GLOBAL BEHAVIOR OF POSITIVE SOLUTIONS
TO A SEMILINEAR EQUATION WITH A NONLINEAR
FLUX CONDITION

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Global Behavior of Positive Solutions to a Semilinear Equation with a Nonlinear Flux Condition

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Abstract. For the equation $u_{xx} = \lambda u^p$ with boundary conditions $u'(0) = 0, u'(R) = u^q(R)$ a complete bifurcation diagram is given. Using these results the global behavior of positive solutions to the associated evolutionary problem is obtained.

1. Introduction. In this paper we consider the following problem

$$
\begin{cases}
  u_t = u_{xx} - \lambda u^p & 0 < t < T, \ 0 < x < R \\
  u_x(0,t) = 0, \ u_x(R,t) = u^q(R) \\
  u(x,0) = u_0(x)
\end{cases}
$$

(1.1)

where $\lambda > 0$, $p, q > 1$ are given constants, and $u_0 > 0$, $u_0 \in C^{2+\alpha}[0,R]$ and satisfies the compatibility condition,

$$
u_0'(0) = 0 \quad u_0'(R) = u^q(R)
$$

Under these conditions a slight modification of the arguments in [LMW] give the existence of a unique maximal classical solution (see also [Am] and [An]).

In the case in which $\lambda = 0$ it was shown in [LMW] (see also [F]) that no global solution exists. In this paper we show that when $\lambda > 0$ a completely different picture arises since positive stationary solutions exist in most cases.

Thus as a first step towards the understanding of the large time behavior of solutions to (1.1) for different initial values we analyze the stationary problem

$$
\begin{cases}
  u_{xx} = \lambda u^p & 0 < x < R \\
  u_x(0) = 0, \ u_x(R) = u^q(R)
\end{cases}
$$

(1.2)

and give a complete bifurcation diagram (see the figures on Section 2).

In section 3 we analyze the stability of the stationary solutions and the large time behavior of the solutions to (1.1). We also prove that when $p \leq q$ blow up may occur only at $x = R$.

Our main results can be formulated in the following way.

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Theorem 1.

(1) \( p < q \)
   For each \( \lambda > 0 \) there exists a unique positive solution \( u_\lambda \) to (1.2). \( u_\lambda(R) \) is differentiable as a function of \( \lambda \) and \( \frac{\partial}{\partial \lambda} u_\lambda(R) > 0 \); \( \lim_{\lambda \to \infty} u_\lambda(R) = \infty \) and \( \lim_{\lambda \to 0} u_\lambda(R) = 0 \)

(2) \( p = q \)
   i) If \( \lambda \leq R^{-1} \) (1.2) has no positive solutions
   ii) If \( \lambda > R^{-1} \) there exists a unique positive solution \( u_\lambda \) of (1.2), \( u_\lambda(R) \) is differentiable as a function of \( \lambda \), \( \frac{\partial}{\partial \lambda} u_\lambda(R) > 0 \) and \( \lim_{\lambda \to R^{-1} +} u_\lambda(R) = 0 \), \( \lim_{\lambda \to \infty} u_\lambda(R) = \infty \)

(3) \( 2q < p + 1 \)
   For each \( \lambda > 0 \) there exists a unique positive solution \( u_\lambda \) to (1.2) differentiable with respect to \( \lambda \), \( \frac{\partial}{\partial \lambda} u_\lambda(R) < 0 \) and \( \lim_{\lambda \to 0} u_\lambda(0) = \infty \), \( \lim_{\lambda \to \infty} u_\lambda(R) = 0 \)

(4) \( 2q = p + 1 \)
   i) If \( \lambda \leq q \) (1.2) has no positive solutions
   ii) If \( \lambda > q \) there exists a unique positive solution \( u_\lambda \) to (1.2) that satisfies \( \frac{\partial}{\partial \lambda} u_\lambda(R) < 0 \) and \( \lim_{\lambda \to q^+} u_\lambda(R) = \infty \), \( \lim_{\lambda \to \infty} u_\lambda(R) = 0 \)

(5) \( 2q > p + 1 \), \( p > q \)
   There exists \( \Lambda > 0 \) depending on \( p, q \) and \( R \) such that
   i) If \( \lambda \leq \Lambda \) (1.2) has no positive solutions
   ii) If \( \lambda = \Lambda \) there exists a unique positive solution to (1.2)
   iii) If \( \lambda > \Lambda \) there exists exactly two solutions \( u_1 > u_2 \) and they satisfy \( \frac{\partial}{\partial \lambda} u_1(R) > 0 \), \( \frac{\partial}{\partial \lambda} u_2(R) < 0 \), \( \lim_{\lambda \to \infty} u_1(R) = \infty \), \( \lim_{\lambda \to \infty} u_2(R) = 0 \), \( \lim_{\lambda \to \Lambda} u_1(R) = u_\Lambda \) uniformly on \([0,R]\).

Theorem 2. Let \( u \) be the maximal solution to (1.1) and \( T(u_0) \leq \infty \) its existence time.

(1) \( p < q \)
   (i) If \( u_0 > u_\lambda \), \( T(u_0) < \infty \) & \( u(R,t) \to \infty \) (\( t \to T(u_0) \))
   (ii) If \( u_0 < u_\lambda \), \( T(u_0) = +\infty \) & \( u(.,t) \to 0 \) (\( t \to \infty \))

(2) \( p = q \)
   (i) \( \lambda > R^{-1} \)
      a) If \( u_0 > u_\lambda \), \( u(R,t) \to \infty \) (\( t \to T(u_0) \))
      b) If \( u_0 < u_\lambda \), \( T(u_0) = +\infty \) & \( u(.,t) \to 0 \) (\( t \to \infty \))
   (ii) \( \lambda \leq R^{-1} \), \( u(R,t) \to \infty \) (\( t \to T(u_0) \)) for every \( u_0 > 0 \). Moreover if \( \lambda < R^{-1} \), \( T(u_0) < \infty \).

(3) \( 2q < p + 1 \)
   \( T(u_0) = +\infty \) for every \( u_0 \) and \( u(.,t) \to u_\lambda \) as \( t \) goes to infinity

(4) \( 2q = p + 1 \)
   (i) \( \lambda > q \), \( T(u_0) = +\infty \) for every \( u_0 \) and \( u(.,t) \to u_\lambda \) (\( t \to \infty \))
   (ii) \( \lambda \leq q \), \( u(R,t) \to \infty \) (\( t \to T(u_0) \)) for every \( u_0 \).

(5) \( 2q > p + 1 \), \( p > q \)
   (i) \( \lambda < \Lambda \), \( u(R,t) \to \infty \) (\( t \to T(u_0) \)) for every \( u_0 \).
(ii) \( \lambda = \Lambda \)
(a) If \( u_0 \leq u_\lambda \), \( T(u_0) = +\infty \) & \( u(. , t) \rightarrow u_\lambda \ (t \rightarrow \infty) \).
(b) If \( u_0 > u_\lambda \), \( u(R, t) \rightarrow \infty \ (t \rightarrow T(u_0)) \).
(iii) \( \lambda > \Lambda \)
(a) If \( u_0 > u_{\lambda}^1 \), \( u(R, t) \rightarrow \infty \) as \( t \rightarrow T(u_0) \)
(b) If \( u_0 < u_{\lambda}^1 \), \( T(u_0) = +\infty \) & \( u(. , t) \rightarrow u_{\lambda}^2 \) as \( t \rightarrow \infty \).

Theorem 1 is proved in section 2 whereas theorem 2 is proved in section 3.

There have been several papers dealing with the balance between different processes that are taken into account. For instance the balance between reaction and a nonlinear diffusion both when the reaction takes place in the interior and at the boundary as well as reaction against dispersion (see among others [Fi],[ChLS], [LPSSt], [L]). But the balance between a reaction in the substance that consumes energy against one at the boundary with the exterior medium that produces energy does not seem to have been studied yet.

2. Stationary Problem.

In this section we analyze problem (1.2) and give a complete bifurcation diagram.

Assume \( u \) is a solution of this problem, multiplying the equation by \( u_x \) and integrating we get

\[
\frac{1}{2} u_x^2 = \frac{\lambda}{p + 1} (u^{p+1} - u_0^{p+1})
\]

where \( u_0 \) stands for \( u(0) \) and the fact that \( u_x(0) = 0 \) was used. The condition \( u_x(R) = u^q(R) \) leads to

\[
u_R^{2q} = \frac{2\lambda}{p + 1} (u_R^{p+1} - u_0^{p+1})
\]

which is equivalent to

\[
u_0^{p+1} = u_R^{p+1} (1 - \frac{p + 1}{2\lambda} u_R^{2q-p-1})
\]

From (2.1) taking square roots and integrating we obtain

\[
x = \sqrt{\frac{p + 1}{2\lambda}} \int_{u_0}^{u} \frac{ds}{\sqrt{s^{p+1} - u_0^{p+1}}} = \sqrt{\frac{p + 1}{2\lambda}} \int_{1}^{u_R} \frac{dt}{\sqrt{t^{p+1} - 1}}
\]

Given \( u_0 > 0 \), (2.3) gives a solution of the differential equation with \( u(0) = u_0 \) and \( u'(0) = 0 \). This function is a solution to our problem if and only if \( u(R) = u_R \) satisfies (2.2). Therefore we are looking for a number \( u_R \) such that if \( u_0 \) is given by (2.2) we have,

\[
R = \sqrt{\frac{p + 1}{2\lambda}} \int_{u_0}^{u_R} \frac{dt}{\sqrt{t^{p+1} - 1}}
\]
in which case (2.3) gives a solution to our problem with \( u(R) = u_R, \ u(0) = u_0 \).

In order to analyze the existence of such a number let us consider the case \( 2q \neq p + 1 \) and rewrite (2.4) in terms of the quotient \( \frac{u_R}{u_0} \) by taking into account (2.2). Let us call \( \theta = \frac{u_R}{u_0} \). It is easy to see that,

\[
(2.5) \quad u_0 = \left( \frac{2\lambda}{p + 1} \right)^{\frac{1}{2q - p - 1}} \frac{1}{\theta} \left( 1 - \frac{1}{\theta^{p+1}} \right)^{\frac{1}{2q - p - 1}}
\]

and thus (2.4) reads

\[
R = \left( \frac{p + 1}{2\lambda} \right)^{\alpha} \frac{\theta^{q\beta}}{(\sqrt{\theta^{p+1}} - 1)^{\beta}} \int_1^\theta \frac{dt}{\sqrt{t^{p+1} - 1}}
\]

where \( \alpha = \frac{q - 1}{2q - p - 1} , \ \beta = \frac{p - 1}{2q - p - 1} \).

Let us call \( f(\theta) = \frac{\theta^{q\beta}}{(\sqrt{\theta^{p+1}} - 1)^{\beta}} \int_1^\theta \frac{dt}{\sqrt{t^{p+1} - 1}} \). We have the relation

\[
(2.6) \quad \lambda^\alpha = c_p R^{-1} f(\theta)
\]

Let us analyze the function \( f(\theta) \).

\[
f'(\theta) = \theta^{q\beta - 1} \left( \frac{1}{\sqrt{\theta^{p+1}} - 1} \right)^\beta \left[ \beta \left( q - \frac{p + 1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) \int_1^\theta \frac{dt}{\sqrt{t^{p+1} - 1}} + \frac{\theta}{\sqrt{\theta^{p+1} - 1}} \right]
\]

It is clear that if the factor between brackets is positive there will be exactly one \( \theta \) for each \( \lambda \) solving (2.6). But this is not always the case. One can see immediately that \( f'(\theta) > 0 \) if \( q - \frac{p + 1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} > 0 \). This is the case if \( \theta^{-p+1} \leq 1 - \frac{p + 1}{2q} \). Thus if \( 2q > p + 1 \), \( f'(\theta) > 0 \) if \( \theta \geq \left( \frac{2q}{2q - p - 1} \right)^{\frac{1}{p+1}} \). In order to analyze the sign of \( f'(\theta) \) for every \( \theta \) we need to consider the whole term in between brackets. The inequality

\[
\int_1^\theta \frac{dt}{\sqrt{t^{p+1} - 1}} < \frac{2}{p + 1} \frac{\theta^{p+1}}{\sqrt{\theta^{p+1} - 1}}
\]

gives when \( q - \frac{p + 1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} < 0 \) the estimate

\[
[ ] > \frac{1}{\sqrt{\theta^{p+1} - 1}} \left( \left( \theta^{p+1} - \frac{2q}{2q - p - 1} \right) \frac{p - 1}{p + 1} + \theta \right)
\]

Since we have the restriction \( \theta > 1 \),

\[
[ ] > \frac{1}{\sqrt{\theta^{p+1} - 1}} \left( \left( \frac{1}{2q - p - 1} \right) \frac{p - 1}{p + 1} + 1 \right)
\]

\[
= \frac{2}{\sqrt{\theta^{p+1} - 1}} \frac{q - p}{2q - p - 1}
\]
Thus we conclude that \( f'(\theta) > 0 \) for all \( \theta > 1 \) if \( q \geq p \) or \( 2q < p + 1 \). Let us first consider the case \( q \geq p \). We have \( \alpha > 0 \) and \( f \) increasing thus there exists a unique \( \theta_\lambda \) for each \( \lambda \) such that

\[
(c_p R^{-1} f_m)^{\frac{1}{\alpha}} < \lambda < (c_p R^{-1} f_M)^{\frac{1}{\alpha}}
\]

where \( f_m = \inf_{\theta > 1} f(\theta) \), \( f_M = \sup_{\theta > 1} f(\theta) \). From the definition of \( f \) and the fact that \( \beta > 0 \), \( q > \frac{p+1}{2} \), \( \sup_{\theta > 1} f(\theta) = +\infty \). In order to compute \( \lim_{\theta \to 1} f(\theta) \) we apply L'Hospital and get

\[
\lim_{\theta \to 1} f(\theta) = \begin{cases} 
0 & p < q \\
\frac{2}{p+1} & p = q
\end{cases}
\]

Thus if \( p < q \) there exists a unique \( \theta_\lambda \) for each \( \lambda > 0 \) whereas if \( p = q \) there exists a unique \( \theta_\lambda \) if \( \lambda > R^{-1} \). In both cases \( \frac{\partial}{\partial \lambda} \theta_\lambda > 0 \). The relation

\[
(2.7) \quad u_R = \left[\frac{2\lambda}{p+1} \left(1 - \frac{1}{\theta^{p+1}}\right)\right]^{\frac{1}{2q-p-1}}
\]

gives the existence of a unique \( u_\lambda(R) \) for each \( \lambda \) in the ranges above with \( \frac{\partial}{\partial \lambda} u_\lambda(R) > 0 \). It is clear from the considerations above that no positive solution exists if \( p = q \) and \( \lambda \leq R^{-1} \).

From (2.7) we see that

\[
\lim_{\lambda \to \infty} u_\lambda(R) = \infty \quad \text{for} \quad p \leq q
\]

\[
\lim_{\lambda \to 0} u_\lambda(R) = 0 \quad \text{for} \quad p < q
\]

\[
\lim_{\lambda \to R^{-1}+} u_\lambda(R) = 0 \quad \text{for} \quad p = q
\]

Thus parts (1) and (2) of Theorem 1 are proven. Let us analyze the case \( 2q < p + 1 \). We already know that \( f'(\theta) > 0 \). Since \( \alpha < 0 \), there exists a unique \( \theta_\lambda \) for each \( \lambda \) in the range

\[
(c_p R^{-1} f_M)^{\frac{1}{\alpha}} < \lambda < (c_p R^{-1} f_m)^{\frac{1}{\alpha}}
\]

and \( \frac{\partial}{\partial \lambda} \theta_\lambda < 0 \). Now since \( \beta < 0 \), \( q < \frac{p+1}{2} \), \( \lim_{\theta \to \infty} f(\theta) = +\infty \) and \( \lim_{\theta \to 1} f(\theta) = 0 \). Therefore there exists a solution for each \( \lambda > 0 \). From (2.7) we see that

\[
\lim_{\lambda \to 0} u_\lambda(R) = \lim_{\lambda \to \infty} \left[\frac{2\lambda}{p+1} \left(1 - \frac{1}{\theta^{p+1}}\right)\right]^{\frac{1}{2q-p-1}} = \infty
\]

since \( 2q - p - 1 < 0 \); and also \( \lim_{\lambda \to \infty} u_\lambda(R) = 0 \).

In order to analyze the sign of \( \frac{\partial}{\partial \lambda} u_\lambda(R) \) it is enough to show that \( \frac{\partial}{\partial \lambda} u_\lambda(0) < 0 \) since \( u_\lambda(R) = u_\lambda(0) \theta_\lambda \) and \( \theta_\lambda \) is decreasing. This last relation also implies that \( \lim_{\lambda \to \infty} u_\lambda(0) = 0 \).

The fact that \( u_\lambda(0) \) is decreasing in \( \lambda \) follows immediately from (2.4). This finishes the proof of part (3) of theorem 1.

Let us now analyze the case in which \( 2q > p + 1 \) but \( p > q \). We already know that \( f'(\theta) > 0 \) if \( \theta \) is large enough. Let us see that there exists exactly one value \( 1 < \theta_0 < \infty \)
such that \( f'(\theta) < 0 \) for \( \theta < \theta_0 \) and \( f'(\theta) > 0 \) for \( \theta > \theta_0 \). To this end we show that
\[
\lim_{\theta \to -1} f(\theta) = +\infty
\]
and that \( f'(\theta) \) cannot vanish more than once. As in the case \( q \leq p \) we apply L'Hôpital. We have,
\[
\lim_{\theta \to -1} f(\theta) = \lim_{\theta \to -1} \frac{1}{\beta \frac{p+1}{2} (\sqrt{\theta^{p+1} - 1})^{\beta-1}} = +\infty
\]
since \( \beta - 1 = 2 \frac{p-q}{q-p} > 0 \).

Let us see that \( f' \) cannot vanish twice. We recall that
\[
f'(\theta) = \theta^{\beta-1} \left( \frac{1}{\sqrt{\theta^{p+1} - 1}} \right)^{\beta} \left[ \beta \left( q - \frac{p+1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) \int_{1}^{\theta} \frac{dt}{\sqrt{t^{p+1} - 1}} + \frac{\theta}{\sqrt{\theta^{p+1} - 1}} \right]
\]
thus \( f'(\theta) = 0 \) if and only if
\[
g(\theta) = \beta \left( q - \frac{p+1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) \int_{1}^{\theta} \frac{dt}{\sqrt{t^{p+1} - 1}} + \frac{\theta}{\sqrt{\theta^{p+1} - 1}} = 0
\]
Let us see that \( g \) is strictly increasing.

\[
g'(\theta) = \frac{1}{\sqrt{\theta^{p+1} - 1}} \left[ \beta \left( q - \frac{p+1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) + 1 - \beta (p+1)^2 \frac{\theta^{p}}{\sqrt{\theta^{p+1} - 1}} \left( 1 - \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) \int_{1}^{\theta} \frac{dt}{\sqrt{t^{p+1} - 1}} \right] - \frac{p+1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1}
\]
As \( \int_{1}^{\theta} \frac{dt}{\sqrt{t^{p+1} - 1}} > \frac{2}{p^{1+\frac{1}{2}}} \sqrt{\frac{\theta^{p+1}-1}{\theta^{p}}} \),
\[
g'(\theta) > \frac{1}{\sqrt{\theta^{p+1} - 1}} \left[ 1 + \beta \left( q - \frac{p+1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) - \beta (p+1) \left( 1 - \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) - \frac{p+1}{2} \frac{\theta^{p+1}}{\theta^{p+1} - 1} - \beta (p+1)^2 \frac{\theta^{p}}{\sqrt{\theta^{p+1} - 1}} \left( 1 - \frac{\theta^{p+1}}{\theta^{p+1} - 1} \right) \int_{1}^{\theta} \frac{dt}{\sqrt{t^{p+1} - 1}} \right] > 0
\]
Here we used that \( \beta - 1 > 0 \). Let \( \Lambda = (c_{p,R}^{-1} f(\theta_0))^\frac{1}{2} \). If \( \lambda < \Lambda \) there is no positive solution. If \( \Lambda < \Lambda \) there is exactly one solution that corresponds to \( \theta = \theta_0 \). When \( \lambda > \Lambda \) there are two different values of \( \theta \), \( \theta_1^\lambda > \theta_2^\lambda \), for which we have \( \frac{\partial}{\partial \lambda} \theta_1^\lambda > 0 \), \( \frac{\partial}{\partial \lambda} \theta_2^\lambda < 0 \). Associated to each of these \( \theta \)'s there is a solution \( u_\lambda^1 \). (2.7) immediately gives that \( \frac{\partial}{\partial \lambda} u_\lambda^1(R) > 0 \). From
(2.4) we obtain $\frac{\partial}{\partial \lambda} u_\lambda^2(0) < 0$ and therefore also $\frac{\partial}{\partial \lambda} u_\lambda^2(R) < 0$. Let us compute the limits of $u_\lambda^2(R)$ and $u_\lambda^1(R)$ as $\lambda \to \infty$. First,

$$\lim_{\lambda \to \infty} u_\lambda^1(R) = \lim_{\lambda \to \infty} \left[ \frac{2\lambda}{p+1} \left( 1 - \frac{1}{\theta_\lambda^{1+p+1}} \right) \right]^\frac{1}{1-q-p-1} = \infty$$

In order to compute the $\lim_{\lambda \to \infty} u_\lambda^2(R)$ let us first observe from (2.4) that the $\lim_{\lambda \to \infty} u_\lambda^2(0) = 0$.

Since $\lim_{\lambda \to \infty} \theta_\lambda^2 = 1$ it follows that $\lim_{\lambda \to \infty} u_\lambda^2(R) = 0$.

This finishes the proof of part (5) of theorem 1. So it only remains to analyze the case $2q = p + 1$. (2.2) becomes

$$u_0^{2q} = u_R^{2q} \left( 1 - \frac{q}{\lambda} \right)$$

Thus it is clear that we must have $\lambda > q$. When this is the case $\frac{u_R}{u_0} = \left( \frac{\lambda}{\lambda-q} \right)^\frac{1}{2q}$ so that (2.4) gives an explicit formula for the value $u_\lambda(0)$,

$$u_\lambda(0) = \left( R^{-1} \sqrt{\frac{q}{\lambda}} \int_1^\lambda \left( \frac{\lambda-t}{\lambda} \right)^{\frac{1}{2q}} dt \right)^{\frac{1}{1-q-p+1}}$$

From this formula it is clear that $\lim_{\lambda \to \infty} u_\lambda(0) = 0$. Since $\lim_{\lambda \to \infty} \frac{u_\lambda(R)}{u_\lambda(0)} = 1$ we find that also $\lim_{\lambda \to \infty} u_\lambda(R) = 0$.

On the other hand $\lim_{\lambda \to q+} \theta_\lambda = +\infty$. Since $\lim_{\lambda \to q+} u_\lambda(0) > 0$ we deduce that $\lim_{\lambda \to q+} u_\lambda(R) = +\infty$.

It is easy to see by analyzing the behavior of $\theta_\lambda$ and $u_\lambda(0)$ that in this case $\frac{\partial}{\partial \lambda} u_\lambda(R) < 0$.

This ends the proof of theorem 1.

We may summarize the results of this theorem in the following bifurcation diagrams.
$2q = p + 1$

$2q > p + 1, \ p > q$

The stability analysis will be a consequence of our results in the next section. But let us first state the following observation.

Let $u_R = \lim_{\lambda \to \lambda_0} u_\lambda(R)$ and $u_0$ given by (2.2). If $u(x)$ is defined by (2.3) $u_\lambda$ converges to $u$ uniformly on $[0, R]$.


In this section we will prove theorem 2. We will also show that every solution remains bounded in time at each point of the interval $[0, R]$ when $p \leq q$.

Let us start with a definition

**Definition 3.1**: Assume $u_0 \in H^1(0, R)$. Given $T > 0$ we say that the function $u : [0, T] \to H^1(0, R)$ is a weak solution of the evolution problem (1.1) on $[0, T]$ if each of the following two conditions is satisfied

1. $u \in L^\infty(0, T; H^1(0, R)) \cap C([0, T]; L^2(0, R)), u_t \in L^2(0, T; L^2(0, R))$
2. The following identity holds

   \[
   \int_0^R u(t)\phi(\cdot, t) = \int_0^R u_0\phi(\cdot, 0) + \int_0^t \int_0^R u\phi_t - \int_0^t \int_0^R u_x\phi_x
   \]

   \[
   + \int_0^t \int_0^R u_t(R, \cdot)\phi(R, \cdot) - \lambda \int_0^t \int_0^R u^{p-1}\phi
   \]

   for every $t \in [0, T]$ and $\phi \in H^1((0, R) \times (0, T))$.

   The function $u = u(t; u_0)$ is said to be a weak global solution of problem (1.1) if conditions W1 and W2 are satisfied for all positive $T$.

For every $u_0 \in H^1(0, R)$ there exists $T = T(u_0) > 0$ such that problem (1.1) has a weak solution on $[0, T]$ (see [Am], [An]). On the other hand arguing as in section 1 in [LMW] the local existence of classical solutions to (1.1) follows when $u_0$ satisfies the compatibility condition. This is
PROPOSITION 3.1. Let $0 < \alpha < 1$ and $u_0 \in C^{2+\alpha}[0, R]$ be such that $u_0'(0) = 0$, $u_0'(R) = u^q(R)$. There exists a maximal $T = T(u_0) > 0$ (it could be infinity) such that (1.1) has a unique solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}([0, R] \times [0, T])$. Moreover if $T < \infty$, $\limsup_{t \to T} |u(R, t)| = \infty$.

To establish the main result of this section we must recall some well known concepts. Given $u_0 \in H^1(0, R)$ such that $u(t; u_0)$ is a weak global solution of (1.1) we denote by $\Gamma^+(u_0)$ the positive orbit of $u_0$ this is

$$\Gamma^+(u_0) = \{u(t; u_0) : t \geq 0\}$$

The $\omega^*$-limit set of $u(t; u_0)$, $\omega^*(u_0)$ is defined as the set of $v \in H^1(0, R)$ for which there exists a sequence in $\Gamma^+(u_0)$ weakly converging to $v$.

We state and prove the following stabilization result.

PROPOSITION 3.2. Let $u_0 \in H^1(0, R)$ be such that $u(t; u_0)$ is a weak global solution of (1.1) Assume $u \in L^\infty(0, \infty; H^1(0, R))$ and $u_t \in L^2(0, \infty; L^2(0, R))$. If $v \in \omega^*(u_0)$, $v$ is a weak solution of (1.2).

PROOF: The original ideas of the proof can be found in [LPh]. Some technical details are taken from [FM].

Let $v \in \omega^*(u_0)$ and $\{t_n\}_{n \geq 1}$, $t_n \to \infty$ as $n \to \infty$ such that $u(t_n; u_0) \to v$ weakly in $H^1(0, R)$ as $n \to \infty$ and define

$$U_n(x, s) = u(x, t_n + s), \quad x \in (0, R), \quad s \in (-1, 1)$$

First we show that

$$U_n \to v \text{ in } L^2((0, R) \times (-1, 1)), \quad (n \to \infty)$$

Applying Cauchy-Schwarz inequality el follows easily that

$$\left| \int_{t_n}^{t_n+s} u_t(x, t) \, dt \right|^2 \leq \int_{t_n}^{t_n+1} u_t^2(x, t) \, dt, \quad s \in (-1, 1)$$

thus

$$\left(3.1\right) \quad \int_0^R \left| \int_{t_n}^{t_n+s} u_t(x, t) \, dt \right|^2 \, dx \leq \int_{t_n}^{t_n+1} u_t^2(x, t) \, dt \, dx$$

and

$$\left(3.2\right) \quad \|U(\cdot, s) - u(\cdot, t_n)\|_{L^2(0, R)}^2 = \int_0^R |u(x, t_n + s) - u(x, t_n)|^2 \, dx$$

$$= \int_0^R \left| \int_{t_n}^{t_n+s} u_t(x, t) \, dt \right|^2 \, dx \leq \int_0^R \int_{t_n}^{t_n+1} u_t^2(x, t) \, dt \, dx$$

and

$$\left(3.3\right) \quad \|U_n(\cdot, s) - u(\cdot, t_n)\|_{L^2((0, R) \times (-1, 1))}^2 = \int_{t_n}^{t_n+1} \int_0^R \left| u(x, t_n + s) - u(x, t_n) \right|^2 \, dx \, ds$$

$$\leq 2 \int_0^R \int_{t_n}^{t_n+1} u_t^2(x, t) \, dt \, dx \to 0, \quad (n \to \infty)$$

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which proves (3.1). In particular for some subsequence which we denote in the same way, \( U_n \to u \) almost everywhere on \((0, R) \times (-1, 1)\).

On the other hand using (3.2) and the definition of \( U_n \) we get

\[
\| (U_n)_s \|^2_{L^2(-1, 1; L^2(0, R))} = \int_{-1}^{1} \int_0^R \| (U_n)_s(x, s) \|^2 \, dx \, ds \\
= \int_0^R \int_{t_n-1}^{t_n+1} u_t^2(x, t) \, dt \, dx \to 0 \quad (n \to \infty)
\]

and thus taking into account that \( u(t; u_0) \) is a weak global solution of (1.1) we deduce the existence of a positive constant \( C \) which is independent of \( n \) such that

\[
\| U_n \|_{H^1((-1, 1) \times (0, R))} \leq C
\]

Now let \( \psi \in C_0^2[0, R] \) and \( \chi \in C_0^2(-1, 1) \), \( \chi \geq 0 \), \( \int_{-1}^{1} \chi(s) \, ds = 1 \) and consider \( \phi_n(x, t) = \psi(x)\chi(t - t_n) \) as a test function in the weak formulation of problem (1.1). Taking \( T = t_n + 1 \), \( T = t_n - 1 \) and substracting the resulting relations it follows that

\[
0 = \int_{t_n-1}^{t_n+1} u(x, t)\psi(x)\chi'(t - t_n) \, dx \, dt - \int_{t_n-1}^{t_n+1} u_x(x, t)\psi'(x)\chi(t - t_n) \, dx \, dt \\
+ \int_{t_n-1}^{t_n+1} u^q(R, t)\psi(R)\chi(t - t_n) \, dt - \lambda \int_{t_n-1}^{t_n+1} \int_0^R u^p(x, t)\psi(x)\chi(t - t_n) \, dx \, dt
\]

which can be as well written as

\[
0 = \int_{-1}^{1} \int_0^R u(x, t_n + s)\psi(x)\chi'(s) \, dx \, ds - \int_{-1}^{1} \int_0^R u_x(x, t_n + s)\psi'(x)\chi(s) \, dx \, ds \\
+ \int_{-1}^{1} u^q(R, t_n + s)\psi(R)\chi(s) \, ds - \lambda \int_{-1}^{1} \int_0^R u^p(x, t_n + s)\psi(x)\chi(s) \, dx \, ds
\]

(3.5)

Now from (3.3), (3.4) we can pass to the limit in (3.5) as \( n \to \infty \) (see [FM] for technical details) to get

\[
0 = \int_{-1}^{1} \int_0^R v(x)\psi(x)\chi'(x) \, dx \, ds - \int_{-1}^{1} \int_0^R v_x(x)\psi'(x)\chi(s) \, dx \, ds \\
+ \int_{-1}^{1} v^q(R)\psi(R)\chi(s) \, ds - \lambda \int_{-1}^{1} \int_0^R v^p(x)\psi(x)\chi(s) \, dx \, ds
\]

Finally taking into account that \( \int_{-1}^{1} \chi(s) \, ds = 1 \), \( \int_{-1}^{1} \chi'(s) \, ds = 0 \) the proof is finished.

Remark 3.1: If \( u \) is a bounded global classical solution of (1.1), \( u \) belongs to \( L^\infty(0, \infty; H^1(0, R)) \) and \( u_t \) to \( L^2(0, \infty; L^2(0, R)) \). This follows from the following identity that is obtained by multiplying the differential equation by \( u_t \) and integrating by parts

\[
\int_0^t \int_0^R u_t^2(x, t) \, dx \, dt + \frac{1}{2} \int_0^R u^2_x(x, t) \, dx \, dt - \frac{1}{q + 1} u^{q+1}(R, t) + \frac{\lambda}{p + 1} \int_0^R u^{p+1}(x, t) \, dx \\
= \frac{1}{2} \int_0^R u_0^2(x) \, dx - \frac{1}{q + 1} u_0^{q+1}(R) + \frac{\lambda}{p + 1} \int_0^R u_0^{p+1}(x) \, dx
\]
Corollary 3.1. Let $u$ be a bounded classical global solution of (1.1). If $v \in \omega^*(u_0)$, $v$ is a classical solution of (1.2).

This follows immediately from remark 3.1, proposition 3.2 and the fact that weak equilibrium solutions are actually classical ones.

We state here the following comparison result that can be found in [An].

Proposition 3.3. Let $u^1$ and $u^2$ be weak solutions of (1.1) with initial values $u^1_0 \leq u^2_0$. Then $u^1 \leq u^2$ for $t < \min(T(u^1_0), T(u^2_0))$.

Corollary 3.2. Let $u$ be a classical solution of (1.1) such that

$$u_{0xx} - \lambda u_0^p < (\text{resp. } >) 0 \quad \text{on } [0, R]$$

then $u_t \leq (\text{resp. } \geq) 0$ for $t < T(u_0)$.

Proof.: It follows by comparison between the two solutions of (1.1), $u$ and $u_\varepsilon$ where

$$u_\varepsilon(x, t) = u(x, t + \varepsilon)$$

and $\varepsilon < \varepsilon_0$.

Corollary 3.3. Let $u$ be a solution of (1.1) with $u_0 = u_\mu$ where $u_\mu$ is a positive stationary solution of

$$u_{xx} = \mu u^p \quad 0 < x < R$$

$$u_x(0) = 0, \ u_x(R) = u^q(R)$$

Then if $\mu > \lambda$, $u_t \geq 0$ and if $\mu < \lambda$, $u_t \leq 0$.

We now prove our main result, Theorem 2.

(1) $p < q$

Let $u_0 < u_\lambda$. There exists $\mu < \lambda$ such that $u_0 < u_\mu$. Let $v$ be the solution of (1.1) with initial value $u_\mu$ then $v_t \leq 0$ and $u \leq v$. This estimate implies that $u$ is a global solution and

$$u(R, t) \leq v(R, t) \leq u_\mu(R) < u_\lambda(R)$$

for every $t > 0$. This in turn implies that $u_\lambda \not\in \omega^*(u_0)$. Thus

$$u(R, t) \to 0 \quad (t \to \infty)$$

Let $u_0 > u_\lambda$. There exists $\mu > \lambda$ such that $u_0 > u_\mu$. If $v$ is the solution of (1.1) with initial value $u_\mu$, $v_t \geq 0$ and $u \geq v$. This in turn implies that $u(R, t) \geq v(R, t)$. It is clear that $v(R, t) \rightarrow \infty$ as $t \rightarrow T(u_\mu)$ since $v$ cannot be bounded and at the same time

$$v(R, t) \geq u_\mu(R) > u_\lambda(R) > 0$$

Let us see that $v$ actually blows up in finite time. Assume $v$ exists globally we claim that there exists a time $t_0$ such that

$$\frac{1}{R} \int_0^R v(x, t_0) \, dx > (\lambda R)^{\frac{1}{q-p}}$$

(3.6)
if not let \( m(t) = \frac{1}{R} \int_0^R v(x, t) \, dx \). \( m \) would be a bounded nondecreasing function and therefore \( m'(t) \to 0 \) as \( t \to \infty \). Let \( t_1 \) be such that \( t > t_1 \) implies \( Rm'(t) < 1 \). We have

\[
Rm'(t) = \int_0^R v_t(x, t) \, dx = \int_0^R v_{xx}(x, t) \, dx - \lambda \int_0^R v^p(x, t) \, dx
\]

\[
= v^q(R, t) - \lambda \int_0^R v^p(x, t) \, dx
\]

Thus \( v^q(R, t) < 1 + \lambda R v^p(R, t) \) since \( v_0 \) increasing implies that \( v \) is increasing in space. This last inequality is impossible since \( p < q \) and \( v(R, t) \to \infty \). Let \( t_0 \) be such that (3.6) holds. Proceeding as before we get from (3.7)

\[
m'(t) \geq \frac{1}{R} (v^q(R, t) - \lambda R v^p(R, t)) = \frac{1}{R} v^p(R, t)(v^{q-p}(R, t) - \lambda R)
\]

\[
\geq \frac{1}{R} v^p(R, t)(m(t)^{q-p} - \lambda R)
\]

thus \( m'(t_0) > 0 \) which implies that

\[
m(t) > m(t_0) > (\lambda R)^{\frac{1}{q-p}}
\]

for \( t > t_0 \), \( t \) close to \( t_0 \). And therefore (3.9) holds for every \( t \). We have

\[
m'(t) \geq \frac{1}{R} v^p(R, t)(m(t_0)^{q-p} - \lambda R) \geq \frac{1}{R} (m(t_0)^{q-p} - \lambda R)[m(t)]^p
\]

so \( m(t) \) cannot exist for all \( t \). Therefore \( u \) does not exist globally and we have

\[
\lim_{t \to T(u_0)} \sup u(R, t) = +\infty
\]

(2) \( p = q \)

(i) \( \lambda \leq R^{-1} \)

Let \( \mu > R^{-1} \) such that \( u_\mu < u_0 \). Since \( \mu > \lambda \) if \( v \) is the solution of (1.1) with \( v(x, 0) = u_\mu \), \( v_t \geq 0 \). Therefore \( u(x, t) \geq u_\mu(x) \geq u_\mu(0) > 0 \). Thus \( u \) cannot remain bounded. Moreover if \( \lambda < R^{-1} \) \( u \) blows up in finite time. In fact from (3.8)

\[
m'(t) \geq \frac{1}{R} (1 - \lambda R)v^p(R, t) \geq \frac{1}{R} (1 - \lambda R)[m(t)]^p
\]

and therefore \( v \) cannot exist for all time.

(ii) \( \lambda > R^{-1} \)

The same arguments as in the case \( p < q \) show that for \( u_0 > u_\lambda \), \( \lim_{t \to T(u_0)} u(R, t) = \infty \) and for \( u_0 < u_\lambda \), \( \lim_{t \to \infty} u(R, t) = 0 \).

(3) \( 2q < p + 1 \)
Let \( u_0 \) \( > 0 \) on \([0, R]\), let \( \mu \) large enough as to have \( \mu > \lambda \) and \( u_\mu < u_0 \) and \( \nu \) small enough as to have \( \nu < \lambda \) and \( u_0 < u_\nu \). Let \( v \) be the solution of (1.1) with initial datum \( u_\mu \) and \( w \) the solution with initial datum \( u_\nu \). We have

\[
0 < u_\mu(0) \leq v(x,t) \leq u(x,t) \leq w(x,t) \leq u_\nu(R)
\]

Therefore \( T(u_0) = +\infty \). Since \( u(x, t_n) \) cannot converge to zero for any sequence \( t_n \to \infty \) we deduce that \( u \) converges to \( u_\lambda \) as \( t \to \infty \).

(i) If \( \lambda \leq q \) we proceed as in the case (2)(i) by taking \( \mu > q \) sufficiently large and comparing \( u \) with \( v \). We deduce that

\[
\limsup_{t \to T(u_0)} u(R,t) = +\infty
\]

(ii) If \( \lambda > q \)

Let \( u_0 > 0 \) on \([0, R]\). There exists \( \lambda > \mu > q \) and a constant \( c > 0 \) such that the function

\[
v_0(x) = cu_\mu(x) + ||u_0||_{L^\infty}
\]

satisfies

\[
v_0'(0) = 0 \quad v_0'(R) = v_0^q(R) \quad \text{and} \quad v_0'' - \lambda v_0^p < 0
\]

In fact the condition at the origin is immediate. Let \( q < \mu < \nu < \lambda \) such that

\[
\frac{||u_0||_{L^\infty}}{u_\mu(R)} < \left[ \left( \frac{\nu}{\lambda} \right)^{\frac{1}{p-1}} \right]^\frac{1}{q} - \left( \frac{\nu}{\lambda} \right)^{\frac{1}{p-1}}
\]

Here we use the fact that \( \lim_{\mu \to q^+} u_\mu(R) = +\infty \). Now let \( \left( \frac{\nu}{\lambda} \right)^{\frac{1}{p-1}} < c < 1 \) be such that

\[
(3.10) \quad \frac{||u_0||_{L^\infty}}{u_\mu(R)} = c^{\frac{1}{p}} - c
\]

Let us see that with this choice of \( c \) and \( \mu \), \( v_0 \) satisfies the conditions above. In fact

\[
(cu_\mu + ||u_0||_{L^\infty})_x x - \lambda (cu_\mu + ||u_0||_{L^\infty})^p = c\mu u_\mu^p - \lambda (cu_\mu + ||u_0||_{L^\infty})^p
\]

\[
= c^{1-p} \mu (cu_\mu)^p - \lambda (cu_\mu + ||u_0||_{L^\infty})^p < 0
\]

because \( c^{1-p} \mu < \lambda \) by construction. Let us check the boundary condition,

\[
(cu_\mu + ||u_0||_{L^\infty})_x (R) = cu_\mu^q(R) = (cu_\mu(R) + ||u_0||_{L^\infty})^q
\]

if and only if

\[
c^{\frac{1}{q}} u_\mu(R) = cu_\mu(R) + ||u_0||_{L^\infty}
\]

and this is equivalent to (3.10).
Since \( v_0 > u_0 \) we have \( v(x,t) > u(x,t) \) if \( v \) is the solution of (1.1) with initial datum \( v_0 \). Also \( v_t \leq 0 \). Therefore \( u \) is bounded. On the other hand we may choose \( \sigma \) large enough as to have \( \sigma > \lambda \) and \( u_\sigma < u_0 \). Therefore
\[
u(x,t) \geq u_\sigma(0) > 0
\]
for every \( t \). This implies that \( u \) converges to \( u_\lambda \) as \( t \) goes to \( \infty \).

Finally we get to
\[
(5) \quad 2q > p + 1, \ p > q
\]
(i) \( \lambda < \Lambda \)

Let \( u_0 > 0 \) and let \( \mu \) large enough as to have \( \mu > \Lambda \), \( u_0 > u_\mu^2 \). Let \( v \) be the solution with initial datum \( u_\mu^2 \). Then
\[
u(x,t) \geq v(x,t) \geq u_\mu^2(0) > 0
\]
As there is no positive stationary solution we deduce that
\[
\limsup_{t \to T(u_0)} u(R,t) = +\infty
\]

(ii) \( \lambda = \Lambda \)
(a) Let \( u_0 > u_\Lambda \) and let \( \mu > \Lambda \) small enough as to have \( u_0 > u_\mu^1 \). We have
\[
u(R,t) \geq u_\mu^1(R) > u_\Lambda(R)
\]
as long as it exists. This implies that \( u(x,t_n) \) cannot converge to \( u_\Lambda \) or 0 as \( n \) goes to infinity. Therefore
\[
\limsup_{t \to T(u_0)} u(R,t) = +\infty
\]
(b) Let \( u_\Lambda \geq u_0 > 0 \) on \( [0,R] \). Let \( \mu \) large enough as to have \( \mu > \Lambda \) and \( u_0 > u_\mu^2 \). Then
\[
u_\Lambda(R) \geq u(x,t) \geq u_\mu^2(0) > 0
\]
and we conclude that \( u \) converges to \( u_\Lambda \) as \( t \) goes to infinity.

(iii) \( \lambda > \Lambda \)
(a) If \( u_0 > u_\lambda^1 \) we proceed as in (5)(ii)(a) to conclude that
\[
\limsup_{t \to T(u_0)} u(R,t) = +\infty
\]
(b) Let \( 0 < u_0 < u_\lambda^1 \) on \( [0,R] \). Let \( \mu \) large enough as to have \( \mu > \lambda \) and \( u_0 > u_\mu^2 \); and
\( \lambda > \nu > \Lambda \) close enough to \( \lambda \) as to have \( u_0 < u_\lambda^1 \). Then
\[
0 < u_\mu^2(0) \leq u(x,t) \leq u_\nu^1(R) < u_\lambda^1(R)
\]
and we conclude that \( u \) converges to \( u_\lambda^2 \) as \( t \) goes to infinity.
This concludes the proof of theorem 2.

Finally we prove that when \( p \leq q \) blow up may occur only at the boundary \( x = R \).
Proposition 3.4. Let $p \leq q$ and let $u$ be a classical positive solution of (1.1) there exists a constant $C = C(u_0)$ such that

$$u(x,t) \leq \frac{C}{(R-x)^{1-q}}$$

as long as it exists.

Proof: The ideas in this proof were first used in [LMW]. Assume first that $u_x \geq 0$. We show that there exists a nonnegative smooth function $g$ with $g(R) > 0$ such that

$$u_x(x,t) \geq g(x)u^q(x,t)$$

In fact let $v = u_x$ and $w = g(x)u^q$ where $g$ is chosen smooth convex with $g(0) = 0$, $g(R) > 0$ and such that

$$\frac{u_t}{u^q} \geq g(x) \quad \text{on } [0, R]$$

then

$$(w-v)_t - (w-v)_{xx} = -\lambda gqu^{p+q-1} - q'u^q - 2g'qu^{q-1}u_x - q(q-1)gu^{p-2}u_x^2 + \lambda pu^{p-1}u_x$$

also

$$v(x,0) \geq w(x,0) \quad 0 < x < R$$
$$v(0,t) = w(0,t) \quad t > 0$$
$$v(R,t) \geq w(R,t) \quad t > 0$$

Thus if $w - v$ where positive somewhere it would attain its positive maximum at an interior point $(x_0, t_0)$. At this point we would have $w = gu^q > u_x = v$. Also $(w - v)_t \geq 0$ and $(w - v)_{xx} \leq 0$. Therefore

$$0 \leq (w-v)_t - (w-v)_{xx} < \lambda gu^{p+q-1}(p-q) \leq 0$$

Therefore $v \geq w$ everywhere. This estimate leads to the inequality

$$u(x,t) \leq \frac{C}{(R-x)^{1-q}}$$

(3.11)

In fact after an integration we get to

$$\frac{1}{(q-1)u^{q-1}(x,t)} \geq \int^R_x g(s) \, ds$$

As $g(R)$ is positive an estimate from below of the integral gives (3.11).

When $u_x$ is not necessarily nonnegative it is possible to show that a nonpositive bound from below of $u'_0$ is a bound from below of $u_x(x,t)$. A slight modification of the arguments above lead to (3.11) (see [LMW]) for the details).

We want to point out that a bound from above of $u$ by a solution of (1.1) that is bounded in time for each $x \in [0, R)$ is not enough to deduce that the same is true for $u$ since its existence time could be larger than that of its bound.
References


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