MULTI-LAYER LOCAL MINIMUM SOLUTIONS OF THE BISTABLE EQUATION IN AN INFINITE TUBE

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Abstract. We construct local minimum solutions of the semilinear bistable equation by minimizing the corresponding functional near some approximate solutions, assuming that some global minimum solutions are isolated. The key step is to prove with the help of a characterization of Palais-Smale sequences that the functional takes higher values away from the approximate solutions than it does near the approximate solutions.

1. Introduction

In this note we continue to study the problem posed in [1]. Consider the semilinear elliptic equation

$$\left\{ \begin{array}{l} -\Delta u + f(u) = 0 \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{array} \right. \tag{1}$$

where f is a bistable function, and Ω is an unbounded tube-shaped domain in R^d. (1) is the Euler-Lagrange equation of the functional E defined by

$$E(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 + W(u)]dx \tag{2}$$

where $W(u) = \displaystyle\int_{-1}^{u} f(w)dw$, and W is a double well potential function of equal depth. The domain of the functional E is taken to be the class

$$\mathcal{A} = \{u \in L^1_{loc}(\Omega) : \ \nabla u \in (L^2(\Omega))^d, \ W(u) \in L^1(\Omega)\}. \tag{3}$$

Here $L^1_{loc}(\Omega)$ is the space of measurable functions that belong to $L^1(K)$ for every compact subset K of Ω.

We present an alternative approach to the construction of multi-layer solutions of (1) by minimizing E near some proper approximate solutions. The solutions being local minima of E reflects the advantage of this approach since in Theorem 4.4 [1] solutions are found through an indirect deformation argument and they are only known as critical points of (2). The hypothesis here that guarantees the existence of the solutions is also weaker than that in [1] (see Remark 2).

The precise conditions on f, W and Ω are given as follows.

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H-1: W is a C^2 function that has exactly two global minima at -1 and 1 where $W(-1)=W(1)=0$, $W''(-1)>0$, and $W''(1)>0$.
H-2: There exists $\Theta>0$ such that $W(u)=\Theta u^2$ for all $|u|>2$.
H-3: Ω is a smooth infinite tube periodic in x^1-direction, i.e., $x=(x^1,x')\in R^1\times R^{d-1}=R^d$ is in Ω if and only if $x+(1,0,...,0)$ is in Ω and x' lies in a bounded subset of R^{d-1}.

Note that H-2 is more or less of technical nature. Indeed for each W satisfying H-1 and $W''(u)>0$ for $|u|>1$, the maximum principle implies that bounded solutions of (1) lie between -1 and 1, so one can always modify W to satisfy H-2 without affecting bounded solutions.

As a consequence of H-1 and H-2 there exists $C>0$ such that for all $u\in R^1$

$$f^2(u)\le CW(u). \tag{4}$$

Define a segment $S(x^1,t)$, x^1 and $t\in R^1$, of Ω by

$$S(x^1,t):=\{(y^1,y')\in\Omega:\ |y^1-x^1|<t\}. \tag{5}$$

For every $u\in\mathcal{A}$ (defined in (3)) define a continuous function $\widehat{u}:R^1\to R^1$ by

$$\widehat{u}(x^1)=\frac{1}{|S(x^1,1/2)|}\int_{S(x^1,1/2)}u(y)dy,\ \ x^1\in R^1 \tag{6}$$

where $|S(x^1,1/2)|$ denotes the Lebesgue measure of $S(x^1,1/2)$ in R^d.

Corollary 2.2 [1] states that for every $u\in\mathcal{A}$, $\lim_{x^1\to-\infty}\widehat{u}(x^1)$ and $\lim_{x^1\to\infty}\widehat{u}(x^1)$ exist and equal -1 or 1. Setting

$$\mathcal{A}_\zeta^\eta=\{u\in\mathcal{A}:\ \lim_{x^1\to-\infty}\widehat{u}(x^1)=\zeta,\ \lim_{x^1\to\infty}\widehat{u}(x^1)=\eta\},\ \zeta,\eta\in\{-1,1\}, \tag{7}$$

we decompose $\mathcal{A}=\mathcal{A}_{-1}^{-1}\cup\mathcal{A}_{-1}^{1}\cup\mathcal{A}_{1}^{-1}\cup\mathcal{A}_1^1$.

If $u\in\mathcal{A}_\zeta^\eta$, then Lemma 2.3 [1] states that every $v\in L^1_{loc}(\Omega)$ is in $\mathcal{A}_\zeta^\eta$ if and only if $v-u\in W^{1,2}(\Omega)$. Therefore each subclass $\mathcal{A}_\zeta^\eta$ is a complete affine space, a translate of $W^{1,2}(\Omega)$, where the tangent space at each point is $W^{1,2}(\Omega)$, the distance of $u,v\in\mathcal{A}_\zeta^\eta$ is $||u-v||_{W^{1,2}(\Omega)}$. In $\mathcal{A}_\zeta^\eta$ we use $\mathcal{B}(u,r)$ to denote $\{v\in\mathcal{A}_\zeta^\eta:||u-v||_{W^{1,2}(\Omega)}<r\}$.

Lemma 2.5 [1] states that $E:\mathcal{A}_\zeta^\eta\to R^1$ belongs to $C^2(\mathcal{A}_\zeta^\eta,R^1)$ where

$$E'(u)\phi=\int_\Omega[\nabla u\cdot\nabla\phi+f(u)\phi]dx,\ \ u\in\mathcal{A}_\zeta^\eta,\phi\in W^{1,2}(\Omega),$$

$$E''(u)(\phi,\psi)=\int_\Omega[\nabla\phi\cdot\nabla\psi+f'(u)\phi\psi]dx,\ \ u\in\mathcal{A}_\zeta^\eta,\phi,\psi\in W^{1,2}(\Omega).$$

We use $||E'(u)||$ ($||E''(u)||$, respectively) to denote the norm of the liner (bilinear, respectively) form $E'(u)$ ($E''(u)$, respectively). It is clear from H-1 and H-2 that $||E''(u)||$ is bounded uniformly in u.

u is a critical point of E if $u\in\mathcal{A}$ and for every $\phi\in W^{1,2}(\Omega)$

$$\int_\Omega[\nabla u\cdot\nabla\phi+f(u)\phi]dx=0.$$

A critical point of E is a classical solution of (1) by the standard elliptic regularity theory. The set of all critical points in $\mathcal{A}$ is denoted $\mathcal{K}$.

The global minima -1 in $\mathcal{A}_{-1}^{-1}$ and 1 in $\mathcal{A}_1^1$ are isolated critical points in the sense of Lemma 2.6 [1] which states that there exists $\lambda_0 > 0$ such that for every $u \in \mathcal{A}_\theta^\theta \cap \mathcal{K}$, $u \neq \theta$, we have $\|u-\theta\|_{W^{1,2}(\Omega)} > \lambda_0$.

Theorem 3.2 [1] asserts that in each $\mathcal{A}_\zeta^\eta$, there is a global minimum of E, i.e., there exists $u \in \mathcal{A}_\zeta^\eta$ such that

$$E(u) = \inf_{v \in \mathcal{A}_\zeta^\eta} E(v).$$

We now take $U_1 \in \mathcal{A}_{-1}^1$, $U_2 \in \mathcal{A}_1^{-1}$, ..., $U_M \in \mathcal{A}_{(-1)^M}^{(-1)^{M+1}}$ to be M global minima of E in their own subclasses. We say that U_1, U_2,...,U_M are isolated global minima if there exists $\mu_0 > 0$ such that for every $u \in \mathcal{B}(U_i, \mu_0)\backslash\{U_i\}$ and every $i = 1, 2, ..., M$, we have $E(u) > E(U_i)$. The U_i's being isolated implies that the domain Ω can not be a cylinder, i.e., $\Omega \neq R^1 \times \Omega'$ where $\Omega' \subset R^{d-1}$, since in a cylinder no global minimum in $\mathcal{A}_{-1}^1$ or $\mathcal{A}_1^{-1}$ is isolated due to the translational invariance.

Two important operators are defined on $\mathcal{A}$. Let k be an integer, and define the shift operator $\tau_k : \mathcal{A}_\zeta^\eta \to \mathcal{A}_\zeta^\eta$ for $\zeta, \eta \in \{-1, 1\}$ by

$$\tau_k u(x) = u(x - (k, 0, ..., 0)),\ x \in \Omega. \tag{8}$$

Define the paste operator $\pi : \mathcal{A}_\zeta^\theta \times \mathcal{A}_\theta^\eta \to \mathcal{A}_\zeta^\eta$ for $\zeta, \theta, \eta \in \{-1, 1\}$ by

$$\pi(u, v) = u + v - \theta. \tag{9}$$

A recursive use of (9) extend π to $\pi : \mathcal{A}_\zeta^{\theta_2} \times \mathcal{A}_{\theta_2}^{\theta_3} \times ... \times \mathcal{A}_{\theta_k}^\eta \to \mathcal{A}_\zeta^\eta$ by

$$\pi(u_1, u_2, ..., u_k) = \pi(u_1, \pi(u_2, \pi(..., \pi(u_{k-1}, u_k)))). \tag{10}$$

These two operators are often used together. If there is no danger of confusion, we write $\pi_j u$ for $\pi(\tau_{j_1} u_1, \tau_{j_2} u_2, ..., \tau_{j_k} u_k)$.

The main result in this paper is the following existence theorem, which improves Theorem 4.4 [1].

Theorem 1. *Let $U_1 \in \mathcal{A}_{-1}^1$, $U_2 \in \mathcal{A}_1^{-1}$,...,$U_M \in \mathcal{A}_{(-1)^M}^{(-1)^{M+1}}$ be isolated global minima in their own subclasses. Then for each $r \in (0, \min\{\mu_0, \lambda_0, 2\sqrt{|S(0,1/2)|}\})$ there exists $L > 0$ such that as long as $\min\{j_2 - j_1, j_3 - j_2, ..., j_M - j_{M-1}\} > L$ there exists $V \in \mathcal{B}(\pi_j U, r/2)$ with*

$$E(V) = \inf_{u \in \mathcal{B}(\pi_j U, r)} E(u),$$

i.e., there is a local minimum of E in $\mathcal{B}(\pi_j U, r) \subset \mathcal{A}_{-1}^{(-1)^{M+1}}$.

Recall that $\pi_j U = \pi(\tau_{j_1} U_1, \tau_{j_2} U_2, ..., \tau_{j_M} U_M)$, μ_0 measures how isolated the U_i's are, λ_0 measures how isolated -1 and 1 are, and $|S(0,1/2)|$ is the Lebesgue measure of $S(0, 1/2)$.

Remark 2. *In Theorem 4.4* [1] *the U_i's are assumed isolated as critical points, while here they are merely isolated as global minima.*

Remark 3. *Each U_i can be regarded as a single layer and $\pi_j U$ as a function of k layers. Since the local minimum V is close to $\pi_j U$, V is a k-layer solution.*

2. Proof of Theorem 1

To make the proof of Theorem 1 more readable, we assume $M = 2$. The general case can be handled along the same line. We use C, C_1, C_2,... to denote generic constants that may vary from line to line. We often do not mention passing to a subsequence when we do so.

Let $U_1 \in \mathcal{A}_{-1}^{1}$ and $U_2 \in \mathcal{A}_{1}^{-1}$ be two isolated global minima of E, and μ_0 be the radius of the balls around U_1 and U_2 in which there is no other global minima. Take two integers j_1 and j_2, $j_1 < j_2$, and look for a local minimum of E in $\mathcal{B}(\pi_j U, r) \subset \mathcal{A}_{-1}^{-1}$ for some $r \in (0, \min\{\mu_0, \lambda_0, 2\sqrt{|S(0,1/2)|}\})$. Here $\pi_j U = \pi(\tau_{j_1} U_1, \tau_{j_2} U_2)$ serves as an approximate solution.

We first show that $E(u)$ is large for all $u \in \mathcal{B}(\pi_j U, r) \backslash \mathcal{B}(\pi_j U, r/2)$.

Lemma 4. *Fix $r \in (0, \min\{\mu_0, \lambda_0, 2\sqrt{|S(0,1/2)|}\})$. There exist $L > 0$ and $\epsilon > 0$ such that for every pair of integers (j_1, j_2) with $j_2 - j_1 > L$*

$$E(u) \geq \inf_{v \in \mathcal{B}(\pi_j U, r)} E(v) + \epsilon$$

for all $u \in \mathcal{B}(\pi_j U, r) \backslash \mathcal{B}(\pi_j U, r/2)$.

Remark 5. *L is a lower bound of the distance between the layers. In general the larger r is, the smaller L can be.*

We postpone the proof of Lemma 4 to next section. Take $u_n \in \mathcal{B}(\pi_j U, r)$ such that

$$\lim_{n \to \infty} E(u_n) = \inf_{v \in \mathcal{B}(\pi_j U, r)} E(v). \tag{11}$$

Because of Lemma 4, we can safely assume $u_n \in \mathcal{B}(\pi_j U, r/2)$.

We now show that $\{u_n\}$ is a Palais-Smale sequence. Recall that a sequence $\{g_n\}$ is a Palais-Smale sequence if $E(g_n) \to c \in R^1$ and $\|E'(g_n)\| \to 0$ as $n \to \infty$. If $\{u_n\}$ is not a Palais-Smale sequence, then we can assume $\|E'(u_n)\| \to \delta > 0$ as $n \to \infty$. Find $\phi_n \in W^{1,2}(\Omega)$ with $\|\phi_n\|_{W^{1,2}(\Omega)} = 1$ such that $E'(u_n)\phi_n \geq \delta/2$. Then consider for $t \in (0, r/2)$

$$E(u_n - t\phi_n) = E(u_n) - tE'(u_n)\phi_n + (t^2/2)E''(u_n - t_n\phi_n)(\phi_n, \phi_n)$$

$$\leq E(u_n) - (\delta/2)t + Ct^2 \leq \inf_{v \in \mathcal{B}(\pi_j U, r)} E(v) + o(1) - (\delta/2)t + Ct^2$$

where $t_n \in (0, t)$ is guaranteed by the Taylor expansion formula and the constant C comes from the fact that E'' is bounded. $o(1)$ stands for a quantity that approaches 0 as $n \to \infty$. Choosing n sufficient large and t sufficiently small, we deduce

$$E(u_n - t\phi_n) < \inf_{v \in \mathcal{B}(\pi_j U, r)} E(v)$$

which is impossible since $u_n - t\phi_n \in \mathcal{B}(\pi_j U, r)$. This proves that $\{u_n\}$ is a Palais-Smale sequence.

We quote a characterization of Palais-Smale sequences from [1].

Proposition 3.1 [1]. *Let $\{u_n\}$ be a Palais-Smale sequence in $\mathcal{A}_\zeta^\eta$, ζ and $\eta \in \{-1,1\}$. If $\lim_{n\to\infty} E(u_n) = 0$, then $\mathcal{A}_\zeta^\eta = A_\theta^\theta$ for some $\theta \in \{-1,1\}$ and*

$$\lim_{n\to\infty} \|u_n - \theta\|_{W^{1,2}(\Omega)} = 0.$$

If $\lim_{n\to\infty} E(u_n) > 0$, then there exist $w_1, w_2, ..., w_k \in \mathcal{K}\backslash\{-1,1\}$, $k \geq 1$, $w_i \in \mathcal{A}_{\theta_i}^{\theta_{i+1}}$, $\theta_1 = \zeta$ and $\theta_{k+1} = \eta$, and k integral sequences $\{l_{1,n}\}$, $\{l_{2,n}\}$,...,$\{l_{k,n}\}$ with $\lim_{n\to\infty}(l_{i+1,n} - l_{i,n}) = \infty$ for each $i = 1, 2, ..., k-1$ such that

$$\lim_{n\to\infty} \|u_n - \pi(\tau_{l_{1,n}} w_1, \tau_{l_{2,n}} w_2, ..., \tau_{l_{k,n}} w_k)\|_{W^{1,2}(\Omega)} = 0,$$

$$\lim_{n\to\infty} E(u_n) = E(w_1) + E(w_2) + ... + E(w_k)$$

along a subsequence of $\{u_n\}$.

Applying Proposition 3.1 [1] to u_n we find k integral sequences $\{l_{1,n}\}$, $\{l_{2,n}\}$, ..., $\{l_{k,n}\}$, and k nontrivial critical points w_1, w_2,...,w_k such that $\|u_n - \pi_{l_n} w\|_{W^{1,2}(\Omega)} = o(1)$. Then $\|\pi_j U - \pi_{l_n} w\|_{W^{1,2}(\Omega)} \leq r/2 + o(1)$. If one of the $l_{i,n}$'s approaches $-\infty$ or ∞, say $l_{k,n} \to \infty$, as $n \to \infty$, then

$$r/2 \geq \|\pi_j U - \pi_{l_n} w\|_{W^{1,2}(\Omega)} + o(1) \geq \|w_k + 1\|_{W^{1,2}(\Omega)} + o(1) \geq \lambda_0 + o(1)$$

by Lemma 2.6 [1], which is inconsistent with the assumption on r. Therefore $k = 1$ and $l_{1,n}$ is bounded in n. We can select a proper subsequence of $\{l_{1,n}\}$ and shift w_1 to assume $l_{1,n} = 0$. Then with the help of (11)

$$\|u_n - w_1\|_{W^{1,2}(\Omega)} \to 0, \; E(w_1) = \lim_{n\to\infty} E(u_n) = \inf_{v\in\mathcal{B}(\pi_j U, r)} E(v),$$

and $w_1 \in \mathcal{B}(\pi_j U, r/2)$, i.e., w_1 is a local minimum of E in $\mathcal{B}(\pi_j U, r)$. The proof of Theorem 1 is complete after we set $V = w_1$.

3. Proof of Lemma 4

Suppose the lemma is not true. Then there exist r satisfying

$$0 < r < \min\{\mu_0, \lambda_0, 2\sqrt{|S(0,1/2)|}\},$$

a sequence of pairs of integers $(j_{1,n}, j_{2,n})$, with $j_{2,n} - j_{1,n} \to \infty$, and a sequence $\{u_n\} \subset \mathcal{B}(\pi_{j_n} U, r)\backslash\mathcal{B}(\pi_{j_n} U, r/2)$ such that

$$E(u_n) - \inf_{v\in\mathcal{B}(\pi_{j_n} U,r)} E(v) = o(1) \tag{12}$$

as $n \to \infty$.

We can find a constant C independent of n such that

$$E(u) < C \tag{13}$$

for all $u \in \mathcal{B}(\pi_{j_n} U, r)$. To see (13) we estimate for each $u \in \mathcal{B}(\pi_{j_n} U, r)$

$$|E(u) - E(\pi_{j_n} U)|$$

$$= |\frac{1}{2}\int_\Omega |\nabla(u - \pi_{j_n} U) + \nabla\pi_{j_n} U|^2 - \frac{1}{2}\int_\Omega |\nabla\pi_{j_n} U|^2 + \int_\Omega [W(u) - W(\pi_{j_n} U)]|$$

$$= |\frac{1}{2}\int_\Omega |\nabla(u-\pi_{j_n}U)|^2 + \int_\Omega \nabla(u-\pi_{j_n}U)\cdot\nabla\pi_{j_n}U + \int_\Omega [W(u)-W(\pi_{j_n}U)]|$$

$$\leq \frac{1}{2}\|\nabla(u-\pi_{j_n}U)\|^2_{L^2(\Omega)} + \|\nabla(u-\pi_{j_n}U)\|_{L^2(\Omega)}\|\nabla\pi_{j_n}U\|_{L^2(\Omega)}$$

$$+\|f(\pi_{j_n}U)\|_{L^2(\Omega)}\|u-\pi_{j_n}U\|_{L^2(\Omega)} + C\|u-\pi_{j_n}U\|^2_{L^2(\Omega)}$$

$$\leq C_1(\|\nabla\pi_{j_n}U\|_{L^2(\Omega)} + \|f(\pi_{j_n}U)\|_{L^2(\Omega)})\|u-\pi_{j_n}U\|_{W^{1,2}(\Omega)} + C_2\|u-\pi_{j_n}U\|^2_{W^{1,2}(\Omega)}$$

$$\leq C_1\sqrt{E(\pi_{j_n}U)}\|u-\pi_{j_n}U\|_{W^{1,2}(\Omega)} + C_2\|u-\pi_{j_n}U\|^2_{W^{1,2}(\Omega)}.$$

The last inequality follows from (4). The last line is bounded independent of n since $E(\pi_{j_n}U) = E(U_1) + E(U_2) + o(1)$ and $\|u-\pi_{j_n}U\|_{W^{1,2}(\Omega)} \leq r$. This proves (13).

If we write $\Omega = \cup_{k=-\infty}^{\infty} S(k, 1/2)$, we can find a sequence $\{m_n\}$ of integers with

$$\lim_{n\to\infty}(m_n - j_{1,n}) = \lim_{n\to\infty}(j_{2,n} - m_n) = \infty \tag{14}$$

and

$$\lim_{n\to\infty}\int_{S(m_n,1)}[\frac{1}{2}|\nabla u_n|^2 + W(u_n)] = 0 \tag{15}$$

by (13). (14) and (15) actually imply

$$\lim_{n\to\infty}\int_{S(m_n,1/2)}[|\nabla(u_n-1)|^2 + |u_n-1|^2] = 0. \tag{16}$$

To see (16) we use Lemma 2.1 [1] to obtain

$$\lim_{n\to\infty}\sup_{x^1\in(m_n-1/2,m_n+1/2)}|\widehat{u}_n(x^1)-\theta| = 0$$

for some $\theta \in \{-1, 1\}$. In particular,

$$\widehat{u}_n(m_n) - \theta = o(1). \tag{17}$$

Consider

$$\int_{S(m_n,1/2)}[|\nabla(u_n-\theta)|^2 + |u_n-\theta|^2].$$

From (15) we know

$$\int_{S(m_n,1/2)}|\nabla(u_n-\theta)|^2 = o(1). \tag{18}$$

About $(u_n-\theta)^2$ we set

$$G_n = \{x\in S(m_n,1/2): |u_n(x)-\theta| < \delta\},\ B_n = S(m_n,1/2)\backslash G_n$$

where δ is so small that for all $u \in (\theta-\delta, \theta+\delta)$, $c_1(u-\theta)^2 \leq W(u) \leq c_2(u-\theta)^2$ for some positive c_1 and c_2. The reader may think G_n as a good set and B_n as a bad set. On the good set G_n by (15) we find

$$\int_{G_n}|u_n-\theta|^2 \leq C\int_{G_n}W(u_n) = o(1). \tag{19}$$

On the bad set B_n we note that $|u_n(x)-\theta| \le 2|u_n(x)-\widehat{u}_n(m_n)|$ if we choose n large enough because of (17). Therefore with the help of the Poincaré inequality

$$(20)\quad \int_{B_n} |u_n-\theta|^2 \le 4\int_{S(m_n,1/2)} |u_n-\widehat{u}_n(m_n)|^2 \le C\int_{S(m_n,1/2)} |\nabla u_n|^2 = o(1)$$

by (15). Then (18), (19) and (20) imply

$$\int_{S(m_n,1/2)} [|\nabla(u_n-\theta)|^2 + |u_n-\theta|^2] = o(1).$$

We need to show $\theta = 1$. Assume $\theta = -1$. Then it follows

$$r^2 \ge \|u_n - \pi_{j_n}U\|^2_{W^{1,2}(\Omega)} \ge \int_{S(m_n,1/2)} [|\nabla(u_n-\pi_{j_n}U)|^2 + |u_n-\pi_{j_n}U|^2]$$

$$= \int_{S(m_n,1/2)} (-1-1)^2 + o(1) = 4|S(0,1/2)| + o(1)$$

with the help of (14). This is inconsistent with our assumption on r, and (16) is proved.

We then truncate u_n at $S(m_n,1/2)$ to define

$$(21)\qquad v_n = \begin{cases} u_n(x) & \text{if} \quad x \notin S(m_n,1/2) \\ (u_n(x)-1)\xi(x-m_n)+1 & \text{if} \quad x \in S(m_n,1/2) \end{cases}$$

where ξ is a smooth function such that

$$\xi(x) = \begin{cases} 0 & \text{if} \quad x^1 \in [-1/4,1/4] \\ 1 & \text{if} \quad x^1 \notin [-1/2,1/2] \end{cases}.$$

It follows from (16) and (21)

$$(22)\qquad E(v_n)-E(u_n) = o(1), \text{ and } \|v_n-u_n\|_{W^{1,2}(\Omega)} = o(1).$$

We set

$$v_{1,n}(x) = \begin{cases} v_n(x) & \text{if} \quad x^1 \le m_n \\ 1 & \text{if} \quad x^1 > m_n \end{cases}, \quad v_{2,n}(x) = \begin{cases} 1 & \text{if} \quad x^1 \le m_n \\ v_n(x) & \text{if} \quad x^1 > m_n \end{cases}.$$

Clearly $v_{1,n} \in \mathcal{A}^1_{-1}$, $v_{2,n} \in \mathcal{A}^{-1}_1$, $v_n = \pi(v_{1,n}, v_{2,n})$ and

$$(23)\qquad E(v_n) = E(v_{1,n}) + E(v_{2,n}).$$

From $E(v_{1,n}) \ge E(U_1)$, $E(v_{2,n}) \ge E(U_2)$, (12), (22) and (23) we deduce

$$(24)\qquad \inf_{v\in\mathcal{B}(\pi_{j_n}U,r)} E(v) \ge E(U_1)+E(U_2)+o(1).$$

On the other hand by (14)

$$E(\pi_{j_n}U) = E(U_1)+E(U_2)+o(1),$$

which implies

$$(25)\qquad \inf_{v\in\mathcal{B}(\pi_{j_n}U,r)} E(v) \le E(U_1)+E(U_2)+o(1).$$

Combing (24) and (25) we deduce

$$\inf_{v\in\mathcal{B}(\pi_{j_n}U,r)} E(v) = E(U_1) + E(U_2) + o(1)$$

and

$$E(v_{1,n}) = E(U_1) + o(1),\ E(v_{2,n}) = E(U_2) + o(1). \tag{26}$$

We turn our attention to the distance between $v_{1,n}$ and $\tau_{j_{1,n}}U_1$, and the distance between $v_{2,n}$ and $\tau_{j_{2,n}}U_2$. Clearly

$$\begin{aligned}&\|v_{1,n}-\tau_{j_{1,n}}U_1\|_{W^{1,2}(\Omega)} + \|v_{2,n}-\tau_{j_{2,n}}U_2\|_{W^{1,2}(\Omega)}\\ &\geq \|v_n-\pi_{j_n}U\|_{W^{1,2}(\Omega)} \geq r/2 + o(1)\end{aligned}$$

by the triangle inequality and (22). Then either

$$\|v_{1,n}-\tau_{j_{1,n}}U_1\|_{W^{1,2}(\Omega)} \geq r/4 + o(1) \tag{27}$$

or

$$\|v_{2,n}-\tau_{j_{2,n}}U_2\|_{W^{1,2}(\Omega)} \geq r/4 + o(1). \tag{28}$$

Assume without the loss of generality that the former occurs. We look for an upper bound for $\|v_{1,n}-\tau_{j_{1,n}}U_1\|_{W^{1,2}(\Omega)}$. Consider, with the help of (14),

$$\begin{aligned}\int_\Omega |\nabla v_n - \nabla\pi_{j_n}U|^2 &\geq \int_{x^1<m_n} |\nabla v_n - \nabla\pi_{j_n}U|^2\\ &= \int_{x^1<m_n} |(\nabla v_{1,n} - \nabla\tau_{j_{1,n}}U_1) + \nabla\tau_{j_{2,n}}U_2|^2\\ &\geq |[\int_{x^1<m_n} |\nabla v_{1,n} - \nabla\tau_{j_{1,n}}U_1|^2]^{1/2} - [\int_{x^1<m_n} |\nabla\tau_{j_{2,n}}U_2|^2]^{1/2}|^2\\ &= |[\int_{x^1<m_n} |\nabla v_{1,n} - \nabla\tau_{j_{1,n}}U_1|^2]^{1/2} + o(1)|^2. \end{aligned}\tag{29}$$

Also note

$$\begin{aligned}\int_\Omega |\nabla v_{1,n} - \nabla\tau_{j_{1,n}}U_1|^2 &= \int_{x^1<m_n} |\nabla v_{1,n} - \nabla\tau_{j_{1,n}}U_1|^2 + \int_{x^1>m_n} |\nabla\tau_{j_{1,n}}U_1|^2\\ &= \int_{x^1<m_n} |\nabla v_{1,n} - \nabla\tau_{j_{1,n}}U_1|^2 + o(1)\end{aligned}\tag{30}$$

again by (14). Then we deduce from (29) and (30)

$$\int_\Omega |\nabla v_{1,n} - \nabla\tau_{j_{1,n}}U_1|^2 \leq \int_\Omega |\nabla v_n - \nabla\pi_{j_n}U|^2 + o(1).$$

A similar argument shows

$$\int_\Omega |v_{1,n} - \tau_{j_{1,n}}U_1|^2 \leq \int_\Omega |v_n - \pi_{j_n}U|^2 + o(1).$$

We then find, with the help of (22),

$$\|v_{1,n}-\tau_{j_{1,n}}U_1\|_{W^{1,2}(\Omega)} \leq \|v_n-\pi_{j_n}U\|_{W^{1,2}(\Omega)} + o(1) \leq r + o(1). \tag{31}$$

We shift $v_{1,n}$ back by $-j_{1,n}$ to consider $\tau_{-j_{1,n}} v_{1,n}$. From (27) and (31) we deduce

$$r/4 + o(1) \leq \|\tau_{-j_{1,n}} v_{1,n} - U_1\|_{W^{1,2}(\Omega)} \leq r + o(1). \tag{32}$$

Note that (26) implies

$$E(v_{1,n}) \to E(U_1) = \inf_{v \in \mathcal{A}^1_{-1}} E(v)$$

as $n \to \infty$, i.e., $\{v_{1,n}\}$ is a global minimizing sequence of E in $\mathcal{A}^1_{-1}$. As in the proof of Theorem 1, $\{v_{1,n}\}$ (as well as $\{\tau_{-j_{1,n}} v_{1,n}\}$) is a Palais-Smale sequence. Applying Proposition 3.1 [1] to $\{\tau_{-j_{1,n}} v_{1,n}\}$, we find k integral sequences $\{l_{1,n}\}$, $\{l_{2,n}\}$,...,$\{l_{k,n}\}$, and k nontrivial critical points w_1, w_2,...,w_k satisfying $\|\tau_{-j_{1,n}} v_{1,n} - \pi_{l_n} w\|_{W^{1,2}(\Omega)} \to 0$ as $n \to \infty$, which implies with the help of (32)

$$r/4 + o(1) \leq \|U_1 - \pi_{l_n} w\|_{W^{1,2}(\Omega)} \leq r + o(1). \tag{33}$$

If one of the $l_{i,n}$'s approaches $-\infty$ or ∞ as $n \to \infty$, say $l_{k,n} \to \infty$, then (33) implies

$$r \geq \|U_1 - \pi_{l_n} w\|_{W^{1,2}(\Omega)} + o(1) \geq \|w_k + 1\|_{W^{1,2}(\Omega)} + o(1) \geq \lambda_0 + o(1),$$

which is again inconsistent with the assumption on r. We conclude that $k = 1$ and $l_{1,n}$ is bounded in n. By passing to a subsequence of $\{l_{1,n}\}$ and shifting w_1 we can assume $l_{1,n} = 0$. Then we deduce with the help of (33)

$$\|\tau_{-j_{1,n}} v_{1,n} - w_1\|_{W^{1,2}(\Omega)} = o(1), \;\; r/4 \leq \|U_1 - w_1\|_{W^{1,2}(\Omega)} \leq r.$$

Since $\{\tau_{-j_{1,n}} v_{1,n}\}$ is a minimizing sequence in $\mathcal{A}^1_{-1}$, we find $E(w_1) = E(U_1)$ and $r/4 \leq \|U_1 - w_1\|_{W^{1,2}(\Omega)} \leq r < \mu_0$. This is inconsistent with the assumption that U_1 is the only minimum in $\mathcal{B}(U_1, \mu_0)$. The proof of Lemma 4 is complete.

References

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