A TIME-DEPENDENT FREE BOUNDARY PROBLEM
MODELING THE VISUAL IMAGE IN
ELECTROPHOTOGRAPHY

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IMA Preprint Series # 975
May 1992
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Abstract. The formation of a visual image in electrophotography can be modeled as a time-dependent free boundary problem. The electric potential \(-u\) satisfies \(\Delta u = 1\) in the toner region and \(\Delta u = 0\) outside this region, whereas on the interface (which is a moving boundary)

\[-\frac{\partial u}{\partial N} = \text{velocity of the interface},\]

\(N\) being the outward normal to the toner region. It is proved that this problem has a smooth solution for a small time interval; furthermore, for a certain version of the free boundary condition, the solution is unique.

Key words. Free boundary problems, electrophotography.

AMS(MOS) subject classifications. 35B45, 35R05, 35R35.

§0. Introduction. The formation of visual images in electrophotography is accomplished by means of a toner injected onto the surface of the photoconductor. The toner is carried by biased carriers and, as a consequence, tends to accumulate in those regions of the photoconductor surface which carry surface charge corresponding to dark spots in the documents which is being copied (the distribution of this charge represents the electric image of the document); for more details see [1,2,5,6,7].

We consider here a 2-dimensional model whereby a pixel is represented by an interval \(-a \leq x \leq a\). We denote by \(-u\) the potential of the electric field which is responsible for the motion and settling of the toner. A small potential difference \(M\) is maintained between boundaries \(y = b\) (where \(u = M\)) and \(y = -h\) (where \(u = 0\)). The surface of the photoconductor is \(\{y = 0\}\), and the electric image is assumed, for simplicity, to be a surface-distribution of uniform density \(\sigma\), \(\sigma > 0\), supported on an interval

\[I = \{(x, 0); -\gamma < x < \gamma\} \quad \text{where} \quad 0 < \gamma < a.\]

Figure 1 shows the formation of the visual image: The domain \(D\) where \(\Delta u = 1\) is precisely the region occupied by the toner. Off \(D \cup I\) the function \(u\) is in harmonic, and it satisfies the boundary conditions shown in the figure; further,

\[ [u_y]_I \equiv u_y(x, 0^+) - u_y(x, 0^-) = -\sigma, \quad -\gamma < x < \gamma.\]

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The domain \( D \) lies in \( \{y > 0\} \) and \( \Gamma = \partial D \cap \{y > 0\} \) is called the free boundary. After the development of the visual image has been completed, \( u \) becomes time-independent, and the equilibrium condition
\[
\frac{\partial u}{\partial N} = 0 \quad \text{on} \quad \Gamma
\]
must hold.

The above model was studied by Friedman and Hu [3] under the conditions
\[
(0.1) \quad h \leq b , \quad M < \sigma h .
\]

Both conditions are satisfied in the physical model. It was proved in [3] that if \( \gamma/a \) is close to 1 then the problem has a unique solution with \( \Gamma \) initiating at \( x = -a \) and terminating at \( x = a \), and \( u \equiv M \) above \( \Gamma \). On the other hand, if \( \gamma/a \) is small then [3] there exist infinitely many solutions for which the toner set \( D \) consists of two symmetric components. The case of small \( \gamma/a \) was also studied, more recently, by Hu and Wang [4] who proved the existence of a solution with \( D \) which is a connected region. It should be noted that both cases, \( \gamma/a \) near 1 and \( \gamma/a \) near 0, are physically not very interesting since most pixels are
neither nearly all dark nor nearly all light. For the case of intermediate $\gamma/a$, no existence results are known.

As the photocopying machines are becoming faster, there may not be enough time for the visual image to fully develop. Thus there is a need to study the time-dependent problem: how does the visual image evolve in time?

In the time-dependent case $u$ is a function of both $(x, y)$ and $t$, and the toner region $D = D(t)$ and the free boundary $\Gamma = \Gamma(t) = \partial D(t) \cap \{y > 0\}$ also depend on $t$. Denote by $V_n$ the velocity of points in $\Gamma(t)$ is the direction of the outward normal $N = N(t)$ to $D(t)$. The continuity equation for the charged toner is $\partial \rho / \partial t = -\nabla \cdot J$ where $J = \mu \rho \vec{E} = -\mu \rho \nabla u$ is the current density; here $\vec{E}$ is the electric field, $\rho$ the charge density and $\mu$ the mobility. Since $\rho \approx \text{const.} = \rho_0 > 0$ in the toner and $\rho = 0$ outside the toner, the continuity equation means that the free boundary $\Gamma(t)$ must move according to the law

$$V_n = -\frac{\partial u}{\partial N} \quad \text{on} \quad \Gamma(t),$$

provided we take $\mu \rho_0 = 1$.

In this paper we consider this evolutionary toner problem and prove the existence and (for a certain version of (0.2)) uniqueness of a solution for a small time interval $0 \leq t \leq t_0$. We also establish some geometric features of the free boundary. The main results are stated, more precisely, in §1, where the structure of the paper is also outlined.

§1. Statement of the main result. Set

$$R = \{(x, y); -a < x < a, -h < h < b\}.$$  

For simplicity we shall take $\gamma = 1$ in Figure 1; then, of course, $a > 1$.

Consider a family of curves

$$\Gamma(t) : y = f(x, t), \quad -x_0(t) < x < x_0(t)$$

for $0 \leq t \leq t_0$ satisfying the following properties:

$$f(x, t) = f(-x, t) \quad \text{and} \quad f(x, t) > 0 \quad \text{if} \quad |x| < x_0(t), \quad f(x_0(t), t) = 0, \quad \text{where}$$

$$1 < x_0(t) < a \quad \text{if} \quad 0 < t \leq t_0 \quad \text{and} \quad x_0(0) = 1,$$

$$|f_x(x, t)| \leq \frac{C}{|\log t|} \quad \text{if} \quad |x| < x_0(t) \quad \text{and}$$
\[ f_x(x,t) \leq -\frac{c}{|\log t|} \quad \text{if} \quad 1 < x < x_0(t) \quad (C > c > 0), \]
\[
[f_x(\cdot,t)]_{0,\alpha} \leq \frac{C_0}{|t\log t|^\alpha} \quad \left(0 < \alpha < \frac{1}{2}, \ C_0 > 0\right),
\]
\[ \forall \ \delta > 0, \ \gamma_1 < f_t(x,t) < \gamma_2 \quad \text{if} \quad |x| < 1 - \delta \quad (\gamma_i = \gamma_i(\delta) > 0), \]
\[ c_1 t|\log t| \leq x_0(t) - 1 \leq C_1 t|\log t| \quad (C_1 > c_1 > 0), \]
\[
\{(x,f(x,t))\} \quad \text{lies between the polygonal lines } \ell_1(t), \ell_2(t) \quad \text{where}
\]
\[ \ell_i(t) \text{ has vertices } (-1 - Q_it|\log t|,0),(-1,A_it), (1,A_it) \text{ and } (1 + Q_it|\log t|,0), \text{ for some constants } A_2 > A_1 > 0, \ Q_2 > Q_1 > 0.\]

Here \([ \quad ]_{0,\alpha}\) denotes the \(\alpha\)-Hölder coefficient in the variable \(x\).

Denote by \(D(t)\) the domain bounded by \(\Gamma(t)\) and the \(x\)-axis, and consider the following problem:

\[
(1.2) \quad \Delta u = \begin{cases} 
1 & \text{in } D(t) \\
0 & \text{in } \mathbb{R}\setminus D(t)
\end{cases},
\]

the limits \(u_y(x \pm 0)\) exist for \(-1 < x < 1\) and

\[
(1.3) \quad u_y(x,0+) - u_y(x,0-) = -\sigma \quad \text{if} \quad -1 < x < 1,
\]

\[
(1.4) \quad u \text{ is continuous in } \overline{\mathbb{R}} \times [0,t_0], \text{ and, } \forall \ t \in [0,t_0], \text{ and } u
\]

is continuously differentiable in \(\mathbb{R}\setminus\{(x,0); -1 \leq x \leq 1\},

\[
(1.5) \quad u(x,b,t) = M, \quad -a < x < a,
\]

\[
(1.6) \quad u(x,-h,t) = 0, \quad -a < x < a,
\]

\[
(1.7) \quad u_x(\pm a,y,t) = 0, \quad -h < y < b.
\]

By uniqueness \(u(x,y,t) = u(-x,y,t)\).

We now wish to consider \(\Gamma(t)\) as unknown, and impose the free boundary condition

\[(0.2). \quad \text{We shall recast (0.2), however, in a way which depends more directly on } u:\]

Suppose we write

\[
(1.8) \quad \Gamma(t) : x = x(t,\lambda), \ y = y(t,\lambda) \quad (0 \leq t \leq t_0)
\]
where $\lambda$ is a parameter such that and

\begin{equation}
(1.9) \quad x(0, \lambda) = \lambda, \; y(0, \lambda) = 0, \quad -\lambda_0(t) < \lambda < \lambda_0(t),
\end{equation}
\[ \lambda_0(0) = 1, \; \lambda_0(t) < 1 \quad \text{if} \quad t > 0. \]

Notice that the condition (0.2) means that

\[ \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \cdot N = -\frac{\partial u}{\partial N}. \]

This condition is satisfied if

\begin{equation}
(1.10) \quad \frac{dx}{dt} = -u_x(x, y, t), \quad \frac{dy}{dt} = -u_y(x, y, t). \]

We shall henceforth replace (0.2) by (1.10).

**Definition 1.1.** If $u, \Gamma$ satisfy (1.1)–(1.10) then we say that they form a solution to the *evolutionary toner problem* for $0 \leq t \leq t_0$.

The main result of this paper is the following:

**Theorem 1.1.** If (0.1) holds then there exists a unique solution to the evolutionary toner problem for some time interval $0 \leq t \leq t_0 \quad (t_0 > 0)$.

In §2 we shall study the function $\bar{u}(x, y)$, which is the initial state $u(x, y, 0)$ of the solution $u$. In §3 we establish interior $C^{2+\alpha}$ estimates for the potential

\[ \int_D \int \log\left[ (x - \xi)^2 + (y - \eta)^2 \right]^{1/2} d\xi d\tau \]

along $\Gamma$, where $D$ is a domain in $\{y > 0\}$ bounded by $\Gamma$ and the $y$-axis; it is assumed to $\Gamma$ is a $C^{1+\alpha}$ curve. These estimates are crucial for the proof of Theorem 1.1. Section 4 outlines the strategy for proving Theorem 1.1: it is based on establishing a fixed point for a mapping $\mathcal{M}$ of a family $\mathcal{A}$ of curves $\bar{\Gamma}(t) : x = \bar{x}(t, \lambda), y = \bar{y}(t, \lambda)$ into itself. The images $\Gamma(t) : x = x(t, \lambda), y = y(t, \lambda)$ are defined by the ODE (1.10) where $u$ is the solution to (1.2)–(1.7) and where $D(t)$ are the domains bounded by $\bar{\Gamma}(t)$ and the $x$-axis. The precise definition of $\mathcal{A}$ is given in §5. In §6 we begin with the analysis of the ODE, constructing barriers that are used to find simple yet sufficiently good approximations to the solution.

Next, in §§7 and 8 we establish, respectively, $C^1$ and $C^{1+\alpha}$ estimates for the curves $\Gamma(t)$, showing that $\mathcal{M}$ maps $\mathcal{A}$ into itself. Finally, in §9 we prove that a sequence of iterates $\mathcal{M}^n D(t)$ converges to a unique fixed point of $\mathcal{M}$, thereby completing the proof of Theorem 1.1.
The estimates

\begin{equation}
\frac{c}{|\log t|} \leq f_x(x,t) \leq \frac{C}{|\log t|} \quad \text{for} \quad 1 < x < x_0(t) \quad (C > c > 0)
\end{equation}

for the free boundary provide an interesting bound for the slope of \( \Gamma(t) \) as it descents toward the \( x \)-axis. The lower bound is critical to the proof of Theorem 1.1. The solution can in fact be continued, in time, as long as such a bound can be a priori established, provided \( f(x,t) \) remains uniformly positive for \(-1 < x \leq 1\).

In §10 we study the shape of the free boundary for \(|x| < 1\). We discover the rather surprising fact that, for any small \( \eta > 0 \), the free boundary \( y = f(x,t) \) for \( 0 \leq x \leq 1 - \eta \) cannot be monotone decreasing in \( x \). In fact, in “average” sense it is actually monotone increasing! On the other hand \( f(x,t) \) is monotone decreasing for \( x > 1 \).

\section*{§2. The initial state.} Initially the toner set is empty, i.e., \( D(0) = \phi \). Hence \( \overline{u}(x,y) \equiv u(x,y,0) \) satisfies:

\begin{equation}
-\Delta \overline{u} = \sigma \chi_{[-1,1]}(x) \delta(y) \quad \text{in} \quad R ,
\end{equation}

\begin{align*}
\overline{u}(x,b) &= M , \quad -a < x < a , \\
\overline{u}(x,-h) &= 0 , \quad -a < x < a , \\
\overline{u}_x(\pm a,y) &= 0 , \quad -h < y < b
\end{align*}

where \( \delta(y) \) is the Dirac function.

\textbf{Theorem 2.1.} The following inequalities hold:

\begin{equation}
\overline{u}_y(x,0+) < 0 \quad \text{if} \quad |x| < 1 ,
\end{equation}

\begin{equation}
\overline{u}_y(x,0) > 0 \quad \text{if} \quad 1 < |x| < a .
\end{equation}

\textbf{Proof.} By the maximum principle, \( \overline{u}(x,y) > 0 \) in \( R \). Introduce the function

\[ v(x,y) = \overline{u}(x,y) - \overline{u}(x,-y) \quad \text{in} \quad -a < x < a , \ 0 < y < h . \]

Notice that

\[ v(x,h) = \overline{u}(x,h) > 0 . \]

Since \( v(x,0) = 0 \) and \( v \) is harmonic, it follows, by the maximum principle, that \( v_y(x,0+) > 0 \). Noting that \( v_y(x,0+) = \overline{u}_y(x,0+) + \overline{u}_y(x,0-) \) and that, for \( |x| > 1, u \) is smooth and thus \( \overline{u}_y(x,0-) = \overline{u}_y(x,0+) \), it follows that

\[ 2\overline{u}_y(x,0) = v_y(x,0+) > 0 \quad \text{if} \quad |x| > 1 , \]
i.e., (2.3) holds.

If \(|x| < 1\) then \(\bar{u}_y(x, 0+) - \bar{u}_y(x, 0-) = -\sigma\), so that

\[
2\bar{u}_y(x, 0+) = -\sigma + v_y(x, 0+) .
\]

(2.4)

In order to estimate \(v_y(x, 0+)\) from above we shall first obtain an upper bound for \(v(x, h)\). Let \(w(x, y)\) be the solution to

\[-\Delta w = \sigma \chi_{(-a, a)}(y) \delta(y) \quad \text{in} \quad R\]

with the same boundary condition as \(\bar{u}\). By comparison

\[
\bar{u}(x, y) \leq w(x, y) .
\]

(2.5)

Observe that \(w\) is independent of \(x\) and, in fact, as easily verified,

\[
w(x, y) = \begin{cases} 
M + \sigma b \frac{y + h}{b + h} & \text{for} \quad -h < y < 0, \\
\frac{M + \sigma b}{b + h} h + \left(\frac{M + \sigma b}{b + h} - \sigma\right) y & \text{for} \quad 0 < y < b .
\end{cases}
\]

Hence, by (2.5),

\[v(x, h) = \bar{u}(x, h) \leq w(x, h) = 2M + \sigma b \frac{h}{b + h} - \sigma h \equiv B .\]

The harmonic function \(V(x, y) = By/h\) majorizes \(v\) on \(y = h\) and \(v(x, 0) = V(x, 0) = 0\). Also \(v_x = V_x = 0\) on \(x = \pm a\). By comparison it then follows that \(v \leq V\) and

\[v_y(x, 0+) < V_y(x, 0) = \frac{B}{h} = -\sigma + 2\frac{M + \sigma b}{b + h} .\]

Recalling (2.4) we get, for \(|x| < 1\),

\[
2\bar{u}_y(x, 0+) < -2\sigma + \frac{2(M + \sigma b)}{b + h} = \frac{2(M - \sigma h)}{b + h} \leq 0 \quad \text{by} \quad (0.1),
\]

and (2.2) follows.

Remark 2.1. Theorem 2.1 implies that the velocity of the free boundary is initially positive for all \(x \in (-1, 1)\), but not for \(|x| > 1\). This means that \(\Gamma(t)\) begins growing only from points of the interval

\[
I = \{(x, 0); -1 < x < 1\} .
\]

(2.6)
Set

\[(2.7) \quad \Phi(x, y) = -\frac{\sigma}{2\pi} \int_{-1}^{1} \log[(x - \xi)^2 + y^2]^{1/2} d\xi. \]

Then \( \Phi \) is harmonic function off \( I \), and it satisfies the jump relation

\[(2.8) \quad [\Phi_y] \equiv \Phi_y(x, 0+) - \Phi_y(x, 0-) = -\sigma, \quad -1 < x < 1. \]

We can write

\[(2.9) \quad \overline{u}(x, y) = \Phi(x, y) + \psi(x, y) \]

where \( \psi \) is harmonic in \( R \).

\section*{§3. \( C^{1+\alpha} \) estimates on \( \nabla G(x, f(x)) \).} In this section we study the function

\[(3.1) \quad G(x, y) = \frac{1}{2\pi} \iint_{D} \log[(x - \xi)^2 + (y - \eta)^2]^{1/2} d\xi d\eta \]

where \( D \) is a domain given by

\[(3.2) \quad D = \{ 0 < y < f(x), \quad -a_0 < x < a_0 \}, \]

for some \( a_0 \in (0, a) \). Denote by \( B_\mu(x_0, y_0) \) the disc \( \{(x - x_0)^2 + (y - y_0)^2 < \mu^2\} \), and set, for simplicity, \( B_\mu(x_0) = B_\mu(x_0, f(x_0)) \). We assume that \( f \) satisfies the following conditions:

\[(3.3) \quad f(x) = f(-x) \quad \text{and} \quad f(x) > 0 \quad \text{if} \quad -a_0 < x < a_0, \quad f(\pm a_0) = 0, \]

and, for some \( (x_0, y_0) \) and \( \mu > 0 \) satisfying

\[(3.4) \quad -a_0 + \mu < x_0 < a_0 - \mu, \quad \mu < y_0 < b - \mu, \]

there holds:

\[(3.5) \quad |f'(x_1)| \leq 1, \]

\[(3.6) \quad \frac{|f'(x_1) - f'(x_2)|}{|x_1 - x_2|^\alpha} \leq \frac{\theta}{\mu^\alpha} \quad (\mu^\alpha < \theta), \]

for any \((x_i, f(x_i)) \quad (i = 1, 2)\) in \( B_\mu(x_0) \), where \( \theta, \alpha \) are positive constants and \( \alpha < 1 \).

Set

\[(3.7) \quad \nabla G(x, f(x)) = \frac{1}{2\pi} \left( \iint_{D} \frac{x - \xi}{(x - \xi)^2 + (f(x) - \eta)^2} d\xi d\eta, \iint_{D} \frac{f(x) - \eta}{(x - \xi)^2 + (f(x) - \eta)^2} d\xi d\eta \right). \]

In this section we prove:
Lemma 3.1. Under the assumptions (3.3)-(3.6), for all \((x, f(x)), (\bar{x}, \bar{f}(\bar{x}))\) in \(B_{\mu/\theta}(x_0)\) there holds:

\begin{equation}
\frac{dA(x)}{dx} + \frac{dB(x)}{dx} \leq C(|\log \mu| + \theta),
\end{equation}

\begin{equation}
\frac{1}{|x - \bar{x}|^\alpha} \left\{ \frac{dA(x)}{dx} - \frac{dA(\bar{x})}{dx} + \frac{dB(x)}{dx} - \frac{dB(\bar{x})}{dx} \right\} \leq \frac{C}{\mu^\alpha} + \frac{C \theta |\log \mu|}{\mu^\alpha}
\end{equation}

where \(C\) is a positive constant independent of \(\theta, \alpha\) and \(\mu\).

Proof. For any small \(\rho > 0\) introduce the truncated integrals

\begin{equation}
A_{\rho}(x) = \iint_{D \setminus B_{\rho}(x)} \frac{x - \xi}{(x - \xi)^2 + (f(x) - \eta)^2} \, d\xi d\eta,
\end{equation}

\begin{equation}
B_{\rho}(x) = \iint_{D \setminus B_{\rho}(x)} \frac{f(x) - \eta}{(x - \xi)^2 + (f(x) - \eta)^2} \, d\xi d\eta.
\end{equation}

As easily verified

\begin{equation}
\frac{dA_{\rho}(x)}{dx} = \iint_{D \setminus B_{\rho}(x)} \frac{-(x - \xi)^2 + (f(x) - \eta)^2 - 2(f(x) - \eta)(x - \xi)f'(x)}{(x - \xi)^2 + (f(x) - \eta)^2} \, d\xi d\eta
\end{equation}

\begin{equation}
+ \iint_{\partial B_{\rho}(x) \cap D} \frac{x - \xi}{(x - \xi)^2 + (f(x) - \eta)^2} N \cdot e_1 \, dS_{\xi \eta} \equiv I_{1\rho}(x) + I_{2\rho}(x)
\end{equation}

where \(N\) is the exterior normal to \(\partial B_{\rho}(x)\) and \(e_1\) is the unit vector in the direction of the \(x\)-axis. Writing

\((\xi, \eta) = (x, f(x)) + \rho(\omega_1, \omega_2)\quad (\omega_1^2 + \omega_2^2 = 1)\)

in \(I_{2\rho}\), we get

\begin{equation}
I_{2\rho} = - \int_{\partial B_1(x) \cap \{ \rho - (x, f(x)) \}} \frac{\omega_1 (N \cdot e_1)}{|\omega|^2} \, d\omega
\end{equation}

\begin{equation}
= - \int_{\partial B_1(x) \cap \{ \omega_2 - f(x) < f'(x)(\omega_1 - x) \}} \frac{\omega_1^2}{|\omega|^2} \, d\omega + \int_{S} \frac{\omega_1^2}{|\omega|^2} \, d\omega
\end{equation}
where
\[ S = \partial B_1(x) \cap \left\{ \frac{D - (x, f(x))}{\rho} \right\} \Delta \{\omega_2 - f(x) < f'(x)(\omega_1 - x)\} \]

and \( A \Delta B = (A \setminus B) \cup (B \setminus A) \). The first integral on the right-hand side of (3.13) is equal to \(-\frac{\pi}{2}\). Since the set \( S \) is contained in the set of points \((\xi, \eta)\) such that
\[
|\eta - f(x) - f'(x)(\xi - x)| \leq \frac{\theta}{\mu^\alpha} |\xi - x|^{1+\alpha},
\]
we deduce from (3.13) that

(3.14)
\[
|I_{2\rho} + \frac{\pi}{2}| \leq C \frac{\theta}{\mu^\alpha} \rho^\alpha.
\]

To estimate \( I_{1\rho} \) assume first that \( f'(x) = 0 \) and denote \( x - \xi \) by \(-\xi\) and \( f(x) - \eta \) by \(-\eta\). Then
\[
I_{1\rho} = \iint_{D \setminus B_\rho(0)} \frac{-\xi^2 + \eta^2}{(\xi^2 + \eta^2)^2} \, d\xi d\eta.
\]

For \( \mu > \rho > \sigma > 0 \) we have
\[
I_{1\sigma} - I_{1\rho} = \iint_{[B_\rho(0) \setminus B_\sigma(0)] \cap D} \frac{-\xi^2 + \eta^2}{(\xi^2 + \eta^2)^2} \, d\xi d\eta = \iint_{[B_\rho(0) \setminus B_\sigma(0)] \cap \{\eta < 0\}} \frac{-\xi^2 + \eta^2}{(\xi^2 + \eta^2)^2} \, d\xi d\eta
\]
\[
+ \left\{ \iint_{B_\rho(0) \setminus B_\sigma(0) \cap D} \frac{-\xi^2 + \eta^2}{(\xi^2 + \eta^2)^2} \, d\xi d\eta - \iint_{[B_\rho(0) \setminus B_\sigma(0)] \cap \{\eta < 0\}} \frac{-\xi^2 + \eta^2}{(\xi^2 + \eta^2)^2} \, d\xi d\eta \right\}
\]
\[
\equiv J_1 + J_2.
\]

Clearly \( J_1 = 0 \). In \( J_2 \) the symmetric difference between the two domains of integration is contained in the set of points \((\xi, \eta)\) such that
\[
|\eta| \leq \frac{\theta}{\mu^\alpha} |\xi|^{1+\alpha}
\]
(recall our assumption that \( f' \) vanishes at \( x \)). Hence
\[
|J_2| \leq C \int_{\sigma}^\rho r \, dr \int_0^{C^\theta/\mu^\alpha} \frac{d\phi}{r^2} = C \frac{\theta}{\mu^\alpha} (\rho^\alpha - \sigma^\alpha)
\]

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where \((r, \varphi)\) are polar coordinates, and thus

\[
|I_{1\rho} - I_{1\sigma}| \leq C \frac{\theta}{\mu^\alpha} (\rho^\alpha - \sigma^\alpha).
\]

It follows that \(\{I_{1\rho}\}\) forms a Cauchy sequence and therefore it has a limit, say \(L\); further,

\[
|I_{1\rho} - L| \leq \frac{C \theta}{\mu^\alpha} \rho^\alpha.
\]

So far we have assumed that \(f'(x) = 0\). If \(f'(x) \neq 0\) then we obtain another term

\[C |f'(x)| (\rho^\alpha - \sigma^\alpha)\]

on the right-hand side of (3.15), and since \(|f'(x)| < 1\), (3.15) and (3.16) remain valid with another constant \(C\).

From what we have proved so far it follows that

\[
\frac{dA(x)}{dx} = -\frac{\pi}{2} + \lim_{\rho \to 0} \iint_{D \setminus B_\rho(x)} \frac{-(x - \xi)^2 + (f(x) - \eta)^2 - 2(f(x) - \eta)(x - \xi) f'(x)}{(x - \xi)^2 + (f(x) - \eta)^2} d\xi d\eta
\]

\[
= -\frac{\pi}{2} + \iint_{D \setminus B_\rho(x)} \cdots + \lim_{\rho \to 0} \iint_{[B_\rho(x) \setminus B_\rho(x)] \cap D} \cdots
\]

where the limit exists, and the last term is actually bounded by

\[
\lim_{\rho \to 0} |I_{1\rho} - I_{1\mu}| = C \theta, \quad \text{by (3.15)}.
\]

Since

\[
\left| \iint_{D \setminus B_\rho(x)} \cdots \right| \leq \iint_{D \setminus B_\rho(x)} \frac{C}{r^2} r dr d\varphi \leq C |\log \mu|,
\]

it follows that \(|dA(x)/dx| \leq C(|\log \mu| + \theta)|.

We next consider \(B_\rho(x)\). We can write

\[
\frac{dB_\rho(x)}{dx} = \iint_{D \setminus B_\rho(x)} \frac{f'(x)(x - \xi)^2 - 2(x - \xi)(f(x) - \eta) - (f(x) - \eta)^2 f'(x)}{(x - \xi)^2 + (f(x) - \eta)^2} d\xi d\eta
\]

\[+ \int_{\partial B_\rho(x) \cap D} \frac{(f(x) - \eta) N \cdot e_1}{(x - \xi)^2 + (f(x) - \eta)^2} dS_\eta \equiv J_{1\rho} + J_{2\rho}.
\]

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We can proceed as in the case of $A_\rho(x)$; the only difference is in evaluating $\lim J_{2\rho}$:

$$\lim_{\rho \to 0} J_{2\rho} = \int_{\partial B_1(x) \cap \{ \omega_2 - f(x) < f'(x)(\omega_1 - x) \}} \frac{\omega_2 \omega_1}{|\omega|^2} \ d\omega_1 d\omega_2 = 0 .$$

We obtain, analogously to (3.17),

$$\frac{dB(x)}{dx} = \lim_{\rho \to 0} \iint_{D \setminus B_\rho(x)} \frac{f'(x)(x - \xi)^2 - 2(x - \xi)(f(x) - \eta) - (f(x) - \eta)^2 f'(x)}{(x - \xi)^2 + (f(x) - \eta)^2} \ d\xi d\eta$$

where the last limit in fact exists; furthermore, $|dB/dx|$ is bounded $C(|\log \mu| + \theta)$.

We now proceed to estimate the $\alpha$-Hölder coefficient of $dA/dx$. Introducing the function

$$\Phi(x, \xi, \eta) = \frac{-(x - \xi)^2 + (f(x) - \eta)^2 - 2(f(x) - \eta)(x - \xi)f'(x)}{(x - \xi)^2 + (f(x) - \eta)^2}$$

we can write

$$\frac{dA_\rho(x)}{dx} \iint_{[B_\rho(x_0) \setminus B_\rho(x)] \cap D} \Phi(x, \xi, \eta) \ d\xi d\eta + \iint_{D \setminus B_\rho(x_0)} \Phi(x, \xi, \eta) \ d\xi d\eta$$

$$+ \int_{\partial B_\rho(x) \cap D} \frac{x - \xi}{(x - \xi)^2 + (f(x) - \eta)^2} N \cdot e_1 dS_\xi \eta = E_1(x, \rho, \mu) + E_2(x, \rho, \mu) + E_3(x, \rho) ,$$

where we shall take $0 < \rho < \mu/8$.

We begin by estimating

$$L = \frac{1}{|x - \bar{x}|^\alpha} |E_1(x, \rho, \mu) - E_1(\bar{x}, \rho, \mu)| ,$$

distinguishing two cases:

- **Case (i):** $[(x - \bar{x})^2 + (f(x) - f(\bar{x}))^2]^{1/2} \leq \frac{\rho}{4}$,
- **Case (ii):** $[(x - \bar{x})^2 + (f(x) - f(\bar{x}))^2]^{1/2} \geq \frac{\rho}{4}$.
By (3.19) and the mean value theorem,

\[ |\Phi(x, \xi, \eta) - \Phi(\bar{x}, \xi, \eta)| \leq \sup |\frac{d}{dx} \left( -\frac{(x - \xi)^2 + (f(x) - \eta)^2}{(x - \xi)^2 + (f(x) - \eta)^2} \right) ||x - \bar{x}| \]

\[ + \sup |\frac{2(f(x) - \eta)(x - \xi)}{(x - \xi)^2 + (f(x) - \eta)^2}||f'(x) - f'(\bar{x})| \]

\[ + \sup |f'(x)| \cdot \sup |\frac{d}{dx} \left( \frac{2(f(x) - \eta)(x - \xi)}{(x - \xi)^2 + (f(x) - \eta)^2} \right) | |x - \bar{x}| \]

where \( \sup |h(x)| \) here means the sup of \( |h(\bar{x})| \) when \( \bar{x} \) varies over the interval with endpoints \( x, \bar{x} \). Setting

\[ R = [(x - \xi)^2 + (f(x) - \eta)^2]^{1/2} \]

and using the assumption of case (i), we easily find that

\[ |\Phi(x, \xi, \eta) - \Phi(\bar{x}, \xi, \eta)| \leq \frac{C}{R^3} |x - \bar{x}| + \frac{C\theta}{\mu^\alpha R^2} |x - \bar{x}|^\alpha \]

(3.22)

\[ \leq \left( \frac{C\rho^{1-\alpha}}{R^3} + \frac{C\theta}{\mu^\alpha R^2} \right) |x - \bar{x}|^\alpha , \]

since \( |x - \bar{x}| < \rho < R \).

For \( x \in B_{\mu/4}(x_0) \) we have

\[ \int \int_{[B_{\mu/4}(x) \setminus B_{\rho}(x)] \cap \{\eta - f(x) < f'(x)(x - \xi)\}} \Phi(x, \xi, \eta) d\xi d\eta = 0 . \]

Therefore

\[ E_1(x, \rho, \mu) = \int \int_{[B_\mu(x_0) \setminus B_{\mu/4}(x)] \cap D} \Phi(x, \xi, \eta) d\xi d\eta \]

(3.23)

\[ + \int \int_{S_\rho(x)} \Phi(x, \xi, \eta) d\xi d\eta \equiv J_1(x) + J_2(x) \]

where

\[ S_\rho(x) = \left\{ [B_{\mu/4}(x) \setminus B_{\rho}(x)] \cap D \right\} \Delta \left\{ [B_{\mu/4}(x) \setminus B_{\rho}(x)] \cap \{\eta - f(x) < f'(x)(x - \xi)\} \right\} . \]

Set

\[ \Omega_1 = [B_\mu(x_0) \setminus B_{\mu/4}(x)] \cap D, \quad \Omega_2 = [B_\mu(x_0) \setminus B_{\mu/4}(\bar{x})] \cap D . \]
To estimate \( J_1(x) - J_1(\bar{x}) \) we use the estimate (3.22) if \((\xi, \eta) \in \Omega_1 \cap \Omega_2\); if \((\xi, \eta) \notin \Omega_1 \cap \Omega_2\) then we simply use the estimate

\[
|\Phi(x, \xi, \eta)| \leq \frac{C}{R^2}
\]

and a similar estimate for \( \Phi(\bar{x}, \xi, \eta) \) (with \( \bar{R} \)). We get

\[
\frac{1}{|x - \bar{x}|^\alpha} |J_1(x) - J_1(\bar{x})| \leq C \int \int_{B_\rho(x) \setminus B_{\rho/4}(x_0)} \left( \frac{\rho^{1-\alpha}}{R^3} + \frac{\theta}{\alpha R^2} \right) d\xi d\eta + \frac{C}{|x - \bar{x}|^\alpha} \int \int_{\Omega_1 \Delta \Omega_2} \left( \frac{1}{R^2} + \frac{1}{\bar{R}^2} \right) d\xi d\eta.
\]

The first integral is bounded by

\[
\frac{C \rho^{1-\alpha}}{\mu^\alpha} + \frac{C \theta}{\mu^\alpha}.
\]

In the second integral \( r \) and \( \bar{R} \) are \( \approx \mu \) and the domain of integration has area \( O(\mu|x - \bar{x}|) \). Hence the integral is bounded by \( C|x - \bar{x}|/\mu \). We conclude that

\[
(3.24) \quad \frac{1}{|x - \bar{x}|^\alpha} |J_1(x) - J_1(\bar{x})| \leq \frac{C}{\mu^\alpha} + \frac{C \theta}{\mu^\alpha}.
\]

To estimate \( |J_2(x) - J_2(\bar{x})| \) we first examine the difference of the sets \( S_\rho(x) \) and \( S_\rho(\bar{x}) \). These sets have the form \( S_\rho(x) = \Omega_1 \cap \tilde{\Omega}_1 \) and \( S_\rho(\bar{x}) = \Omega_2 \cap \tilde{\Omega}_2 \) where

\[
\tilde{\Omega}_1 = [B_{\rho/4}(x) \setminus B_\rho(x)] \cap \{ \eta - f(x) \leq f'(x)(x - \xi) \},
\]

\[
\tilde{\Omega}_2 = [B_{\rho/4}(\bar{x}) \setminus B_\rho(\bar{x})] \cap \{ \eta - f(\bar{x}) < f'(\bar{x})(\bar{x} - \xi) \}.
\]

We can write

\[
|J_2(x) - J_2(\bar{x})| \leq \int \int_{S_\rho(x) \cap S_\rho(\bar{x})} |\Phi(x, \xi, \eta) - \Phi(\bar{x}, \xi, \eta)|
\]

(3.25)

\[
\int \int_{S_\rho(x) \Delta S_\rho(\bar{x})} (|\Phi(x, \xi, \eta)| + |\Phi(\bar{x}, \xi, \eta)|).
\]
Using (3.22) we find that the first integral is bounded by
\[
C|x - \bar{x}|^\alpha \int_\rho^{\mu/4} \int_0^{\theta R^\alpha/\mu^\alpha} \left( \frac{\rho^{1-\alpha}}{R^3} + \frac{\theta}{\mu^\alpha R^2} \right) \, d\varphi
\]
\[
\leq C|x - \bar{x}|^\alpha \left( \frac{1}{\mu^\alpha} + \frac{\theta}{\mu^\alpha} \right).
\]
In the second integral the integrand is bounded by \(C/R^2\). Furthermore, the domain of integration, \((\Omega_1 \cap \bar{\Omega}_1) \Delta (\Omega_2 \cap \bar{\Omega}_2)\), has interior piece with \(R \approx \rho\) and area \(\approx |x - \bar{x}|^{\rho^{1+\alpha}}\) and exterior piece with \(R \approx \mu\) and area \(\approx |x - \bar{x}|^{\mu^{1+\alpha}}\). Hence this integral is bounded by
\[
C|x - \bar{x}| \left( \frac{\rho^{1+\alpha}}{\rho^2} + \frac{\mu^{1+\alpha}}{\mu^2} \right) \leq C|x - \bar{x}|^\alpha.
\]
We conclude that
\[
\frac{1}{|x - \bar{x}|^\alpha} |J_2(x) - J_2(\bar{x})| \leq \frac{C}{\mu^\alpha} + \frac{C\theta}{\mu^\alpha}.
\]
Combining this with (3.24) and recalling (3.23), we obtain
\[
\frac{1}{|x - \bar{x}|^\alpha} |E_1(x, \rho, \mu) - E_1(\bar{x}, \rho, \mu)| \leq \frac{C(1 + \theta)}{\mu^\alpha},
\]
in case (i) holds.

Consider next case (ii) and set
\[
\lambda = 4[(x - \bar{x})^2 + (f(x) - f(\bar{x}))^2]^{1/2}.
\]
The same argument that was used to prove (3.24) shows that
\[
\frac{1}{|x - \bar{x}|^\alpha} \int \int \Phi(x, \xi, \eta) \, d\xi \, d\eta - \int \int \Phi(\bar{x}, \xi, \eta) \, d\xi \, d\eta \leq \frac{C(1 + \theta)}{\mu^\alpha}.
\]
On the other hand, by the proof of (3.15),
\[
\frac{1}{|x - \bar{x}|^\alpha} \int \int \Phi(x, \xi, \eta) \, d\eta \, d\xi - \int \int \Phi(\bar{x}, \xi, \eta) \, d\eta \, d\xi \leq \frac{1}{|x - \bar{x}|^\alpha} \int \int |\Phi(x, \xi, \eta)| + \int \int |\Phi(\bar{x}, \xi, \eta)| \leq C \frac{1}{|x - \bar{x}|^\alpha} \frac{\theta}{\mu^\alpha} \lambda \leq C \frac{\theta}{\mu^\alpha}.
\]
We conclude that (3.26) holds also in case (ii).

From the definition of $E_2$ in (3.20) we see that we can apply (3.22) provided $(x, f(x))$ and $(\bar{x}, f(\bar{x}))$ belong to $B_{\mu/2}(x_0)$. We then get

$$
\frac{1}{|x-\bar{x}|^\alpha} |E_2(x, \rho, \mu) - E_2(\bar{x}, \rho, \mu)| \leq \int_{D \setminus B\mu(x_0)} C \left( \frac{\mu^{1-\alpha}}{R^3} + \frac{\theta}{\mu^\alpha R^2} \right) d\xi d\eta
$$

(3.27)

$$
\leq C \left( \frac{1}{\mu^\alpha} + \frac{\theta}{\mu^\alpha |\log \mu|} \right).
$$

Consider finally

$$
E_3(x, \rho) - E_3(\bar{x}, \rho) = \int_{\partial B\mu(x) \cap D} G(x, \xi, \eta) dS_{\xi\eta}
$$

$$
- \int_{\partial B\mu(\bar{x}) \cap D} G(\bar{x}, \xi, \eta) dS_{\xi\eta}
$$

where

$$
G(x, \xi, \eta) = \frac{x - \xi}{(x - \xi)^2 + (f(x) - \eta)^2} N \cdot e_1
$$

$$
= -\frac{(x - \xi)^2}{[(x - \xi)^2 + (f(x) - \eta)^2]^{3/2}}
$$

in the first integral, and $G(\bar{x}, \xi, \eta)$ is similarly defined in the second integral. Notice that

(3.28)

$$
\int_{\partial B\mu(x) \cap \{\eta - f(x) < f'(x)(\xi-x)\}} G(x, \xi, \eta) dS_{\xi\eta} = -\frac{\pi}{4}.
$$

Also, with $v = (v_1, v_2) = (\bar{x} - x, f(\bar{x}) - f(x))$,

$$
E_3(x, \rho) - E_3(\bar{x}, \rho) = \int_{\partial B\mu(x) \cap D(t)} G(x, \xi, \eta) dS_{\xi\eta} - \int_{\partial B\mu(x) \cap \{D(t) - v\}} G(\bar{x}, \xi + v_1, \eta + v_2) dS_{\xi\eta}.
$$

Since, as immediately seen,

$$
G(\bar{x}, \xi + v_1, \eta + v_2) = G(x, \xi, \eta),
$$

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we deduce, after using (3.28), that

\[ |E_3(x, \rho) - E_3(\overline{x}, \rho)| \leq \int_{\partial B(x) \cap \{ f(x) < f(x)(\xi - x) \}} |G(x, \xi, \eta)| dS_{\xi \eta} \]

where \( \Omega \) is the symmetric difference of the sets \( D(t) \) and \( D(t) - v \). In the last integral, the integrand is bounded by \( C/\rho \) and the domain of integration has measure

\[ \leq \frac{C\theta}{\mu^\alpha} |x - \overline{x}|^\alpha \rho . \]

Hence

\[ (3.29) \quad \frac{1}{|x - \overline{x}|^\alpha} |E_3(x, \rho) - E_3(\overline{x}, \rho)| \leq \frac{C\theta}{\mu^\alpha} . \]

Combining (3.26), (3.27), (3.29) and using (3.20), we deduce the Hölder estimate (3.9) for \( dA/dx \). The same analysis can be used to derive the Hölder estimate (3.9) for \( dB/dx \).

§4. Outline of the proof of the main result.

To prove Theorem 1.1 we shall proceed as follows:

Choose a family of curves

\[ \tilde{\Gamma}(t) : x = \tilde{x}(t, \lambda), \ y = \tilde{y}(t, \lambda), \ 0 \leq t \leq t_0 , \]

(4.1)

\[ \tilde{x}(0, \lambda) = \lambda, \ \tilde{y}(0, \lambda) = 0, \ \lambda_0(t) < \lambda < \lambda_0(t) \]

with \( \lambda_0(0) = \lambda \), \( \lambda_0(t) > 1 \) if \( t > 0 \) and \( \tilde{y}(t, \lambda) \geq 0 \), and denote by \( \tilde{D}(t) \) the set bounded by \( \tilde{\Gamma}(t) \) and the \( x \)-axis.

Recall that the function \( \bar{u}(x, y) \) (= \( u(x, y, 0) \)) studied in §1 has the form (2.9) where \( \Phi(x, y) \) is defined in (2.7), and

\[ \Delta \psi = 0 \ \text{in} \ R , \]

\[ \psi(x, b) = M - \Phi(x, b) , \]

(4.2)

\[ \psi(x, -h) = -\Phi(x, -h) , \]

\[ \psi_x(\pm a, y) = -\Phi_x(\pm a, y) . \]

Let \( w \) be the solution of

\[ \Delta w = \chi_{\tilde{D}(t)}(x, y) \ \text{in} \ R , \]

(4.3)

\[ w(x, b) = 0 , \]

\[ w(x, -h) = 0 , \]

\[ w_x(\pm a, y) = 0 . \]
We can write

\[ w(x, y, t) = G(x, y, t) + \tilde{\zeta}(x, y, t) \]  

where

\[ G(x, y, t) = \frac{1}{2\pi} \iint_{D(t)} \log[(x - \xi)^2 + (y - \eta)^2]^{1/2} d\xi d\eta \]  

and

\[ \Delta \tilde{\zeta} = 0 \quad \text{in} \quad R, \]

\[ \tilde{\zeta}(x, b, t) = -G(x, b, t), \]

\[ \tilde{\zeta}(x, -h, t) = -G(x, -h, t), \]

\[ \tilde{\zeta}_x(\pm a, y, t) = -G_x(\pm a, y, t). \]

Set

\[ u = \bar{u} + w = \Phi + \psi + G + \tilde{\zeta}. \]

and consider the differential equation (1.10).

In the following sections it will be shown that this system has a solution \( x = x(t, \lambda), y = y(t, \lambda) \) for \( 0 \leq t \leq t_0 \); it determines a family of curves \( \Gamma(t) \) as in (1.8) and a family of sets \( D(t) \):

\[ D(t) \text{ is bounded by } \Gamma(t) \text{ and } \{y = 0\}. \]

Our plan is to show that the mapping from \( \{\tilde{D}(t)\} \) to \( \{D(t)\} \) has a unique fixed point. The solution \( u = u(x, y, t) \) corresponding to this fixed point, together with the corresponding family \( \Gamma(t) \) form a solution to the evolutionary toner problem, i.e., to (1.1)–(1.10).

The existence proof will also establish an asymptotic behavior of the curves \( \Gamma(t) \) near the critical points \((\pm 1, 0), \) for \( t \to 0.\)

§5. The ODE. We assume that the function \( \tilde{x}(t, \lambda), \tilde{y}(t, \lambda) \) are defined in

\[ Q_{t_0} = \{(t, \lambda); -1 < \lambda < 1, 0 < t < \min(t_0, \tilde{t}(\lambda))\} \]
for some \( t_0 > 0 \), and satisfy the following conditions:

\[
\begin{align*}
\bar{x}(0, \lambda) &= \lambda, \quad \bar{y}(0, \lambda) = 1, \quad \bar{y}(t, \lambda) > 0 \quad \text{if} \quad 0 < t < \bar{t}(\lambda), \quad \bar{y}(\bar{t}(\lambda), \lambda) = 0, \\
\bar{x}(t, -\lambda) &= -\bar{x}(t, \lambda), \quad \bar{y}(t, -\lambda) = \bar{y}(t, -\lambda),
\end{align*}
\]

\( \lambda \to \bar{x}(t, \lambda) \) and \( \lambda \to \bar{y}(t, \lambda) \) belong to \( C^{1+\alpha} \) for each \( t \in [0, t_0] \) and \( \frac{1}{2} < \frac{\partial}{\partial \lambda} \bar{x}(t, \lambda) < 2 \);

consequently, the inverse function \( \lambda = \bar{\lambda}(x, t) \) of \( x = \bar{x}(t, \lambda) \) is well defined. Setting

\[ y = \bar{y}(t, \lambda) = \bar{y}(t, \bar{\lambda}(x, t)) \equiv f(x, t), \]

the function \( f(x, t) \) is then defined for \( -x_0(t) < x < x_0(t) \), for some \( x_0(t) > 1 \), and

\[
\begin{align*}
f(x, t) &= f(-x, t), \\
f(x, t) &> 0 \quad \text{if} \quad -x_0(t) < x < x_0(t) \\
&= 0 \quad \text{if} \quad |x| = x_0(t),
\end{align*}
\]

\[
1 + Q_1 t |\log t| < x_0(t) < 1 + Q_2 t |\log t| \quad (Q_2 > Q_1 > 0),
\]

the curve \( \{(x, f(x, t))\} \) lies between the polygonal lines \( \ell_1(t), \ell_2(t) \) where \( \ell_i(t) \) has vertices \((-1 - Q_i t |\log t|, 0),
\((-1, \gamma_i t), (1, \gamma_i t) \) and \((1 + Q_i t |\log t|, 0)\)

where \( \gamma_2 > \gamma_1 > 0 \),

\[
|f_x(x, t)| < \frac{L_1}{|\log t|} \quad \text{if} \quad |x| \leq x_0(t),
\]

\[
f_x(x, t) > \frac{L_0}{|\log t|} \quad \text{if} \quad 1 < x < x_0(t)
\]

where \( 0 < L_0 < L_1 \), and

\[
\frac{|f_x(x, t) - f_x(\bar{x}, t)|}{|x - \bar{x}|^{\alpha}} \leq \frac{L_2}{\min[f(x, t), f(\bar{x}, t)]^{\alpha}} \quad \text{if} \quad -x_0(t) < x, \bar{x} < x_0(t)
\]
where $L_2 > 0$, $0 < \alpha < \frac{1}{2}$.

Set

$$W = (Q_0, Q_1, Q_3, L_0, L_1, L_2, \alpha)$$

and denote by $A_W$ the class of functions $(\bar{x}, \bar{y})$ satisfying (5.1)-(5.9).

We shall need an explicit form for the first derivatives of the function $\Phi$ defined in (2.7):

$$\frac{\partial \Phi(x, y)}{\partial x} = -\frac{\sigma}{4\pi} \int_{-1}^{1} \frac{\partial}{\partial x} \log[(x - \xi)^2 + y^2] d\xi$$

$$= \frac{\sigma}{4\pi} \log[(x - \xi)^2 + y^2]_{-1}^{1} = \frac{\sigma}{4\pi} \log \frac{(x - 1)^2 + y^2}{(x + 1)^2 + y^2}$$

and

$$\frac{\partial \Phi(x, y)}{\partial y} = -\frac{\sigma}{2\pi} \int_{-1}^{1} \frac{y}{(x - \xi)^2 + y^2} d\xi$$

$$= -\frac{\sigma}{2\pi} \left[ \arctan \frac{1 - x}{y} + \arctan \frac{1 + x}{y} \right]$$

where arctan is taken to have values in $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$ (arctan$(-x) = -\text{arctan} x$).

Recalling (4.7), the ODE (1.10) can then be written in the form

$$\dot{x} = -\frac{\sigma}{4\pi} \log \frac{(1 - x)^2 + y^2}{(1 + x)^2 + y^2} - \psi_x(x, y) - w_x(x, y, t),$$

$$\dot{y} = \frac{\sigma}{2\pi} \left\{ \arctan \frac{1 - x}{y} + \arctan \frac{1 + x}{y} \right\} - \psi_y(x, y) - w_y(x, y, t).$$

From (4.5) we have

$$|\nabla G(x, y, t)| \leq C \int_{\tilde{D}(t)} \frac{d\xi d\eta}{R} \leq \left( \int_{\tilde{D}(t)} d\xi d\eta \right)^{1/p'} \left( \int_{\tilde{D}(t)} \frac{d\xi d\eta}{R^p} \right)^{1/p}$$

where $1/p + 1/p' = 1$. Since $|\tilde{D}(t)| \leq Ct$ and the last integral is bounded if $1 < p < 2$, we get

$$|\nabla G(x, y, t)| \leq C_\delta t^{\frac{1}{2} - \delta} \quad \forall \delta > 0.$$
The same bound holds also for $G(x, y, t)$ and thus, by the definition of $\tilde{\zeta}$ in (4.6), also for $\nabla \tilde{\zeta}$. We conclude that

$$|\nabla w(x, y, t)| \leq C_\delta t^{\frac{1}{2} - \delta}$$

(5.12)

Set

$$B = \psi_y(1, 0).$$

(5.13)

We wish to consider the behavior of the solution of the ODE (5.11) near the point $(1,0)$. For simplicity we set $\xi = x - 1$. Then, by (5.12), (5.13),

$$\dot{\xi} = -\frac{\sigma}{4\pi} \log(\xi^2 + y^2) + o(1)$$

(5.14)

$$\dot{y} = \frac{\sigma}{2\pi} \left\{ -\arctan \frac{\xi}{y} + \arctan \frac{2 + \xi}{y} \right\} - B + o(1)$$

as $t \to 0, (\xi, y) \to (0, 0)$.

By Theorem 2.1,

$$B > 0, \quad \frac{\sigma}{2} - B > 0.$$  

(5.15)

We want to approximate (5.14) by the system

$$\dot{\xi} = -\frac{\sigma}{4\pi} \log \xi^2,$$

(5.16)

$$\dot{y} = \begin{cases} \frac{\sigma}{2} - B & \text{if} \quad \xi < 0 \\ -B & \text{if} \quad \xi > 0. \end{cases}$$

(5.17)

Setting

$$F(z) = \int_0^z \frac{d\xi}{|\log \xi^2|},$$

(5.18)

the solution to (5.16) with $\xi(0) = \xi_0$ is given by

$$F(\xi(t)) - F(\xi_0) = \frac{\sigma}{4\pi} t.$$  

(5.19)

In the next section we shall use this approximate solution to construct barriers (or invariant domains) for solutions of the complete system (5.11), and then derive a simple but sufficiently effective approximation to the solution of the system.

§6. Construction of barriers. We shall prove that the solution of (5.11) with $\xi = x - 1, \xi(0) = \xi_0 < 0, |\xi_0|$ small, must remain in the region $R_{\xi_0}$ indicated in Figure 2.
We need to construct upper barriers $\Gamma_1$ and $\Gamma_3$ and lower barriers $\Gamma_2$ and $\Gamma_4$.

From (5.17) we see that

$$y = \left(\frac{\sigma}{2} - B\right) t + \gamma \quad (\gamma \text{ constant})$$

for $\xi(t) < 0$. Combining this with (5.19) we get the function solution

(6.1) \[ \lambda_0 y = F(\xi) + \gamma_0, \quad \lambda_0 = \frac{\sigma}{2\pi} \cdot \frac{1}{\sigma - 2B} \]

where $\gamma_0$ is a constant. We shall now construct upper and lower barriers $\Gamma_1, \Gamma_2$ having a form similar to (6.1).
We begin with an upper barrier $\Gamma_1$ defined as follows:

$$\Gamma_1 : y = \frac{1}{\lambda} [F(\xi - \epsilon \xi_0) - F(\xi_0 - \epsilon \xi_0)] , \quad \xi_0 \leq \xi \leq \epsilon |\xi_0|$$

(6.2)

where \( \lambda = \frac{\sigma}{2\pi} \cdot \frac{1}{\sigma - 2B} - \epsilon' > 0 \),

for any small $\epsilon' > 0$ and for sufficiently small $\epsilon > 0$.

We need to show that the vector field along $\Gamma_1$ as determined by (5.14) has smaller slope than $\Gamma_1$. The slope of this vector field is

$$\frac{dy}{d\xi} = \frac{\frac{\sigma}{2\pi} \left\{ -\arctan \frac{\xi}{y} + \arctan \frac{2\xi}{y} \right\} - B + o(1)}{-\frac{\sigma}{4\pi} \log \left\{ \xi^2 + \frac{1}{\lambda^2} [F(\xi - \epsilon \xi_0) - F(\xi_0 - \epsilon \xi_0)]^2 \right\}}$$

$$< \frac{\frac{4\pi}{\sigma}}{B + o(1)} \left( \frac{\xi^2 + \frac{1}{\lambda^2} [F(\xi - \epsilon \xi_0) - F(\xi_0 - \epsilon \xi_0)]^2}{\log \left\{ \xi^2 + \frac{1}{\lambda^2} [F(\xi - \epsilon \xi_0) - F(\xi_0 - \epsilon \xi_0)]^2 \right\}} \right)$$

$$< \frac{1}{\lambda} \left( \frac{1}{\log(\xi - \epsilon \xi_0)^2} \right) = \text{slope of } \Gamma_1 \quad (\text{by 6.2}).$$

The last inequality is a consequence of

$$\xi^2 + \frac{1}{\lambda^2} [F(\xi - \epsilon \xi_0) - F(\xi_0 - \epsilon \xi_0)]^2 > (\xi - \epsilon \xi_0)^2$$

which is clearly valid if $\xi_0 \leq \xi \leq \epsilon \xi_0$; if $\epsilon \xi_0 < \xi < \epsilon |\xi_0|$ then the second term on the left-hand side majorizes the right-hand side.

We next construct the lower barrier $\Gamma_2$:

$$\Gamma_2 : y = \frac{1}{\lambda} [F(\xi + \epsilon |\xi_0|) - F(\xi_0 + \epsilon |\xi_0|)] , \quad \xi_0 \leq \xi \leq -4\epsilon |\xi_0| ,$$

(6.3)

where \( \lambda = \frac{\sigma}{2\pi} \cdot \frac{1}{\sigma - 2B} + \epsilon' , \quad \epsilon' > 0 \).

We compute the slope $dy/d\xi$ from (5.14), along $\Gamma_2$:

$$\frac{dy}{d\xi} = \frac{\frac{\sigma}{2\pi} \left\{ -\arctan \left( \frac{\lambda \xi}{F(\xi + \epsilon |\xi_0|) - F(\xi_0 + \epsilon |\xi_0|)} \right) + \frac{\pi}{2} \right\} - B + o(1)}{-\frac{\sigma}{4\pi} \log \left\{ \xi^2 + \frac{1}{\lambda^2} [F(\xi + \epsilon |\xi_0|) - F(\xi_0 + \epsilon |\xi_0|)]^2 \right\}}$$

If $\xi_0 \leq \xi \leq -\epsilon |\xi_0|$ then the expression in the arctan is negative and large in absolute value, if $|\xi_0| \to 0$ ($\epsilon$ is small but fixed). Hence the arctan $(-\cdots)$ is approximately equal to $-\pi/2$. At the same time the denominator is positive and smaller than

$$(1 + \eta_1) \frac{\sigma}{4\pi} |\log \xi^2| \leq \frac{\sigma}{4\pi} (1 + \eta_1)(1 + \eta_2) |\log(\xi + \epsilon |\xi_0|)^2|$$
if \( \xi_0 \leq \xi \leq -4\varepsilon|\xi_0| \), where \( \eta_1, \eta_2 \) are positive numbers which converge to zero if \( \xi_0 \to 0 \). Since

\[
\frac{\sigma}{2\pi} \frac{\pi - B}{\frac{\sigma}{4\pi}} > \frac{1}{\lambda}
\]

we deduce that along \( \Gamma_2 \),

\[
\frac{dy}{d\xi} > \frac{1}{\lambda} \frac{1}{\log(\xi + \varepsilon|\xi_0|)^2} = \text{slope of } \Gamma_2
\]

if \( \xi_0 \leq \xi \leq -4\varepsilon|\xi_0| \).

We proceed to construct the barrier \( \Gamma_3, \Gamma_4 \). From (5.17) we have that \( y = -Bt + \gamma (\gamma \text{ constant}) \) for \( \xi(t) > 0 \). Recalling (5.19) we see that

\[
(6.4) \quad \frac{\sigma}{4\pi B} y = F(\xi) + \gamma_0, \quad \gamma_0 \text{ constant}
\]

is an approximate solution. This suggests the upper barrier

\[
(6.5) \quad \Gamma_3 : y = -\frac{1}{\lambda} [F(\xi - \varepsilon\xi_0) - F(\xi_0 - \varepsilon\xi_0)] \quad \text{for } \varepsilon|\xi_0| \leq \xi \leq \xi_0
\]

where \( \lambda = \frac{\sigma}{4\pi B} - \varepsilon', \xi_0 \text{ small; } \)

\( \varepsilon' \) is any small number and \( \varepsilon \) is positive and sufficiently small. To verify this we compute along \( \Gamma_3 \) the slope of the vector field (5.14):

\[
\frac{dy}{d\xi} = \frac{\frac{\sigma}{2\pi} \left\{ - \arctan \left( \frac{\xi}{-\frac{1}{\lambda} [F(\xi - \varepsilon\xi_0) - F(\xi_0 - \varepsilon\xi_0)]} + \frac{\pi}{2} \right) \right\} - B + o(1)}{
- \frac{\sigma}{4\pi} \log \left\{ \xi^2 + \frac{1}{\lambda^2} [F(\xi - \varepsilon\xi_0) - F(\xi_0 - \varepsilon\xi_0 - \varepsilon\xi_0)]^2 \right\}
}\]

For \( \xi > \varepsilon|\xi_0| \) the expression in \arctan\ is positive and large so that \arctan\ is approximately \( \pi/2 \). The denominator is negative and

\[
> -(1 + \eta_1) \frac{4\pi B}{\sigma} \frac{1}{\log(\xi - \varepsilon\xi_0)}
\]

(cf. the argument in the case of \( \Gamma_2 \)); \( \eta_1 \to 0 \) if \( \xi_0 \to 0 \). It follows that

\[
\frac{dy}{d\xi} < -\frac{1}{\lambda} \frac{1}{\log(\xi - \varepsilon\xi_0)} = \text{slope of } \Gamma_3.
\]
Consider next

\[ \Gamma_4 : y = -\frac{1}{\lambda} [F(\xi + \varepsilon_0) - F(\bar{\xi}_0 + \varepsilon\xi_0)] \quad \text{for} \quad -\varepsilon|\xi_0| \leq \xi \leq \bar{\xi}_0 \]

(6.6)

where \( \lambda = \frac{\sigma}{4\pi B} + \varepsilon' \)

with any small \( \varepsilon' > 0 \) and sufficiently small \( \varepsilon > 0 \). We can argue similarly to the case of \( \Gamma_1 \) to deduce that along \( \Gamma_4 \)

\[ \frac{dy}{d\xi} > -\frac{\sigma}{4\pi} \log \left\{ \xi^2 + \frac{1}{\lambda^2} [F(\xi + \varepsilon\xi_0) - F(\bar{\xi}_0 + \varepsilon\xi_0)]^2 \right\} \]

\[ > -\frac{1}{\lambda} \frac{1}{|\log(\xi + \varepsilon\xi_0)|^2} = \text{slope of } \Gamma_4 ; \]

hence \( \Gamma_4 \) is a lower barrier.

We finally have to fit endpoints of \( \Gamma_1 \) with \( \Gamma_3 \) and \( \Gamma_2 \) with \( \Gamma_4 \). This gives the approximate equations for \( \bar{\xi}_0, \bar{\xi}_0 \):

\[ (\sigma - 2B) \frac{2\pi}{\sigma} F(|\xi_0|) \approx \frac{4\pi B}{\sigma} F(\bar{\xi}_0) , \]

(6.7)

\[ (\sigma - 2B) \frac{2\pi}{\sigma} F(|\xi_0|) \approx \frac{4\pi B}{\sigma} F(\bar{\xi}_0) ; \]

the ratio of the left-hand side by the right-hand side (in both \( \approx \)) goes to 1 as \( \xi_0 \to 0 \).

From the relation

\[ \int_0^x \frac{d\xi}{\log \xi} = \frac{x}{\log x} + \int_0^x \frac{d\xi}{(\log \xi)^2} \]

we deduce that

(6.8) \[ F(x) \equiv \int_0^x \frac{d\xi}{|\log \xi|^2} = \frac{x}{|\log x|^2} \left( 1 + O\left( \frac{1}{|\log x|} \right) \right) \quad \text{for} \quad x \to 0 . \]

Hence (6.7) implies that

(6.9) \[ \bar{\xi}_0 \approx \frac{\sigma - 2B}{2B} |\xi_0| , \quad \bar{\xi}_0 \approx \frac{\sigma - 2B}{2B} |\xi_0| . \]

From the form of the barriers we conclude that the solution of (5.11), or (5.14), with \( \xi(0) = \xi_0, y(0) = 0 \) behaves approximately like a solution to (5.16), (5.17). More precisely:
**Lemma 6.1.** The solution to (5.11) with $\xi(0) = \xi_0 < 0$, $y(0) = 0$ satisfies:

$$F(\xi + o(|\xi_0|)) - F(\xi_0 + o(|\xi_0|)) = \frac{\sigma}{4\pi} t ,$$

(6.10)

$$y = \left( \frac{\sigma}{2} - B \right) t(1 + o(1)) \quad \text{if} \quad \xi < 0 ,$$

$$y = -Bt(1 + o(1)) + \gamma \quad \text{if} \quad \xi > 0$$

where $o(1) \to 0$, $\frac{1}{\xi_0} o(|\xi_0|) \to 0$ if $\xi_0 \to 0$, uniformly in $(\xi, y)$.

**§7. C^1 estimate of $\Gamma(t)$.** By Lemma 3.1 $\nabla w(x, y, t)$ is $C^{1+\alpha}$ on $y = f(x, t)$; it is not however that smooth elsewhere. Since we shall need to work with a $C^{1+\alpha}$ functions of $(x, y)$ on the right-hand sides in (5.11), we are forced to modify the function $\nabla w(x, y, t)$. Figure 3 is a visual aid in describing the modification. The internal longitudinal curve is the curve $y = f(x, t)$.

![Figure 3](image_url)

Each transversal line segment $\ell$ is of length $\varepsilon_0 t$, where $\varepsilon_0$ is small enough. Along each $\ell$ we take a cutoff function $\zeta_\ell$, $\zeta_\ell = 1$ on $\ell \cap \{y = f(x, t)\}$, and define

(7.1)

$$\tilde{A}(x, y, t) = w(x, f(x, t), t)\zeta_\ell .$$
We construct the $\ell$ and $\zeta_\ell$ to be symmetric with respect to the $y$-axis. Along the segment $\ell$ the directional derivative $D_\ell \zeta_\ell$ of $\zeta_\ell$ is $O\left(\frac{1}{t}\right)$ and therefore by (5.12),

$$|D_\ell \tilde{A}| = O\left(\frac{1}{t^{\frac{1}{2}+\delta}}\right) \text{ along } \ell,$$

for any small $\delta > 0$. On the other hand, the derivative of $\tilde{A}$ along the curve $\Gamma(t)$ is bounded by $C(|\log y| + \theta)$, by Lemma 3.1 with $\mu = \varepsilon_1 |\log y|$, $\varepsilon_1 > 0$ and small (the assumptions (5.6)-(5.8) enable us to choose $\varepsilon_0$ and $\varepsilon_1$). The angle between $\ell$ and $\Gamma$ is at least $\geq L_0/|\log t|$ by (5.8), and therefore

$$|\nabla \tilde{A}(x, y, t)| \leq \frac{C_\delta}{t^{\frac{1}{2}+\delta}} + C|\log y| \cdot |\log t|$$

(7.2)

for any small $\delta > 0$. Similarly we define

$$\tilde{B}(x, y, t) = w_y(x, f(x, t), t)\zeta_\ell$$

(7.3)

and, then,

$$|\nabla \tilde{B}(x, y, t)| \leq \frac{C_\delta}{t^{\frac{1}{2}+\delta}} + C|\log y| \cdot |\log t|.$$

(7.4)

The reason we take the line segments $\ell$ to be horizontal near $y = 0$ is that otherwise, the singularity in the $C^{1+\alpha}$ estimates at $(x_0(t), t)$ (which occurs as we take $\mu \to 0$ in Lemma 3.1) will propagate into entire subinterval of $\ell$ lying in $\{y > 0\}$ (with one endpoint at $(x_0(t), 0)$). The factor $|\log t|$ on the right-hand sides of (7.2), (7.4) which results from this choice of $\ell$ will not cause any difficulties.

We shall henceforth replace (5.11) by

$$\dot{x} = -\frac{\sigma}{4\pi} \log \frac{(1-x)^2 + y^2}{(1+x)^2 + y^2} - \psi_x(x, y) - \tilde{A}(x, y, t),$$

(7.5)

$$\dot{y} = \frac{\sigma}{2\pi} \left\{ \arctan \frac{1-x}{y} + \arctan \frac{1+x}{y} \right\} - \psi_y(x, y) - \tilde{B}(x, y, t).$$

Suppose we follow the procedure outlined in §4 and establish a fixed point, using (7.5) instead (5.11). At the fixed point, $\tilde{x}(t, \lambda) = x(t, \lambda)$ and $\tilde{y}(t, \lambda) = y(t, \lambda)$ and therefore $\tilde{A}$ and $\tilde{B}$ coincide with $w_x$ and $w_y$ respectively; i.e., (7.5) reduces to (5.11). This fixed point will then be a fixed point also for the mapping based on the ODE (5.11), and thus the existence proof will be completed.
We proceed to prove that the curves $\Gamma(t)$ belong to the class $A_W$. We begin by studying the $C^1$ nature of these curves near $x = 1, y = 0$, and again resort to change variable $\xi = x - 1$. If we differentiate (7.5) with respect to the parameter $\lambda$ and set $X = \partial \xi / \partial \lambda, Y = \partial y / \partial \lambda$, we obtain

$$
\dot{X} = -\frac{\sigma}{2\pi} \frac{\xi}{\xi^2 + y^2} X - \frac{\sigma}{2\pi} \frac{y}{\xi^2 + y^2} Y + g_1(\xi, y)X + g_2(\xi, y)Y
$$

$$
- \frac{\partial \tilde{A}}{\partial \xi} X - \frac{\partial \tilde{A}}{\partial y} Y,
$$

(7.6)

$$
\dot{Y} = -\frac{\sigma}{2\pi} \frac{y}{\xi^2 + y^2} X + \frac{\sigma}{2\pi} \frac{\xi}{\xi^2 + y^2} Y + g_3(\xi, y)X + g_4(\xi, y)Y
$$

$$
- \frac{\partial \tilde{B}}{\partial \xi} X - \frac{\partial \tilde{B}}{\partial y} Y
$$

and

(7.7)

$$
X(0) = 1, \ Y(0) = 0;
$$

the functions $g_i$ have uniformly bounded derivatives.

The slope of $\Gamma(t)$ is given by $W = Y/X$, as $\lambda$ varies. From (7.6) we easily obtain an equation for $W(t) \equiv W(\xi(t), y(t)) \equiv W(\xi(t), y(t), \lambda)$:

$$
\dot{W} = \left( -\frac{\sigma}{2\pi} \frac{y}{\xi^2 + y^2} + g_3 - \frac{\partial \tilde{B}}{\partial \xi} \right) + \left( \frac{\sigma}{2\pi} \frac{\xi}{\xi^2 + y^2} + (g_4 - g_1) + \left( \frac{\partial \tilde{A}}{\partial \xi} - \frac{\partial \tilde{B}}{\partial y} \right) \right) W
$$

(7.8)

$$
+ \left( \frac{\sigma}{2\pi} \frac{y}{\xi^2 + y^2} + \frac{\partial \tilde{A}}{\partial y} - g_2 \right) W^2, \ W(0) = 0.
$$

Hence

$$
W(t) = \int_0^t \left[ -\frac{\sigma}{2\pi} \frac{y(s)}{\xi^2(s) + y^2(s)} + g_3(\xi(s), y(s)) - \frac{\partial \tilde{B}}{\partial \xi}(\xi(s), y(s), s) \right]
$$

(7.9)

$$
\times \exp \left\{ \int_s^t \left[ \frac{\sigma}{2\pi} \frac{\xi(\tau)}{\xi^2(\tau) + y^2(\tau)} + (g_4 - g_1)(\xi(\tau), y(\tau)) + \left( \frac{\partial \tilde{A}}{\partial \xi} - \frac{\partial \tilde{B}}{\partial y} \right)(\xi(\tau), y(\tau), \tau) \right] d\tau 
$$

$$
+ \int_s^t \left[ \frac{\sigma}{2\pi} \frac{y(\tau)}{\xi^2(\tau) + y^2(\tau)} + \frac{\partial \tilde{A}}{\partial y}(\xi(\tau), y(\tau), \tau) - g_2(\xi(\tau), y(\tau)) \right] W(\xi(\tau), y(\tau)) \right\} ds.
$$
By Lemma 6.1, the trajectories of (7.5) with $|\xi_0|$ small satisfy approximately the equations (see (6.10))

\[
F(\xi) - F(\xi_0) = \frac{\sigma}{4\pi} t ,
\]

(7.10)

\[
y = \left(\frac{\sigma}{2} - B\right) t \quad \text{if} \quad \xi < 0 ,
\]

\[
y = -Bt + \gamma \quad \text{if} \quad \xi > 0 , \quad \gamma \quad \text{constant} .
\]

where $F$ is given by (6.8).

Choose any $M$ large. Then, by (7.10) (or, more precisely, by (6.10)), if $|\xi_0|$ is sufficiently small then we have:

\[
\text{if} \quad |\xi(t)| \leq My(t) \quad \text{then} \quad |F(\xi(t))| \ll |F(\xi_0)| , \quad \text{so}
\]

(7.11)

\[
\text{that} \quad |F(\xi_0)| \approx \frac{\sigma}{4\pi} t \quad \text{and} \quad y(t) \approx \lambda |F(\xi_0)|
\]

where $\lambda$ is a positive constant, $\lambda_1$, if $\xi(t) < 0$ and another positive constant, $\lambda_2$, if $\xi(t) > 0$.

**Lemma 7.1.** There holds:

\[
\int_0^t \frac{|\xi(s)|}{\xi^2(s) + y^2(s)} \, ds \leq C \frac{\log |\log |\xi_0||}{|\log |\xi_0||} .
\]

(7.12)

**Proof.** We split the integral into two parts, according to $\{ |\xi(s)| > y(s) \}$ and $\{ |\xi(s)| \leq y(s) \}$ and use (7.11). We obtain

\[
\int_0^t \frac{|\xi(s)|}{\xi^2(s) + y^2(s)} \, ds \leq C \int_{\{ |\xi(s)| > y(s) \}} \frac{ds}{|\xi(s)|} + \frac{C}{|F(\xi_0)|} |\{ s ; |\xi(s)| < y(s) \}| .
\]

(7.13)

Here and in the sequel, when we write integrals such as

\[
\int_{\{ |\xi(s)| \geq y(s) \}} \quad \text{or} \quad \int_{\{ |\xi(s)| \leq y(s) \}} ,
\]

it is always to be understood that $s$ varies in the interval $0 \leq s \leq t$.

By (5.14)

\[
\dot{\xi} \approx -\frac{\sigma}{4\pi} (\xi^2 + y^2) ,
\]
and \( \log(\xi^2 + y^2) \approx \log \xi^2 \) if \( y(s) \leq |\xi(s)| \). Hence, in this range,

\[
\frac{\sigma}{4\pi} \int ds \approx \frac{d\xi}{|\log \xi^2|}.
\]

Also, from (6.10) we see (using (6.8) that the range of \(|\xi(s)|\) when \(|\xi(s)| > y(s)\) lies in an interval \( \approx [F(\xi_0)], c_0 |\xi_0|, c_0 > 0 \). Hence

\[
\int_{\{|\xi(s)| > y(s)\}} \frac{ds}{|\xi(s)|} \leq C \int_{c_1 |\xi_0|/|\log |\xi_0||}^{c_0 |\xi_0|} \frac{d\xi}{|\xi||\log \xi|}
\]

\[
= \log \frac{\log c_0 |\xi_0|}{\log c_1 |\xi_0| - \log |\log |\xi_0||} \leq C \frac{\log |\log |\xi_0||}{|\log |\xi_0||}.
\]

On the complementary set \( \{|\xi(s)| \leq y(s)\} \) we have

\[
\frac{4\pi}{\sigma} \xi \approx -\log(\xi^2 + y^2) \sim -\log y^2 \sim -\log |\xi_0|
\]

by (7.11), whereas \(|\xi(s)| \leq y(s) \leq |F(\xi_0)|\). It follows that

\[
|\{s; |\xi(s)| < y(s)\}| \leq C \frac{|F(\xi_0)|}{|\log |\xi_0||}.
\]

Using this and (7.14) in (7.13), the assertion (7.12) follows.

**Lemma 7.2.** The following estimates hold:

\[
\int_0^t \frac{y(s)}{\xi^2(s) + y^2(s)} \, ds \leq \frac{C}{|\log t|},
\]

\[
\int_0^t \frac{y(s)}{\xi^2(s) + y^2(s)} \, ds \geq \frac{c}{|\log t|} \text{ at points } (\xi(t, \lambda), y(t, \lambda)) \text{ with } \xi(t, \lambda) > 0,
\]

where \( C, c \) are positive constants.

**Proof.** We first establish both (7.16), (7.17) in case \( \xi(t, \lambda) > 0 \). We write, for any large \( M > 0 \),

\[
\int_0^t \frac{y(s)}{\xi^2(s) + y^2(s)} \, ds = \int_{\{|\xi(s)| > M_0(s)\}} + \int_{\{|\xi(s)| \leq M_0(s)\}} 
\]

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Then (cf. (7.14))

\[
\int_{\{|\xi(s)|>M y(s)\}} \frac{y(s)}{\xi^2(s) + y^2(s)} \, ds \leq C|F(\xi_0)| \int_{\{|\xi(s)|>M y(s)\}} \frac{ds}{|\xi(s)|^2}
\]

\[
\leq C|F(\xi_0)| \int_{C_1 M|\xi_0|/|\log|\xi_0||}^{c_0|\xi_0|} \frac{d\xi}{\xi^2 |\log \xi|}.
\]

Since

\[
\int_{e}^{1} \frac{d\xi}{\xi^2 |\log \xi|} \approx \frac{1}{e |\log e|} \quad \text{if} \quad e \to 0
\]

(cf. the proof of (6.8)), it follows that

\[
(7.19) \quad \int_{\{|\xi(s)|>M y(s)\}} \frac{y(s)}{\xi^2(s) + y^2(s)} \, ds \leq \frac{C}{M} \frac{1}{|\log|\xi_0||}.
\]

Next, by (7.11),

\[
(7.20) \quad \int_{\{|\xi(s)|\leq M y(s)\}} \frac{y(s)}{\xi^2(s) + y^2(s)} \, ds \approx \tilde{C}|F(\xi_0)| \int_{\{|\xi(s)|\leq M y(s)\}} \frac{ds}{\xi^2(s) + \tilde{C}^2|F(\xi_0)|^2}
\]

where \(\tilde{C}\) is a constant independent of \(M\); here \(\approx\) means that \(\leq\) holds with constant \(\tilde{C}\) and \(\geq\) holds with another positive constant \(\tilde{C}\), of course not the same. Also, by (7.11)

\[
\frac{4\pi}{\sigma} \dot{\xi} = \log(\xi^2 + y^2) \approx -\log(F(\xi_0))^2 \approx -\log t^2.
\]

Hence

\[
(7.21) \quad \int_{\{|\xi(s)|\leq M y(s)\}} \frac{y(s)}{\xi^2(s) + y^2(s)} \, ds \approx \frac{C|F(\xi_0)|}{|\log t|} \int_{0<|\xi|<\tilde{C}|F(\xi_0)|}^{\tilde{C} F(\xi_0)} \frac{d\xi}{\xi^2 + \tilde{C}^2(F(\xi_0))^2}
\]

\[
\approx \frac{C}{|\log t|}.
\]

(Since \(\xi(t, \lambda) > 0\), the integral on the right-hand side should actually extend also to a range of \(\xi\)'s with \(\xi > 0\); however this portion of the integral is bounded by the same integral taken only over \(\xi < 0\).) Using (7.21) and (7.19) in (7.18), and choosing \(M\) large enough, both (7.16) and (7.17) follow.
Notice that the condition \( \xi(t, \lambda) > 0 \) was implicitly used in the proof of (7.17). This condition implies that the trajectory \((\xi(s), y(s))\) for \(0 \leq s \leq t\) contains at least the part of the full trajectory in \(\{y > 0\}\) which lies in \(\{\xi < 0\}\), and this allows us to assert that the domain of integration for the \(\xi\)-variable in (7.21) contains an interval \(-\tilde{C}|F(\xi)| < \xi < 0\) and is contained in another interval \(-\tilde{C}|F(\xi_0)| < \xi < \tilde{C}|F(\xi_0)|\).

If we drop the restriction that \(\xi(t, \lambda) > 0\), we can only establish (7.4) with "\(\leq\)" instead of "\(\approx\)", and this together with (7.19), (7.21) establish the assertion (7.16).

We now proceed to estimate \(W\) from (7.9).

We shall prove that

(7.22) \[ |W(t)| \leq \frac{C}{|\log t|} , \]

and

(7.23) \[ |W(t)| > \frac{c}{|\log t|} \text{ at } (\xi(t, \lambda), y(t, \lambda)), \text{ if } \xi(t, \lambda) > 0 . \]

Recall that \(W = W(t, \lambda)\) is a function of both \(t\) and \(\lambda\), and \(W(0, \lambda) = 0\). Hence for any \(\lambda\) there exists a constant \(C(\lambda)\) such that

(7.24) \[ |W(t, \lambda)| \leq \frac{C(\lambda)}{|\log t|} . \]

We are considering here values of \(\lambda\) near \(x = 1\), which corresponds to values of \(\xi_0 < 0\) with \(|\xi_0|\) small. Although \(C(\lambda)\) may possibly become unbounded as \(\lambda \to 1\), the inequality (7.24) will still be useful.

By (7.2), (7.4),

\[ \int_0^t (|\nabla \tilde{A}| + |\nabla \tilde{B}|) ds \leq C t^{\frac{1}{2}-\delta} + C \int_0^* \log s \log y(s) ds . \]

The asterisk over the last integral indicates that we only integrate over the range of \(s\) for which \(\nabla \tilde{A}\) or \(\nabla \tilde{B}\) do not vanish. From the definition of \(\tilde{A}, \tilde{B}\) it follows that, in \(\int^*\), \(s \approx t\). The last integral can now be estimated by substituting \(y = y(s)\) and using the relation \(dy/ds \approx 1\). We get the bound

\[ \exp[-1/(t^{\frac{1}{4}+\delta} |\log t|)] \]

\[ C |\log t| \int_0^\infty \log y |dy \leq C t^{\frac{1}{2}-\delta} . \]

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Hence

\begin{equation}
\int_0^t (|\nabla \tilde{A}| + |\nabla \tilde{B}|) \leq C t^{\frac{1}{2} - \delta} .
\end{equation}

Using (7.25), the bounds $|g_i| \leq C$ and Lemmas 7.1, 7.2, we see that the exponent in (7.9) is bounded by

$$\frac{C}{|\log t|} + \frac{C(\lambda)}{|\log t|} \frac{C}{|\log t|} .$$

If $0 \leq t \leq t_\lambda$, where $t_\lambda$ is small enough so that $C(\lambda) \leq |\log t|$ in (7.24), then we deduce, using (7.9) and Lemma 7.2, that

\begin{equation}
|W(t)| \leq \left(1 + \frac{C}{|\log t|} \left(1 + \frac{C(\lambda)}{|\log t|}\right)\right) \frac{C}{\log t} .
\end{equation}

We may decrease $C(\lambda)$ if necessary so that it actually satisfies

$$C(\lambda) = \sup_{0 \leq t \leq t_\lambda} |W(t, \lambda) \log t| .$$

Then (7.25) implies that $C(\lambda) \leq C$ where $C$ is independent of $\lambda$. This allows us to repeat the above argument with $t_0 < t \leq 2t_\lambda$, etc. We conclude that (7.22) holds for small $t$, say $0 \leq t \leq t_\ast$ where $t_\ast$ is positive and independent of $\lambda$.

The proof of (7.23) follows from (7.9) upon using (7.22), (7.25) and (7.17).

So far we considered only what happens if $|\xi_0|$ is small, i.e., if $\lambda$ is near 1. The same considerations apply also to the case where $\lambda$ is near $-1$. We therefore know how the ODE trajectories behave in $\{y \geq 0\}$ when $1 - \delta_1 < |\lambda| < 1$ and $0 \leq t \leq t_1$, where $\delta_1$, and $t_1$ are sufficiently small positive numbers.

All the other trajectories, i.e., those $-1 + \delta_1 \leq \lambda \leq 1 - \delta_1$ are very smooth if $0 \leq t \leq t_0$ where $t_0$ is a sufficiently small positive number such, that all these trajectories stay in a region $-1 + \delta_0 < x < 1 - \delta_0$ for some $0 < \delta_0 < \delta_1$; here we take $0 < t_0 < t_1$. For all these trajectories

\begin{equation}
\frac{\partial x}{\partial \lambda} \sim 1 , \quad \frac{\partial y}{\partial \lambda} \sim 0 \quad \text{if} \quad t \to 0 ,
\end{equation}

and the corresponding portions of the $\Gamma(t)$ are uniformly $C^{1+\alpha}$. Furthermore, by continuity

\begin{equation}
|f_x(x, t)| \leq Ct \quad \text{if} \quad |x| < 1 - \delta_0,
\end{equation}

\begin{equation}
[f_x(\cdot, t)]_{0, \alpha} \leq Ct
\end{equation}
One can easily check that with the exception of the Hölder condition (5.9), the family \((x(t, \lambda), y(t, \lambda))\) satisfies all the conditions imposed on \((\bar{x}(t, \lambda), \bar{y}(t, \lambda))\) in §5, provided \(0 \leq t \leq t_0\). Furthermore, the new constants \(L_i, Q_i, t_0\), are actually independent of the constants \(L_i, Q_i, t_0\) in §5. By appropriate choice of these initial constants, we conclude that, with the exception of (5.9), the mapping

\[
M\{\bar{x}(t, \lambda), \bar{y}(t, \lambda)\} \rightarrow \{x(t, \lambda), y(t, \lambda)\}
\]

maps \(A_W\) into itself.

The proof of (5.9) is given in §8. Here again it suffices to consider only the portion of \(\Gamma(t)\) for which the trajectories initiate at \((\xi_0, 0)\) with \(\xi_0 < 0\) and \(|\xi_0|\) sufficiently small.

§8. C\(1+\alpha\) estimate of \(\Gamma(t)\). Writing \(Y = XW\) in the first differential equation of (7.6), we can immediately solve for \(X\):

\[
X(t) = \exp \left\{ \int_0^t \left\{ -\frac{\sigma}{2\pi} \frac{\xi(s)}{\xi^2(s) + y^2(s)} - \frac{\sigma}{2\pi} \frac{y(s)}{\xi^2(s) + y^2(s)} W(s) + (g_1 + g_2)(\xi(s), y(s)) \right. \\
+ g_2(\xi(s), y(s))W(s) - \frac{\partial \tilde{A}}{\partial \xi}(\xi(s), y(s), s) - \frac{\partial \tilde{A}}{\partial y}(\xi(s), y(s), s)W(s) \right\} ds.
\]

Using (7.22) and Lemmas 7.1, 7.2 we find that \(X(t) \approx 1\) if \(\lambda \rightarrow 1\) (i.e., if \(\xi_0 < 0, |\xi_0| \rightarrow 0\), that is,

\[
\frac{\partial \xi(t, \lambda)}{\partial \lambda} \approx 1, \text{ or } \xi(t, \lambda) - \xi(t, \bar{\lambda}) \approx \lambda - \bar{\lambda}.
\]

This implies that

\[
\text{if } \lambda_1 < \lambda_2 \text{ then } \xi(t, \lambda_1) < \xi(t, \lambda_2) \text{ along } \Gamma(t).
\]

The right end-point \((x_0(t), t)\) of \(\Gamma(t)\) corresponds to some value \(\lambda = \lambda_0(t)\). From (8.2) it follows that \(\Gamma(t)\) is formed precisely by the points \((x(t, \lambda), y(t, \lambda))\) with \(-\lambda_0(t) < \lambda < \lambda_0(t)\).

To prove the Hölder continuity of the slope of \(\Gamma(t)\), we take two values \(\lambda\) and \(\bar{\lambda}\) near \(\lambda_0(t)\), but smaller than \(\lambda_0(t)\) and their corresponding trajectories. For simplicity we set

\[
\xi(t) = \xi(t, \lambda), \ y(t) = y(t, \lambda), \ W(t) = W(t, \lambda)
\]

and

\[
\tilde{\xi}(t) = \xi(t, \bar{\lambda}), \ \tilde{y}(t) = y(t, \bar{\lambda}), \ \tilde{W}(t) = W(t, \bar{\lambda}).
\]

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Set
\[ \Phi(\xi, y) = \frac{y}{\xi^2 + y^2}, \quad G(\xi, y) = \frac{\xi}{\xi^2 + y^2}. \]

We easily compute that
\[ (8.3) \quad \Phi_\xi = G_y = -\frac{2\xi y}{(\xi^2 + y^2)^2}, \quad \Phi_y = -G_\xi = \frac{\xi^2 - y^2}{(\xi^2 + y^2)^2}. \]

From (7.8) we deduce that
\[ \frac{d}{dt} \frac{W - \bar{W}}{|\lambda - \bar{\lambda}|^\alpha} = \left( \frac{\sigma}{\pi} \frac{\xi}{\xi^2 + y^2} + \mu(\xi, y) \right) \frac{W - \bar{W}}{|\lambda - \bar{\lambda}|^\alpha} \]
\[ + \left( \frac{\sigma}{2\pi} \frac{y}{\xi^2 + y^2} + \frac{\partial \tilde{A}}{\partial y}(\xi, y, t) \right) (W + \bar{W}) \frac{W - \bar{W}}{|\lambda - \bar{\lambda}|^\alpha} + J(t) \]

where
\[ J(t) = -\frac{\sigma}{2\pi} \frac{1}{|\lambda - \bar{\lambda}|^\alpha} [\Phi(\xi, y) - \Phi(\bar{\xi}, \bar{y})] + \frac{1}{|\lambda - \bar{\lambda}|^\alpha} [g_3(\xi, y) - g_3(\bar{\xi}, \bar{y})] \]
\[ - \frac{1}{|\lambda - \bar{\lambda}|^\alpha} \left[ \frac{\partial \tilde{B}}{\partial \xi}(\xi, y, t) - \frac{\partial \tilde{B}}{\partial \xi}(\bar{\xi}, \bar{y}, t) \right] \]
\[ + \frac{\bar{W}}{|\lambda - \bar{\lambda}|^\alpha} \left[ \left( \frac{\partial \tilde{A}}{\partial \xi} - \frac{\partial \tilde{B}}{\partial y} \right)(\xi, y, t) - \left( \frac{\partial \tilde{A}}{\partial \xi} - \frac{\partial \tilde{B}}{\partial y} \right)(\bar{\xi}, \bar{y}, t) \right] \]
\[ + \frac{\bar{W}}{|\lambda - \bar{\lambda}|^\alpha} \left[ G(\xi, y) - G(\bar{\xi}, \bar{y}) \right] + \frac{\bar{W}}{|\lambda - \bar{\lambda}|^\alpha} \left[ \mu(\xi, y) - \mu(\bar{\xi}, \bar{y}) \right] \]
\[ + \frac{\sigma}{2\pi} \frac{\bar{W}^2}{|\lambda - \bar{\lambda}|^\alpha} [\Phi(\xi, y) - \Phi(\bar{\xi}, \bar{y})] + \frac{\bar{W}^2}{|\lambda - \bar{\lambda}|^\alpha} \left[ \frac{\partial \tilde{A}}{\partial y}(\xi, \eta) - \frac{\partial \tilde{A}}{\partial y}(\bar{\xi}, \bar{y}) \right] \]
\[ - \frac{\bar{W}^2}{|\lambda - \bar{\lambda}|^\alpha} [g_2(\xi, y) - g_2(\bar{\xi}, \bar{y})] \]

where \( \mu = g_4 - g_1 \). Hence
\[ \frac{W(t) - \bar{W}(t)}{|\lambda - \bar{\lambda}|^\alpha} = \int_0^t ds J(s) \exp \left[ \int_s^t dt \left\{ \frac{\sigma}{2\pi} \frac{\xi}{\xi^2 + y^2} + \mu(\xi, y) \right\} \right] \]
\[ \left( \frac{\sigma}{2\pi} \frac{y}{\xi^2 + y^2} + \frac{\partial \tilde{A}}{\partial y} \right) (W + \bar{W}) \right] . \]

(8.5)
From (8.1) and (7.22) we have

\[
(8.6) \quad \left| \frac{\partial y(t, \lambda)}{\partial \lambda} \right| \leq \frac{C}{|\log t|}.
\]

Hence

\[
(8.7) \quad |\xi - \bar{\xi}|^\alpha + |y - \bar{y}|^\alpha \leq C|\lambda - \bar{\lambda}|^\alpha.
\]

Since \(g_i\) is in \(C^1\) it follows that

\[
(8.8) \quad \frac{1}{|\lambda - \bar{\lambda}|^\alpha} |g_i(\xi, y) - g_i(\tilde{\xi}, \tilde{y})| \leq C.
\]

We next show that

\[
(8.9) \quad \frac{1}{|\lambda - \bar{\lambda}|^\alpha} \left\{ \left[ |\nabla \bar{A}(\xi, y, t) - \nabla \bar{A}(\tilde{\xi}, \tilde{y}, t)| \right] + \left[ |\nabla \bar{B}(\xi, y, t) - \nabla \bar{B}(\tilde{\xi}, \tilde{y}, t)| \right] \right\} \leq \frac{C}{t^{\frac{1}{2} + \delta + \alpha}} + \frac{C}{|\bar{y}|^\alpha} |\log \bar{y}| |\log t|
\]

where \(\bar{y} = \min(y, \tilde{y})\). Indeed, from the definition of \(\bar{A}\) in (7.1) we have deduced (7.2), and similarly we can show, using Lemma 3.1, that

\[
[\nabla \bar{A}]_{0, \alpha} \leq \frac{C_\delta}{t^{\frac{1}{2} + \delta + \alpha}} + \frac{C|\log \bar{y}| |\log t|}{|\bar{y}|^\alpha}.
\]

where \([ \quad ]_{0, \alpha}\) stands here for the \(\alpha\)-Hölder coefficient in \((\xi, \eta)\). Recalling (8.7), the assertion (9.7) for \(\bar{B}\) follows. The proof for \(\bar{B}\) is the same.

From Lemmas 7.1, 7.2 and the estimate (7.24) we deduce that the expression in the exponent in (8.5) is uniformly bounded if \(|\xi_0|\) is small enough.

Notice that if \(\lambda < \bar{\lambda}\) then \(\bar{y} \approx y\), and, as in the proof of (7.25),

\[
\int_0^t \frac{1}{\bar{y}(s)^\alpha} |\log \bar{y}(s)||\log s|ds \quad \text{if} \quad t \to 0.
\]

Using this in (8.9), and then using also (8.8), we conclude from (8.5), (8.4) that

\[
\frac{|W(t) - \bar{W}(t)|}{|\lambda - \bar{\lambda}|^\alpha} \leq C + \int_0^t ds \left\{ \frac{1}{|\lambda - \bar{\lambda}|^\alpha} \int_0^1 \frac{\partial \Phi(x_r)}{\partial \xi} |\xi(s) - \bar{\xi}(s)| + \frac{|y(s) - \bar{y}(s)|}{|\log s|} \right\}
\]

\[
+ \frac{1}{|\lambda - \bar{\lambda}|^\alpha} \int_0^1 \frac{\partial \Phi(x_r)}{\partial y} |\xi(s) - \bar{\xi}(s)| + \frac{|y(s) - \bar{y}(s)|}{|\log s|} dr \}
\]

\[
(8.10)
\]

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where \( x_r = r \xi(s) + (1 - r) \tilde{\xi}(s), ry(s) + (1 - r) \tilde{y}(s) \); here we used the mean value theorem and (8.3) in evaluating the difference of the \( \Phi \)'s and \( G \)'s in (8.4), and we also used the fact that \( \frac{1}{2} + \delta + \alpha < 1 \) in (8.9) (which is valid if \( 0 < \alpha < \frac{1}{2} \) and \( \delta \) is sufficiently small).

By (8.1), (8.6),

\[
\frac{|\xi(s) - \tilde{\xi}(s)|}{|\lambda - \tilde{\lambda}|^\alpha} \leq C|\xi(s) - \tilde{\xi}(s)|^{1-\alpha},
\]

\[
\frac{|y(s) - \tilde{y}(s)|}{|\lambda - \tilde{\lambda}|^\alpha} \leq \frac{C}{|\log s|^\alpha}|y(s) - \tilde{y}(s)|^{1-\alpha} \leq \frac{C}{|\log s|}|\xi(s) - \tilde{\xi}(s)|^{1-\alpha}.
\]

Consequently, using (8.3),

\[
(8.11) \quad \frac{|W(t) - \tilde{W}(t)|}{|\lambda - \tilde{\lambda}|^\alpha} \leq C + C \int_0^t ds \left\{ \int_0^s \frac{|\xi_r y_r|}{(\xi_r^2 + y_r^2)^2} dr + \frac{C}{|\log t|} \int_0^s \frac{|\xi_r - \tilde{\xi}_r|}{(\xi_r^2 + y_r^2)^2} dr \right\} |\xi(t) - \tilde{\xi}(t)|^{1-\alpha};
\]

here we used the fact (which follows from (8.1)) that

\[
(8.12) \quad |\xi(t) - \tilde{\xi}(t)| \approx |\xi(s) - \tilde{\xi}(s)|.
\]

To prove the Hölder continuity we only need to show, in view of (8.12), that

\[
(8.13) \quad \frac{|W(t) - \tilde{W}(t)|}{|\lambda - \tilde{\lambda}|^\alpha} \leq \frac{C}{\min[\|y(t), \tilde{y}(t)\]|^\alpha}.
\]

The initial point of \( \Gamma(t) \) is given by \( (\tilde{\xi}_0(t), 0) \) where \( \tilde{\xi}_0(t) = \lambda_0(t) - 1 \). Recall also that

\[
(8.14) \quad \tilde{\xi}_0(t) \approx t \log t.
\]

To prove (8.13) we consider two cases:

Case (i) : \( |\xi_0(0) - \tilde{\xi}_0(0)| > \tilde{c}|\tilde{\xi}_0(t)|, \)

Case (ii) : \( |\xi_0(0) - \tilde{\xi}_0(0)| \leq \tilde{c}|\tilde{\xi}_0(t)| \)

where \( \tilde{c} \) is a small positive number.

In case (i),

\[
(8.15) \quad \frac{|W(t) - \tilde{W}(t)|}{|\lambda - \tilde{\lambda}|^\alpha} \leq \frac{|W(t)| + |\tilde{W}(t)|}{|\lambda - \tilde{\lambda}|^\alpha} \leq \frac{C}{|\log t| |t \log t|^\alpha}
\]

by (8.14), (7.22).
In case (ii) we have, by (8.12) and the approximate equations (6.10), that

\[ \xi_r(t) \approx \xi(t), \quad y_r(t) \approx y(t) \]

in (8.11) and, therefore,

\[
\frac{|W(t) - \tilde{W}(t)|}{|\lambda - \tilde{\lambda}|^\alpha} \leq C + C|\tilde{\xi}_0(t)|^{1-\alpha} \left\{ \int_0^t \frac{|\xi(s)y(s)|}{(\xi^2(s) + y^2(s))^2} ds \right\} 
+ \frac{1}{|\log t|} \int_0^t \frac{|\xi^2(s) - y^2(s)|}{(\xi^2(s) + y^2(s))^2} ds \right\}.
\]

(8.16)

We shall evaluate the integrals by the same method used to prove Lemmas 7.1, 7.2.

First

\[
\int_0^t \frac{|\xi(s)y(s)|}{(\xi^2(s) + y^2(s))^2} ds = \int_{\{|\xi(s)| \geq y(s)\}} + \int_{\{|\xi(s)| \leq y(s)\}} \cdots = J_1 + J_2.
\]

(8.17)

By (7.11), (7.15),

\[
J_2 \leq C \int_{\{|\xi(s)| \leq y(s)\}} \frac{ds}{y^2(s)} \leq \frac{C}{|F(\bar{\xi}_0(t))|^2} \frac{|F(\bar{\xi}_0(t))|}{|\log |\xi_0(t)||} = \frac{C}{|\xi_0(t)||}. \tag{8.18}
\]

Next, since |y(s)| \leq C|F(\bar{\xi}_0(t))|,

\[
J_1 \leq C|F(\bar{\xi}_0(t))| \int_{\{|\xi(s)| \geq y(s)\}} \frac{ds}{|\xi(s)|^3}
\]

\[
\leq C|F(\bar{\xi}_0(t))| \int_{\{|C_0 \bar{\xi}_0(t)| \leq C|\xi_0(t)|| \log |\xi_0(t)||} \frac{d\xi}{|\xi^3| \log |\xi|}.
\]

The last integral is

\[
\approx 1/ \left\{ \left( \frac{C_0 \bar{\xi}_0(t)}{|\log |\xi_0(t)||} \right)^2 \log \left| \frac{|C_0 \xi_0(t)|}{|\log |\xi_0(t)||} \right| \right\}
\]

(by integration by parts; cf. the proof of (6.8)) and, therefore,

\[
J_1 \leq \frac{C}{|\xi_0(t)||}.
\]

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Combining this with (8.18), we conclude from (8.17) that

\[(8.19)\]
\[
\int_0^t \frac{\left|\xi(s)y(s)\right|}{(\xi^2(s) + y^2(s))^2} \, ds \leq \frac{C}{|\xi_0(t)|}.
\]

Next we estimate

\[
\int_0^t \frac{\xi^2(s)}{(\xi^2(s) + y^2(s))^2} \, ds = \int_{\{\xi(s) \geq y(s)\}} \cdots + \int_{\{\xi(s) \leq y(s)\}} \equiv L_1 + L_2.
\]

\[
L_1 \leq C \int_{\{\xi(s) \geq y(s)\}} \frac{ds}{\xi^2(s)} \leq C \int_{\{\xi(s) \geq y(s)\}} \frac{d\xi}{|\xi|^3 \log |\xi|} \leq \frac{C}{|\xi_0(t)|}.
\]

Also,

\[
L_2 \leq C \int_{\{\xi(s) \leq y(s)\}} \frac{ds}{y^2(s)} \leq \frac{C}{|\xi_0(t)|}
\]

by (8.18). It follows that

\[(8.20)\]
\[
\int_0^t \frac{\xi^2(s)}{(\xi^2(s) + y^2(s))^2} \, ds \leq \frac{C}{|\xi_0(t)|}.
\]

Finally,

\[(8.21)\]
\[
\int_0^t \frac{y^2(s)}{(\xi^2(s) + y^2(s))^2} \, ds \leq \frac{C}{|\xi_0(t)|}.
\]

Indeed, the left-hand side is bounded by

\[
C \int_{\{\xi(s) \geq y(s)\}} \frac{ds}{\xi^2(s)} + C \int_{\{\xi(s) \leq y(s)\}} \frac{ds}{y^2(s)}.
\]

The first integral is bounded by

\[
\frac{c|\xi_0(t)|}{C|\xi_0(t)| |\log |\xi_0(t)||} \leq \frac{C}{|\xi_0(t)|},
\]
and the second integral has already been estimated in (8.18) by \( C/|\tilde{\xi}_0(t)| \).

Using (8.19)–(8.21) in (8.16) we find that

\[
(8.22) \quad \frac{|W(t) - \tilde{W}(t)|}{|\lambda - \tilde{\lambda}|^{\alpha}} \leq \frac{C}{|\tilde{\xi}_0(t)|^{\alpha}} \leq \frac{C}{|t \log t|^{\alpha}},
\]

where we have used also (8.14). Recalling (8.15), we conclude that (8.22) holds in both cases. This implies (8.13), since the right-hand side of (8.13) is larger than \( C/t^{\alpha} \).

So far we have (tacitly) assumed in the above analysis that \( \xi(0) \) and \( \tilde{\xi}(0) \) are near 0, i.e., \( x(0) \) and \( \tilde{x}(0) \) are near +1. The same estimate holds if \( x(0) \) and \( \tilde{x}(0) \) are near −1. Finally if \( x(0), \tilde{x}(0) \) are in some interval \([-1 + \delta_1, 1 - \delta_1]\) with \( \delta_1 > 0 \) then the \( C^{1+\alpha} \) estimate is rather immediate (see the paragraph containing (7.27)).

A review of the proof shows that the constant \( C \) in (8.13) is independent of the constant \( L_2 \) in (5.9). Hence by choosing \( L_2 \) larger than this constant \( C \), we conclude that the mapping (7.27) maps \( A_W \) into itself.

Remark 8.1. Having proved (8.22), which is stronger than (8.13), it follows that if we replace (5.9) by

\[
(8.23) \quad [f_x(\cdot, t)]_{0, \alpha} \leq \frac{L_2}{|t \log t|^{\alpha}}
\]

then \( M \) still maps the new class \( A_W \) into itself; we denote this new class by \( A^0_W \).

§9. A fixed point. We have shown that \( M \) maps a family \( \{\tilde{x}(t, \lambda), \tilde{y}(t, \lambda)\} \) in \( A^0_W \) into another family \( \{x(t, \lambda), y(t, \lambda)\} \) in \( A^0_W \), provided \( 0 \leq t \leq t_0 \) where \( t_0 \) is a sufficiently small positive constant. Denote by \( \Gamma(t) \) and \( D(t) \) the boundaries and domains corresponding to \( \{\tilde{x}(t, \lambda), \tilde{y}(t, \lambda)\} \), and define \( \Gamma(t), D(t) \) similarly with respect to \( \{x(t, \lambda), y(t, \lambda)\} \). Then we write

\[ D(t) = M\tilde{D}(t). \]

Choose any element \( \{x(t, \lambda), y(t, \lambda)\} \) in \( A^0_W \) with the corresponding domains \( D(t) \) and boundaries \( \Gamma(t) \), and define the iterates

\[ D^{n+1}(t) = MD^n(t) \quad (n = 1, 2, \ldots) \]

where \( D^1(t) = D(t) \).

The modification of \( \nabla w \) as described in connection with Figure 3 is somewhat arbitrary. We could for instance replace the transversal segments \( \ell_{\gamma} \) (of direction \( \gamma \)) with concentric segments of larger length; the length being \( \approx \varepsilon_0 t \) if the midpoint is in \( |x| < 1 \), and it grows to \( \approx \varepsilon_0 t|\log t| \) as the directions of \( \ell_{\gamma} \) become horizontal. Let us choose the cutoff functions \( \zeta_\gamma \) such that they are equal to 1 on each symmetrically situated subinterval of \( \ell_{\gamma} \) of length
\[ \frac{1}{2} |\ell_\gamma|. \] From the approximate behavior of the trajectories as described in §6 (see Lemma 6.1) we can deduce that if \( t \) is small enough and the internal longitudinalis are the curves \( \Gamma(t) \) corresponding, say, to the domains \( D^2(t) \), then we can carry out the modifications of \( w_x, w_y \) by (7.1), (7.3) with the same \( \ell_\gamma, \zeta_\gamma \) for all the \( D^n(t), \) \( n \geq 2 \). Furthermore, for each \( n \geq 2 \), each line segment \( \ell_\gamma \) (or its extension) forms angle \( \leq \theta_0 \leq \frac{\pi}{2} \) (\( \theta_0 \) constant) with the direction \((x(t,\lambda), y(t,\lambda))\) of the trajectory at the point \((x(t,\lambda), y(t,\lambda))\).

Since
\[ |\nabla \zeta_\gamma| \leq \frac{C}{t|\log t|} \leq \frac{\widetilde{C}}{t}, \]
the analysis in §§7,8 remains unchanged.

Now take two domains
\[ D_1(t) = D^{n1}(t), \quad D_2(t) = D^{n2}(t) \]
and introduce in addition to the usual Hausdorff distance \( \delta_1(t) \equiv \delta_1(D_1(t), D_2(t)) \) another distance function based on measuring distances along the half lines \( \hat{\ell}_\gamma \) in \( \{x \geq 0\} \) or in \( x \leq 0 \) containing the line segments \( \ell_\gamma \):
\[ \delta(t) \equiv \delta(D_1(t), D_2(t)) = \sup_{\gamma} \text{diam}\{\hat{\ell}_\gamma \cap [\partial D_1(t) \cup \partial D_2(t)]\}. \]

Clearly (since \( \theta_0 < \frac{\pi}{2} \)),
\[ \delta_1(t) \leq \delta(t) \tag{9.1} \]

Denote by \( r_i = (x_i, y_i) = (x_i(t,\lambda), y_i(t,\lambda)) \) the trajectories defining the boundaries \( \Gamma_i(t) \) of \( \mathcal{M}D_i(t) \) and by \( \lambda_i = (\bar{A}_i, \bar{B}_i) \) the modifications of \( \nabla w_i \).

The ODE for \( r_i \) is then
\[ \dot{r}_i(t) = -\nabla \bar{u}(r_i(t)) - \lambda_i(r_i(t), t). \tag{9.2} \]

Recall also that
\[ \nabla w_i(x, y, t) = -\frac{1}{2\pi} \int_{D_i(t)} \left( \frac{x - \xi}{\rho^2}, \frac{y - \eta}{\rho^2} \right) d\xi d\tau + \nabla \psi_i \tag{9.3} \]
where \( \rho^2 = (x - \xi)^2 + (y - \eta)^2 \)
and \( \nabla \psi_i \) is a smooth function.
We can write

\[
|\tilde{A}_1(r,t) - \tilde{A}_2(r,t)| \leq \iint_{[D_1(t) \cup D_2(t)] \cap B_{c \delta}(r)} |\cdots| + \iint_{[D_1(t) \cap D_2(t)] \setminus B_{c \delta}(r)} |\cdots| + \iint_{[D_1(t) \Delta D_2(t)] \cap B_{c \delta}(r)} |\cdots|
\]

(9.4) \[= J_1 + J_2 + J_3.\]

Here \(c\) is any fixed large positive constant. The integrand in \(J_1\) is bounded by \(C/\rho\) (using (9.3)). Hence

(9.5) \[J_1 \leq \iint_{B_{c \delta}} \frac{C}{\rho} \leq C \delta.\]

In \(J_3\) we estimate the integrand also by \(C/\rho\) and conclude that

(9.6) \[J_3 \leq C \delta_1(t)|\log \delta_1(t)|.\]

Finally, using the definition of \(\tilde{\lambda}_i\) above and setting \(\Sigma(t) = [D_1(t) \cap D_2(t)] \setminus B_{c \delta}(r)\) we see that

\[J_2 \leq C \iint_{\Sigma(t)} \left| \frac{x_1(t) - \xi}{\rho_2^2} - \frac{x_2(t)}{\rho_2^2} \right| d\xi dy\]

where \((x_i(t), y_i(t)) = r_i(t)\) and \(\rho_i^2 = (x_i(t) - \xi)^2 + (y_i(t) - \eta)^2\). Applying the mean value theorem we get

\[J_2 \leq C \delta_1(t) \iint_{\Sigma(t)} \frac{C}{\rho_2^2} \leq C \delta_1(t)|\log \delta_1(t)|.\]

Combining this estimate with (9.6), (9.5) we get, after using (9.1),

(9.7) \[|\tilde{A}_1(r, t) - \tilde{A}_2(r, t)| \leq C \delta(t)|\log \delta(t)|.\]

We now take the difference of (9.2) with \(i = 1\) and \(i = 2\) to obtain after using (9.7) and the correspondence estimate for the \(\vec{B}_i\),

\[|r_1 - r_2|_t \leq |\nabla \vec{u}(r_1(t)) - \nabla \vec{u}(r_2(t))| + |\tilde{\lambda}_1(r_1(t), t) - \tilde{\lambda}_1(r_2(t), t)| + C \delta(t)|\log \delta(t)|.\]
By the mean value theorem and (7.2), (7.4),

\[ |\lambda_1(r_1(t), t) - \lambda_1(r_2(t), t)| \leq \gamma(t)|r_1(t) - r_2(t)| \]

where

\[ \gamma(t) = \frac{C}{t^{1+\delta}} + C|\log y(t)||\log t|, \tag{9.9} \]

and where \( y(t) \) belongs to the interval \( (y_1(t), y_2(t)) \).

By the mean value theorem, also

\[ |\nabla \bar{u}(r_1(t)) - \nabla \bar{u}(r_2(t), t)| \leq C|\nabla^2 \bar{u}||r_1(t) - r_2(t)| \]

and, as easily verified,

\[ |\nabla^2 \bar{u}(x, y)| \leq \frac{C}{|x| + y}. \tag{9.10} \]

We can then write

\[ |r_1(t) - r_2(t)| \leq C \int_0^t \delta(s)||\log \delta(s)||e^s ds \tag{9.11} \]

where the expression in \( \cdots \) is the sum of \( \gamma(t) \) plus the right-hand side of (9.10) evaluated at a point in the interval \( (r_1(t), r_2(t)) \). By the results of §8 ((7.22) and Lemmas 7.1, 7.2) it follows that

\[ \sup_{0 < s < t} \left\{ \exp \int_0^s \cdots \right\} \to 1 \quad \text{if} \quad t \to 0. \]

Hence (9.11) implies that

\[ \delta(MD_1(t), MD_2(t)) \leq C \int_0^t \delta(s)||\log \delta(s)||ds. \tag{9.12} \]

We shall apply (9.12) to the domain \( D_1(t) = D^n(t), D_2(t) = D^{n+1}(t) \). Setting

\[ g_n(t) = \delta(D^n(t), D^{n+1}(t)), \]

we get

\[ g_{n+1}(t) \leq C \int_0^t g_n(s)||\log g_n(s)||. \tag{9.13} \]
We claim that

\begin{equation}
(9.14) \\
g_n(t) \leq A^n t^n |\log t|^n.
\end{equation}

Indeed, this is true for \( n = 1 \) if \( A \) is sufficiently large. Proceeding by induction we shall assume that (9.14) holds for some \( n \) and prove it for \( n + 1 \).

Applying (9.13) we get

\[
g_{n+1}(t) \leq CA^n \int_0^t s^n |\log s|^n |\log A + \log s + \log |\log s||ds.
\]

Since \( \log s \) is negative whereas \( A + \log |\log s| \) is positive if \( s \) is small,

\[
0 < \log |A + \log s + \log |\log s|| \leq |\log s|.
\]

It follows that

\begin{equation}
(9.15) \\
g_{n+1}(t) \leq CA^n \int_0^t s^n |\log s|^{n+1} ds.
\end{equation}

But

\[
\int_0^t s^n |\log s|^{n+1} = \frac{t^{n+1} |\log t|^{n+1}}{n+1} + \int_0^t s^n |\log s|^n ds
\]

and therefore

\[
\int_0^t s^n |\log s|^{n+1} \leq \frac{2t^{n+1} |\log t|^{n+1}}{n+1}
\]

if \( t \) is small. Substituting this into (9.15) and choosing \( A > 2C \), the proof of (9.14) for \( n + 1 \) follows.

By compactness, any subsequence of \( \{\Gamma^n(t)\} = \{\partial D^n(t) \cap \{y \geq 0\}\} \) has a subsequence which converges to a \( C^{1+\alpha} \) curve \( \Gamma(t) \). We claim that the complete family \( \{\Gamma^n(t)\} \) has a unique limit. Indeed, this follows from the fact that for any two sequences \( \{\Gamma^{n_1}(t)\}, \{\Gamma^{n_2}(t)\} \)

\[
\delta(D^{n_1}(t), D^{n_2}(t)) \to 0 \quad \text{as} \quad n_1, n_2 \to \infty,
\]

by (9.14). If we denote by \( D(t) \) the limit, in the \( \delta \)-metric, of the \( D^n(t) \), then

\[
MD(t) = D(t).
\]

It can be easily check that \( u(x, y, t) \) is continuous in \((x, y, t)\) and this completes the existence part of Theorem 1.1.
To prove uniqueness suppose we have another solution with domains \( \tilde{D}(t) \) and set

\[
\delta(t) = \delta(D(t), \tilde{D}(t)).
\]

We can choose the same \( \ell_\gamma, \zeta_\ell \gamma \) for both domains, and therefore (9.12) can be applied. We thus get

\[
\delta(t) \leq C \int_0^t \delta(s)|\log \delta(s)|ds.
\]

We deduce as before that

\[
\delta(t) \leq A^n t^n |\log t|^n \quad \forall \ n \geq 1,
\]

so that \( \delta(t) \equiv 0 \), i.e., the two solutions coincide.

\section{10. The shape of the free boundary.}

\textbf{Theorem 10.1.} The function \( \bar{u}(x,y) \) satisfies:

\[
(10.1) \quad \bar{u}_{xy}(x,0) < 0 \quad \text{if} \quad 0 < x < a, \quad x \neq 1.
\]

\textbf{Proof.} Consider \( \bar{u}_x \) in

\[
R_\delta = \{0 < x < a, -h < y < b\} \setminus B_\delta(1,0), \quad \delta > 0.
\]

Notice that \( \bar{u}_x \) and \( \bar{u}_{xy} \) are continuous across \( \{(x,0), 0 \leq x \leq 1 - \delta\} \) \( (\bar{u}_{xy}(x,0^+) - \bar{u}_{xy}(x,0^-)) = -\sigma_x = 0 \) so that \( \bar{u}_x \) is actually harmonic in \( R_\delta \). By (2.9) and (5.10) we see that

\[
\bar{u}_x < 0 \quad \text{in} \quad B_\delta(1,0)
\]

if \( \delta \) is small enough. Since, further, \( \bar{u}_x = 0 \) on \( \partial R_\delta \setminus \partial B_\delta(1,0) \), it follows by the maximum principle that

\[
(10.2) \quad \bar{u}_x < 0 \quad \text{in} \quad 0 < x < a, \quad -h < y < b.
\]

Consider the harmonic function

\[
w(x,y) = \bar{u}_x(x,y) - \bar{u}_x(x,-y)
\]

in \( R_0^h = \{0 < x < a, 0 < y < h\} \). This function is bounded in a neighborhood of \( (1,0) \), as seen by (2.9) and the fact that \( \Phi_x(x,y) = \Phi_x(x,-y) \) (cf. (5.10)). Also \( w = 0 \) on \( x = 0, x = a \) and \( y = 0 \) and, by (10.2), \( w(x,h) < 0 \). Therefore, by the maximum principle, \( w < 0 \) in \( R_0^h \) and \( w_y(x,0+) < 0 \). This yields (10.1).
Let us define curves $\Gamma_0(t)$ in $\{y > 0\}$ by

$$x = x_0(t, \lambda), \quad y = y_0(t, \lambda)$$

where

$$\dot{x}_0 = -\overline{u}_x(x_0, y_0), \quad \dot{y}_0 = -\overline{u}_y(x_0, y_0),$$

(10.3)

$$x_0(0, \lambda) = \lambda, \quad y_0(0, \lambda) = 0 \quad (-1 < \lambda < 1).$$

Since

$$\frac{\partial x_0}{\partial \lambda} = 1, \quad \frac{\partial y_0}{\partial \lambda} = 0 \quad \text{at} \quad t = 0,$$

(10.4) \Gamma_0(t) can be written in the form

$$\Gamma_0(t) = \{y = f_0(x, t)\}$$

(10.5) provided $t$ is small. From (5.11), (5.12) it follows that

$$|f(x, t) - f_0(x, t)| \leq C t^{\frac{\alpha}{2} - \delta} \quad \delta > 0.$$  

(10.6) Set

$$g_0(x) = -\frac{\partial^2 \overline{u}(x, 0)}{\partial x \partial y}, \quad g(x) = \int_0^x g_0(y) dy.$$  

(10.7) By Theorem 10.1, $g'(x) > 0$ if $0 < x < 1.$

By (10.4)

$$\frac{\partial}{\partial x} f_0(x, t) = -\frac{\partial y_0}{\partial x_0} \frac{\partial \lambda}{\partial x} = \frac{\partial y_0}{\partial \lambda} (1 + O(t)).$$

On the other hand, for any $\eta > 0,$

$$\frac{d}{dt} \frac{\partial y_0}{\partial \lambda} = -\frac{\partial^2 \overline{u}}{\partial x \partial y} \frac{\partial x_0}{\partial \lambda} - \frac{\partial^2 \overline{u}}{\partial y^2} \frac{\partial y_0}{\partial \lambda} = g_0(x)(1 + O(t))$$

if $0 \leq x \leq 1 - \eta,$ using again (10.4). It follows that

$$\frac{\partial}{\partial x} f_0(x, t) = g_0(x)(1 + O(t))$$

and, by (10.6),

$$f(x, t) = g(x)t + O \left( t^{\frac{\alpha}{2} - \delta} \right) \quad \text{if} \quad 0 \leq x \leq 1 - \eta.$$  

(10.8) We summarize:
Theorem 10.2. For any $\eta > 0$ the free boundary $y = f(x, t)$ satisfies (10.8), where $g'(x) > 0$ if $0 < x < 1$.

Thus $f(x, t)$ is increasing in $x$ in “average” sense.

From (10.8) we see the precise linear growth in $t$, of the free boundary, when $0 < x \leq 1 - \eta$.

We recall that

$$\frac{C}{t|\log t|} \leq f_x(x, t) < \frac{c}{|\log t|} \quad \text{if} \quad 1 \leq x < x_0(t)$$

where $C > c > 0$. Thus $f(x, t)$ decreases in $x$ for $1 < x < x_0(t)$, at rate $\approx 1/|\log t|$.

Acknowledgement. (1) The authors wish to express their thanks to Dr. John Spence from Eastman Kodak for suggesting the problem studied in this paper.

(2) The first author is partially supported by the National Science Foundation Grant DMS–86–12880. The second author is partially supported by CICYT Research Grant PB90–0235 and Fulbright Fellowship.

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