GLOBAL EXISTENCE OF WEAK SOLUTIONS
FOR INTERFACE EQUATIONS COUPLED
WITH DIFFUSION EQUATIONS

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GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR INTERFACE EQUATIONS COUPLED WITH DIFFUSION EQUATIONS

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Abstract. A weak formulation for an interface dynamics coupled with a diffusion equation is introduced. A global-in-time weak solution is constructed for an arbitrary initial data under a periodic boundary condition. The result applies to the interface equation obtained as a certain singular limit of some reaction-diffusion systems including the activator-inhibitor model.

Key words. Interface equation with diffusion equation, global existence, viscosity solution

AMS(MOS) subject classifications. 35K55, (35K57 35K65)

1. Introduction. This paper is concerned with interface equations coupled with diffusion equations. A typical example is formally obtained as a certain singular limit of a class of reaction diffusion systems [XYC]. Our main objective is to construct a global-in-time weak solution for the initial value problem of these interface equations.

Let \( \Omega_+ (t) \) be two disjoint open sets in \( \mathbb{R}^n \) depending on time \( t \). The complement of the union of \( \Omega_+ (t) \) and \( \Omega_- (t) \) is called the interface and denoted by \( \Gamma (t) \). To write down the equation we assume that the interface \( \Gamma (t) \) is a smooth hypersurface so that \( \Gamma (t) \) is the boundary of \( \Omega_\pm (t) \). Let \( V = V(t,x) \) denote the speed of \( \Gamma (t) \) at \( x \in \Gamma (t) \) in the normal direction \( \vec{n} \) from \( \Omega_+ (t) \) to \( \Omega_- (t) \). Let \( \kappa = \text{div} \vec{n} \) denote \((n - 1)\) times the mean curvature of \( \Gamma (t) \) at \( x \in \Gamma (t) \). We consider an interface equation for \( \Gamma (t) \):

\[
V = W(v) - ck \quad \text{on} \quad \Gamma (t)
\]

(1.1) coupled with a diffusion equation for \( v = v(t,x) \):

\[
v_t = D\Delta v + g_\pm (v), \quad x \in \Omega_\pm (t), \quad t > 0,
\]

(1.2) where \( c \geq 0 \) and \( D > 0 \) are constants. Here \( g_\pm \) and \( W \) are given bounded continuous functions on \( \mathbb{R} \). We also impose a condition that \( v(t) = v(t,\cdot) \) is continuous in \( \mathbb{R}^n \) with its first derivatives, i.e.,

\[
v(t) = v(t,\cdot) \in C^1(\mathbb{R}^n) \quad \text{for} \quad t > 0.
\]

(1.3)

Our goal is to construct a global solution of the initial value problem for interface equations coupled with diffusion equations — a typical example of which is (1.1)–(1.3). There is an intrinsic difficulty to construct a global solution \((\Omega_\pm (t), v(t))_{t \geq 0} \) since \( \Gamma (t) \) may have singularities in a finite time. If \( v \) is a constant so that \( g_\pm (v) = 0 \), (1.1)–(1.3) becomes

\[
V = C - ck,
\]

(1.4)

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where $C$ is a constant. If $C = 0$ and $c > 0$, (1.4) becomes

$$V = -ck,$$

which is called the mean curvature flow equation. Even for (1.5) Grayson [Gr] gives an example of a barbell in $\mathbb{R}^3$ with a long, thin handle that actually pinches off in a finite time. To track whole evolution of interface we interpret $\Gamma(t)$ as a level set of viscosity solution of some second order evolution equations as in [CGG]. In fact Y.-G. Chen and the first two authors [CGG] constructed a unique global weak solution with arbitrary initial data for a class of interface equations including (1.4) and (1.1) where $v$ only depends on time; see [GG1] for interface equations that the theory in [CGG] applies to. Almost at the same time Evans and Spruck [ES1] constructed the same solution but only for (1.5). Another formulation closely related to [CGG] and [ES1] is given in [S]. We refer to [ES2] and [GG2] for further development of the theory and references. We note that idea using level of viscosity solutions for $V = C$ is also found in an unpublished paper of Barles [B].

Although the interface equation admits a global weak solution, $\Gamma(t)$ may develop an interior (Remark 2.5). We introduce a generalized formulation of (1.2)–(1.3). For technical reasons we impose a periodic boundary condition. In this paper among other results we construct a global weak solution of the initial value problem for (1.1)–(1.3) with arbitrary initial data $(\Gamma(0), v(0, x))$, $v(0, x) \in C^2$ under the periodic boundary condition (Theorem 4.6). (We may assume $D = 1$ without the loss of generality.) For this purpose for continuous $v$ we construct a unique global weak solution for (1.1). The basic idea is the same as in [CGG] but we are forced to use results in [GIS] since $v$ may depend on $x$ as well as $t$. We solve (1.2) – (1.3) with above $v$ and $\Omega(t)$ determined by (1.1) with the initial condition. If we write the solution by $w$ we have a mapping $v \mapsto w$. Since our weak formulation forces us to interpret the mapping $v \mapsto w$ as a multi-valued mapping, we use Kakutani (-Ky Fan's) fixed point theory (see e.g. [AF]) to get a global generalized solution as a fixed point of the mapping $v \mapsto w$. Our results applies to the system (1.2) – (1.3) with (1.1) replaced by more general interface equations including anisotropic motion (cf [Gu1,2], [C]). We do not know the uniqueness of our solutions.

Let us mention some results on (1.1) – (1.3) related to ours. In [XYC] X.-Y. Chen constructed a unique local smooth solution for a smooth initial data $(\Gamma(0), v(0, x))$ in $\mathbb{R}^n$ when $c > 0$. When $n = 1$, the curvature term in (1.1) disappears. Hilhorst, Nishiura and Mimura [HN] constructed a global unique solution for (1.1) – (1.3) when the interface is a point and $n = 1$ under the Neumann boundary condition. Their interface is $C^1$ in time. After this work was completed, we learned of the recent paper of X. Chen [XC2] which extends the local existence results in [XYC] to the case $c = 0$. Our result seems to be a first global result even for (1.1) – (1.3) with $c > 0$ or $c = 0$ when $n > 1$.

Interface equations and reaction-diffusion equations are closely related; see [F]. Typical examples of the system (1.1) – (1.3) is provided as a singular limit of reaction-diffusion equations at least formally; see [OMK] and [XYC]. We will explain it more explicitly following [XYC]. We consider a reaction-diffusion system describing the activator-inhibitor model:

\begin{align*}
(1.6) & \quad u_t = \varepsilon \Delta u + \frac{1}{\varepsilon} f(u, v), \quad x \in \mathbb{R}^n, t > 0, \\
(1.7) & \quad v_t = D \Delta v + g(u, v), \quad x \in \mathbb{R}^n, t > 0,
\end{align*}
with

\[ f(u, v) = f_0(u) - v, f_0(u) = u(1 - u)(u - a), \]
\[ g(u, v) = u - \gamma v, \]

where \( \gamma > 0, 0 < a < 1 \) and \( \varepsilon \) is a small positive parameter. The zero set of \( f \) consists of three branches

\[ u = h_-(v) \quad \text{for } u < a_-, \]
\[ u = h_+(v) \quad \text{for } a_- < u < a_+, \]
\[ u = h_0(v) \quad \text{for } a_- < u < a_+, \]

where \( a_- < a_+ \) and \( f_0'(a_-) = f_0'(a_+) = 0 \). When \( \varepsilon \to 0 \), it is expected that \( u \) tends to \( h_\pm(v) \) in some region \( \Omega_\pm(t) \) in \( \mathbb{R}^n \) since \( h_\pm(v) \) is a stable zero of \( u_t = f(u, v) \). From (1.6) it is also expected that the interface \( \Gamma(t) \) moves by (1.1) with \( c = \varepsilon \). Here \( W(b) \) for \( b, f_0(a_-) < b < f_0(a_+) \) is the speed of the travelling wave of

\[ u_t = \Delta u + f(u, b) \]

and is given by

\[ W(b) = \frac{1}{\sqrt{2}} (h_+(b) + h_-(b) - h_0(b)); \]

see Aronson and Weinberger [AW]. The equation (1.7) now becomes (1.2) as \( \varepsilon \to 0 \) by taking \( g_\pm(v) = g(h_\pm(v), v) \). For more details we refer to [OMK] and [XJC] and references therein. We note that anistropic interface equations are also derived by a singular limit of some reaction-diffusion equation [C].

There is an extensive literature on the behavior \( u^\varepsilon \) as \( \varepsilon \to 0 \) in (1.6) and its relation to the solutions of interface equation when \( v \) is given and the space dimension \( n = 1 \). See e.g. [FII], [BK], [CP]. Recently, some results are extended to the case \( n > 1 \) where the curvature effect comes in. If \( v \) is a constant and \( W(v) = 0 \), (1.6) is called the Allen-Cahn equation whose relation to (1.5) with \( c > 0 \) is rigorously analyzed by Bronsard and Kohn [BK] and DeMottoni and Schatzman [DS]. X. Chen [XC2] extended results of [DS] and simplified the argument. After this work was completed, we learned that X. Chen [XC2] derived (1.1) - (1.3) with \( c = 0 \) rigorously as a singular limit of (1.6) - (1.7). There is also an argument to interpret the case \( c = \varepsilon > 0 \) in [XC2]. His method is an extension of his work [XC1]. All results in [BK, DS, XC1, XC2] assume that the solution of the interface equation is smooth to get the behavior of \( u^\varepsilon \) as \( \varepsilon \to 0 \). Very recently we learned that Evans, Soner and Songanidis [ESS] obtained the behavior of \( u^\varepsilon \) even after singularities appear on the interface for the Allen-Cahn equation.

In §2 we solve a general interface equation including (1.1) for given continuous function \( v \) globally in time under a periodic boundary condition. In §3 we give a generalized formulation of (1.2) - (1.3). In §4 we state our main existence results and prove them by a fixed point argument. In Appendix we state a stability property of viscosity solutions used in §4.

After this work was completed, X.-Y. Chen kindly informed that he found another proof for global existence for (4.1), (4.2), (4.3') with \( c > 0 \) without using a fixed point argument for multi-valued mappings.
2. Interface equations. We consider interface equations under periodic boundary conditions. The periodic boundary condition is important because it is often used in numerical experiments. For \( \alpha_i > 0 (1 \leq i \leq n) \) let \( R \) be a rectangle in \( \mathbb{R}^n \) of the form
\[
R = \{(x_1, \cdots, x_n) \in \mathbb{R}^n; 0 \leq x_i \leq \alpha_i, 1 \leq i \leq n\}.
\]
We identify faces \( x_i = 0 \) and \( x_i = \alpha_i (1 \leq i \leq n) \) of \( R \) to obtain an \( n \)-dimensional flat torus \( T \). Motion of interfaces in \( R \) under periodic boundary conditions is interpreted as the motion in \( T \). We consider \( T \) rather than \( \mathbb{R}^n \) for later technical convenience because \( T \) is compact and has no boundary.

Let \( \Omega_{\pm}(t) \) be an open set in \( T \) depending on time \( t \geq 0 \) such that \( \Omega_{+}(t) \cap \Omega_{-}(t) = \emptyset \). Let \( \Gamma(t) \) denote the complement of \( \Omega_{+}(t) \cup \Omega_{-}(t) \) in \( T \). Physically speaking, \( \Gamma(t) \) is called an interface bounding two phases \( \Omega_{\pm}(t) \) of material, e.g. solid and liquid region. Suppose that \( \Gamma(t) \) is a smooth hypersurface and let \( \vec{n} \) denote the unit normal vector field pointing from \( \Omega_{+}(t) \) to \( \Omega_{-}(t) \). Let \( V = V(t,x) \) denote the speed of \( \Gamma(t) \) at \( x \in \Gamma(t) \) in the direction \( \vec{n} \). It is convenient to extend \( \vec{n} \) to a vector field (still denoted by \( \vec{n} \)) on a tubular neighborhood of \( \Gamma(t) \) such that \( \vec{n} \) is constant in the normal direction of \( \Gamma(t) \). The equation for \( \Gamma(t) \) we consider here is of the form
\[
V = \xi(t,x,\vec{n},\nabla\vec{n})
\]
\[
= \eta(\vec{n},\nabla\vec{n}) + \omega(t,x,\vec{n}) \text{ on } \Gamma(t),
\]
where \( \eta \) and \( \omega \) are given functions and \( \nabla \) stands for the spatial gradient in \( T \). A typical example is
\[
V = c \text{ div } \vec{n} + \omega(t,x),
\]
where \( c \) is a nonnegative constant and \( \omega \) is independent of \( \vec{n} \). The equation (2.2) is called the mean curvature flow equation if \( c > 0 \) and \( \omega \equiv 0 \). A reason we consider general (2.1) is to include anisotropic motion as in [Gu1,2].

We next introduce a weak formulation for (2.1) following [CGG, GG1]. For \( \eta \) we set
\[
F_\eta(p,X) := -|p|\eta(-\vec{p},-Q_p(X)), \bar{p} = p/|p|,
\]
\[
Q_p(X) = R_pX R_p \text{ with } R_p = I - \vec{p} \otimes \vec{p},
\]
where \( p \in \mathbb{R}^n \setminus \{0\} \) and \( X \in S_n \), the space of \( n \times n \) real symmetric matrices. We also set
\[
F_\xi(t,x,p,X) := F_\eta(p,X) - \omega(t,x,-\vec{p})|p|.
\]
For example a calculation shows
\[
F_\eta(p,X) = -c \text{ trace } ((I - \vec{p} \otimes \vec{p})X)
\]
if
\[
\eta(\vec{n},\nabla\vec{n}) = -c \text{ div } \vec{n}
\]
as in (2.2). The following definition of weak solutions for (2.1) is a variant of that in [CGG, GG1]. For the definition of (viscosity) sub- and supersolutions and viscosity solutions; see e.g. [GGIS].
DEFINITION 2.1. Let \( \{\Omega_{\pm}(t)\}_{0 \leq t < T} \) be a one parameter family of open sets in \( \mathbb{T} \) such that \( \Omega_{+}(t) \cap \Omega_{-}(t) = \emptyset \). Suppose that there is a viscosity solution \( u \in C([0,T) \times \mathbb{T}) \) of
\[
 u_t + F_\xi(t, x, \nabla u, \nabla^2 u) = 0 \text{ in } (0, T) \times \mathbb{T}
\]
such that
\[
 \Omega_{\pm}(t) = \{ x \in \mathbb{R}^n ; u(t, x) \gtrless 0 \} \text{ for } 0 \leq t < T.
\]
We say \( \{\Omega_{\pm}(t)\}_{0 \leq t < T} \) is a weak solution of (2.1) in \( (0, T) \) with initial data \( \Omega_{\pm}(0) \). Here \( F_\xi \) is defined by (2.3) – (2.4).

Roughly speaking, if (2.1) is parabolic (not necessarily strictly parabolic), \( \eta \) grows linearly in \( \nabla \bar{\eta} \) then one can claim the unique global existence of weak solutions for (2.1) with given initial data \( \Omega_{\pm}(0) \) provided that \( \eta \) and \( \omega \) are continuous. If \( \omega \) is independent of \( x \) and \( \mathbb{T} \) is replaced by \( \mathbb{R}^n \), the unique global existence is now well known if one of \( \Omega_{\pm}(0) \) is bounded (cf. [CGG, GG1]). We now list our assumptions on \( \eta \) and \( \omega \).

\( \eta \) is a real valued continuous function on the vector bundle
\[
 E = \{ (\bar{p}, Q_{\bar{p}}(X)) ; \bar{p} \in S^{n-1}, X \in \mathbb{S}_n \}
\]
over a unit sphere \( S^{n-1} \).

\( \eta(-\bar{p}, -Q_{\bar{p}}(X)) \geq \eta(-\bar{p}, -Q_{\bar{p}}(Y)) \) for \( X \geq Y, \bar{p} \in S^{n-1} \),
where \( \mathbb{S}_n \) is equipped with usual ordering.

\[
 \liminf_{\rho \downarrow 0} \inf_{|\bar{p}|=1} \eta \left( -\bar{p}, \frac{I - \bar{p} \otimes \bar{p}}{\rho} \right) > -\infty
\]
\[
 \limsup_{\rho \downarrow 0} \sup_{|\bar{p}|=1} \eta \left( -\bar{p}, \frac{I + \bar{p} \otimes \bar{p}}{\rho} \right) < \infty.
\]
\( \omega \) is continuous from \( [0, T) \times \mathbb{T} \times S^{n-1} \) to \( \mathbb{R} \).

All assumptions on \( \eta \) is found in [GG1]; (2.10) means that \( -\eta \) is degenerate elliptic and (2.11) restricts the growth of \( \eta \) in \( \nabla \bar{\eta} \). The only assumption for \( \omega \) is (2.12).

THEOREM 2.2. Assume (2.9) – (2.12) for \( \eta \) and \( \omega \). Let \( \Omega_{\pm}(0) \) be mutually disjoint open sets in \( \mathbb{T} \). Then there is a unique global weak solution \( \{\Omega_{\pm}(t)\}_{0 \leq t < T} \) of (2.1) in \( (0, T) \) with initial data \( \Omega_{\pm}(0) \). (The case \( T = \infty \) is included.)

The basic idea of the proof is the same as [CGG, Theorems 6.8, 7.1]; see also [GG1] for relation between assumptions on \( \eta \) and \( F_\eta \). The major technical difference is that the comparison theorem in [CGG] does not apply to (2.7) because \( F_\xi \) depends on \( x \). Instead we apply [GGIS, Theorem 4.1] to get a comparison principle for (2.7).

For the reader’s convenience we state a version of the comparison principle which follows from [GGIS, Theorem 4.1] and give a brief proof of Theorem 2.2.
Proposition 2.3. Assume (2.9) – (2.12). Let \( u \) and \( v \) be, respectively, (viscosity) sub- and supersolutions of (2.7). Assume that \( u \) and \( v \) are, respectively, upper and lower semicontinuous functions on \( [0,T) \times T \). If

\[
u(0,x) \leq v(0,x) \text{ on } T,
\]

then \( u(t,x) \leq v(t,x) \) on \( [0,T) \times T \).

Proof. To apply [GGIS, Theorem 4.1] we extend \( u, v \) and \( \omega \) periodically in space variables outside \( R \) and regard (2.7) as

\[
u_t + F_\xi(t,x,\nabla u, \nabla^2 u) = 0 \text{ in } (0,T') \times \mathbb{R}^n,
\]

where \( T' \) is an arbitrary positive number less than \( T \).

We first check assumptions of equation in [GGIS]. By (2.9) – (2.10) we know \( F_\eta \) satisfies all assumptions on \( F \) in [GGIS, Theorem 4.1]. Except the boundedness of \( F_\eta(p,X) \) on a bounded set in \( (\mathbb{R}^n \setminus \{0\}) \times S_n \) the proof is found in [GG1]. This boundedness of \( F_\eta \) can be proved similarly to the proof of [GG1, Lemma 3.5].

By (2.12) we see \( \omega \) is uniformly continuous in \( [0,T'] \times \mathbb{R}^n \times S^{n-1} \) so \( F_\xi \) satisfies the uniform continuity assumption in \( x \) of [GGIS, (F8)]: there is a modulus \( \sigma \) (i.e. \( \sigma : [0,\infty) \to [0,\infty) \) is continuous, nondecreasing and \( \sigma(0) = 0 \)) such that

\[
|F_\xi(t,x,p,X) - F_\xi(t,y,p,X)| \leq \sigma(|x - y|(|p| + 1))
\]

for \( x, y \in \mathbb{R}^n \), \( t \in [0,T'] \), \( p \in \mathbb{R}^n \setminus \{0\} \), \( X \in S_n \). All other assumptions on \( F \) in [GGIS, Theorem 4.1] are fulfilled since \( \omega \) satisfies (2.12) and \( F_\eta \) satisfies all assumptions on \( F \).

Since \( u \) and \( v \) is extended periodically and \( R \) is bounded, it is not difficult to see that \( u \) and \( v \) satisfy all the assumptions of [GGIS, Theorem 4.1].

We now apply [GGIS, Theorem 4.1] to conclude \( u \leq v \) on \( [0,T') \times \mathbb{R}^n \). Since \( T' < T \) is arbitrary, this completes the proof. \( \square \)

Proof of Theorem 2.2. (Uniqueness). Suppose that \( u,v \in C([0,T) \times T) \) solves (2.7) such that

\[
\Omega_\pm(0) = \{ x \in T, u(0,x) \geq 0 \} = \{ x \in T; v(0,x) \geq 0 \}.
\]

By [CGG, Lemma 7.2] there is a continuous nondecreasing function \( \theta : \mathbb{R} \to \mathbb{R} \) with \( \theta(0) = 0 \) such that

\[
u(0,x) \leq \theta(v(0,x)).
\]

Since \( F_\xi \) is geometric, i.e.,

\[
F_\xi(t,x,\lambda p, \lambda X + \sigma p \otimes p) = \lambda F_\xi(t,x,p,X)
\]

for \( \lambda > 0, \sigma \in \mathbb{R}, t \in (0,T), x \in T, p \in \mathbb{R}^n \setminus \{0\}, X \in S_n \),

by [CGG, Theorem 5.2] we see \( \theta(v(0,x)) \) also solves (2.7). From Proposition 2.3 it follows \( u \leq \theta(v) \) on \( [0,T) \times T \). We thus observe that \( u > 0 \) implies \( v > 0 \) and \( v < 0 \) implies \( u < 0 \). A parallel argument yields the converse implication so \( \Omega_\pm(0) \) is determined by \( \Omega_\pm(0) \) and is independent of the choice of \( u \). This proves the uniqueness of weak solutions.
(Existence). For given $\Omega_{\pm}(0)$ we take $u_0(x) \in C(T)$ such that

$$\Omega_{\pm}(0) = \{ x \in T; u_0(x) \geq 0 \}.$$ 

Since (2.11) is assumed, one may apply [CGG, Proposition 6.4] to (2.7) in $[0, T') \times \mathbb{R}^n$ with periodic initial data and find sub- and supersolutions $v_-, v_+ \text{ of } (2.7)$ on $[0, T') \times \mathbb{R}^n$ such that

$$v_\pm(0, x) = u_0(x) \text{ on } \mathbb{R}^n,$$

$$v_-(t, x) \leq u_0(x) \leq v_+(t, x) \text{ on } [0, T') \times \mathbb{R}^n,$$

where $T' < T$. The dependence of $x$ in $F_\xi$ is allowed in [CGG, Proposition 6.4]. A trivial modification of the argument enables us to take $v_-, v_+$ as functions on $[0, T') \times T$.

Existence of $v_\pm$ yields a viscosity solution $u \in C([0, T') \times T)$ of (2.7) with $u(0, x) = u_0(x)$ by Perron's method and Proposition 2.3. Since $T' < T$ is arbitrary and the solution is unique, we now obtain a weak solution $\{\Omega_\pm(t)\}_{0 \leq t < T}$ for initial data $\Omega_{\pm}(0)$.

Note that the scaling property (2.13) also used to construct $v_\pm$.

**Remark 2.4.** The family $\{\Omega_\pm(t)\}$ is determined by $\Omega_+(0)$ and is independent of $\Omega_-(0)$. Indeed, if $u$ solves (2.7) with (2.8) then $\theta(u)$ solves (2.7) for continuous nondecreasing $\theta : R \to R$ since $F_\xi$ is geometric. Take $\theta(\sigma) = \sigma_+ = \max(\sigma, 0)$ to observe that $u_+ = \theta(u)$ solves (2.7). By (2.8) $u_+$ gives a weak solution $\{\Omega_\pm(t)\}_{0 \leq t < T}$ with initial data $(\Omega_+(0), \phi)$. By definition of $u_+$ we see $\Omega_+'(t) = \Omega_+(t)$ and $\Omega_-'(t) = \phi$. We thus observe that $\Omega_+(t)$ is determined by $\Omega_+(0)$.

**Remark 2.5.** The interface $\Gamma(t)$ is defined by the complement of $\Omega_+(t) \cup \Omega_-(t)$ in $T$. There is a chance that $\Gamma(t)$ develops an interior even if $\Gamma(0)$ is a smooth hypersurface in $T$. For example consider the equation $V = -1$ and

$$R = \{(x_1, x_2) \in \mathbb{R}^2; 0 \leq x_1, 0 \leq x_2 \leq 2\}.$$

Suppose that

$$\Omega_+(0) = \{ x \in T; x_1 \neq 1 \}, \Omega_-(0) = \phi$$

so that $\Gamma(0) = \{ x_1 = 1 \}$. Then $\Omega_+(t) = \{ x \in T; 0 \leq x_1 \leq 2, |x_1 - 1| > t \}$ and $\Omega_-(t) = \phi$. Indeed equation (2.7) for this example is

$$u_t + |\nabla u| = 0.$$ 

(2.14)

By definition of viscosity solutions, one can check that

$$u(t, x) = \begin{cases} 
0 & \text{for } |x_1 - 1| \leq t \\
x_1 - 1 - t & \text{for } x_1 - 1 > t \quad (x \in \mathbb{R}^2) \\
1 - x_1 - t & \text{for } x_1 - 1 < -t 
\end{cases}$$

(2.15)

is a viscosity solution of (2.14) on $(0, \infty) \times T$. We now observe that $\Omega_\pm(t)$ is given by (2.8) with $u$ of (2.15).

For the mean curvature flow equation

$$V = -\operatorname{div} \bar{n}$$

we do not know whether or not $\Gamma(t)$ develops an interior if $\Gamma(0)$ is a smooth hypersurface. As pointed out in [ESI] we know $\Gamma(t)$ may develop an interior if $\Gamma(0)$ has a singularity.
3. Diffusion equations across interfaces. This section gives a generalized formulation of

\begin{align}
(3.1) \quad v_t &= \Delta v + g_{\pm}(v) \quad \text{in } Q_T^\pm = \bigcup_{0 < t < T} \{ t \} \times \Omega_{\pm}(t), \\
(3.2) \quad v(t) &= v(t, \cdot) \in C^1(\Gamma) \quad \text{for } 0 < t < T,
\end{align}

where

\begin{equation}
(3.3) \quad Q_T^\pm = \{ (t, x) \in Q_T; u(t, x) \geq 0 \}, \quad Q_T = (0, T) \times \Gamma
\end{equation}

with some function \( u \in C(\overline{Q_T}) \). The interpretation of the equation on the interface is crucial.

We introduce a multi-valued function \( \Phi \) associated with continuous functions \( g_{\pm}(\sigma) \). For \((s, \sigma) \in \mathbb{R}^2\) we define a closed interval \( \Phi(s, \sigma) \) such that

\begin{equation}
(3.4) \quad \Phi(s, \sigma) = \begin{cases} 
\{g_{+}(\sigma)\} & \text{if } s > 0 \\
[g(\sigma), \overline{g}(\sigma)] & \text{if } s = 0 \\
\{g_{-}(\sigma)\} & \text{if } s < 0
\end{cases}
\end{equation}

where \( g(\sigma) = \min(g_{+}(\sigma), g_{-}(\sigma)), \overline{g} = \max(g_{+}, g_{-}) \). This correspondence defines a mapping \( \Phi : \mathbb{R}^2 \to 2^\mathbb{R} \). For \( u, v \in C(\overline{Q_T}) \) we define a subset \( G(u, v) \) such that

\begin{equation}
(3.5) \quad G(u, v) = \{ q \in L^\infty(Q_T); q(z) \in \Phi(u(z), v(z)) \text{ a.e. } z \in Q_T \},
\end{equation}

where \( z = (t, x) \). This correspondence defines a mapping \( G : C(\overline{Q_T}) \times C(\overline{Q_T}) \to 2^L^\infty(Q_T) \).

**Definition 3.1.** Suppose that \( u \in C(\overline{Q_T}) \) is given and that \( Q_T^\pm \) is defined by (3.3). Suppose that \( g_{\pm} : \mathbb{R} \to \mathbb{R} \) is continuous. We say \( v \in C(\overline{Q_T}) \) is a generalized solution of (3.1) - (3.2) if

\[ v_t - \Delta v \in G(u, v) \text{ in } Q_T \]

i.e., there is \( q \in G(u, v) \) such that

\[ v_t - \Delta v = q \text{ in } Q_T \]

in the distribution sense. Since \( G(u, v) \) depends on \( u \) only through its signature, this definition only depends on \( Q_T^\pm \) and is independent of the choice of \( u \).

**Proposition 3.2.** For \( u, v \in C(\overline{Q_T}) \) the set \( G(u, v) \) is a nonempty, bounded convex subset of \( L^\infty(Q_T) \).

**Proof.** Since \( \Phi(s, \sigma) \) is convex in \( \mathbb{R} \),

\[ \lambda q_1(z) + (1 - \lambda)q_2(z) \in \Phi(u(z), v(z)) \text{ for a.e. } z \]

if \( q_1, q_2 \in G(u, v) \) and \( 0 < \lambda < 1 \). This implies that \( \lambda q_1 + (1 - \lambda)q_2 \in G(u, v) \) so \( G(u, v) \) is convex in \( L^\infty(Q_T) \).

The Borel measurable function

\[ \psi(s, \sigma) = \chi_{(-\infty,0)}(s)g_-(\sigma) + \chi_{[0,\infty)}(s)g_+(\sigma) \]
on $\mathbb{R}^n$ satisfies $\psi(s, \sigma) \in \Phi(s, \sigma)$ for all $s, \sigma \in \mathbb{R}$, and therefore $\psi(u, v) \in G(u, v)$.

Since $g^\pm_\cdot$ is locally bounded, we see $G(u, v)$ is bounded in $L^\infty(Q_T)$. 

**Lemma 3.3.** Suppose that $u_m \to u$ in $C(\overline{Q_T})$ and that $v_m \to v$ in $C(\overline{Q_T})$. Suppose that $q_m \in G(u_m, v_m)$. Then there is a subsequence $\{m_j\}$ and $q \in G(u, v)$ such that $q_{m_j} \to q$ *-weakly in $L^\infty(Q_T)$.

**Proof.** Since $g^\pm_\cdot$ is continuous, $\bigcup_{m=1}^{\infty} G(u_m, v_m)$ is bounded in $L^\infty(Q_T)$. In particular $\{q_m\}$ is bounded in $L^\infty(Q_T)$. By the Banach-Alaoglu theorem $\{q_m\}$ has a *-weak convergent subsequence (still denoted $\{q_m\}$), i.e.,

$$q_m \to q \text{ *-weakly in } L^\infty(Q_T).$$

In particular $q_m \to q$ weakly in $L^2(Q_T)$ since $Q_T$ is bounded. Applying Mazur’s theorem (see e.g. [Y]) we see there is $\lambda^m_1, \ldots, \lambda^m_t \geq 0$ with

$$\sum_{j=m}^{t_m} \lambda^j_m = 1$$

such that

$$q_m := \sum_{j=m}^{t_m} \lambda^j_m q_j \to q \text{ strongly in } L^2(Q_T) \text{ as } m \to \infty.$$

Taking a subsequence if necessary we may conclude

$$q_m(z) \to q(z) \quad (m \to \infty) \text{ for a.e. } z. \quad (3.6)$$

We fix $z \in Q_T$ such that (3.6) and

$$q_m(z) \in \Phi(u_m(z), v_m(z)) \text{ for all } m \geq 1. \quad (3.7)$$

Suppose that $u(z) = 0$. By (3.4) and (3.7)

$$q_m(z) \in [g(v_m(z)), \overline{g}(v_m(z))] \quad (3.8)$$

since $\{g^\pm(v_m(z))\}$ lies in the interval in (3.8). Since $g$ and $\overline{g}$ are continuous and $v_m(z) \to v(z)$, for each $\varepsilon > 0$ there is $m_0$ such that if $m \geq m_0$ then

$$[g(v_m(z)), \overline{g}(v_m(z))] \subset (a - \varepsilon, b + \varepsilon)$$

with

$$[a, b] := [g(v(z)), \overline{g}(v(z))].$$

By (3.8) we now observe that

$$q_m(z) \in (a - \varepsilon, b + \varepsilon).$$

From (3.6) it follows that

$$q(z) \in (a - \varepsilon, b + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, this implies

$$q(z) \in [g(v(z)), \overline{g}(v(z))]. \quad (3.9)$$
Suppose that \( u(z) > 0 \). For sufficiently large \( m \), say \( m \geq m_0 \), we may assume \( u_m(z) > 0 \). It follows that

\[
\Phi(u_m(z), v_m(z)) = \{g_+(v_m(z))\} \text{ for } m \geq m_0.
\]

By (3.7) we have

\[
q_m(z) = g_+(v_m(z)) \text{ for } m \geq m_0.
\]

Since \( \tilde{q}_m(z) \to q(z) \) by (3.6) and \( g_+ \) is continuous, (3.10) yields

\[
q(z) = q_+(v(z)).
\]

The proof for \( u(z) < 0 \) parallels that for \( u(z) > 0 \). By (3.9) and (3.11) one can conclude that

\[
q(z) \in \Phi(u(z), v(z)), \text{ a.e. } z \in Q_T,
\]

which completes the proof. \( \square \)

**Corollary 3.4.** The set \( G(u, v) \) is weak \(*\) compact in \( L^\infty(Q_T) \).

**Proof.** By Lemma 3.3 we see \( G(u, v) \) is weak \(*\) sequentially closed. Since \( G(u, v) \) is bounded by Proposition 3.2 and since the predual \( L^1(Q_T) \) is separable, one can drop the word "sequentially". The boundedness of \( G(u, v) \) now implies that \( G(u, v) \) is weak \(*\) compact. \( \square \)

**Remark 3.5.** The condition (3.2) is implicit in Definition 3.1. We will see that all generalized solution \( v \) has the regularity property (3.2).

**4. Main results.** We consider a system (3.1) – (3.2) coupled with an interface equation:

\[
\begin{align*}
(4.1) & \quad v_t = \Delta v + g_\pm(v) \quad \text{in } Q_T^\pm = \bigcup_{0 < t < T} \{t\} \times \Omega_\pm(t) \\
(4.2) & \quad v(t) = v(t, \cdot) \in C^1(T) \quad \text{for } 0 < t < T \\
(4.3) & \quad V = \eta(\vec{n}, \nabla \vec{n}) + W(v)\alpha(\vec{n}) \quad \text{on } \Gamma(t) = T \setminus (\Omega_+(t) \cup \Omega_-(t))
\end{align*}
\]

with given initial data

\[
\begin{align*}
(4.4) & \quad v(0, x) = v_0(x) \text{ in } T \\
(4.5) & \quad \Omega_\pm(t)|_{t=0} = \Omega_\pm(0).
\end{align*}
\]

Here we assume that

\[
\begin{align*}
(4.6a) & \quad g_\pm : \mathbb{R} \to \mathbb{R} \text{ is continuous and bounded} \\
(4.6b) & \quad \eta \text{ satisfies (2.9) – (2.11)} \\
(4.6c) & \quad W : \mathbb{R} \to \mathbb{R} \text{ is continuous} \\
(4.6d) & \quad \alpha : S^{n-1} \to \mathbb{R} \text{ is continuous}.
\end{align*}
\]

We say \( (\Omega_\pm(t), v(t)) \) is a \textit{weak solution} of (4.1)–(4.5) if \( \Omega_\pm(t)_0 \leq t < T \) is a weak solution of (3.3), (4.5) with \( v \in C(Q_T) \), and \( v \) is a generalized solution of (4.1)–(4.2) with (4.4); see Definitions 2.1 and 3.1. We now state one of our main results.
THEOREM 4.1. Let $T > 0$. Assume that $g_\pm, \eta, W, \alpha$ satisfy (4.6a-d). Suppose that $\Omega_+(0)$ and $\Omega_-(0)$ are mutually disjoint open sets in $\mathbb{T}$ and that $v_0(x) \in C^2(\mathbb{T})$. Then there exists a (global) weak solution $(\Omega_\pm(t), v(t))$ of (4.1)–(4.5) such that $v \in C^{1,0}(\overline{Q_T}) = \{v \in C(\overline{Q_T}); \nabla v \in C(\overline{Q_T})\}$.

Remark 4.2. We note that (4.3) includes

$$(4.3') \quad V = -c \text{div } \mathbf{n} + W(v), \quad c \geq 0$$

as a special example. If (4.3') replaces (4.3) in (4.1) – (4.5), then it is known that there is a unique smooth local solution. This is proved by X.-Y. Chen [XYC] for $c > 0$ and by X. Chen [XC2] for $c = 0$ where $\mathbb{R}^n$ replaces $\mathbb{T}$. Our result is the first global existence result even for this special system if the space dimension $n \geq 2$. For $n = 1$ see [HN].

We shall construct a solution using Kakutani’s fixed point theory for a multi-valued mapping. We take a Banach space

$$X := C^{1,0}(\overline{Q_T}).$$

For $v \in X$ we solve (4.3), (4.5) by applying Theorem 2.2. Since $v$ can be extended continuously for $t > T$ we have a unique weak solution $\{\Omega_\pm(t)\}_{0 \leq t \leq T}$ for (4.3), (4.5) with given $\Omega_\pm(0)$. If we set

$$\tilde{Q}_T^\pm = \bigcup_{0 \leq t \leq T} \{t\} \times \Omega_\pm(t),$$

then we have a mapping

$$\mathcal{T}: X \rightarrow \mathcal{O}, v \mapsto (\tilde{Q}_T^+, \tilde{Q}_T^-),$$

where $\mathcal{O}$ denotes the set of disjoint pair of open sets in $[0, T] \times \mathbb{T}$.

For $q \in L^\infty(\overline{Q_T})$ let $w = E(q)$ be the unique solution of

$$(4.7) \quad \begin{cases} w_t - \Delta w = q & \text{in } Q_T \\ u(0, x) = v_0(x) \in C^2(\mathbb{T}). \end{cases}$$

By the parabolic theory [LUS] $E$ defines a continuous linear operator from $L^\infty(\overline{Q_T})$ to $\cap_{p > 1} W_p^{2,1}(\overline{Q_T})$, which is continuously embedded in $X$ by the Sobolev inequality. Thus

$$E : L^\infty(\overline{Q_T}) \rightarrow X$$

is a bounded linear operator. For $u, v \in C(\overline{Q_T})$ we define a subset of $X$ by

$$\mathcal{P}(u, v) = \{E(q); q \in G(u, v)\}.$$
Since $G$ depends on $u$ through its signature, one may regard the mapping $\mathcal{P}$ as
\[ \mathcal{P} : \emptyset \times C(\mathcal{Q}_T) \to 2^X. \]

For given $v_0$ and $\Omega_\pm(0)$ we define
\[ \mathcal{S} : X \to 2^X \]
by $\mathcal{S}(v) = \mathcal{P}(\mathcal{I}(v), v)$. If $\mathcal{S}$ has a fixed point $\overline{v} \in X$, i.e.,
\[ \overline{v} \in \mathcal{S}(\overline{v}) \]
it is easy to see that $(\Omega_\pm(t), \overline{v}(t))$ is a weak solution of (4.1) - (4.5) where
\[ \mathcal{I}(\overline{v}) = \bigcup_{0 \leq t \leq T} \{t\} \times \Omega_\pm(t). \]

We shall prove that $\mathcal{S}$ has a fixed point.

**Proposition 4.3.** The set $\mathcal{P}(u, v)$ is nonempty, compact and convex in $C(\mathcal{Q}_T)$ (and in $X$).

**Proof.** Since $E$ defined by (4.7) is linear and $G(u, v)$ is nonempty and convex by Proposition 3.2 we see $\mathcal{P}(u, v)$ is convex.

We next observe that $E$ is continuous from a bounded set of $L^\infty(\mathcal{Q}_T)$ (equipped with weak * topology) to $X$. Indeed, if $q_m \to q$ weakly in $L^\infty(Q_T)$ then $\{E(q_m)\}$ has a weakly convergent subsequence in $W^{2,1}_p(Q_T)$ for $p > 1$. Since the inclusion
\[ W^{2,1}_p(Q_T) \to X = C^{1,0}(\mathcal{Q}_T) \text{ is compact if } p > n + 1 \]
(see e.g. [LUS]), $E(q_m) \to w$ strongly in $X$ by taking a subsequence. Since $w_m = E(q_m)$ satisfies
\[ (\partial_t - \Delta)w_m = q_m \text{ in } Q_T, \quad w_m(0, x) = v_0(x), \]
w solves
\[ (\partial_t - \Delta)w = q \text{ in } Q_T \]
in the distribution sense with $w(0, x) = v_0(x)$. This implies $w = E(q)$. Since the limit $w$ is independent of the choice of subsequences, we observe
\[ E(q_m) \to E(q) \text{ in } X. \]

This sequential continuity implies the continuity on a bounded set of $L^\infty(Q_T)$.

Since $G(u, v)$ is weak * compact in $L^\infty(Q_T)$, the continuous image of $G(u, v)$ is compact. The above continuity of $E$ implies that $\mathcal{P}(u, v)$ is compact in $C^0(\mathcal{Q}_T)$ as well as in $X$. \[ \square \]

Since $g_{\pm}$ is bounded by (4.6a), we see
\[ \bigcup_{u, v \in C(\mathcal{Q}_T)} G(u, v) \]
is bounded in $L^\infty(Q_T)$. Therefore, by the parabolic theory for (4.7) [LUS]
\[ \mathcal{S}(v) \subset K = \{w \in W^{2,1}_p(Q_T); \|w\|_{W^{2,1}_p} \leq M\}, p > 1 \]
if $M$ is taken sufficiently large. We fix $p > n + 1$ so that $K$ is compact in $X$ by (4.8). The mapping $S$ is now interpreted as

$$S : X \to 2^K.$$ 

The graph of $S$ is defined by

$$\text{gr} S = \{(v, w); w \in S(v)\} \subset X \times K.$$ 

Since $K$ is compact, $\text{gr} S$ is closed if and only if $S$ is upper semicontinuous. For the definition of upper semicontinuity see [AF].

**Proposition 4.4.** The set $\text{gr} S$ is closed in $X \times K$.

**Proof.** Suppose that $v_m, v \in X, w_m \in S(v_m), w \in X$ such that $v_m \to v$ in $X$ and $w_m \to w$ in $X$. Our goal is to prove $w \in S(v)$. By the definition of weak solutions for (4.3) there is a viscosity solution $u_m \in C(\overline{Q_T})$ of

$$u_t + F_m(t, x, \nabla u, \nabla^2 u) = 0 \text{ in } Q_T$$

with

$$F_m(t, x, p, X) = F_\eta(p, X) - W(v_m(t, x))\alpha(-p/|p|)|p|$$

such that

$$\mathcal{T}(v_m) = \{\{u_m(t, x) > 0\}, \{u_m(t, x) < 0\}\}.$$ 

One can arrange $u_m(0, x) = u_0(x)$ independent of $m$ such that

$$\Omega_\pm(0) = \{x \in \Gamma; u_0(x) \geq 0\}.$$ 

Since $v_m \to v$ in $C(\overline{Q_T})$, by the stability of viscosity solutions there is $u \in C(\overline{Q_T})$ such that $u_m \to u$ in $C(\overline{Q_T})$ and $u$ solves (in the viscosity sense)

$$u_t + F(t, x, \nabla u, \nabla^2 u) = 0 \text{ in } Q_T$$

with $F(t, x, p, X) = F_\eta(p, X) - W(v(t, x))\alpha(-p/|p|)|p|$ (see Theorem in Appendix), where $u(0, x) = u_0(x)$. This implies

$$\mathcal{T}(v) = \{\{u > 0\}, \{u < 0\}\}.$$ 

By the definition of $P$ there is $q_m \in G(u_m, v_m)$ such that

$$(\partial_t - \Delta)u_m = q_m \text{ in } Q_T,$$

$$w_m(0, x) = v_0(x) \text{ on } \Gamma.$$

Applying Lemma 3.3, we may conclude that

$$q_m \to q \ast \text{-weakly in } L^\infty(Q_T)$$

with some $q \in G(u, v)$ by taking a subsequence if necessary. Since $w_m \to w$ in $X$, (4.10) implies that

$$(\partial_t - \Delta)w = q \text{ in } Q_T$$
in the distribution sense and
\[ w(0, x) = v_0(x). \]
This yields \( w \in S(v) \) by (4.9) so the proof is now complete. \( \square \)

Proof of Theorem 4.1. Since \( K \) is compact and convex by Propositions 4.3 and 4.4 one can apply the following fixed point theorem to conclude that there is \( \bar{v} \in S(\bar{v}) \cap K \). By the definition of \( S \) we see \( \bar{v} \) together with \( T(\bar{v}) \) is a desired weak solution of (4.1)–(4.5).

Kakutani's fixed point theorem [AF, Theorem 3.2.3]. Let \( K \) be a convex compact subset of a Banach space \( X \) and \( S : X \to 2^K \). If \( S \) is upper semicontinuous and \( S(v) \) is a convex closed set in \( K \) for \( v \in X \), then \( S \) has a fixed point \( \bar{v} \in K \cap S(\bar{v}) \).

Remark 4.5. The assumption \( v_0(x) \in C^2(\mathcal{T}) \) in Theorem 4.1 is weakened as \( v_0(x) \in W^{2-2/p}(T) \), \( p > n + 1 \) because the regularity condition on \( v_0 \) is only used to solve (4.7) in \( W^{p,1}_p(\mathcal{Q}_T) \).

We conclude this paper by stating an existence result of a global solution on the time interval \( (0, \infty) \).

Theorem 4.6. Assume the same hypotheses of Theorem 4.1 for \( g, \eta, w, \alpha, \Omega_{\pm}(0) \). Suppose that \( v_0 \in W^{2-2/p}(T) \) for \( p > n + 1 \). Then there exists \( \{\Omega_{\pm}(t), v(t)\}_{t \geq 0} \) which is a weak solution of (4.1) – (4.5) for arbitrary \( T > 0 \).

Proof. For fixed \( T > 0 \) by Remark 4.5 there is \( v_T \) such that \( v_T \in S(v_T) \). This implies
\[
(\partial_t - \Delta)v_T = q_T \text{ in } Q_T,
\]
\[
v_T(0, x) = v_0(x),
\]
with
\[
q_T \in \bigcup_{u, v \in C(\bar{Q}_T)} G(u, v).
\]
Since \( g_{\pm} \) is bounded by (4.6) we observe
\[
|q_T|_{L^\infty(Q_T)} \leq M = \sup_{\sigma} |g_{\pm}(\sigma)|
\]
By the parabolic regularity theory [LUS] \( \{v_T\}_{T \geq 1} \) is bounded in \( W^{2,1}_p(\bar{Q}_{t_0}) \) for \( p > n + 1 \) for each \( t_0 > 0 \). By (4.8) and a diagonal argument there is a subsequence \( \{v_{T'}\} \) and \( v \in C([0, \infty) \times T) \) such that
\[
v_{T'} \to v \text{ in } X_{t_0} = C^{1,0}(\bar{Q}_{t_0}).
\]
Since \( v_{T'} \in S(v_{T'}) \subset X_{t_0} \) and \( \text{gr} S \) is closed, (4.11) implies \( v \in S(v) \subset X_{t_0} \) where \( S \) depends on \( t_0 \). Since \( t_0 \) is arbitrary, this yields a desired global solution on \( [0, \infty) \). \( \square \)

Appendix. We shall state stability properties of viscosity solutions used in the proof of Proposition 4.4 for the reader's convenience. We use a following notation. For \( h_m : L \to \mathbb{R}, L \subset Z \) we define
\[
\lim h_m : \mathcal{L} \to \mathbb{R} \cup \{-\infty\}
\]
\[
\lim h_m : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}
\]
by
\[ (\lim h_m)(\cdot) = \lim_{m \to \infty} \inf_{y \in Z} \{ h_j(y), d(z, y) < \epsilon, j \geq m, y \in L \} \]
and \( \lim h_m = -\lim(-h_m) \),

where \( Z \) is a metric space with the metric \( d \). If \( h \) is independent of \( m \), we write \( h_* = \lim h_m, h^* = \lim^* h_m \). We shall suppress the word “viscosity”.

**Lemma.** Suppose that \( F_m : \Omega_T \times \mathbb{R}^n \times \mathbb{S}_n \to \mathbb{R} \) is lower semicontinuous and that \( F = \lim F_m \). Suppose that \( u_m \) is a subsolution of
\[ u_t + F_m(t, x, \nabla u, \nabla^2 u) = 0 \text{ in } \Omega_T. \]
Then \( u = \lim^* u_m \) is a subsolution of
\[ u_t + F(t, x, \nabla u, \nabla^2 u) = 0 \text{ in } \Omega_T \]
provided that \( u \) does not take \(+\infty\) in \( \Omega_T \).

Similar results are proved by Barles and Perthame [BP] for first order differential equations and formulated in Ishii [I] in the general case. Since the proof is easily modified for our setting, we omit the proof. The following is a simple application of Lemma, the comparison Proposition 2.3, and construction of sub and supersolutions.

**Theorem.** Suppose that \( \eta, W, \alpha \) satisfy (4.6 b-d). Suppose that \( v_m \to v \) in \( C(\overline{\Omega_T}) \). We set
\[
F_m = F_\eta(p, X) - W(v_m(t, x))\alpha(-p/|p|)|p|,
\]
\[
F = F_\eta(p, X) - W(v(t, x))\alpha(-p/|p|)|p|,
\]
where \( F_\eta \) is defined by (2.3). Suppose that \( u_m \in C(\overline{\Omega_T}) \) is a solution of
\[
(1) \quad u_t + F_m(t, x, \nabla u, \nabla^2 u) = 0 \text{ in } \Omega_T
\]
with \( u_m(0, x) = u_0(x) \in C(\Omega) \). Then \( u_m \to u \) in \( C(\overline{\Omega_T}) \) for some \( u \in C(\overline{\Omega_T}) \) and \( u \) is a solution of
\[
(2) \quad u_t + F(t, x, \nabla u, \nabla^2 u) = 0 \text{ in } \Omega_T
\]
with \( u(0, x) = u_0(x) \).

**Proof.** Since \( v_m \to v \) in \( C(\overline{\Omega_T}) \) there are sub and supersolutions \( w_\pm \) of (1) such that
\[
(3) \quad w_\pm(0, x) = u_0(x), \quad w_-(t, x) \leq u_0(x) \leq w_+(t, x) \text{ in } \Omega_T
\]
and that \( w_\pm \) is independent of \( m \); see [CGG, Proposition 6.4] and the proof of Theorem 2.2. By Proposition 2.3 we see
\[
(4) \quad w_- \leq u_m \leq w_+ \in \Omega_T.
\]
Since \( u_m \) is a subsolution of
\[
  u_t + (F_m)_*(t, x, \nabla u, \nabla^2 u) = 0 \quad \text{in } Q_T
\]
by definition, applying Lemma yields that \( \overline{u} = \lim^* u_m \) is a subsolution of
\[
  u_t + F_*(t, x, \nabla u, \nabla^2 u) = 0 \quad \text{in } Q_T.
\]
(This is the definition that \( \overline{u} \) is a subsolution of (2)). Similarly \( \underline{u} = \lim_* u_m \) is a supersolution of
\[
  u_t + F^*(t, x, \nabla u, \nabla^2 u) = 0 \quad \text{in } Q_T.
\]
By (3) we observe that
\[
  \overline{u}(0, x) = \underline{u}(0, x) = u_0(x).
\]
Applying Proposition 2.3 implies \( \overline{u} = \underline{u} \) and \( \underline{u} = \overline{u} \) is a solution of (2). The property \( \overline{u} = \underline{u} \) implies that \( u_m \to u \) in \( C(Q_T) \). The proof is now complete. \( \square \)

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