A FREE BOUNDARY PROBLEM ARISING
IN SMOULDER COMBUSTION

By

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1. Introduction. There are many flame propagations models, see [1, 2, 6] and the references therein. The smouldering combustion is a slow burning process such as burning of a paper.

![Diagram of a free boundary problem in smoulder combustion](image)

Figure 1.

Assuming that the thickness of the reaction zone is negligible, a two dimensional flame propagation model is derived in [1] as follows:

\[ \frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{for } y < f(x,t), \ x > 0, \ t > 0, \]

with the boundary conditions

\[ u = 1 \quad \text{on } y = 0, \]

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(1.3) \[ u = 0 \quad \text{on} \quad y = f(x, t) \]

and the free boundary conditions:

(1.4) \[ f(0, t) = 0 \]
(1.5) \[ -\frac{\partial f}{\partial x}(x, t) = \frac{\partial u}{\partial y}(x, f(x, t)) - \varepsilon \frac{\partial u}{\partial x}(x, f(x, t)) \frac{\partial f}{\partial x}(x, t) \]

where \( u \) is the oxidizer concentration and \( y = f(x, t) \) is the reaction front (as in figure 1); \( \sqrt{\varepsilon} \) is proportional to \( \rho_0/\rho_f \), \( \rho_0 \) is the mean density of the oxidizer, and \( \rho_f \) is the density of the solid reactant. All the quantities have been nondimensionalized and \( \sqrt{\varepsilon} \) is typically small.

In §2, we study the travelling wave solutions. The Duvaut’s transform reduces the problem to a variational inequality, and existence and uniqueness are therefore obtained. For this variational inequality, however, the solution will not solve the original problem (1.1)–(1.5), which will be shown in §3. In fact, the system (1.1)–(1.5) has no solutions if \( \varepsilon > 0 \); this is due to the assumption that the oxidizer concentration is discontinuous at the reaction front on the \( x \)-axis. After modifying the assumption so that the oxidizer at the reaction front is continuous, we obtain travelling wave solutions. In §3, we also establish the monotonicity of the free boundary. In §4, we show that the free boundary will converge to the corresponding free boundary of the Stefan problem as \( \varepsilon \to 0 \).

Finally in §6, we study the time dependent problem and show that the solution will converge to the travelling wave solution as \( t \to \infty \) under certain assumptions.

2. Travelling wave solutions. We look for travelling wave solutions, namely, the solution which is a function of the variable \( (x + t, y) \) only. As in [1], by making a change of variable \( x + t = \xi \), and renaming \( \xi \) as \( x \), the system (1.1)–(1.5) becomes

(2.1) \[ \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{for} \quad y < f(x), \quad x > 0, \]

with the boundary conditions

(2.2) \[ u = 1 \quad \text{on} \quad y = 0, \]
(2.3) \[ u = 0 \quad \text{on} \quad y = f(x), \]

and the free boundary conditions:

(2.4) \[ f(0) = 0, \]
(2.5) \[ -\frac{df}{dx}(x) = \frac{\partial u}{\partial y}(x, f(x)) - \varepsilon \frac{\partial u}{\partial x}(x, f(x)) \frac{df}{dx}(x). \]
When $\varepsilon = 0$, this problem is the Stefan problem with the explicit solution

\begin{align}
(2.6) \quad u_0(x, y) &= 1 - \int_0^{y/\sqrt{\varepsilon}} \exp \left(-\sigma^2/4\right) d\sigma / \int_0^M \exp \left(-\sigma^2/4\right) d\sigma, \\
(2.7) \quad f_0(x) &= M\sqrt{x} \quad \text{for } x \geq 0
\end{align}

where the constant $M > 0$ is the unique solution of

\begin{equation}
(2.8) \quad \frac{1}{2} M \int_0^M \exp \left(-\sigma^2/4\right) d\sigma = \exp \left(-M^2/4\right).
\end{equation}

When $\varepsilon > 0$ and small, a similarity solution is found in [1] for the first order expansion in $\varepsilon$. Actually, it is more convenient to write the free boundary as $x = g(y)$. Then (2.5) becomes

\begin{equation}
(2.9) \quad -1 = g'(y) \frac{\partial u}{\partial y}(g(y), y) - \varepsilon \frac{\partial u}{\partial x}(g(y), y).
\end{equation}

The Duvaut's transform

\begin{align}
w &= \int_{g(y)}^x u(\sigma, y) d\sigma \quad \text{for } x > g(y) \\
&= 0 \quad \text{for } -K \leq x \leq g(y)
\end{align}

reduces (2.1)–(2.5) to an obstacle problem:

\begin{equation}
(2.10) \quad w_{xx} - w_{yy} + w_x = -1 \quad \text{for } x > g(y), \\
w = 0 \quad \text{for } -K \leq x \leq g(y),
\end{equation}

$w \in C^1(\{x > -K, y > 0\})$ and satisfies the boundary condition

\begin{equation}
(2.11) \quad w = (x - g(0))^+ \quad \text{on } y = 0.
\end{equation}

Here $g(0)$ should be solved as part of the solution and should not be predetermined. In fact, we can show later that if $g(0) = 0$ then $g(0) < 0$ (Theorem 3.1) and therefore we get a contradiction. On the other hand, if we do not assume $g(0) = 0$, we can make a change of variable $x - g(0) \rightarrow x$ and get $g(0) < g(0)$, which is still a contradiction. This means that there is no solution if we assume (2.11). This result is not surprising since (2.11) implies that there is a jump on the oxidizer concentration $u$ on the reaction front $x = g(y)$ near $y = 0$. We shall modify the boundary condition (2.11) to the following condition

\begin{equation}
(2.12) \quad w = x^+ \quad \text{on } y = 0,
\end{equation}

which means that we have assumed $u(x, 0) = 0$ in $g(0) \leq x \leq 0$ instead of $u(x, 0) = 1$ in $g(0) \leq x \leq 0$. It also makes sense if we replace $x^+$ by a monotone decreasing, concave-up, $C^{1,1}$ function vanishing on $x < 0$ (which means that $u$ is continuous on the entire $x$-axis); we shall discuss this case in §5.
We expect that for the solution \((w, g)\), we have \(g(y) > -K\) if \(K\) is large enough. Since \(0 \leq u(x, y) \leq 1\), we have \(0 \leq w(x, y) \leq x + K\). So the obstacle problem can be formulated as follows:

\begin{align}
(2.13) & \quad -\varepsilon w_{xx} - w_{yy} + w_x \geq -1 \quad \text{in } \Omega, \\
(2.14) & \quad w \geq 0 \quad \text{in } \Omega, \\
(2.15) & \quad w(-\varepsilon w_{xx} - w_{yy} + w_x + 1) = 0 \quad \text{in } \Omega, \\
(2.16) & \quad w(x, 0) = x, \quad w(-K, y) = 0, \\
(2.17) & \quad 0 \leq w(x, y) \leq x + K \quad \text{in } \Omega
\end{align}

where \(\Omega = \{(x, y); \ x > -K, y > 0\}\).

**Theorem 2.1.** For each \(K\), the system (2.13)–(2.17) has a unique solution \(w \in C(\bar{\Omega}) \cap W^{2,\infty}_{loc}(\Omega)\). Furthermore,

\begin{align}
(2.18) & \quad 0 \leq w_x \leq 1, \quad w_{xx} \geq 0 \quad \text{in } \Omega.
\end{align}

**Proof.** The proof follows from the penalization method for the obstacle problem. For completeness, we present the proof here. The elliptic estimates used in this section can be found, for example, [4].

We first prove the uniqueness. Suppose that \(w_i\) \((i = 1, 2)\) are two solutions to the system (2.13)–(2.17). Let \(\tilde{w}_i = w_i e^{-\eta(x+y)}\), then we have

\begin{align}
(2.19) & \quad \mathcal{L} [\tilde{w}_i] + e^{-\eta(x+y)} \geq 0, \quad \tilde{w}_i \geq 0, \quad \tilde{w}_i \left( \mathcal{L} [\tilde{w}_i] + e^{-\eta(x+y)} \right) = 0 \quad \text{in } \Omega,
\end{align}

where

\[\mathcal{L} = -\varepsilon \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + (1 - 2\varepsilon\eta) \frac{\partial}{\partial x} - 2\eta \frac{\partial}{\partial y} + (\eta - \varepsilon\eta^2 - \eta^2).\]

We take \(\eta > 0\) to be small enough so that \(\eta - \varepsilon\eta^2 - \eta^2 > 0\). The functions \(\tilde{w}_i\) are uniformly bounded and converges to 0 when \((x, y) \to \infty\). Therefore if the supremum of \(\tilde{w}_1 - \tilde{w}_2\) is positive, it must be obtained at a finite point \((x_0, y_0)\), i.e.,

\[
\sup_{(x,y) \in \Omega} (\tilde{w}_1(x, y) - \tilde{w}_2(x, y)) = \tilde{w}_1(x_0, y_0) - \tilde{w}_2(x_0, y_0) > 0.
\]

The point \((x_0, y_0)\) can not be on the boundary by the boundary conditions. Therefore by the maximum principle (see [5], for example),

\begin{align}
(2.20) & \quad \text{ess lim inf}_{(x, y) \to (x_0, y_0)} \mathcal{L}[\tilde{w}_1 - \tilde{w}_2] \geq (\eta - \varepsilon\eta^2 - \eta^2)(\tilde{w}_1(x_0, y_0) - \tilde{w}_2(x_0, y_0)) > 0.
\end{align}

Since \(\tilde{w}_1(x_0, y_0) - \tilde{w}_2(x_0, y_0) > 0\), there is a neighborhood \(B_\delta(x_0, y_0)\) in which \(\tilde{w}_1(x, y) - \tilde{w}_2(x, y) > 0\), and therefore \(\tilde{w}_1(x, y) > \tilde{w}_2(x, y) \geq 0\) on \(B_\delta(x_0, y_0)\). It follows that

\[\mathcal{L}[\tilde{w}_1 - \tilde{w}_2] = - \left( \mathcal{L}[\tilde{w}_2] + e^{-\eta(x+y)} \right) \leq 0 \quad \text{in } B_\delta(x_0, y_0),\]
which contradicts (2.20).

For the existence, we can approximate the domain $\Omega$ with $\Omega_R = \{(x,y); -K < x < R, 0 < y < R\}$. Take functions $\beta_{\delta} \in C^\infty$ such that

\[
\begin{align*}
\beta_{\delta}'(s) &\geq 0, \quad \beta_{\delta}''(s) \geq 0 \quad \text{for all } s \in (-\infty, \infty), \\
\beta_{\delta}(s) &= 0, \quad \text{for } s < -\delta, \quad \beta_{\delta}(0) = 1, \\
\beta_{\delta}(s) &\to \infty \quad \text{as } \delta \to 0 \text{ for each } s > 0,
\end{align*}
\]

and consider the corresponding penalization problem on $\Omega_R$:

\[(2.22) \quad -\varepsilon w_{xx} - w_{yy} + w_x = -1 + \beta_{\delta}(-w) \quad \text{in } \Omega_R,
\]

with boundary condition:

\[(2.23) \quad w(x,y) = p_R(x)(1 - y/R) \quad \text{on } \partial \Omega_R,
\]

where

\[p_R(x) = \begin{cases} x & \text{for } x > 1/R, \\ 0 & \text{for } x < -1/R \end{cases}, \quad \begin{aligned} p_R'(x) &\geq 0, \\
p_R''(x) &\geq 0, \quad p_R \in C^\infty(R).\end{aligned}
\]

This system has a unique solution $w_{R,\delta}$. By the maximum principle $0 \leq w_{R,\delta} \leq \frac{x + K}{R + K} R \left(1 - \frac{y}{R}\right)$ on $\Omega_R$. Hence

\[(2.24) \quad 0 \leq \frac{\partial w_{R,\delta}}{\partial x} \leq 1 \quad \text{on } \partial \Omega_R.
\]

Differentiating the equation (2.22) once in $x$ and using the maximum principle, we get

\[(2.25) \quad 0 \leq \frac{\partial w_{R,\delta}}{\partial x} \leq 1 \quad \text{in } \Omega_R.
\]

Since $w_{R,\delta}(x,y) \geq 0$ in $\Omega_R$,

\[
\sup_{(x,y) \in \Omega_R} \beta_{\delta}(-w_{R,\delta}(x,y)) = 1.
\]

Next,

\[
\frac{\partial^2 w_{R,\delta}}{\partial x^2} \geq 0 \quad \text{on } y = 0 \text{ and } y = R,
\]

and, by the equation,

\[
\varepsilon \frac{\partial^2 w_{R,\delta}}{\partial x^2} = \frac{\partial w_{R,\delta}}{\partial x} + 1 - \beta_{\delta}(-w_{R,\delta}) \geq 0 \quad \text{on } x = 0 \text{ and } x = R.
\]
After differentiating (2.22) twice in \(x\) and recalling (2.21), we conclude, by the maximum principle,

\[
\frac{\partial^2 w_{R,\delta}}{\partial x^2} \geq 0 \quad \text{in } \Omega_R.
\]

Now by the standard estimates for the penalization problem, we have the \(W^{2,\infty}\) interior estimates. Thus by compactness, we can take the limit for \(R \to \infty\) and then \(\delta \to 0\) along subsequences. This completes the proof of existence. \(\Box\)

Lemma 2.18 implies that the free boundary is a curve \(x = g(y)\).

**Lemma 2.2.**

\[
g(y) \leq \frac{y^2}{M^2}
\]

where the constant \(M\) is defined in (2.8).

**Proof.** Let

\[
\varphi = \int_{y^2/M^2}^x u_0(\sigma, y) d\sigma \quad \text{for } x > y^2/M^2
\]

\[
= \begin{cases} 
0 & \text{for } -K \leq x \leq y^2/M^2
\end{cases}
\]

where \(u_0\) is the solution for the Stefan problem defined by (2.6). Then

\[
\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial u_0}{\partial x} \geq 0 \quad \text{for } x > y^2/M^2;
\]

therefore

\[
-\varepsilon \varphi_{xx} - \varphi_{yy} + \varphi_x = -1 - \varepsilon \varphi_{xx} \leq -1 \quad \text{for } x > y^2/M^2.
\]

Notice that \(\varphi \in C^{1,1}(\Omega) \cap C(\overline{\Omega})\) and satisfy the same boundary conditions as that of \(w\). Although we have an unbounded region here, we still have the comparison principle for this variational inequality (the proof is essentially the same as the uniqueness proof in Lemma 2.1). Thus \(\varphi(x, y) \leq w(x, y)\) in \(\Omega\) and hence \(g(y) \leq y^2/M^2\). \(\Box\)

**Lemma 2.3.** If \(K > \varepsilon\), then

\[
g(y) \geq -\frac{1}{2} \varepsilon \quad \text{for all } y > 0.
\]

**Proof.** Construct the one dimensional auxiliary function

\[
\varphi(x, y) = \varphi(x) = \begin{cases} 
x & \text{for } x \geq \frac{\varepsilon}{2} \\
\frac{1}{2\varepsilon} \left( x + \frac{\varepsilon}{2} \right)^2 & \text{for } |x| \leq \frac{\varepsilon}{2} \\
0 & \text{for } x \leq -\frac{\varepsilon}{2}
\end{cases}
\]
Then \( \varphi \in C^{1,1}(\Omega_R) \). Clearly \(-\varepsilon \varphi_{xx} - \varphi_{yy} + \varphi_x \geq -1\) whenever \( \varphi > 0 \). On the boundary \( y = 0 \),
\[
\varphi(x) \geq \frac{1}{2\varepsilon} \left( x^2 + \varepsilon x + \frac{\varepsilon^2}{4} \right) \geq x \geq w(0, x).
\]

It follows form the comparison principle for the variational inequality that \( w(x, y) \leq \varphi(x) \), and therefore \( g(y) \geq -\varepsilon/2 \). \( \square \)

**Remark 2.1.** Since \( w \in W_{loc}^{2,\infty}(\Omega) \), the curve \( x = g(y) \) is Lipschitz continuous for all \( y > 0 \), by [3, Theorem 6.1, page 177].

**Remark 2.2.** The curve \( x = g(y) \) has a local analytic representation, by the general theory of free boundary problems. Thus \( g(y) \) is analytic for all \( y > 0 \).

**Remark 2.3.** It is obvious that \( w \) is smooth up to the free boundary from the side \( \{ x > g(y) \} \). Hence \( w_x \) solves the problem consisting of (2.1), (2.9), \( u = 0 \) on \( x = g(y) \), and \( u(0, x) = H(x) \), where \( H(x) \) is the Heaviside function.

### 3. Properties of the free boundary.

**Proposition 3.1.** There exists \( \eta = \eta_\varepsilon > 0 \) such that
\[
(3.1) \quad g(y) \leq -\eta \quad \text{for } 0 < y < \eta.
\]

**Proof.** We shall prove that
\[
(3.2) \quad w_y(x, y) \geq 1 \quad \text{for } |x| \leq \eta, \ 0 \leq y \leq \eta, \ \text{for some } \eta > 0.
\]

The theorem will then follow since \( \nabla w(g(y), y) \equiv 0 \). Set
\[
(3.3) \quad \varphi(\xi, y) = \frac{y}{2\pi} \log \frac{(\xi - 1)^2 + y^2}{\xi^2 + y^2} + \frac{\xi}{\pi} \left( \arctan \frac{1 - \xi}{y} + \arctan \frac{\xi}{y} \right).
\]

Then \( \varphi \) is continuous in \( \Omega_s \equiv \{ (\xi, y); \ 0 \leq y \leq \sqrt{s^2 - \xi^2} \} \ (0 < s < 1) \),
\[
(3.4) \quad \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{in } \Omega_s,
\]

and
\[
(3.5) \quad \lim_{y \to 0^+} \varphi(\xi, y) = \xi^+ \quad \text{uniformly for } |\xi| \leq \frac{3}{4}.
\]

We next set \( \xi = x/\sqrt{\varepsilon} \) and define
\[
\psi(x, y) = \sqrt{\varepsilon} \varphi \left( \frac{x}{\sqrt{\varepsilon}}, y \right).
\]
then we have
\[ \varepsilon \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{in } D_s \equiv \left\{ (x, y); \ 0 \leq y \leq \sqrt{s^2 - \frac{x^2}{\varepsilon}} \right\}, \]
and
\[ \lim_{y \to 0^+} \psi(x, y) = x^+ \quad \text{uniformly for } |x| \leq \frac{3}{4} \sqrt{\varepsilon}. \]

We now go back to the equation (2.22) for the solution \( w_{R,\delta} \). Since \( 0 \leq (w_{R,\delta})_x \leq 1 \) and \( 0 \leq \beta_\delta(-w_{R,\delta}) \leq 1 \), we have
\[ 0 \leq \varepsilon \frac{\partial^2 w_{R,\delta}}{\partial x^2} + \frac{\partial^2 w_{R,\delta}}{\partial y^2} \leq 2 \quad \text{in } D_s \equiv \left\{ (x, y); \ 0 \leq y \leq \sqrt{s^2 - \frac{x^2}{\varepsilon}} \right\} \quad (0 < s < 1). \]

Therefore the limit \( w \) satisfies
\[ \varepsilon \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x, y) \quad \text{in } D_s, \quad \|f\|_{L^\infty} \leq 2, \quad f(x, y) \geq 0. \]

Clearly \( w - \psi \) is continuous on \( D_{3/4} \), \( w - \psi = 0 \) on \( y = 0 \) and
\[ \varepsilon \frac{\partial^2 (w - \psi)}{\partial x^2} + \frac{\partial^2 (w - \psi)}{\partial y^2} = f(x, y) \quad \text{in } D_s, \quad \|f\|_{L^\infty} \leq 2, \quad f(x, y) \geq 0. \]

It follows by interior-boundary estimates for the elliptic equation that
\[ (3.6) \quad |\nabla (w - \psi)(x, y)| \leq C_\varepsilon \quad \text{for } (x, y) \in D_{1/2}. \]

A direct calculation shows that
\[ \frac{\partial \psi}{\partial y} = \sqrt{\varepsilon} \left[ \frac{1}{2\pi} \log \frac{(\xi - 1)^2 + y^2}{\xi^2 + y^2} + \frac{1}{\pi} \left( \frac{y^2}{(\xi - 1)^2 + y^2} - \frac{y^2}{\xi^2 + y^2} \right) \right. \]
\[ + \frac{1}{\pi} \left( \frac{\xi(\xi - 1)}{(\xi - 1)^2 + y^2} - \frac{\xi^2}{\xi^2 + y^2} \right) \]
\[ = \frac{\sqrt{\varepsilon}}{2\pi} \left( \log \frac{1}{\xi^2 + y^2} + O(1) \right) \quad \text{in } D_{1/2} \quad (\xi = x/\sqrt{\varepsilon}). \]

Combining this with (3.6), we obtain (3.2). \( \square \)

This Proposition shows that, even if \( \varepsilon \) is small, it still played a dramatic role for the local behavior of the free boundary near \((0, 0)\). Since \( g(y) \) is away from \((0, 0)\), \( D^2_{x,y} w \) is uniformly bounded near the free boundary. Therefore we have

**Proposition 3.2.** The function \( g(y) \) is continuous on \([0, \infty)\).

**Proof.** We need only to show that \( g(y) \) is continuous at \( y = 0 \). It suffices to show that the oscillation of \( g(y) \) converges to 0 as \( y \to 0^+ \). Suppose that \( g(y_1) = g(y_2) \equiv b \)
and \( a \leq g(y) \leq b \) for \( y_1 < y < y_2 \). Then the function \( \frac{1}{2\varepsilon}(x-a)^2 \) in the region \( a \leq x \leq b, y_1 \leq y \leq y_2 \) is a supersolution for the equation \(-\varepsilon w_{xx} - w_{yy} + w_x = -1\). Since \(|w_{yy}| \leq C_\varepsilon \) and \( w = w_y = 0 \) at \((b,y_1)\),

\[
w(b,y) \leq C_\varepsilon(y_2 - y_1)^2 = \frac{1}{2\varepsilon} (b - a)^2
\]

provided \( \sqrt{C_\varepsilon(y_2 - y_1)} = \frac{1}{\sqrt{2\varepsilon}} (b - a) \). Thus \( |g(y) - g(y_1)| \leq b - a \equiv \sqrt{2C_\varepsilon} (y_2 - y_1) \) for all \( y_1 \leq y \leq y_2 \). It follows that there are no fingers for the free boundary and therefore \( g(y) \) is continuous at 0. \( \Box \)

Now define \( N = \{(x,y); \ g(y) < x < \infty, \ 0 < y < \infty\} \) and let \( \Gamma = \{(g(y),y); \ 0 \leq y < \infty\} \).

**Proposition 3.3.** There exists \( y_0 > 0 \) such that

\[
(3.7) \quad g'(y) < 0 \quad \text{for} \ 0 < y < y_0, \quad g'(y_0) = 0, \quad \text{and} \ g'(y) > 0 \quad \text{for} \ y_0 < y < \infty,
\]

furthermore

\[
(3.8) \quad \liminf_{y \to 0^+} \frac{g(y) - g(0)}{y} = -\infty.
\]

**Proof.** First we notice that \( w_x(x,0) = 0 \) for \( g(0) < x < 0 \) is the minimum of \( w_x \) in \( N \). The maximum principle implies that \( w_{xy}(x,0) > 0 \) for \( g(0) < x < 0 \) and therefore \( w_y(x,0) \) is monotone increasing. Since \( w_y(g(0),0) = 0, w_y(x,0) > 0 \) for \( g(0) < x < 0 \).

Similarly, \( w_x(x,0) = 1 \) for \( x > 0 \), which is the maximum of \( w_x \) in \( N \). By the maximum principle \( w_{xy}(x,0) < 0 \) for \( x > 0 \). It follows that \( w_y(x,0) \) is monotone decreasing along the positive \( x \) axis. From the proof of Lemma 2.3, we have already obtained that \( w(x,y) \leq x \) for \( x \geq \varepsilon/2 \). Hence \( w_y(x,0) \leq 0 \) for \( x \geq \varepsilon/2 \). Note also that \( w_y(x,y) > 0 \) in a neighborhood of \((0,0)\), from the proof of Proposition 3.1. Therefore there exists a \( \tau \in (0,\varepsilon/2] \) such that \( w_y(x,0) > 0 \) for \( 0 < x < \tau \) and \( w_y(x,0) < 0 \) for \( \tau < x < \infty \).

Differentiating \( w_x(g(y),y) = 0 \) and \( w_y(g(y),y) = 0 \), we get

\[
g'(y)w_{xx}(g(y),y) + w_{xy}(g(y),y) = 0, \quad g'(y)w_{yx}(g(y),y) + w_{yy}(g(y),y) = 0.
\]

Combining these relations with the equation

\[
\varepsilon w_{xx}(g(y),y) + w_{yy}(g(y),y) = 1,
\]

we obtain

\[
(3.9) \quad w_{xx} = \frac{1}{\varepsilon + (g'(y))^2}, \quad w_{xy} = -\frac{g'(y)}{\varepsilon + (g'(y))^2}, \quad w_{yy} = \frac{(g'(y))^2}{\varepsilon + (g'(y))^2} \quad \text{on} \ \Gamma.
\]
Next, we claim that
\[(3.10) \quad \text{if } g'(\tilde{y}) < 0, \text{ then } g'(y) < 0 \text{ for all } 0 < y < \tilde{y}.\]

Since \(g'(\tilde{y}) < 0\), \(w_{yx}(g(\tilde{y}), \tilde{y}) > 0\), by (3.9). It follows that \(w_y(x, \tilde{y})\) is positive and strictly monotone increasing for \(g(x) < x \leq \delta\) for some small \(\delta > 0\). We can take \(h > 0\) to be sufficiently small such that \(h\) is not a critical value for \(w_y\) in \(N\), by Sard’s lemma. \(w_y(x_h, \tilde{y}) = h\) for some \(x_h \in (g(\tilde{y}), \delta)\) and we can initiate a curve \(\gamma_h\) starting from \((x_h, \tilde{y})\). By the theory of ODE, the curve \(\gamma_h\) must either hit the boundary or exit any bounded domain in a finite time. The curve \(\gamma_h\) can not hit the free boundary \(x = g(y)\) since \(w_y(g(y), y) = 0 \neq h\).

If the curve \(\gamma_h\) goes to infinity, then \(\gamma_h, \) the line segment \(\{(x, \tilde{y}); g(\tilde{y}) \leq x \leq x_h\}\) and \(\{x = g(y); y \geq \tilde{y}\}\) enclose a region \(D\). Clearly \(w_y\) is bounded by a polynomial near \(\infty\). Thus the maximum principle implies that \(w_y > 0\) in \(D\), which implies \(w_{yx}(g(\tilde{y}), y) > 0\) for \(y > \tilde{y}\) and hence \(g'(y) < 0\) for all \(y > \tilde{y}\) by (3.9). This is a contradiction to the fact that \(\lim_{y \to \infty} g(y) = \infty\) (this fact will be proved in §4, and it is valid for any \(\varepsilon > 0\)).

Therefore the curve \(\gamma_h\) can only hit the positive \(x\)-axis at a point \(x = \tau_h < \tau\). Thus the curve \(\gamma_h\), the \(x\)-axis for \(g(0) \leq x \leq \tau_h\) and \(\{x = g(y); 0 \leq y \leq \tilde{y}\}\) enclose a domain \(D\). \(w_y\) is bounded from below in \(D\). The maximum principle implies that \(w_y > 0\) in \(D\), which in turn implies that \(w_{yx}(g(y), y) > 0\) for \(0 < y < \tilde{y}\). Now by (3.9), (3.10) follows.

Similarly, we claim that
\[(3.11) \quad \text{if } g'(\tilde{y}) > 0, \text{ then } g'(y) > 0 \text{ for all } y > \tilde{y}.\]

The proof is essentially the same. Since \(w_{yx}(g(\tilde{y}), \tilde{y}) < 0\), \(w_y(x, \tilde{y})\) is negative and strictly monotone decreasing for \(g(\tilde{y}) < x \leq \delta\) for some small \(\delta > 0\). Take \(h > 0\) to be sufficiently small such that \(-h\) is not a critical value for \(w_y\) in \(N\). \(w_y(x_h, \tilde{y}) = -h\) for some \(x_h \in (g(\tilde{y}), \delta)\) and we can initiate a curve \(\gamma_h\) starting from \((x_h, \tilde{y})\). The curve \(\gamma_h\) will either extend to infinity or hit the positive \(x\)-axis at \(x = x_h > \tau\). In both cases the curve \(\gamma_h\), the line segment \(\{(x, \tilde{y}); g(\tilde{y}) \leq x \leq x_h\}\) and \(\{x = g(y); y \geq \tilde{y}\}\) enclose a region \(D\). \(w_y < 0\) in \(D\) by the maximum principle, which implies that \(w_{yx}(g(y), y) < 0\) for \(y > \tilde{y}\). Using (3.9), we conclude (3.11).

Note that there will be no horizontal segment for the free boundary since \(g(y)\) is analytic. It follows from (3.10) and (3.11) that
\[(3.12) \quad \text{either: } \begin{cases} 
\text{there exists } y_0 > 0 \text{ such that } \\
g'(y) < 0 \text{ in } (0, y_0), \quad g'(y_0) = 0, \quad g'(y) > 0 \text{ in } (y_0, \infty),
\end{cases}\]
\[(3.13) \quad \text{or: } \begin{cases} 
\text{if } g'(y) > 0 \text{ for all } y > 0.
\end{cases}\]

To complete the proof, it remains to establish (3.8), which will then imply that (3.13) will not happen.
We claim that, there exists $C > 0$ such that
\begin{equation}
(3.14) \quad g(y) - g(0) \leq Cy \quad \text{for small } y > 0.
\end{equation}
Take a curve $\gamma$ in $N$ connecting $(g(y_1), y_1)$ $(0 < y_1 < 1)$ to a point $(x_1, 0)$ $(g(0) < x_1 < 0)$ such that $\gamma$ hits the free boundary $\Gamma$ in the reverse direction of the $x$-axis. Consider the domain $D$ bounded by $\gamma$, the free boundary $\{x = g(y); 0 \leq y \leq y_1\}$ and $x$-axis for $g(0) \leq x \leq x_1$. In a neighborhood of $(x_1, 0)$, $w_y \geq 0$. Since $w_{xx}(g(y_1), y_1) > 0$ and all second derivatives of $w$ are bounded in $D$, $Cw_x + w_y \geq 0$ on $\gamma$ in a neighborhood of $(g(y_1), y_1)$ if $C$ is large enough. Since $w_x > 0$ in $N$, $Cw_x + w_y \geq 0$ on the rest of the boundary of $\partial D$ if $C$ is taken large enough. By the maximum principle, $Cw_x + w_y > 0$ in $D$. From which (3.14) follows.

Suppose that (3.8) is not true. Then there exists $\theta_0 \in (\pi/2, \pi)$ such that $x = g(y)$ lies in the sector $S \equiv \{0 < \theta < \theta_0, 0 < \rho < \rho_0\}$ ($\rho_0$ is a small constant); here we used the polar coordinate defined as follows:
\[ \frac{1}{\sqrt{\epsilon}}(x - g(0)) = \rho \cos \theta, \quad y = \rho \sin \theta. \]
Let $\alpha = \pi/\theta_0$ ($> 1$) and define $\varphi(x, y) = C\rho^\alpha \sin(\alpha \theta)$. Then $\epsilon\varphi_{xx} + \varphi_{yy} = 0$ in $S$. Clearly $w_x = 0$ on $\{ \theta = 0; g(0) \leq x < g(0) + \sqrt{\epsilon}\rho_0 \}$; $u_x = 0$ on the free boundary, which lies in the sector $S$ for $\rho < \rho_0$. Since all the second derivatives of $w$ is bounded in $S$, we can choose $C$ to be large enough so that $\varphi \geq w_x$ on $\rho = \rho_0$.

Since $w_{xx} \geq 0$, $\epsilon w_{xxx} + w_{xyy} = w_{xx} \geq 0$ in $S \cap N$. By the maximum principle $w_x \leq \varphi$ in $S \cap N$. It follows that $w_x \leq C \left( \frac{1}{\epsilon} (x - g(0))^2 + y^2 \right)^\alpha$ for $\rho < \rho_0$. Integrating in $x$, we obtain,
\begin{equation}
(3.15) \quad w(x, y) \leq C(x - g(y)) \left( \frac{1}{\epsilon} (x - g(0))^2 + y^2 \right)^\alpha \\
\leq C \left( |x - g(0)| + |\cot \theta_0 y| \right) \left( \frac{1}{\epsilon} (x - g(0))^2 + y^2 \right)^\alpha \quad \text{for } \rho < \frac{\rho_0}{\sqrt{2}}.
\end{equation}

Now we define
\[ \tilde{w}_\delta(x, y) = \frac{1}{\delta^2} w(\delta x + g(0), \delta y). \]
Then by (3.15) (notice that $\alpha > 1$),
\[ \lim_{\delta \to 0} \tilde{w}_\delta(x, y) \equiv 0. \]
On the other hand $D^2 \tilde{w}_\delta(x, y)$ is uniformly bounded in any bounded domain, and $\tilde{w}_\delta$ satisfies the equation
\[ -\epsilon (\tilde{w}_\delta)_{xx} - (\tilde{w}_\delta)_{yy} + \delta (\tilde{w}_\delta)_x = -1 \quad \text{for } 0 < Cy < x, \]
by (3.14). Letting $\delta \to 0$ along a subsequence, we get a contradiction. \qed
4. Asymptotic behavior for small \( \varepsilon \). Physically, \( \varepsilon \) is a small constant. In this section we prove that the free boundary of the problem (2.1)–(2.5) converges to the free boundary corresponding to \( \varepsilon = 0 \), namely, \( x = y^2/M^2 \), where \( M \) is given in (2.8). To emphasis the dependence on \( \varepsilon \) we denote \( g_\varepsilon(y) = g(y) \).

**Theorem 4.1.**

\[
(4.1) \quad \frac{y^2}{M^2} - O(\sqrt{\varepsilon}) y - O(\sqrt{\varepsilon}) \leq g_\varepsilon(y) \leq \frac{y^2}{M^2}.
\]

**Proof.** We have already proved that \( g_\varepsilon(y) \leq \frac{y^2}{M^2} \) in \S 2. What we need to establish is the other half of the inequality in (4.1). Let \( \eta \in (0,1/2] \) and let \( N_\eta \) be the unique solution of

\[
(4.2) \quad \frac{1}{2} (1-\eta) N \int_0^N \exp \left( -\sigma^2/4 \right) d\sigma = \exp \left( -N^2/4 \right).
\]

Then the pair

\[
(4.3) \quad u_\eta(x,y) = 1 - \int_0^{y/\sqrt{x}} \exp \left( -\sigma^2/4 \right) d\sigma + \int_0^{N_\eta} \exp \left( -\sigma^2/4 \right) d\sigma,
\]

\[
(4.4) \quad f_\eta(x) = N_\eta \sqrt{x} \quad \text{for } x \geq 0
\]

is the explicit solution to the Stefan problem

\[
(4.5) \quad \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2} \quad \text{for } y < f(x), \ x > 0, \ t > 0,
\]

\[
(4.6) \quad u = 1 \quad \text{on } y = 0,
\]

\[
(4.7) \quad u = 0 \quad \text{on } y = f(x),
\]

\[
(4.8) \quad f(0) = 0,
\]

\[
(4.9) \quad (1-\eta) \frac{df}{dx}(x) = - \frac{\partial u}{\partial y}(x,f(x)).
\]

It follows that

\[
(4.10) \quad \varphi_\eta = \begin{cases} u_\eta(\sigma,y) d\sigma & \text{for } x > y^2/N^2_\eta \\ 0 & \text{for } -K \leq x \leq y^2/N^2_\eta \end{cases}
\]

is in \( C^1(\Omega) \) and satisfies

\[
(4.11) \quad \frac{\partial \varphi_\eta}{\partial x} - \frac{\partial^2 \varphi_\eta}{\partial y^2} = -1 + \eta \quad \text{for } x > y^2/N^2_\eta.
\]

Now let \( \psi(x,y) = \varphi_\eta(x+\delta,y) \), then \( \psi(-\varepsilon/2,y) \geq 0 = w(-\varepsilon/2,y) \), \( \psi(x,0) = (x+\delta)^+ > w(x,0) \), and \( 0 \leq \psi(x,y) \leq x + \delta \). Clearly

\[
-\varepsilon \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial \psi}{\partial x}
\]
\[ = -1 + \eta - \frac{\varepsilon}{2(x + \delta)} \frac{y}{\sqrt{x + \delta}} \exp \left( - \frac{y^2}{4(x + \delta)} \right) / \int_0^{N_0} \exp \left( - \sigma^2/4 \right) d\sigma \]
\[ \geq -1 + \eta - \frac{\varepsilon}{2\delta - \varepsilon} \sqrt{2} e^{-1/2} \int_0^{N_0} \exp \left( - \sigma^2/4 \right) d\sigma \geq -1, \]
provided \( \varepsilon \leq \delta \) and
\[ (4.12) \varepsilon \leq \frac{\sqrt{2}}{2} \sqrt{\varepsilon} \eta \delta \int_0^M \exp \left( - \sigma^2/4 \right) d\sigma \left( \leq \frac{\sqrt{2}}{2} \sqrt{\varepsilon} \eta \delta \int_0^{N_0} \exp \left( - \sigma^2/4 \right) d\sigma \right). \]
Notice that (4.12) is satisfied if \( \varepsilon \leq \eta \delta \). We thus conclude that \( w(x, y) \leq \psi(x, y) \) in \( \Omega \) and hence
\[ (4.13) \quad g_\varepsilon(y) \geq y^2/N_\eta^2 - \delta \geq y^2/M^2 - C\eta y^2 - \delta, \]
where we used (4.2) for the inequality \( 1/N_\eta^2 \geq 1/M^2 - C\eta \). The inequality (4.13) is valid as long as (4.12) holds. For \( y \leq 1 \) we can take \( \delta = \sqrt{\varepsilon} \) and \( \eta = \sqrt{\varepsilon} \). For \( y \geq 1 \) we take \( \delta = \sqrt{\varepsilon} y \) and \( \eta = \sqrt{\varepsilon}/y \). Thus the theorem follows from (4.13). \( \Box \)

Remark 4.1. Theorem 4.1 implies that
\[ (4.14) \quad \lim_{y \to \infty} \frac{g_\varepsilon(y)}{y^2} = \frac{1}{M^2}, \]
A careful examination of the proof indicates that this equality is valid without assuming \( \varepsilon \) to be small.

Remark 4.2. Theorem 4.1 also implies that the first zero \( y_1 > 0 \) of the function \( g(y) \) satisfies
\[ y_1 \leq O(\varepsilon^{1/4}). \]

5. Continuous oxidizer concentration for the boundary conditions. Now let's consider the case where \( x^+ \) is replace by \( p(x) \) in (2.12). Here \( p(x) \) is a \( C^{1,1} \) function such that
\[ \begin{align*}
p(x) &= 0 \quad \text{for } x \leq 0, \\
p(x) &= \frac{1}{2} \quad \text{for } x \geq 1, \quad p''(x) \geq 0.
\end{align*} \]
The existence and uniqueness results of §2 extend to this case without any difficulties. The analog of Lemmas 2.2 and 2.3 is

**Lemma 5.1.** Suppose that \( x = \bar{g}(y) \) is the free boundary, then
\[ -\frac{1}{2} \varepsilon \leq \bar{g}(y) \leq g^*(y), \]
where \( g^*(y) \) is the free boundary of the Stefan problem
\[
\begin{align*}
\varphi_x &= \varphi_{yy} \quad \text{in} \; g^*(y) < x < \infty, \\
\varphi(x, 0) &= p'(x) \quad \text{for} \; x \geq 0, \\
\varphi(x, y) &= 0 \quad \text{on} \; x = g^*(y), \\
g^*(0) &= 0, \quad (g^*)'(y)\varphi_y(g^*(y), y) = -1. \quad \square
\end{align*}
\]

Next, we claim that the free boundary is monotone increasing if \( \varepsilon \) is small enough.

**Theorem 5.2.** If \( \varepsilon \leq 1/\|p''\|_{L^\infty} \), then
\[
\tag{5.2}
0 \leq w_x \leq 1, \quad w_y \leq 0, \quad 0 \leq \varepsilon w_{xx} \leq 2, \quad 0 \leq w_{yy} \leq 2,
\]
and the free boundary is monotone increasing. Furthermore, \( \bar{g}(y) \) is analytic for \( y > 0 \) and \( \bar{g}(0) = 0 \).

**Proof.** As before, we can derive that
\[
\tag{5.3}
0 \leq w_x \leq 1, \quad w_{xx} \geq 0.
\]

From the proof of Theorem 2.1, we see that the estimate can be extended to \( (w_{\varepsilon,R})_{yy} \) to derive that \( (w_{\varepsilon,R})_{yy} \geq -\varepsilon \|p''\|_{L^\infty} \). Using the equation we find that both \( (w_{\varepsilon,R})_{xx} \) and \( (w_{\varepsilon,R})_{yy} \) are uniformly bounded. Hence after taking the limit,
\[
\sup_{\Omega} \left| \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2} \right| < \infty,
\]
where \( \Omega = \{-K < x < \infty, \; 0 < y < \infty\} \). By the assumption, we have \( w_{yy}(x, 0) = 1 + p'(x) - \varepsilon p''(x) \geq 0 \) for \( x > g(0) \). By the general theory of free boundary problems, \( w_{yy} \geq 0 \) on the free boundary \( x = g(y), \; y > 0 \). Thus by the maximum principle (for the unbounded domain), we obtain, \( w_{yy} \geq 0 \) in \( \Omega \). In particular, \( -(w_y)_y \leq 0 \) on the \( x \)-axis. It follows by the maximum principle that \( w_x \leq 0 \) in \( \Omega \). Combining these estimates and the equation, (5.2) follows, which also implies that the free boundary is monotone increasing and \( \bar{g}(0) = 0 \). The analyticity of \( \bar{g} \) is the same as before. \( \square \)

**6. Time dependent problem.** Now we go back to the time dependent problem (1.1)–(1.5). One can use the Duvaut's transform integrate either in \( t \) or in \( x \) direction. If we integrate the equation in \( t \), we get a quasi-variational inequality with the equation itself involving the derivative of the free boundary on the right-hand-side. So, we will integrate in \( x \) direction, as before. Naturally, the burnt region should be growing. We also assume that the free boundary is given by \( x = g(y,t) \), and \( g(y,t) \) is differentiable. Let
\[
\begin{align*}
\tag{6.1}
w &= \int_{g(y,t)}^x u(\sigma, y, t) d\sigma \quad \text{for} \; x > g(y,t) \\
&= 0 \quad \text{for} \; -K \leq x \leq g(y,t).
\end{align*}
\]
Assume that $0 \leq u(x, y, 0) = u_0(x, y) \leq 1$ and set $Q_T = R_K \times [0, T]$. Then $w \in C^1(Q_T)$ and satisfies

\begin{align}
(6.2) & \quad w_t - \varepsilon w_{xx} - w_{yy} \geq -1 \quad \text{in } Q_T, \\
(6.3) & \quad w \geq 0 \quad \text{in } Q_T, \\
(6.4) & \quad w \left( w_t - \varepsilon w_{xx} - w_{yy} + 1 \right) = 0 \quad \text{in } Q_T, \\
(6.5) & \quad 0 \leq w(x, y, t) \leq x + K \quad \text{in } Q_T,
\end{align}

where $R_K \equiv \{(x, y); -K < x < \infty, 0 < y < \infty\}$, with the initial and boundary conditions:

\begin{align}
(6.6) & \quad w(x, y, 0) = u_0(x, y) \quad \text{for } (x, y) \in R_K, \\
(6.7) & \quad \begin{cases}
  w(x, 0, t) = p \left( x - g(0, t) \right) \quad \text{for } x \in R, t \in [0, T], \\
  w(-K, y, t) = 0 \quad \text{for } y > 0, t \in [0, T]
\end{cases} \\
(6.8) & \quad g(y, 0) = g_0(y),
\end{align}

where $p(x) = x^+$. However, like the travelling wave solution case in §2, the existence of a solution will very unlikely be valid if $p(x) = x^+$. We shall make a little modification by assuming that the oxidizer concentration is continuous, namely, we assume that $p \in C^{1,1}$ and

\begin{align}
(6.9) & \quad p(x) \left\{ \begin{array}{ll}
  = 0 & \text{for } x \leq 0 \\
  > 0 & \text{for } x > 0 \\
\end{array} \right.,
\quad p(x) = x - \frac{1}{2} \quad \text{for } x \geq 1, \quad p''(x) \geq 0.
\end{align}

We expect that if $K = K_T$ is large enough, then $g(y, t) > -K$ for $0 \leq t \leq T$. From the equation (6.1),

\begin{align}
(6.10) & \quad w_0(x, y) = \int_{g_0(y)}^{x} u_0(\sigma, y) d\sigma.
\end{align}

For this variational inequality, the solution is not unique. In fact, there are infinitely many solutions, as we shall see below.

We assume that

\begin{enumerate}
  \item[(H1)] $g_0 \in C^2[0, \infty)$, \quad $g_0(0+) = 0$, \quad for $y > 0$,
\end{enumerate}

that $u_0$ satisfies

\begin{align}
(\text{H2}) & \quad \left\{ \begin{array}{l}
  u_0 \in C^\infty(\{x \geq g_0(y)\} \setminus (0, 0)), \quad 0 \leq u_0(x, y) \leq 1 \quad \text{in } \{x \geq g_0(y)\}, \\
  \text{and the compatibility conditions:} \\
  u_0(g_0(y), y) = 0, \quad u_0(x, 0) = p'(x), \\
  -1 = g_0'(y) \frac{\partial u_0}{\partial y}(g_0(y), y) - \varepsilon \frac{\partial u_0}{\partial x}(g_0(y), y)
\end{array} \right.,
\end{align}
that $\varepsilon$ satisfies

$$\textbf{(H3)} \qquad \varepsilon \leq 1/\|p''\|_{L^\infty},$$

and that

$$\textbf{(H4)} \quad \begin{cases} 
\|u_0\|_{W^{2,\infty}(\{x > g_0(y)\})} < \infty, & \frac{\partial u_0}{\partial y}(x, y) \leq 0, \\
0 \leq \varepsilon \frac{\partial^2 u_0}{\partial x^2}(x, y) + \frac{\partial^2 u_0}{\partial y^2}(x, y) \leq C^* & \text{in } \{x > g_0(y)\}.
\end{cases}$$

**Existence.** Let

$$\mathcal{M} = \{\eta \in C^1[0, T]; \ \eta(0) = 0, \eta'(t) \leq 0 \ \text{for } t \in [0, T]\}.$$  

**Theorem 6.1.** Let the assumptions (6.9) and (H1)-(H4) be in force. Then for each $\eta \in \mathcal{M}$, there exists a unique solution $(w, g)$ to the system (6.2)-(6.8), with $w \in W^{2,1,\infty}_{loc}(Q_T) \cap C(\overline{Q}_T)$, $g(y, t)$ is monotone decreasing in $t$, monotone increasing in $y$ and $g(0+, t) = \eta(t)$.

**Proof.** The uniqueness proof is essentially the same as in Theorem 2.1. One can establish that solution $w \in W^{2,1,n+1}_{loc}(Q_T) \cap C(\overline{Q}_T)$ $(n = 2$ here) which is bounded by a polynomial at infinity is unique. In fact, One can use the change of variable $w = e^{-\alpha t}e^{x+y\tilde{w}}$ to overcome the difficulties at $\infty$, and apply a parabolic version of the maximum principle for functions in $W^{2,1,n+1}$ [7, Theorem 3.1].

For the existence, we can approximate the domain $Q_T$ with $Q_{T,R} = \Omega_R \times [0, T]$ where $\Omega_R = \{(x, y); -K < x < R, 0 < y < R\}$. Take functions $\beta_\delta \in C^\infty$ such that

$$\beta_\delta(s) \geq 0, \quad \beta_\delta''(s) \geq 0 \quad \text{for all } \ s \in (-\infty, \infty),$$

$$\beta_\delta(s) = 0, \quad \text{for } s < -\delta, \quad \beta_\delta(0) = 1,$$

$$\beta_\delta(s) \to \infty \quad \text{as } \delta \to 0 \text{ for each } s > 0,$$

and consider the corresponding penalization problem on $Q_{T,R}$:

$$w_t - \varepsilon w_{xx} - w_{yy} = -1 + \beta_\delta(-w) \quad \text{in } Q_{T,R},$$

with boundary and initial conditions:

$$w(x, y, t) = p \left(x - \eta(t) \right) \left(1 - \frac{y}{R}\right) \quad \text{on } \partial \Omega_R \times [0, T],$$

$$w(x, y, 0) = w_0(x, y) \quad \text{in } \Omega_R.$$

We denote the solution by $w_{R,\delta}(x, y, t)$. By (H2) and (H4),

$$\frac{\partial w_{R,\delta}}{\partial t}(x, y, 0) \begin{cases} 
\geq \varepsilon \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} - 1 \geq 0 & \text{for } x > g_0(y), y > 0, \\
= -1 + \beta_\delta(0) = 0 & \text{for } x \leq g_0(y), y > 0,
\end{cases}$$
It is obvious that
\[
\frac{\partial w_{R,\delta}}{\partial t}(x, y, t) \geq 0 \quad \text{on } \partial Q_R \times [0, T].
\]
Differentiating the equation in \(t\) and applying the maximum principle, we get
\[
(6.16) \quad \frac{\partial w_{R,\delta}}{\partial t} \geq 0 \quad \text{in } Q_T,
\]
which also implies that \(w_{R,\delta} \geq 0\). Similarly, we have
\[
\frac{\partial w_{R,\delta}}{\partial t}(x, y, 0) \begin{cases} \leq \varepsilon \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \leq 1 + C^*(x + K) \quad \text{for } x > g_0(y), y > 0, \\ = -1 + \beta_\delta(0) = 0 \quad \text{for } x \leq g_0(y), y > 0,
\end{cases}
\]
and, notice that \(0 \leq p'(x) \leq 1,\)
\[
\frac{\partial w_{R,\delta}}{\partial t}(x, y, t) \leq \|\eta'\|_{L^\infty[0, T]} \quad \text{on } \partial Q_R \times [0, T].
\]
It follows that
\[
(6.17) \quad \frac{\partial w_{R,\delta}}{\partial t} \leq \max \left(\|\eta'\|_{L^\infty[0, T]}, 1\right) + C^*(x + K) \quad \text{in } Q_T,
\]
by the maximum principle. Differentiating the equation in \(x\) twice and applying the maximum principle, we obtain
\[
(6.18) \quad \frac{\partial^2 w_{R,\delta}}{\partial x^2} \geq -\left\|\left( (u_0)_x \right)^- \right\|_{L^\infty(\{x > g_0(y)\})} \quad \text{in } Q_T.
\]
Similarly,
\[
(6.19) \quad \frac{\partial^2 w_{R,\delta}}{\partial y^2} \geq \min \left( -1, -\left\|\left( 1 - \varepsilon (u_0)_x \right)^- \right\|_{L^\infty(\{x > g_0(y)\})} \right) \quad \text{in } Q_T,
\]
where (H2) and (H4) are used. Now using the equation we find that \((w_{R,\delta})_{xx}\) and \((w_{R,\delta})_{yy}\) are also bounded from above by \(C(x + K)\). Now we can use the compactness to pass the limit for \(R \to \infty\) and \(\delta \to 0\) along a subsequence, and a solution \(w\) is therefore obtained.

Using (H3) and the fact that \(\eta'(t) \geq 0\), we find that \(p(x - \eta(t))\) is a supersolution. It follows that \(w(x, y, t) \leq p(x - \eta(t))\) in \(Q_T\), and thus \(w_y(x, 0, t) \leq 0\). By the maximum principle, \(w_y \leq 0\) in \(Q_T\). Similarly, \(w_x(x, 0, t) = p'(x - \eta(t)) \geq 0\), and so \(w_x \geq 0\) in \(Q_T\), by the maximum principle. We established:
\[
(6.20) \quad w_t \geq 0, \quad w_x \geq 0, \quad w_y \leq 0 \quad \text{in } Q_T.
\]
These inequalities imply that the free boundary is a graph $x = g(y, t)$, $g(y, t)$ is monotone decreasing in $t$, monotone increasing in $y$ and $g(0+, t) = \eta(t)$. Therefore we can take $K > \|\eta\|_{L^\infty[0,T]}$, and the free boundary will not touch $x = -K$. \hfill \Box

Theorem 6.1 asserts the existence of a solution. The solution is uniquely determined by the propagation speed of the reaction front on the $x$-axis ($g(0, t), 0$). Next, we shall study the asymptotic behavior of the solution when $t \to \infty$. We first study the case $g(0, t) \equiv -t$. In this case the solution should go to the travelling wave solution, as we shall show below.

**Lemma 6.2.** Let the assumptions of Theorem 6.1 be in force, if

$$g(0, t) \equiv -t,$$

then

$$\lim_{t \to \infty} (g(y, t) + t) = \bar{g}(y),$$

where we denote by $(\bar{w}, \bar{g})$ the solution of the elliptic variational inequality in Lemma 5.1 corresponding to the boundary condition $p(x)$.

**Proof.** Let $\varphi(x, y, t) = w(x - t, y, t)$, then $\varphi$ satisfies the variational inequality:

$$\varphi_t - \varepsilon \varphi_{xx} - \varphi_{yy} + \varphi_x \geq -1$$

for $0 < x < \infty, 0 < y < \infty, t > 0$  

$$\varphi \geq 0$$

for $0 < x < \infty, 0 < y < \infty, t > 0$

$$\varphi \cdot (\varphi_t - \varepsilon \varphi_{xx} - \varphi_{yy} + \varphi_x + 1) = 0$$

for $0 < x < \infty, 0 < y < \infty, t > 0$

$$\varphi(x, y, 0) = w_0(x, y)$$

for $0 < x < \infty, 0 < y < \infty$

$$\varphi(x, 0, t) = p(x)$$

for $0 < x < \infty, t > 0$

$$\varphi(0, y, t) = 0$$

for $0 < y < \infty, t > 0$

$$0 \leq \varphi(x, y, t) \leq x$$

for $0 < x < \infty, 0 < y < \infty, t > 0$,

with its free boundary given by $x = g(y, t) + t$. Like the system in Theorem 6.1, this system can also be approximated by

$$\varphi_t - \varepsilon \varphi_{xx} - \varphi_{yy} + \varphi_x = -1 + \beta_\delta (\varphi)$$

for $0 < x < R, 0 < y < R, t > 0$,

with the initial and boundary conditions:

$$\varphi(x, y, 0) = w_0(x, y)$$

for $0 < x < \infty, 0 < y < \infty$,

$$\varphi(x, 0, t) = p(x) \left(1 - \frac{y}{R}\right)$$

for $(x, y) \in \partial([0, R] \times [0, R]), t > 0$.

Denote the solution by $\varphi_{R, \delta}$, then by (H2) and (H4),

$$\left| \frac{\partial \varphi_{R, \delta}}{\partial t}(x, y, 0) \right| = |-1 + \beta_\delta (-w_0) + \varepsilon (w_0)_{xx} + (w_0)_{yy} - (w_0)_x|$$

$$\leq 1 + C^* x$$

for $0 \leq x \leq g_0(y), y > 0$. 

Next, we fix $\lambda > 0$ to be small enough such that $\lambda - \varepsilon \lambda^2 > \lambda/2$ and set
\[
\psi = \left(t + \frac{2}{\lambda}\right) e^{-\lambda x} \frac{\partial \varphi_{R_\delta}}{\partial t}.
\]
Then
\[
\psi_t - \varepsilon \psi_{xx} - \psi_{yy} + (1 - 2\varepsilon \lambda)\psi_x + \beta'_e \cdot \psi + \left(\lambda - \varepsilon \lambda^2 - \frac{1}{t + 2/\lambda}\right)\psi = 0.
\]
By (6.24),
\[
|\psi(x, y, 0)| \leq \frac{2}{\lambda} (1 + C^* x) e^{-\lambda x} \leq \tilde{C}.
\]
From the boundary conditions,
\[
\psi = 0 \quad \text{on } (\partial([0, R] \times [0, R])) \times [0, \infty).
\]
Thus by the maximum principle, $|\psi| \leq \tilde{C}$ for $0 < x < R, 0 < y < R, t > 0$. Hence, after taking the limit as $R \to \infty$ and $\delta \to 0$, we obtain,
\[
(6.25) \quad \left|\frac{\partial \varphi}{\partial t}\right| \leq \tilde{C} e^{\lambda x} \frac{1}{t + 2/\lambda} \quad \text{for } x > 0, y > 0, t > 0.
\]
We can derive estimates for $\varphi_{xx}$ and $\varphi_{yy}$ in any bounded domain in $x$-$y$ plane, as in Theorem 6.1. The estimates is independent of $t$. Thus, there exists a subsequence $\{t_j\}$ such that $\lim_{t_j \to \infty} \varphi(x, y, t_j) = h(x, y)$. Recalling (6.25), we derive that $h(x, y)$ satisfies the elliptic variational inequality in §5. By uniqueness, we conclude that $h(x, y) \equiv \tilde{w}(x, y)$. Since the limit is independent of the choice of the sequence $\{t_j\}$, the limit as $t \to \infty$ exists, i.e.,
\[
(6.26) \quad \lim_{t \to \infty} \varphi(x, y, t) = \tilde{w}(x, y).
\]
Noticing that $\tilde{w}(x, y) > 0$ for $x > \tilde{g}(y)$, we derive
\[
(6.27) \quad \limsup_{t \to \infty} (g(y, t) + t) \leq \tilde{g}(y).
\]
By the nondegeneracy lemma (a parabolic version nondegeneracy lemma similar to Lemma 3.1 of [3, page 154]), $\varphi(x, y, t)$ can not be uniformly small in a neighborhood of any point $(x_0, y_0, t_0)$ such that $\varphi(x_0, y_0, t_0) > 0$. It follows that
\[
(6.28) \quad \liminf_{t \to \infty} (g(y, t) + t) \geq \tilde{g}(y),
\]
and the lemma follows. □
COROLLARY 6.3. Let the assumptions of Theorem 6.1 be in force, if

\begin{equation}
\lim_{t \to \infty} (t + g(0, t)) = 0,
\end{equation}

then

\begin{equation}
\lim_{t \to \infty} (g(y, t) + t) = \tilde{g}(y).
\end{equation}

Proof. We will take $T$ large enough so that $|t + g(0, t)| \leq \delta$ for $t > T$. Then we use the comparison function with $g(0, t)$ replaced by $-t \pm \delta$. Applying Lemma 6.2, we immediately obtain

\begin{equation}
\limsup_{t \to \infty} |g(y, t) + t - \tilde{g}(y)| \leq \delta.
\end{equation}

Letting $\delta \to 0$, we conclude. \qed

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