrix}
\begin{bmatrix}
\phi_- \\
\phi_+
\end{bmatrix} = 0 \tag{54}
\]

where \( E_{11} := \mathcal{F}_+ - \mathcal{V}_+ \mathcal{L} \), and \( E_{21} := \mathcal{F}_- - \mathcal{V}_- \mathcal{L} \). Now eliminating \( \phi_+ = -E_{11}^{-1}\phi_- \) from the first \( n \) equations of (54) the second set of \( n \) equations in (54) becomes
\[
\left( E_{21} - \mathcal{V}_- \mathcal{V}_+^{-1}E_{11} \right) \phi_- = 0. \tag{55}
\]

We can further simplify (55) to obtain
\[
\left( \mathcal{V}_-^{-1} \mathcal{F}_- - \mathcal{V}_+^{-1} \mathcal{F}_+ \right) \phi_- = 0. \tag{56}
\]

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The above formula appears in Özbay (1990), and it can be summarized as follows:

**Theorem 4.6** (Özbay, 1990): Under the assumptions of Theorem 4.5, $\gamma_{opt}$ is the largest value of $\rho$ for which there exists a non-zero $\phi_- \in \mathbb{C}^n$ such that

$$R_\rho \phi_- = 0,$$

where

$$R_\rho := \left( V_-^{-1} F_- - V_+^{-1} F_+ \right).$$

**Proof.** The result follows from Theorem 4.5 (Foias et al., 1988) and the fact that the matrices $V_+$ and $\Lambda$ (used in the transformation from $\phi_+$ to $\phi'_+$) are invertible. Invertibility of the Vandermonde matrix $V_+$ is obvious from Assumption 3. Again by the same assumption $\Lambda$ is invertible. □

Note that all the singular values (not just the norm) of $w_1(T)$ (and of $\Gamma_{m \cdot w_1}$) are given by the values of $\rho$ which makes $R_\rho$ singular. In order to construct $R_\rho$ we first need to find the roots of $\lambda_\rho(z) = 0$. But when $n$ is large it is not possible to compute the roots as explicit functions of $\rho$. On the other hand we can compute these roots and hence $R_\rho$ numerically for each fixed value of $\rho$. Therefore, we can search for $\gamma_{opt}$ by decreasing $\rho$ from an upper bound e.g. $\|w_1\|_\infty$. At each step we check whether the matrix $R_\rho$ is singular (or “close” to being singular). This can be done by computing the smallest singular value $\sigma_{\min}(R_\rho)$. Then the zeros in the plot of $\sigma_{\min}(R_\rho)$ versus $\rho$ indicate the location of the singular values of $w_1(T)$, largest of which is the norm, i.e. $\gamma_{opt}$. We will illustrate this point via an example in Section 5.

Another interesting point to remark here is that $z_i$ is a zero of $\lambda_\rho(z)$ if and only if it is a pole of $(\rho^2 - w_1(z^{-1})w_1(z))^{-1}$. Therefore, $z_i = \frac{s_i}{s_i + 1}$ where $s_i$ is a pole of the transfer function (defined in the $s$-plane)

$$(\rho^2 - W_1(-s)W_1(s))^{-1}, \quad \text{where} \quad W_1(s) = \frac{s - 1}{s + 1}.$$

Let $[A, B, C, d]$ be a minimal realization of the function $W_1(s)$, i.e. $W_1(s) = d + C(sI - A)^{-1}B$, where the dimension of $A$ is $n \times n$. Then, one can easily prove that $s_i$'s are given by the eigenvalues of the Hamiltonian matrix

$$H_\rho = \begin{bmatrix} (A + \frac{BdC}{\rho^2}) & -\frac{BB^*}{\rho^2} \\ C^*(I + \frac{d^2}{\rho^2})C & -(A + \frac{BdC}{\rho^2})^* \end{bmatrix}.$$

This formula for the roots of $\lambda_\rho(z)$ suggests that there may be interesting connections between the set of $n$-equations given above and the same number of equations obtained in Lypchuk et al. (1988), Smith (1989a), Zhou and Khargonekar (1987). In each of these
references the set of \( n \)-equations are obtained in terms of certain Hamiltonian matrices constructed from the state space realizations of \( W_1(s) \), like \( H_p \).

After obtaining the optimal performance level \( \gamma_{opt} \) from the plot of \( \sigma_{min}(R_\rho) \) versus \( \rho \) we can obtain the optimal controller from a non-zero \( \phi_{-}^{opt} \) satisfying

\[
R_{\gamma_{opt}} \phi_{-}^{opt} = 0.
\]

Note that from equation (54) \( \phi_{-}^{opt} \) gives \( \Lambda \phi_{+}^{opt} = -(V_+ F_+ - L) \phi_{-}^{opt} \). These define the optimal singular vector from (47) as

\[
u_{opt}(z) = \frac{V_+(z)((F(z) - L) \phi_{-}^{opt} + \Lambda \phi_{+}^{opt})}{\lambda_{\gamma_{opt}}(z)}.
\]

Then this gives \( s_{opt}(z) = w_1(z) - m(z) q_{1}^{opt}(z) \) from Theorem 4.4:

\[
s_{opt}(z) = \frac{(b(T)u_{opt}(z))}{(k(T)u_{opt}(z))} = \frac{b(z)u_{opt}(z) - m(z)V_+(z)\beta \phi_{-}^{opt}}{k(z)u_{opt}(z) - m(z)V_+(z)\kappa \phi_{-}^{opt}}.
\]

Then solving for \( q_{1}^{opt} \) we obtain \( c_{opt} \) from (36). These computations lead to the following simplified formula.

**Theorem 4.7**: (Özbay, 1990) Under the assumptions of Theorem 4.5 the optimal controller for \( p(z) = m(z)n_1(z) \) is given by

\[
c_{opt}(z) = \left(\frac{w_1(z)w_1(z)^{-1}}{\gamma_{opt}^2} - 1\right) \frac{g_{opt}(z)}{1 + m(z) g_{opt}(z)} n_1^{-1}(z),
\]

where

\[
g_{opt}(z) := \gamma_{opt}^2 \frac{\bar{k}(z)}{b(z)} \frac{V_+(z)(b(z)\kappa - k(z)\beta) \phi_{-}^{opt}}{V_+(z)V_+^{-1} F_+ \phi_{-}^{opt}},
\]

and non-zero \( \phi_{-}^{opt} \) satisfies \( R_{\gamma_{opt}} \phi_{-}^{opt} = 0 \). □

The structure of the controller given by (58) was first observed in Özbay and Tannenbaum (1991), and the general form for \( g_{opt}(z) \) was obtained in Özbay (1990).

With Theorems 4.6 and 4.7 we have an explicit solution to the 1-block \( H^\infty \) control problem for stable distributed plants. To recap, we have seen that

1. \( \gamma_{opt} \) is the norm of the operator \( w_1(T) \),

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2. singular value singular vector equation implies that there exists a singular vector 
\( u \in H(m) \) for a singular value candidate \( \rho \) if and only if 
\[
A_\rho u = 0,
\]

3. the equation \( A_\rho u = 0 \) is equivalent to 
\[
\lambda_\rho(z)u(z) = R_\rho(z)\phi,
\]
where \( \phi \) is a constant in \( C^{2n} \), \( \lambda_\rho(z) \) is a polynomial of degree \( 2n \) and the entries of \( 1 \times 2n \) vector \( R_\rho(z) \) can be explicitly computed,

4. from 3, we have \( 2n \) linear equations for the existence of a non-zero \( u \in H(m) \)
\[
R_\rho(z_i)\phi = 0, \quad \text{where } 0 \neq \phi \in C^{2n}, \quad \lambda(z_i) = 0, \quad i = 1, 2, \ldots, 2n,
\]

5. the largest value of \( \rho \) for which there is such a non-zero \( \phi \) gives \( \gamma_{\text{opt}} \), and finally

6. the optimal controller can be obtained from a non-zero \( \phi_{\text{opt}} \) satisfying the above \( 2n \)-equations for \( \rho = \gamma_{\text{opt}} \).

Essentially the same outline applies for the 2-block \( H^\infty \) control of a class of distributed plants with finitely many unstable modes. This is the subject of the next section.

### 4.2 Two-block \( H^\infty \) control of unstable distributed systems

Let us recall the two-block \( H^\infty \) control problem defined in Section 3.

\[
\gamma_{\text{opt}} = \inf_{c \text{ stabilizing } p} \left\| \begin{bmatrix} w_1(1 + pc)^{-1} \\
 w_2pc(1 + pc)^{-1} \end{bmatrix} \right\|_\infty,
\]

where the weight \( w_1 \) comes from the sensitivity reduction condition, and \( w_2 = w_m \) comes from the multiplicative uncertainty bound. If we apply the stabilization technique of Section 3.1 to the plant \( p = n/d \), where \( n, d \in H^\infty \), then controller has to be of the form

\[
c = \frac{x + dq}{y - nq},
\]

with \( x, y \in H^\infty \) satisfying
\[
x(z)n(z) + y(z)d(z) = 1.
\]
Recall that \( d(z) = m_d(z) \) is a rational inner function, i.e. a finite Blaschke product of the form

\[
m_d(z) = \prod_{k=1}^{\ell} \left( \frac{z - a_k}{1 - \overline{a}_k z} \right) \quad |a_k| < 1, \quad \forall \ k = 1, 2, \ldots, \ell,
\]

where we have assumed (without loss of generality) that \( P(s) \) does not have a pole at \( s = 1 \), (i.e. \( p(z) \) does not have a pole at \( z = 0 \)), and the poles are distinct. Recall also from Section 3 that in \((61)\) \( x(z) \) can be chosen as a rational function.

With this remark and Assumptions 1 and 2 we see that the two-block problem \((60)\) is equivalent to

\[
\gamma_{opt} = \inf_{q \in H^\infty(D)} \left\| \begin{bmatrix} w_1 \\ 0 \end{bmatrix} - \begin{bmatrix} w_1 n_2 \\ -w_2 n_2 \end{bmatrix} m_n n_1 (x + m_d q) \right\|_\infty
\]

\[
= \inf_{q \in H^\infty(D)} \left\| \begin{bmatrix} w_1 \\ 0 \end{bmatrix} - \begin{bmatrix} w_1 n_2 \\ -w_2 n_2 \end{bmatrix} m_n (x n_1 + m_d q_1) \right\|_\infty,
\]

where \( q_1 = n_1 q \) or \( q = q_1/n_1 \). Note that \( q \in H^\infty(D) \) if and only if \( q_1 \in H^\infty(D) \).

Recall that rationality of \( m_d \) implied existence of a rational \( x \in H^\infty(D) \). Similarly we can show that, since \( m_d \) is rational, there exists a rational function \( r \in H^\infty(D) \) such that

\[
g_1(z) := \frac{r(z) - x(z)n_1(z)}{m_d(z)} \in H^\infty(D).
\]

Then, defining \( q_2 = q_1 - g_1 \) (there is an invertible relationship between \( q_1 \) and \( q_2 \), hence between \( q \) and \( q_2 \)) we have

\[
\gamma_{opt} = \inf_{q \in H^\infty(D)} \left\| \begin{bmatrix} w_1 \\ 0 \end{bmatrix} - \begin{bmatrix} w_1 n_2 \\ -w_2 n_2 \end{bmatrix} m_n (r + m_d q_2) \right\|_\infty.
\]

(62)

We remark that the only infinite dimensional part in \((62)\) is the inner function \( m_n \). Also recall from Assumption 2 that \( w_3 := w_2 n_2 \) is in \( H^\infty(D) \) and so is \( w_3^{-1} \). The problem \((62)\) can further be reduced as follows. First we perform a spectral factorization:

\[
|w_1(e^{j\theta}) n_2(e^{j\theta})|^2 + |w_3(e^{j\theta})|^2 = |g(e^{j\theta})|^2
\]

(63)

where \( g(z) \) is a rational function such that \( g, g^{-1} \in H^\infty(D) \). Such function \( g \) exists because \( w_3^{-1} \in H^\infty(D) \), and \( g \) can be obtained from \( w_1, n_2 \) and \( w_2 \), see e.g. Francis (1987) pp. 90–94, where the algorithm of Bart et al. (1979) is presented. By using \( g \) we can define a \( 2 \times 2 \) unitary matrix

\[
L(e^{j\theta}) = \begin{bmatrix}
\frac{w_1(e^{j\theta}) n_2(e^{j\theta})}{g(e^{j\theta})} & \frac{w_3(e^{j\theta})}{g(e^{j\theta})} \\
\frac{-w_2(e^{j\theta})}{g(e^{j\theta})} & \frac{w_1(e^{j\theta}) n_2(e^{j\theta})}{g(e^{j\theta})}
\end{bmatrix}.
\]

(64)

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It is important to note that for any vector $\begin{bmatrix} a \\ b \end{bmatrix}$ whose entries $a, b$ are in $L^\infty(T)$ we have

$$\|L^* \begin{bmatrix} a \\ b \end{bmatrix}\|_\infty = \| \begin{bmatrix} a \\ b \end{bmatrix}\|_\infty.$$  \hfill (65)

Therefore multiplying the expression inside $\| \cdots \|_\infty$, on the right hand side of (62), by $L(e^{j\theta})^*$ we obtain

$$\gamma_{opt} = \inf_{q_2 \in H^\infty(D)} \left\| \begin{bmatrix} \frac{w_1 w_2^* n_2^*}{g} \\ -\frac{w_1 w_3}{g} \end{bmatrix} - \begin{bmatrix} gm_n \\ 0 \end{bmatrix} (r + m_d q_2) \right\|_\infty. \hfill (66)$$

In (66) the second block,

$$g_0 := -w_1 w_3 / g,$$

is independent of the free parameter $q_2$. Also, since $w_3^{-1} \in H^\infty(D)$, we have that $g_0, g_0^{-1} \in H^\infty(D)$. In order to express (66) in terms of $H^\infty(D)$ functions we find a rational inner function $m_w(z)$ such that the rational function

$$w_0 := m_w \frac{w_1 w_2^* n_2^*}{g^*} \text{ is in } H^\infty(D).$$

Since $w_1, n_2, g$ are rational we can always find such $m_w(z)$. In fact one can construct $m_w(z)$ by choosing its zeros as $1/\overline{p_i}$ and $1/\overline{z_i}$: the poles $p_i$ of $w_1 n_2$, and the zeros $z_i$ of $g$, reflected around the unit circle (note that $|p_i| > 1 < |z_i|$ and hence $|1/\overline{p_i}| < 1 > |1/\overline{z_i}|$).

Then, multiplying the expression inside $\| \cdots \|_\infty$, on right hand side of (66), by the unitary matrix $\begin{bmatrix} m_w & 0 \\ 0 & 1 \end{bmatrix}$ we obtain

$$\gamma_{opt} = \inf_{q_3 \in H^\infty(D)} \left\| \begin{bmatrix} w_0 - m_1 \hat{w}_0 - m_1 m_2 q_3 \\ g_0 \end{bmatrix} \right\|_\infty, \hfill (67)$$

where

$$g_0 := -w_1 w_3 / g,$$

$$w_0 := m_w w_1 w_2^* n_2^* / g^*,$$

$$\hat{w}_0 := g r,$$

$$m_1 := m_n m_w,$$

$$m_2 := m_d,$$

$$q_3 := g q_2 \text{ or } q_2 = q_3 / g,$$

with $w_3 := (w_2 n_2)$. Note once more that $w_0, g_0, \hat{w}_0, m_2$ are rational functions in $H^\infty(D)$, with $m_2$ inner, and $m_1$ is arbitrary inner. Moreover, when the plant is stable we can choose $m_d = 1, y = 1, x = 0$ in (61) and in this case we have $\hat{w}_0 = 0.$

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Set \( m_0 := m_1 m_2 = m_w m_r m_d \). Then, the commutant lifting theorem gives the following result.

**Theorem 4.8:** Let us define the operator \( A : H^2(D) \rightarrow H(m_0) \oplus H^2(D) \) by

\[
A := \left[ \frac{P_{H(m_0)}(w_0(S) - m_1(S)\tilde{w}_0(S))}{g_0(S)} \right].
\]

Then,

\[
\gamma_{opt} = \| A \|. \tag{68}
\]

**Proof:** For notational convenience we introduce \( w := w_0 - m_1 \tilde{w}_0 \). Observe that for any \( q_3 \in H^\infty(D) \) we have

\[
\left\| \begin{bmatrix} w - m_0 q_3 \\ g_0 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} P_+(w(S) - m_0(S)q_3(S)) \\ g_0(S) \end{bmatrix} \right\| \\
\geq \left\| \begin{bmatrix} P_{H(m_0)}(w(S) - m_0(S)q_3(S)) \\ g_0(S) \end{bmatrix} \right\| \\
\geq \left\| \begin{bmatrix} P_{H(m_0)}w(S) \\ g_0(S) \end{bmatrix} \right\| = \| A \|.
\]

Therefore, \( \gamma_{opt} \geq \| A \| \). In order to complete the proof we need to show the existence of \( q_3^{opt} \in H^\infty(D) \) such that

\[
\left\| \begin{bmatrix} w - m_0 q_3^{opt} \\ g_0 \end{bmatrix} \right\|_\infty = \| A \|,
\]

so that we can conclude \( \gamma_{opt} \leq \| A \| \). The key observation for this step is

\[
AS = \left[ \begin{bmatrix} TP_{H(m_0)}w(S) \\ g_0(S) \end{bmatrix} \right],
\]

where \( T \) is the compression of \( S \) on \( H(m_0) \). Then the result follows from the commutant lifting theorem. The details can be found in Foias and Tannenbaum (1988). \( \Box \)

With Theorem 4.8 the two-block problem is reduced to a norm computation problem. The optimal performance level \( \gamma_{opt} \) is the norm of \( A \). Moreover, assuming the norm is achieved at a singular value, the optimal controller can be computed from a non-zero singular vector \( x_o \in H^2(D) \) satisfying

\[
A^* A x_o = \gamma_{opt}^2 x_o. \tag{69}
\]

Therefore, we need to understand the "action of" \( A^* A \) on an element of \( H^2(D) \), in order to derive necessary and sufficient conditions for the existence of a candidate singular vector \( x_o \) associated with the singular value \( \gamma_{opt} \).
One can check that \( \mathbf{A}^* \mathbf{A} \) can be expressed in terms of Hankel and Toeplitz operators:
\[ \mathbf{A}^* \mathbf{A} = \Gamma_{m_0}^* \Gamma_{m_0} + \mathbf{Y}_{g_0}^* \mathbf{Y}_{g_0}. \]
This type of "Hankel + Toeplitz" operators for several different special cases of \( m_0, w \) and \( g_0 \) have been studied to obtain solutions to two block \( H^\infty \) control problems for several different special types of plants, see e.g. Flamm and Yang (1990), Verma and Jonckheere (1984), Zames and Mitter (1988) and references therein. We will follow the same idea, but rather than computing the action of the Hankel + Toeplitz operator directly, we will study the action of a "skew Toeplitz" operator obtained from (69). This approach will neither simplify nor complicate the final formulae for \( \gamma_{opt} \) and \( x_o \), (interested reader may want to compare the formulae in Foias et al. (1988) and Zames and Mitter (1988) to get an idea).

We can solve the two block problem (60) by reducing (69) to an equation of the type (38), (see also Verma and Jonckheere (1984), where the two block problem is reduced a broadband matching problem of Helton (1981)). First step in this procedure is to write the singular value singular vector equation (69): \( \rho^2 \) is an eigenvalue, with finite multiplicity, of \( \mathbf{A}^* \mathbf{A} \) if and only if there exists a non-zero \( x \in H^2(D) \) such that
\[
(\rho^2 \mathbf{I} - \mathbf{A}^* \mathbf{A}) x = 0
\]
which is equivalent to
\[
(\rho^2 \mathbf{I} - g_0(\mathbf{S})^* g_0(\mathbf{S})) x = w(\mathbf{S})^* \mathbf{P}_{H(m_0)} w(\mathbf{S}) x. \tag{70}
\]
It is easy to see from (67) that \( \alpha := \|g_0\|_\infty \) is a lower bound for \( \gamma_{opt} \). Since we are interested in the largest singular value for \( \mathbf{A} \), we will assume that \( \rho > \alpha \). Then, there exists a rational function (which depends on \( \rho \)) \( f_\rho \in H^\infty(D) \) with \( f_\rho^{-1} \in H^\infty(D) \) such that
\[
f_\rho(\mathbf{S})^* f_\rho(\mathbf{S}) = \rho^2 \mathbf{I} - g_0(\mathbf{S})^* g_0(\mathbf{S}).
\]
In fact \( f_\rho \) can be computed from rational spectral factorization techniques, see e.g. Francis (1987). Now defining \( y = f_\rho(\mathbf{S}) x \), i.e. \( x = f_\rho(\mathbf{S})^{-1} y \), the equation (70) becomes
\[
y = w_\rho(\mathbf{S})^* \mathbf{P}_{H(m_0)} w_\rho(\mathbf{S}) y, \tag{71}
\]
where \( w_\rho := f_\rho^{-1} w = w_{0,\rho} - m_1 \hat{w}_{0,\rho} \), with \( w_{0,\rho} := f_\rho^{-1} w_0 \) and \( \hat{w}_{0,\rho} := f_\rho^{-1} \hat{w}_0 \). Let \( \mathbf{T}_0 \) be the compression of the shift operator, defined on \( H(m_0) \). Since \( \mathbf{T}_0^* = \mathbf{S}^* \) on \( H(m_0) \), the right hand side of (71) is in \( H(m_0) \), this implies that \( y \) has to be in \( H(m_0) \), and hence (71) is equivalent to
\[
(1 \cdot \mathbf{I} - w_\rho(\mathbf{T}_0)^* w_\rho(\mathbf{T}_0)) y = 0. \tag{72}
\]
In other words \( 1 \) has to be a singular value of \( w_\rho(\mathbf{T}_0) \).

**Stable plants case:** Recall that when the plant is stable we have \( \hat{w}_0 = 0 \). Therefore, in this case \( w_\rho = w_{0,\rho} \) which is rational. Hence there exist polynomials \( b_\rho(z) \) and \( k_\rho(z) \)
with $k_{p}^{-1}(z) \in H^{\infty}(D)$ such that $w_{\rho} = b_{\rho}(z)/k_{\rho}(z)$. By defining $y' = k_{p}^{-1}(T_{0})y$ we see that (72) is equivalent to

$$(k_{p}(T_{0})^{*}k_{p}(T_{0}) - b_{p}(T_{0})^{*}b_{p}(T_{0}))y' = 0. \tag{73}$$

Note that the left hand side of (73) is a skew Toeplitz operator acting on an element $y'$ of $H(m_{0})$. In Section 4.1 we have solved a problem, namely (39), which is the same as (73). Thus, the results of Section 4.1 can be used in order to solve the two block problem for stable plants. □

In more general cases where the plant satisfies Assumption 1, with $m_{d}$ being a finite Blaschke product, we have $\hat{w}_{0,\rho} \neq 0$, and hence $w_{\rho}$ is not a rational function, so we cannot directly apply the results of Section 4.1. Nevertheless, $w_{\rho}$ is not arbitrary, it has a special structure:

$$w_{\rho} = w_{0,\rho} - m_{1}\hat{w}_{0,\rho}$$

with $w_{0,\rho}$ and $\hat{w}_{0,\rho}$ rational functions in $H^{\infty}(D)$, and $m_{1}$ related to $m_{0}$ by: $m_{0} = m_{1}m_{2}$, where $m_{2}$ is a finite Blaschke product. So, there exist polynomials $b_{\rho}(z)$, $c_{\rho}(z)$ and $k_{\rho}(z)$ such that $w_{0,\rho} = b_{\rho}/k_{\rho}$ and $\hat{w}_{0,\rho} = c_{\rho}/k_{\rho}$. In this case defining $y' = k_{p}^{-1}(T_{0})y$ we see that (72) is equivalent to

$$
\left(k_{p}(T_{0})^{*}k_{p}(T_{0}) - b_{p}(T_{0})^{*}b_{p}(T_{0})\right)y' \\
= \left(-\left(m_{1}(T_{0})^{*}c_{\rho}(T_{0})^{*}(b_{p}(T_{0}) - m_{1}(T_{0})c_{\rho}(T_{0})) - b_{p}(T_{0})^{*}c_{\rho}(T_{0})m_{1}(T_{0})\right)\right)y'. \tag{74}
$$

The left hand side of (74) is the same as the left hand side of the equation (73): a skew Toeplitz operator acting on an element of $H(m_{0})$. From Section 4.1 we know how this operator acts on $y'$. As for the right hand side of (74), again similarly to the formulae given in Section 4.1 we can explicitly write down the action of $b_{p}(T_{0})$, $b_{p}(T_{0})^{*}$, $c_{\rho}(T_{0})$ and $b_{p}(T_{0})^{*}$. The only term we need to understand is $m_{1}(T_{0})$ and $m_{1}(T_{0})^{*}$. Because of the special relation $m_{0} = m_{1}m_{2}$, where $m_{2}$ is a finite Blaschke product, both of these operators are of finite rank. We can explicitly compute their action on elements of $H(m_{0})$, in terms of finitely many unknown coefficients (similar to unknowns in $\phi$ of Section 4.1). Therefore, (74) leads to an equation of the form (46), and similarly to Section 4.1 this equation leads to a finite set of equations, (necessary and sufficient conditions for 1 to be a singular vector of $w_{\rho}(T_{0})$). The number of equations to be solved increases by $2\ell$ in this case, where $\ell$ is the dimension of $m_{2} = m_{d}$, i.e. the number of RHP poles of the plant. The details of the above arguments and the final formulae for the optimal performance and the optimal controller can be found in Özbaý et al. (1990) and Özbaý et al. (1992).
5 A numerical example

This section contains an example which illustrates the numerical computation of $\gamma_{opt}$ from the formula obtained in Theorem 4.6. The example deals with a stable plant with one non-minimum phase zero and a delay element. The weight is a second order low-pass filter.

Recall from Sections 4.1 and 4.2 that the generic part of the one and two block $H^\infty$ control problems with stable plants amounts to finding

$$\gamma_{opt} = \inf_{q \in H^\infty(D)} \|w - mq\|_\infty,$$  \hspace{1cm} (75)

where $w$ is rational in $H^\infty$ and $m$ is arbitrary inner.

Consider the following example

$$M(s) = e^{-hs} \frac{s - 0.1}{s + 0.1} \quad \text{or} \quad m(z) = e^{h \frac{z + 1}{z - 1}} \frac{z + 9/11}{1 + 9z/11}.$$  \hspace{1cm} (76)

and

$$W(s) = \frac{0.4(s + 0.5)(s + 100)}{(s + 2)(s + 10)} \quad \text{or} \quad w(z) = \frac{b(z)}{k(z)} = \frac{60.6 - 39.2z - 19.8z^2}{33 - 38z + 9z^2}.$$  \hspace{1cm} (77)

The magnitude plot for the weight is given in Figure 2, which shows that the magnitude of the weight is relatively large in the frequency range between 1rd/sec and 10rd/sec.

When $h > 0$ the essential norm is $\|w(T)\|_e = w(1) = 0.4$, because $z = 1$ is the only essential singularity for $m(z)$. One can find another lower bound for $\gamma_{opt}$ from (35) as follows: Recall (75) and note that $m(a) = 0$ at $a = -9/11$, so $\gamma_{opt} \geq |w_1(-9/11)| = 1.1327$, which is larger than the essential norm 0.4, in this special case. An upper bound for $\gamma_{opt}$ is $\|w\|_\infty = 3.3578$, which can be seen from Figure 2. The matrix $\mathcal{R}_\rho$ is constructed for the values of $\rho$ between 0.4 (the essential norm) and 3.3578, and then its minimum singular value is plotted. For example when $h = 0.1$ (respectively $h = 0.3$) this plot is as shown in Figure 3, where $\gamma_{opt} = 1.7962$ (respectively Figure 4, $\gamma_{opt} = 2.6473$).
Figure 2: $|W(j\omega)|$ versus $\log(\omega)$

Figure 3: $\sigma_{\min}(R_{\rho})$ versus $\rho$: $h = 0.1$

Figure 4: $\sigma_{\min}(R_{\rho})$ versus $\rho$: $h = 0.3$
There are three interesting points to observe from these plots. First of all we can see the accumulation of singular values near the essential norm 0.4 (if we take a finer look at these plots near $\rho = 0.4$ we can see these accumulations more clearly, see e.g. Özbay (1990). This illustrates infinite dimensionality of the problem. Secondly, we see that as $h$ increases $\gamma_{\text{opt}}$ increases, (i.e. the best achievable performance gets worse). This is a measure of the effect of time delays on system performance. Also from Fig.3 we see clearly that the function $\sigma_{\text{min}}(R_{\rho})$ dips down to zero at $\rho = 1.1321$, which is the lower bound of $\gamma_{\text{opt}}$ coming from the zero of $M(s)$ in the right half plane. It is interesting to see that whenever this zero is within a frequency band where $|W|$ is large, the lower bound we obtain becomes large. In such a case $\gamma_{\text{opt}}$ is guaranteed to be relatively large. So, there is a severe limitation in the system performance due to the zero in $RHP$. It is quite well known that time delays and $RHP$ zeros degrade the best achievable performance of the closed loop system. This is verified in our example.

6 Approximations of the optimal $H^\infty$ controllers

As we have seen in Section 3, solutions to robust control problems (under additive or multiplicative uncertainty) related to infinite dimensional unstable systems, with finitely many unstable modes, require controllers with finitely many closed RHP poles. However, there is no a priori guarantee for this to happen, if the optimal controllers are obtained from the parametrization of Theorem 3.3, by solving (67). Because, we don\'t know if the optimal solution $Q_{3,\text{opt}}$ will lead to an optimal controller with finitely many unstable modes. Therefore, one may want to approximate the denominator $(Y - NQ_{\text{opt}})$ of the optimal controller by a rational $H^\infty$ function, so that the resulting controller satisfies this condition. An obvious way to do this is to approximate each term $Y$, $N$, $Q_{\text{opt}}$ separately, and then combine the approximations. On the other hand, even if such an approximation is possible, we must make sure that it will not destabilize the closed loop system, and will yield a performance $\gamma$ close to $\gamma_{\text{opt}}$. This issue will be discussed below.

Also, for practical implementation purposes we may want to obtain a finite dimensional controller to start with. It is obvious that the optimal controllers for infinite dimensional systems will be infinite dimensional. One way to obtain a finite dimensional controller is to reduce the problem data (plant and the weights) to a finite dimensional one, and to find the corresponding optimal solution, which will be finite dimensional, (we should then check the stability and performance of this controller), see e.g. Rodriguez and Dahleh (1990) and Wu (1990). Another method to obtain a finite dimensional controller is to approximate the infinite dimensional parts $Y$, $N$, $Q_{\text{opt}}$ of the optimal controller as in Özbay (1992). Conditions for "uniform approximability" of these terms will be summarized below.

Let us consider the 2-block $H^\infty$ control problem

$$
\gamma_{\text{opt}} = \inf_{C \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1 + PC)^{-1} \\ W_2C(1 + PC)^{-1} \end{bmatrix} \right\|_\infty,
$$

where $W_1 = W_d^{-1}$ is rational, $W_2 = W_a$ satisfies Assumption 2, and $P = N/D$, ($N =$
\( M_nN_1N_2, D = M_d \) is the nominal plant, satisfying Assumption 1. Recall the structure of the optimal controller

\[
C_{opt}(s) = \frac{X(s) + D(s)Q_{opt}(s)}{Y(s) - N(s)Q_{opt}(s)},
\]

where \( X, Y \) satisfy (6), and \( Q_{opt} \in H^\infty \) is the optimal solution of the 2-block \( H^\infty \) problem. As explained earlier, when the plant has finitely many unstable modes we can choose \( D(s) \) and \( X(s) \) as rational functions. Therefore, the infinite dimensional parts of \( C_{opt}(s) \) are \( N(s), Y(s), Q_{opt}(s), \) and a finite dimensional controller can be obtained by simply replacing these irrational functions by their \( k \)th order rational approximates \( N_k(s), Y_k(s), Q_k(s) \)

\[
C_k(s) = \frac{X(s) + D(s)Q_k(s)}{Y_k(s) - N_k(s)Q_k(s)},
\]

(77)

(here by a slight abuse of notation we write \( C_k \) for the finite dimensional controller (77) whose order is larger than \( k \)). We can re-write \( C_k \) as

\[
C_k = \frac{(X + DQ_{opt}) + \Delta^\|_k}{(Y - NQ_{opt}) + \Delta^\|_k},
\]

(78)

where

\[
\Delta^\|_k = D(Q_k - Q_{opt})
\]

(79)

and

\[
\Delta^\_k = (Y_k - Y) - N_k(Q_k - Q_{opt}) - Q_{opt}(N_k - N).
\]

(80)

We know from Georgiou and Smith (1990) that convergence of \( \|\Delta^\|_k\|_\infty \) and \( \|\Delta^\_k\|_\infty \) to zero as \( k \to \infty \) is equivalent to the convergence of the approximate controller \( C_k \) to \( C_{opt} \) in the so called gap metric. We will see that this guarantees that for \( k \) sufficiently large \( C_k \) stabilizes \( P \), and the performance of \( C_k \)

\[
\gamma_k := \left\| \begin{bmatrix} W_1(1 + PC_k)^{-1} \\ W_2C_k(1 + PC_k)^{-1} \end{bmatrix} \right\|_\infty,
\]

is “close” to the optimal performance \( \gamma_{opt} \). Obviously the convergence of \( \|\Delta^\|_k\|_\infty \) and \( \|\Delta^\_k\|_\infty \) to zero depend on whether the infinite dimensional parts \( Y, N, Q_{opt} \) are uniformly approximable (in \( H^\infty \)) or not. If \( N(j\omega) \) is uniformly continuous on the extended imaginary axis \( j\mathbb{R}_+ \) then \( N \) and \( Y \) are uniformly approximable by rational functions in \( H^\infty \). On the other hand, \( Q_{opt} \) depends on \( N, D \) and the weights \( W_1 \) and \( W_2 \). So, uniform continuity of \( Q_{opt}(j\omega) \) depends on the plant and the weights. In Özbay (1992) certain
conditions are given for $Q_{opt}$ to be uniformly approximable by rational functions. The precise statements of these conditions are given below.

**Lemma 6.1** (Özbay, 1992): Consider the 2-block $H^\infty$ mixed sensitivity minimization problem (76), with the plant $P = N/D$, satisfying Assumption 1, and the weights $W_1 = W_d^{-1}$ and $W_2 = W_e$, satisfying Assumption 2. Suppose that $N(j\omega)$ is continuous on $j\mathbb{R}_+$, $N \in H^2$, and $\frac{d}{ds}N \in H^1$. Further assume that either (i) $N_1$ is rational, or (ii) $N_1(j\omega)$ and $M_n(j\omega)N_2(j\omega)$ are continuous on $j\mathbb{R}_+$, and $N_1, (M_nN_2) \in H^2$, with $\frac{d}{ds}N_1, \frac{d}{ds}(M_nN_2) \in H^1$. Then the optimal controller $C_{opt}$ can be uniformly approximated, in the gap metric, by rational functions $C_k$ of the form (78):

$$C_k = \frac{X + DQ_k}{Y_k - N_kQ_k},$$

where the rational function $X \in H^\infty$ is determined from (6), $Y_k, N_k$ and $Q_k$ are rational ($k$th order) approximations of $Y$, $N$ and $Q_{opt}$ respectively, such that

$$\|Y - Y_k\|_\infty \to 0 \quad \text{as} \quad k \to \infty,$$

$$\|N - N_k\|_\infty \to 0 \quad \text{as} \quad k \to \infty,$$

$$\|Q_{opt} - Q_k\|_\infty \to 0 \quad \text{as} \quad k \to \infty.$$

**Proof:** For the case (i), where $N_1$ is rational, the result is given in Özbay (1992), which uses certain facts from Gu et al. (1989) and Power (1982). If $N_1$ is not rational, but (ii) is satisfied, then, by arguments similar to ones used in Lemmas 3.1 and 3.2 of Özbay (1992), we can still show that $Y$, $N$ and $Q_{opt}$ are uniformly approximable by rational functions in $H^\infty$: Uniform approximability of $Y$ and $N$ can be deduced from Theorem 2.12 of Gu et al. (1989) Also by the same theorem we can show that if the optimal solution $Q_{3, opt}$ of (67) is uniformly approximable by rational functions in $H^\infty$ then so is $Q_{opt}$. In order to show uniform approximability of $Q_{3, opt}$ it is sufficient to check (see Power, 1982, pp. 48–53) that $m_{\ast}w_{\ast}^\dagger$ of (71) has an absolutely summable power series expansion; and this also is guaranteed by Theorem 2.12 of Gu et al. (1989), provided $N_1$ and $M_nN_2$ satisfy the conditions stated in the lemma. □

With the above lemma stability of the closed loop system $(C_k, P)$ is established as follows.

**Theorem 6.2** (Özbay, 1992): Assume that the hypotheses of Lemma 6.1 hold. Then there exists a sufficiently large number $K$ such that the closed loop system $(C_k, P)$ is stable for all $k \geq K$.

**Proof:** With Lemma 6.1 we have that $C_k$ converges to $C_{opt}$ in the gap metric. But $C_{opt}$ stabilizes $P$. Therefore, by Georgiou and Smith (1990) $C_k$ stabilizes the plant $P$, if $k$ is sufficiently large. □
Convergence of the performance level $\gamma_k$ is given by the following.

**Theorem 6.3** (Özbay, 1992): Assume that the hypotheses of Lemma 6.1 hold. Then

$$\left\| \left[ \begin{array}{c} W_1(1 + P C_k)^{-1} \\ W_2 C_k(1 + P C_k)^{-1} \end{array} \right] \right\|_\infty =: \gamma_k \rightarrow \gamma_{opt} := \left\| \left[ \begin{array}{c} W_1(1 + P C_{opt})^{-1} \\ W_2 C_{opt}(1 + P C_{opt})^{-1} \end{array} \right] \right\|_\infty,$$

as $k \rightarrow \infty$.

**Proof:** First define $C^n_k := X + D Q_k$, $C^d_k := Y_k - N_k Q_k$, and $S_{opt} := (1 + P C_{opt})^{-1}$. Then, one can re-write $\gamma_k$ as

$$\gamma_k = \left\| \left[ \begin{array}{c} W_1 D C^d_k (D C^d_k + N C^n_k)^{-1} \\ W_2 D C^n_k (D C^d_k + N C^n_k)^{-1} \end{array} \right] \right\|_\infty.$$

It is easy to see that

$$\gamma_k = \left\| \left[ \begin{array}{c} W_1 S_{opt} + W_1 D \Delta^d_k \\ W_2 C_{opt} S_{opt} + W_2 D \Delta^n_k \end{array} \right] \left( \frac{1}{1 + (D \Delta^d_k + N \Delta^n_k)} \right) \right\|_\infty.$$

Recall that $D$ is inner, so we have

$$\gamma_k \leq \left( \gamma_{opt} + \left\| \left[ \begin{array}{c} W_1 \Delta^d_k \\ W_2 \Delta^n_k \end{array} \right] \right\|_\infty \right) \left( \frac{1}{1 - \left( D \Delta^d_k + N \Delta^n_k \right)} \right).$$

(82)

On the other hand, by Lemma 6.1, $\|\Delta^n_k\|_\infty \rightarrow 0$, and $\|\Delta^d_k\|_\infty \rightarrow 0$, as $k \rightarrow \infty$. Since $W_1$, $W_2$, $N$ and $D$ are in $H^{\infty}$, we conclude that $\gamma_k \rightarrow \gamma_{opt}$ as $k \rightarrow \infty$. □

The above theorem gives an explicit bound (82) for the performance degradation ($\gamma_k - \gamma_{opt}$). This bound is expressed in terms of the approximation errors $\Delta^n_k$ and $\Delta^d_k$, which depend on the specific methods used in the approximations $Q_k$, $Y_k$ and $N_k$.

### 7 Conclusions

We have seen that the frequency domain method described in this paper provides a framework for the solution of $H^{\infty}$ control problems arising from disturbance attenuation, and from robust control under unstructured uncertainties, for distributed nominal plant models. The key point is to use the fact that the weights modelling the multiplicative uncertainties and the signals of interest are finite dimensional. Then, results from operator theory help us to reduce the $H^{\infty}$ optimal control problems to a problem of finding the singular values and vectors of an infinite rank operator. Singular value singular vector equation associated with this operator can be expressed as an eigenvalue
eigenvector equation for a Hankel + Toeplitz, or skew Toeplitz, operator. Special structures of these types of operators allow us to compute the $H^\infty$ optimal performance and the controller from a finite set of linear equations.

Optimal $H^\infty$ controllers for distributed plants are infinite dimensional. Therefore, their implementation is an important issue. We have seen in Section 6 that approximating the infinite dimensional parts of the optimal controller gives a finite dimensional controller stabilizing the infinite dimensional nominal plant and achieving a performance level close to optimum. For the finite dimensional controllers obtained this way, a bound on performance degradation is obtained in terms of approximation errors, which depend on the specific approximation method used.

In order to keep the discussion at a minimum technical level we did not discuss the multivariable situations. The skew Toeplitz method is applied to MIMO systems too. However, certain restrictive assumptions on the structure of the plant and the weights are needed in order to apply this theory to such systems, see e.g. Bercovici et al. (1988), Khargonekar et al. (1989), Özbay (1991), Özbay and Tannenbaum (1990), etc. For certain MIMO plants one can compute the optimal performance level form the results of these references. On the other hand computation of the optimal controller for MIMO distributed plants requires more work in this direction.

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**Appendix A: Notation and Some Properties of Hardy Spaces**

**Notation:**

- **R**: Real numbers,
- **C**: Complex numbers,
- **RHP**: open right half plane in **C**, $\{ s \in \mathbb{C} : \text{Re} s > 0 \}$,
- **D**: open unit disc, $\{ z \in \mathbb{C} : |z| < 1 \}$,
- **jR**: imaginary axis, $\{ s \in \mathbb{C} : \text{Re} s = 0 \}$,
- **jRe**: extended imaginary axis: $\{ j\omega : \omega \in \mathbb{R} \cup \{\infty\} \}$,
- **T**: unit circle, $\{ \xi \in \mathbb{C} : |\xi| = 1 \}$,
- **$L^\infty$**: Banach space of essentially bounded functions on **jR**,
$H^\infty$: $L^\infty$ functions which admit bounded analytical extensions to $RHP$,
$L^2$: Hilbert space of square integrable functions on $j\mathbb{R}$,
$H^2$: $L^2$ functions which admit analytical extensions to $RHP$,
$H^1$: Banach space of absolutely integrable functions on $j\mathbb{R}$ which admit analytical extensions to $RHP$,
$L^\infty(T), H^\infty(D), L^2(T), H^2(D), H^1(D)$: replace $j\mathbb{R}$ with $T$ and $RHP$ with $D$ in the above definitions of $L^\infty$, $H^\infty$, $H^2$, and $H^1$, respectively,
$\|G\|_n$: norm of $G$, when $G \in L^n, H^n, L^n(T)$, or $H^n(D)$, $n = 1, 2, \infty$,
$H_1 \ominus H_2$: orthogonal complement of $H_2$ in $H_1$, ($H_1, H_2$ are Hilbert spaces),
$I$: identity operator,
$P_-$: orthogonal projection operator from $L^2(T)$ to $L^2(T) \ominus H^2(D)$,
$P_+ := I - P_-$

Some Properties of Hardy Spaces:

A function $G_n(s)$, $s \in \mathbb{C}$, is in $H^n$, $n = 1, 2, \infty$, if (i): $G_n$ is analytic in $RHP$, (ii): defined almost everywhere on $j\mathbb{R}$, and (iii): its $n-$norm defined by

$$\|G_n\|_n = \sup_{\sigma > 0} \left( \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} |g_n(\sigma + j\omega)|^n d\omega \right)^{1/n}, \ (n = 1, 2),$$

$$= (\text{ess} \sup_{\sigma > 0, j\omega \in j\mathbb{R}} |g_n(\sigma + j\omega)|) \quad (n = \infty)$$

is finite. If $G_n$ does not satisfy (i) but satisfies (ii) and (iii) with $\sigma = 0$, then it is in $L^n$.
A function $g_n(z)$, $z \in \mathbb{C}$ is in $H^n(D)$, $n = 1, 2, \infty$, if (i): $g_n$ is analytic in $D$, (ii): defined almost everywhere on $T$, and (iii): its $n-$norm defined by

$$\|G_n\|_n = \sup_{r < 1} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |g_n(re^{i\theta})|^n d\theta \right)^{1/n}, \ (n = 1, 2),$$

$$= (\text{ess} \sup_{r < 1, \theta \in [0, 2\pi]} |g_n(re^{i\theta})|) \quad (n = \infty)$$

is finite. If $g_n$ does not satisfy (i) but satisfies (ii) and (iii) with $r = 1$, then it is in $L^n(T)$.

The spaces $L^2(T)$ and $H^2(D)$ are Hilbert spaces, with the inner product

$$<g_2, f_2> := \frac{1}{2\pi} \int_{0}^{2\pi} g_2(e^{i\theta})(f_2(e^{i\theta}))^* d\theta.$$

We can see $L^2(T)$ as the discrete Fourier transforms of two-sided finite energy sequences, e.g. $g \in L^2(T)$ has an expansion

$$g(e^{i\theta}) = \sum_{k=-\infty}^{\infty} g_k e^{jk\theta}, \quad \text{with,} \quad g_k \in \mathbb{C}, \quad \text{and} \quad \|g\|_2 = \sum_{k=-\infty}^{\infty} |g_k|^2 < \infty.$$
The second Hardy space $H^2(D)$, (a subspace of $L^2(T)$), is the space of discrete Fourier transforms of one-sided finite energy sequences, i.e. $g \in H^2(D)$ if and only if $g \in L^2(T)$ and $g_k = 0$ for $k < 0$, in this case $g(z) = \sum_{k=0}^{\infty} g_k z^k$ converges for $z \in D$.

It is also important to note that

$$H^\infty(D) = L^\infty(T) \cap H^2(D).$$

We can define a vector valued function $U(z) := \begin{bmatrix} a(z) \\ b(z) \end{bmatrix}$, from two $H^\infty(D)$ functions $a(z)$ and $b(z)$. Then the $\infty$–norm of $U$ is given by

$$\|U\|_\infty := \text{ess sup}_{\theta \in [0,2\pi]} \sqrt{|a(e^{i\theta})|^2 + |b(e^{i\theta})|^2}.$$ 

Any $2 \times 2$ matrix $L(e^{i\theta})$ whose entries are in $L^\infty(T)$ with the property $L^*L = LL^* = I$ is called unitary; and such matrices preserve the norm, i.e. $\|LU\|_\infty = \|L^*U\|_\infty = \|U\|_\infty$, where $U$ is an arbitrary $2 \times 1$ vector valued function whose entries are in $L^\infty(T)$.

**Appendix B: Essential Norm**

Consider an infinite rank, bounded linear operator $A$ from a separable Hilbert space $H_1$ to another separable Hilbert space $H_2$. Let $\langle \cdot, \cdot \rangle_1$ denote the inner product on $H_1$. Then, we say that the sequence $x_n \in H_1$, $n = 1, 2, 3, \ldots$, converges to zero weakly if

$$\langle y, x_n \rangle_1 \to 0, \quad \text{as} \quad n \to \infty, \quad \forall \quad y \in H_1.$$ 

The essential norm of $A$ is given by

$$\|A\|_e = \max \{ \sqrt{\lambda} : \lambda \in \sigma_e(A^*A) \},$$

where $\sigma_e(A^*A)$ denotes the essential spectrum of $A^*A$ which consists of those $\lambda \in \mathbb{C}$, for which there exists a sequence $x_n \in H_1$, with $\langle x_n, x_n \rangle_1 = 1$ for all $n = 1, 2, \ldots$ and $x_n \to 0$ weakly as $n \to \infty$, such that

$$(\lambda I - A^*A)x_n \to 0 \quad \text{as} \quad n \to \infty.$$
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