NOTE ON A GENERALIZED SYLVESTER EQUATION

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Note on a Generalized Sylvester Equation*

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ABSTRACT

In this note we show how to compute the minimum-norm, least squares solution of the generalized Sylvester equation

\[ AX + YB = C, \]

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In a personal communication, Richard I. Shrager inquired about the problem of solving the generalized Sylvester equation

\[ AX + YB = C, \]  

(1)

for \( X \) and \( Y \). Here \( A \) is \( m \times k \), \( B \) is \( l \times n \), and \( C \) is \( m \times n \) with \( k < m \) and \( l < n \). The equation arises in a generalization of a technique for measuring chemical transitions by spectra [1].

In general the system (1) is inconsistent, and we will ask for a least squares solution:

\[ \| C - AX - YB \|_2^2 = \min. \]  

(2)

Here \( \| \cdot \| \) denotes the Frobenius norm defined by

\[ \| C \|_2^2 = \sum_{i,j} \gamma_{ij}. \]

Since the least squares problem (2) is in general underdetermined, we shall also require that

\[ \| X \|_2^2 + \| Y \|_2^2 = \min; \]  

(3)

i.e., that the combined solution solution be of minimal norm.

The solution can be expressed in terms of the singular value decomposition of \( A \) and \( B \). Specifically, let

\[ U_A^T A V_A = \begin{pmatrix} \hat{A} & 0 \\ 0 & 0 \end{pmatrix}, \]

where \( U_A \) and \( V_A \) are orthogonal and

\[ \hat{A} = \text{diag}(\hat{\alpha}_i) \]

has positive diagonal entries. Similarly let

\[ U_B^T B V_B = \begin{pmatrix} \hat{B} & 0 \\ 0 & 0 \end{pmatrix}, \]
where $U_B$ and $V_B$ are orthogonal and

$$B = \text{diag}(\hat{\beta}_i)$$

has positive diagonal entries. If (with an obvious partitioning) we write

$$V_A^T X V_B = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix},$$

$$U_A^T Y U_B = (\hat{Y}_1 \ \hat{Y}_2),$$

and

$$U_A^T C V_B = \begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{pmatrix},$$

then we have

$$\begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{pmatrix} = \begin{pmatrix} \hat{A} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} + (\hat{Y}_1 \ \hat{Y}_2) \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

From (4) we see that the values of $X_2$ and $Y_2$ do not affect the norm (2). Consequently, by (3) they must be zero.

Now let us further partition

$$\hat{X}_1 = (\hat{X}_{11} \ \hat{X}_{12})$$

and

$$\hat{Y}_1 = \begin{pmatrix} \hat{Y}_{11} \\ \hat{Y}_{21} \end{pmatrix}.$$

Then our problem becomes one finding the minimum norm solution of

$$\|\hat{C}_{11} - \hat{A}\hat{X}_{11} - \hat{Y}_{11}\hat{B}\|^2 + \|\hat{C}_{12} - \hat{A}\hat{X}_{12}\|^2 + \|\hat{C}_{21} - \hat{Y}_{21}\hat{B}\|^2 = \min.$$ 

Clearly, we must have $\hat{X}_{21} = \hat{A}^{-1}\hat{C}_{21}$ and $\hat{Y}_{21} = \hat{C}_{21}\hat{B}^{-1}$. Thus the problem becomes one of determining minimal $\hat{X}_{11}$ and $\hat{Y}_{11}$ satisfying

$$\hat{C}_{11} = \hat{A}\hat{X}_{11} - \hat{Y}_{11}\hat{B}. \tag{5}$$

Because $\hat{A}$ and $\hat{B}$ are diagonal, the problem (5) uncouples into the independent problems

minimize $\hat{\xi}_{ij}^2 + \hat{\eta}_{ij}^2$

subject to $\hat{\alpha}_{ij}\hat{\xi}_{ij} + \hat{\eta}_{ij}\hat{\beta}_{ij} = \hat{\gamma}_{ij}$. 

A Generalized Sylvester Equation
But this scalar problem is easily seen to have the solution

\[ \hat{\xi}_{ij} = \frac{\hat{\gamma}_{ij}\hat{\alpha}_i}{\hat{\alpha}_i^2 + \hat{\beta}_j^2} \quad \text{and} \quad \hat{\eta}_{ij} = \frac{\hat{\gamma}_{ij}\hat{\beta}_j}{\hat{\alpha}_i^2 + \hat{\beta}_j^2}. \]

This completes the solution of the original problem.

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