QUASISTATIC LINEAR THERMOELASTICITY
ON THE UNIT DISK

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QUASISTATIC LINEAR THERMOELASTICITY ON THE UNIT DISK

Dedicated to Professor R. P. Gilbert on the Occasion of his 60th Birthday.

PETER SHI* and YONGZHI XU**

Abstract. The quasi-static, linearized thermoelastic system on the unit disk is decoupled, such that the temperature satisfies an integro-differential equation. The result, based on the function theoretic method, is of both theoretical and numerical interest.

Key words. Thermoelasticity, boundary problem, decoupling, functional theoretic methods


1. Introduction.

This paper demonstrates a new application of function theoretic methods to the linear, quasistatic system of thermoelasticity on the unit disk. Using the results in Xu [9], and Gilbert and Lin [2], we show that the temperature is decoupled from the system and satisfies an integro-differential equation independent of the displacement. The existence and the analyticity of the solution is proved for the decoupled problem, and consequently for that of the coupled problem.

We assume that the elastic body in consideration is homogeneous and isotropic, with the reference configuration being the unit disk. The word quasistatic refers to the negligibility of the inertia term in the system, which is usually the case when the acceleration of the system is sufficiently slow. The field equations in the quasi-static approximation of linearized thermoelasticity can be written as

\begin{equation}
\lambda \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right) = \beta \frac{\partial \theta}{\partial x_i}, \quad i = 1, 2, \tag{1.1}
\end{equation}

and

\begin{equation}
q \Delta \theta = \rho s \frac{\partial \theta}{\partial t} + \beta c \frac{\partial}{\partial t} \text{div} \mathbf{u}. \tag{1.2}
\end{equation}

Here we used the summation convention, with \( \mathbf{u} = (u_1, u_2) \) denoting the displacement and \( \theta \) denoting the temperature. The constants \( \lambda \) and \( \mu \) are Lamé constants, \( \beta \) is the
interaction constant, \( q, \rho, s \) and \( c \) denote, respectively, the heat conductivity, the density, the specific heat, and the absolute reference temperature.

The quasi-static theory of linear thermoelasticity in one space dimension is thoroughly studied in Day [1] via the decoupled temperature equation. In contact problems of one dimensional thermoelasticity, the decoupled temperature equation enjoys an interesting nonlocal nonlinearity (Shi and Shillor [6]). However, the decoupling in two space dimension is much more difficult because of the structure of (1.1) and (1.2). For clarity of exposition, we limit our consideration to the homogeneous Dirichlet boundary data. Since the result of this paper is valid, at this stage, only on the unit disk, it seems worthwhile to extend it to more general configurations, possibly using the conformal mapping technique.

The rest of the paper is organized as follows. In section 2 we review some results in function theory that will be used in our approach. In section 3 we rewrite the system (1.1)-(1.2) in complex form. The detail steps of the decoupling are presented in section 4, where the decoupled temperature equation is obtained. We then briefly discuss the decoupled problem and prove the existence of the solution and the analyticity of the solution in space variables.

We remark that the decoupled equation for the temperature carries some nonstandard features. In this respect, some open questions naturally arise, but it is impossible to address all the questions in detail and it is not the aim of this paper to do so.

2. Some Results on \((\lambda, 1)\) Bi-analytic Functions.

In this section we give a brief account of some relevant results in [9]. The results here are presented less general than the original work for the purpose of clarity. The principle theory used in this paper is quite popular in the east. We refer to the book by Hua, Lin and Wu [3], and Wen and Begehr [8] for a detail account of the theory.

We begin with notation. Any notation that is not explained in the following can be found in Vekua [7] and Ladyzhenskaya, Solonikov and Ural'eva [5]. Let \( \Omega \) be a bounded domain in \( R^2 \) with \( C^2 \) boundary. Consider the elliptic system on the \((x, y)\)–plane in the matrix form
(2.1) \[
\begin{pmatrix}
1 & 0 \\
0 & -\lambda
\end{pmatrix}
\frac{\partial^2}{\partial x^2} + \begin{pmatrix}
0 & \lambda - 1 \\
1 - \lambda & 0
\end{pmatrix}
\frac{\partial^2}{\partial x \partial y} + \begin{pmatrix}
\lambda & 0 \\
0 & -1
\end{pmatrix}
\frac{\partial^2}{\partial y^2}
\begin{pmatrix}
u \\
u
\end{pmatrix}
= \begin{pmatrix} f_1 \\
f_2 \end{pmatrix}, \quad (x, y) \in \Omega.
\]

Here \((f_1, f_2)^T\) is a given vector field on \(\Omega\), \(u\) and \(v\) are unknowns. We introduce the complex notation

(2.2) \[ w = u + iv, \quad z = x + iy, \quad f = (f_1 + if_2)/4, \]

(2.3) \[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \]

and operators \(D\) and \(D^*\) defined by

(2.4) \[ D w = (1 - \lambda) \frac{\partial^2 w}{\partial \bar{z}^2} + (1 + \lambda) \frac{\partial^2 \bar{w}}{\partial \bar{z} \partial z}, \]

(2.4') \[ D^* w = (1 - \lambda) \frac{\partial^2 w}{\partial \bar{z} \partial z} + (1 + \lambda) \frac{\partial^2 \bar{w}}{\partial \bar{z}^2}. \]

It turns out that the system of equations (2.1) can be equivalently written in the beautiful form

(2.5) \[ Dw = f. \]

In the literature, functions satisfying \(D w = 0\) are called \((\lambda, 1)\) bi-analytic functions, whose theory has been thoroughly studied in [3]. We also refer the reader to [9] and [2] for generalizations and applications. The solution to (2.5) on the unit disk, with the Dirichlet boundary condition on \(w\), can be represented in a closed form. Following [9], we introduce the integral operators

(2.6) \[ (\Gamma f)(z) = \frac{\lambda - 1}{4\lambda} \frac{1}{\pi} \iint_{\Omega} \frac{f(\zeta)}{\bar{\zeta} - \bar{z}} d\sigma_\zeta + \frac{\lambda + 1}{4\lambda} \frac{1}{\pi} \iint_{\Omega} \frac{f(\zeta)}{\ln(\bar{\zeta} - \bar{z})(\zeta - z)} d\sigma_\zeta, \]
and
\[
(\Gamma f)(z) = \frac{\lambda - 1}{4\lambda} \int_{\Omega} f(\zeta) \frac{\bar{\zeta} - \bar{z}}{\zeta - 1/\bar{z}} \, d\sigma_\zeta + \frac{\lambda + 1}{4\lambda} \int_{\Omega} \frac{f(\zeta)}{\zeta} \ln(1 - z\bar{\zeta})(1 - \bar{z}\zeta) \, d\sigma_\zeta,
\]
\[
- \frac{\lambda - 1}{\lambda + 1} (1 - z\bar{z}) \left\{ \frac{\lambda - 1}{4\lambda} \int_{\Omega} f(\zeta) \left[ \frac{1 - \zeta}{1 - z\zeta} + \frac{1 - \bar{\zeta}}{(1 - z\zeta)^2} \right] \, d\sigma_\zeta \right. \\
- \frac{\lambda + 1}{4\lambda} \int_{\Omega} f(\zeta) \frac{\bar{\zeta}^2}{1 - z\zeta} \, d\sigma_\zeta \left. \right\}.
\]
Here $d\sigma_\zeta$ denotes $d\xi d\eta$ for $\zeta = \xi + i\eta$. Some properties of these operators and their relations to the equation (2.5) are given in the following theorem.

**Theorem 1.** The operators $\Gamma$ and $\Gamma_1$ are bounded operators from $L^2(\Omega)$ into $H^2(\Omega)$. Suppose now that $f \in L^2(\Omega)$ and in addition, $\Omega$ is the unit disk. Then the unique solution to the problem
\[
\mathbf{D}w = f \quad \text{in } \Omega, \quad w|_{\partial \Omega} = 0, \quad (2.8)
\]
is given by
\[
w(z) = (\Gamma f)(z) - (\Gamma_1 f)(z), \quad (2.9)
\]
where (2.8) holds in the Sobolev sense and the sense of trace.

**Proof.** The result is in fact proved in [9] as an intermediate step in the discussion of the nonlinear problem
\[
\mathbf{D}w = F(z, w, w_z, w_{\bar{z}}, \mathbf{D}^* w) \quad \text{in } \Omega, \quad w|_{\partial \Omega} = g.
\]
The original proof was done in the $L^p(\Omega)$-$W^{2,p}(\Omega)$ framework, with $p > 2$, but this assumption was only used to deal with the nonlinearity. The restriction to $p > 2$ is unnecessary in our case and the same proof is valid.

**3. The Complex Form of the Thermoelastic System.**

An interesting observation due to Gilbert and Lin [2] is to put the two dimensional elastic system in a matrix form so the analytic function theory applies. In our case for (1.1)-(1.2), one obtains
\[ (3.1) \quad \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & -\mu \end{pmatrix} \frac{\partial^2}{\partial x_1^2} + \begin{pmatrix} 0 & \lambda + \mu \\ \lambda + \mu & 0 \end{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_2} + \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix} \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \beta \begin{pmatrix} \theta_{x_1} \\ \theta_{x_2} \end{pmatrix}, \quad (x_1, x_2) \in \Omega. \]

Multiplying (3.1) on the left by
\[ \begin{pmatrix} \frac{1}{\lambda + 2\mu} & 0 \\ 0 & \frac{1}{\lambda + 2\mu} \end{pmatrix} \]
and letting \( \hat{\lambda} = \mu/(\lambda + 2\mu) \), \( \hat{\beta} = \beta/(\lambda + 2\mu) \), we obtain
\[ (3.2) \quad \begin{pmatrix} 1 & 0 \\ 0 & -\hat{\lambda} \end{pmatrix} \frac{\partial^2}{\partial x_1^2} + \begin{pmatrix} 0 & 1 - \hat{\lambda} \\ \hat{\lambda} - 1 & 0 \end{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_2} + \begin{pmatrix} \hat{\lambda} & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial^2}{\partial x_2^2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \hat{\beta} \begin{pmatrix} \theta_{x_1} \\ -\theta_{x_2} \end{pmatrix}, \quad (x_1, x_2) \in \Omega. \]

We now make a change of variables
\[ x = -x_1 \quad u(x, y) = u_1(-x_1, x_2), \]
\[ y = x_2 \quad v(x, y) = u_2(-x_1, x_2), \]
and invoke the notation (2.2)-(2.4). This rewrites (3.2) in the form of (2.5) with
\[ f_1 = -\theta_x, \quad f_2 = -\theta_y, \quad \text{and} \quad f = -\hat{\beta}(\theta_x + i\theta_y)/4. \]

The only change is that \( \lambda \) in the definition of the operator \( D \) in (2.4) has to be replaced by \( \hat{\lambda} \).

We now summarize our result in this section, taking into account the adaption of the new notation (2.2)-(2.4) and (3.2)-(3.4).
Theorem 2. The thermoelastic system (1.1) and (1.2) can be equivalently written as, respectively,

\begin{align}
(3.5) \quad D_w &= -\frac{\beta}{2} \frac{\partial \theta}{\partial \bar{z}}, \\
(3.6) \quad q \Delta \theta &= \rho_s \frac{\partial \theta}{\partial t} + 2\beta c \frac{\partial}{\partial t} \text{Re} \left\{ \frac{\partial w}{\partial \bar{z}} \right\}.
\end{align}

Proof. To verify (3.6), we utilize the relation

\[ 2 \text{Re} \left\{ \frac{\partial w(z)}{\partial \bar{z}} \right\} = u_x(x, y) - v_y(x, y) = -u_{1x}(-x_1, x_2) - u_{2x}(-x_1, x_2). \]

The rest of the proof has been derived in this section.

4. The Decoupled Equation for the Temperature.

We are now in a position to decouple the temperature \( \theta \) of the system (3.5) and (3.6), which in turn yields the decoupled equation for \( \theta \) in the original variables \( \{x_1, x_2\} \). Besides Theorem 2.1, we also need the Green’s formula in the complex form (see e.g. Vekua [7])

\begin{align}
(4.1) \quad \iint_{\Omega} \frac{\partial \Theta}{\partial \bar{z}} \Psi \, dx dy &= \frac{1}{2i} \int_{\partial \Omega} \Theta \Psi \, dz - \iint_{\Omega} \frac{\partial \Psi}{\partial \bar{z}} \Theta \, dx dy,
\end{align}

and

\begin{align}
(4.2) \quad \iint_{\Omega} \frac{\partial \Theta}{\partial z} \Psi \, dx dy &= -\frac{1}{2i} \int_{\partial \Omega} \Theta \Psi \, d\bar{z} - \iint_{\Omega} \frac{\partial \Psi}{\partial z} \Theta \, dx dy,
\end{align}

which hold in the Sobolev sense for functions in \( H^1(\Omega) \).

In the following, the time variable \( t \) will be frequently omitted in \( w(z, t) \) and \( \theta(\zeta, t) \) etc., when no confusion is caused. Assuming \( w = 0 \) on \( \partial \Omega \), we now apply Theorem 2.1 to (3.5) to obtain

\begin{align}
(4.3) \quad w(z) &= -\frac{\beta}{2} \left\{ \Gamma \left( \frac{\partial \theta}{\partial \bar{z}} \right)(z) - \Gamma_1 \left( \frac{\partial \theta}{\partial \bar{z}} \right)(z) \right\}.
\end{align}

Thanks to (2.6), (2.7) and the Green’s formula (4.1) and (4.2), we can simplify (4.3) a
great deal under the assumption that \( \theta = 0 \) on \( \partial \Omega \). In fact,

\[
\Gamma_1 \left( \frac{\partial \theta}{\partial \bar{z}} \right) (z) = \frac{\lambda - 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\partial \theta}{\partial \bar{z}} \frac{\zeta - \bar{z}}{\zeta - 1/\bar{z}} \, d\sigma_\zeta + \frac{\lambda + 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\partial \theta}{\partial \zeta} \ln(1 - z\bar{\zeta})(1 - \bar{z}\zeta) \, d\sigma_\zeta,
\]

\[
- \frac{\lambda - 1}{\lambda + 1} (1 - \bar{z}z) \left\{ - \frac{\lambda - 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\partial \theta}{\partial \zeta} \frac{1 - \zeta\bar{z}}{1 - z\bar{\zeta}} + \frac{1 - \zeta\bar{z}}{(1 - z\bar{\zeta})^2} \right\} \, d\sigma_\zeta
\]

\[
- \frac{\lambda + 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\partial \theta}{\partial \zeta} \frac{\bar{\zeta}^2}{1 - z\bar{\zeta}} \, d\sigma_\zeta \right\}
\]

\[
= \frac{\lambda - 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \theta(\zeta) \frac{\bar{z}}{1 - z\bar{\zeta}} \, d\sigma_\zeta + \frac{\lambda + 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \theta(\zeta) \frac{\bar{z}}{1 - z\bar{\zeta}} \, d\sigma_\zeta
\]

\[
- \frac{\lambda - 1}{\lambda + 1} (1 - \bar{z}z) \left\{ - \frac{\lambda - 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \theta(\zeta) \left[ \frac{\bar{\zeta}}{1 - z\bar{\zeta}} + \frac{\bar{\zeta}}{(1 - z\bar{\zeta})^2} \right] \right\} \, d\sigma_\zeta
\]

\[
+ \frac{\lambda + 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \theta(\zeta) \frac{\bar{\zeta}[2 - z\bar{\zeta}]}{(1 - z\bar{\zeta})^2} \, d\sigma_\zeta \right\}
\]

After a straightforward simplification, one obtains

\[
(4.4) \quad \Gamma_1 \left( \frac{\partial \theta}{\partial \bar{z}} \right) (z) = \frac{1}{2\pi} \int_\Omega \int_\Omega \frac{\bar{z} \theta(\zeta)}{1 - z\bar{\zeta}} \, d\sigma_\zeta - \frac{\lambda - 1}{\lambda + 1} \frac{1}{2\pi} \int_\Omega \int_\Omega \frac{\bar{\zeta}[2 - z\bar{\zeta}]}{(1 - z\bar{\zeta})^2} \theta(\zeta) \, d\sigma_\zeta.
\]

Similarly,

\[
(4.5) \quad \Gamma_1 \left( \frac{\partial \theta}{\partial \zeta} \right) (z) = \frac{\lambda - 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\partial \theta(\zeta)}{\partial \zeta} \frac{\bar{z} - \bar{\zeta}}{\zeta - z} \, d\sigma_\zeta + \frac{\lambda + 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\partial \theta(\zeta)}{\partial \zeta} \ln(\bar{\zeta} - \bar{z})(\zeta - z) \, d\sigma_\zeta
\]

\[
- \frac{\lambda - 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\theta(\zeta)}{\zeta - z} \, d\sigma_\zeta - \frac{\lambda + 1}{4\lambda} \frac{1}{\pi} \int_\Omega \int_\Omega \frac{\theta(\zeta)}{\zeta - z} \, d\sigma_\zeta
\]

\[
= - \frac{1}{2\pi} \int_\Omega \int_\Omega \frac{\theta(\zeta)}{\zeta - z} \, d\sigma_\zeta.
\]

Combining (4.4) and (4.5), yields

\[
(4.6) \quad w(z) = \frac{\beta}{2} \left\{ \frac{1}{2\pi} \int_\Omega \int_\Omega \frac{\theta(\zeta)}{\zeta - z} \, d\sigma_\zeta + \frac{1}{2\pi} \int_\Omega \int_\Omega \frac{\bar{z} \theta(\zeta)}{1 - z\bar{\zeta}} \, d\sigma_\zeta
\]

\[
- \frac{\lambda - 1}{\lambda + 1} \frac{1}{2\pi} \int_\Omega \int_\Omega \frac{\bar{\zeta}[2 - z\bar{\zeta}]}{(1 - z\bar{\zeta})^2} \theta(\zeta) \, d\sigma_\zeta \right\}
\]
By now we are able to decouple $\theta$ from the system (3.5) and (3.6), simply substituting (4.6) into (3.6) to obtain the desired equation

\[
(\rho s + \beta \hat{\beta} c/2) \frac{\partial \theta}{\partial t} - q \Delta \theta \\
= -2\beta c \frac{\partial}{\partial t} \Re \left\{ \frac{\hat{\beta}}{4\pi} \iint_{\Omega} \left[ \frac{1}{(1 - z \zeta)^2} + \frac{\lambda - 1}{\lambda + 1} \frac{z \zeta[2 - z \zeta]}{(1 - z \zeta)^2} \right] \theta(\zeta, t) \, d\sigma \zeta \right\}
\]

where we have used the formula [7]

\[
\frac{\partial}{\partial z} \frac{1}{\pi} \iint_{\Omega} \frac{\theta(\zeta)}{\zeta - z} \, d\sigma \zeta = \theta(z).
\]

Our result so far is based on the formal reduction. However, it is not difficult to verify that all steps are valid with appropriate regularity assumptions on $w$ and $\theta$. To be precise, let

\[
W_{p,1}^2(\Omega_T) = \{ w \in W^{1,p}(\Omega_T); \ w_{xx}, \ w_{xy}, \ w_{yy} \in L^p(\Omega_T) \}
\]

where $\Omega_T = \Omega \times (0, T)$, $T > 0$ is given, and $1 < p < +\infty$. Functions in $W_{p,1}^2(\Omega_T)$ are complex valued. In this aspect, there is no loss of generality to identify $W_{p,1}^2(\Omega_T)$ with the product of two real-valued function spaces with the corresponding degree of differentiability. Therefore, it is legitimate to write

\[
w \in W_{p,1}^2(\Omega_T) \quad \text{or} \quad (u, v) \in W_{p,1}^2(\Omega_T).
\]

**Theorem 3.** Let $\theta$ and $u = (u_1, u_2)$ be functions in the space $W_{p,1}^2(\Omega_T)$. Then the pair \( \{u, \theta\} \) satisfies the thermoelastic system (3.1) and (3.2) subject to the initial-boundary conditions

\[
u = 0, \quad \theta = 0 \quad \text{on } \partial \Omega \times (0, T), \quad \text{and} \quad \theta(\cdot, 0) = \varphi \quad \text{on } \Omega
\]

if and only if the pair \( \{w, \theta\} \) in complex form satisfies (4.6) and (4.7), and

\[
\theta = 0 \quad \text{on } \partial \Omega \times (0, T), \quad \theta(\cdot, 0) = \varphi \quad \text{on } \Omega.
\]

**Proof.** The necessity has been proved since the our argument leading to (4.6) and (4.7) is valid for functions in $W_{p,1}^2(\Omega_T)$. Conversely, the function $w(z)$ given by (4.6) vanishes
on $\partial \Omega$. Moreover, (4.6) and (4.5) are equivalent for any $\theta \in W_p^{1,0}(\Omega_T)$. Therefore the argument leading to (4.6) and (4.7) is reversible.

In the remaining part of the paper, we address two questions pertaining to (4.7) and (4.9), the existence of the solution in the space $W_2^{2,1}(\Omega_T)$ and the analyticity of the solution in $x$ and $y$. In fact, many questions considered in Day [1] enjoy similar answers here and the authors believe that many standard properties for the two dimensional heat equation can be analogously carried over to (4.7) and (4.9), while some other properties may not trivially hold. However, we will not pursue all of these aspects in this short note.

Equation (4.7) can be viewed as a perturbation of the heat equation by a nonlocal source term. To establish the existence of the solution, we rewrite (4.7) in a more compact form, and for this, we introduce the operator

\[
(M\theta)(z) = \frac{1}{\pi} \iint_{\Omega} \text{Re} \left[ \frac{1}{(1 - \bar{z}\zeta)^2} + \frac{\lambda - 1}{\lambda + 1} \frac{z\bar{z} \zeta[2 - z\zeta]}{(1 - z\zeta)^2} \right] \theta(\zeta) \, d\sigma_\zeta.
\]

Therefore, equation (4.7) can be written as

\[
\frac{\partial \theta}{\partial t} - a\Delta \theta = b \frac{\partial}{\partial t} M\theta,
\]

where

\[
a = \frac{q}{\rho s + \beta \hat{\beta} c / 2}, \quad b = -\frac{\beta \hat{\beta} c}{2\rho s + \beta \hat{\beta} c}.
\]

It is shown in Wen and Begehr [8] that the operator $N$ defined by

\[
(N\theta)(z) = \frac{1}{\pi} \iint_{\Omega} \frac{\theta(\zeta)}{(1 - \bar{z}\zeta)^2} \, d\sigma_\zeta
\]

is bounded from $L^p(\Omega)$ into $L^p(\Omega)$ and the norm of $N$ is estimated by

\[
\|N\|_{L^p(\Omega)}^p \leq \left[ 4\pi^{1-1/p} \|\theta\|_{L^p(\Omega)} \right]^p + \max\{1, 2^{2p-4}\} \left[ \Lambda_p \|\theta\|_{L^p(\Omega)} \right]^p,
\]

where $\Lambda_p$ depends only on $p$ with $\Lambda_2 = 1$. This yields

\[
\|N\|_{L^2(\Omega)} \leq \sqrt{(4/\pi) + 1} \cdot \|\theta\|.
\]
and in turn, one obtains
\begin{equation}
(4.14) \quad \|M\|_{L^2(\Omega)} \leq C_\lambda \equiv \left(1 + 2\frac{\lambda - 1}{\lambda + 1}\right) \sqrt{(4/\pi) + 1}.
\end{equation}

**Theorem 4.** Assuming $\varphi \in H^1_0(\Omega)$, there is a unique solution in $W^{2,1}_2(\Omega_T)$ to (4.9) and (4.11), provided that $|b| < C_\lambda$. Moreover, the solution is analytic in the space variables for almost all fixed $t \in (0, T)$.

**Proof.** We apply the contraction principle. To this end, we define the operator $F$ from $W^{2,1}_2(\Omega_T)$ to $W^{2,1}_2(\Omega_T)$ via $\Theta = F \theta$, such that
\begin{align}
(4.15) \quad & \Theta_t - a\Delta \Theta = b M \theta_t \quad \text{in } \Omega_T, \\
(4.16) \quad & \Theta(\cdot, 0) = \varphi \quad \text{in } \Omega, \\
(4.17) \quad & \Theta = 0 \quad \text{on } \partial \Omega \times (0, T).
\end{align}

Using the standard estimate (see e.g. Ladyzhenskaya [4, p.111]) for (4.15)-(4.17), it can be shown that $F$ defines a contraction mapping from $W^{2,1}_2(\Omega_T)$ into itself, under the assumption that $|b| < C_\lambda$, thus the proof of the existence and uniqueness of the solution is complete. The proof for the analyticity of the solution is based on the following observation. Let $\theta$ be the solution to (4.9) and (4.11). Introducing the new function $\theta^* = \theta - b M \theta$ we obtain
\begin{equation}
\theta^*_t = a\Delta \theta^* \quad \text{on } \Omega_T.
\end{equation}

Indeed, the kernel of the integration in the definition of $M$ is harmonic in $\Omega$, which can be verified using the complex form $\Delta = 4\partial^2 / \partial x \partial y$ for the Laplace operator. It is well known that functions satisfying the heat equation are analytic in the space variables $x$ and $y$. This implies that $\theta = \theta^* + b \cdot M \theta$ is analytic in $x$ and $y$ because $M \theta$ is harmonic on $\Omega$ for almost all fixed $t$. The proof is complete.

In concluding the paper, we would like to propose some questions that may be of interest for further considerations.

(1) The estimate on the norm of the operator $M$ may not be optimal, which led to the restriction $|b| < C_\lambda$. Although it is not restrictive from the point of view of
applications, since \( \beta \) is usually very small, it would be worthwhile to remove it from Theorem 4.

(2) What should be the integral representations for problems under non-Dirichlet boundary conditions?

(3) How should one generalize the result in this paper to a more general domain? How can the decoupled temperature equation be analyzed numerically?

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