THE THERMISTOR PROBLEM
WITH ONE-ZERO CONDUCTIVITY

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Abstract. Thermistors are modeled as a system of two nonlinear elliptic equations for the electric potential $\varphi$ and the temperature $u$. The electric conductivity $\sigma(u)$ is temperature dependent. The case $\sigma(u) = 1$ if $u < u^*$, $\sigma(u) = 0$ if $u \geq u^*$ is a good approximation for many thermistors. In this case the system becomes degenerate. We prove existence and uniqueness of solution for such $\sigma(u)$, and characterize the set of infinite resistance $\{u = u^*\}$.

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§0. Introduction. A thermistor is an electric circuit device made of ceramic material (typically a cylinder of diameter $\sim 5\text{mm}$ and height $\sim 2\text{mm}$) where the electrical resistivity $1/\sigma(u)$ increases 5 orders of magnitude as the temperature increases beyond a critical number $u^*$ (typically between $100^\circ C$ and $200^\circ C$). If there is a current surge in the circuit, the thermistor will act as a circuit breaker. In comparison with circuit breaker such as a fuse, the thermistor has the advantage that when the surge has fallen off the thermistor will cool down and the circuit will resume its normal function without needing to replace or reset the thermistor. For more details see, for instance, [10] [11] [13].

The case where $\sigma(u)$ is uniformly positive was considered by several authors, who proved existence of a solution, and under special boundary conditions, also uniqueness; see Cimatti [3] and the references therein. More recently Chen and Friedman [2] considered the case where

$$\sigma(u) > 0 \text{ if } u < u^*, \quad \sigma(u) = 0 \text{ if } u \geq u^*,$$

(0.1)

with $\sigma(u)$ continuous across $u = u^*$.

They established the existence of a weak solution. Under special boundary conditions (as in [3]) they also proved uniqueness, and showed that the set of infinite resistance

$$S = \{\sigma(u(x)) = 0\}$$

is a level surface of a harmonic function.

In the present paper we assume that

$$\sigma(u) = \begin{cases} 1 & \text{if } u < u^*, \\ 0 & \text{if } u \geq u^* \end{cases}$$

(0.2)

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Note that in contrast with (0.1), \( \sigma(u) \) is discontinuous at \( u = u^* \). We also consider different and probably more realistic boundary conditions than in [2].

We establish existence and uniqueness of a solution, and derive a characterization of the set \( S \) of infinite resistance, as a suitable curve lying in a compact subset of the thermistor. All our results are for 2-dimensional domains only.

In §1 we state the thermistor problem and the main results. In §2 we transform the problem into a simpler one for the electric potential and an auxiliary function \( \psi, \psi = u + \frac{1}{2} \varphi^2 \) where \( u \) is the temperature. Next, in §3, we use the conformal mapping \( x + iy \to \bar{\varphi} + i\varphi \) (\( \bar{\varphi} = \) harmonic conjugate of \( -\varphi \)) to transform the problem in \( (\varphi, \psi) \) to a problem for \( \psi \) only. (The use of such a conformal mapping was suggested by Howison [9].) In fact, \( \psi \) turns out to be a solution to a variational inequality whose properties are studied in §4. In §5 we go back from \( \psi \) as a function of \( (\varphi, \bar{\varphi}) \) to recover \( \psi \) as well as \( \varphi \) as functions of \( (x, y) \). This step is quite delicate and requires a special choice of the additive constant in the definition of \( \bar{\varphi} \); it completes the proof of existence and uniqueness of a solution, and of characterization of \( S \). A more general uniqueness result is proved in §6.

All the above results are proved in case the thermistor is a rectangle. In §7 we extend the results to general 2-dimensional domains.

§1. The problem and the main results. Let \( R \) be a rectangle

\[
R = \{(x, y) \mid -a < x < a, -b < y < b\}.
\]

The thermistor problem consists of the system of elliptic equations

\[
\begin{align*}
(1.1) \quad & \nabla(\sigma(u)\nabla\varphi) = 0 \quad \text{in} \quad R, \\
(1.2) \quad & \nabla(\nabla u + \sigma(u)\varphi\nabla\varphi) = 0 \quad \text{in} \quad R
\end{align*}
\]

and boundary conditions

\[
\begin{align*}
(1.3) \quad & \varphi(x, b) = A, \varphi(x, -b) = A \quad \text{if} \quad -a < x < a, \\
& \varphi_x(\pm a, y) = 0 \quad \text{if} \quad -b < y < b,
\end{align*}
\]

\[
(1.4) \quad u = 0 \quad \text{on} \quad \partial R,
\]

where

\[
\begin{align*}
\varphi & = \text{electric potential}, \\
u & = \text{temperature}, \\
\sigma(u) & = \text{conductivity}.
\end{align*}
\]
(0.1). By approximating \( \sigma(u) \) by uniformly positive \( \sigma_\varepsilon(u) \) \( (\varepsilon > 0) \) they show that the corresponding solutions \( (\varphi_\varepsilon, u_\varepsilon) \) converge to a weak solution \( (\varphi, u) \), and \( u \leq u^* \). Although the boundary conditions taken in [2] are different from those in (1.3), (1.4), their method applies as well to the present boundary conditions.

In this paper we assume that

\[
\sigma(u) = \begin{cases} 
1 & \text{if } u < u^* \\
0 & \text{if } u \geq u^* 
\end{cases}, \quad u^* > 0.
\]

If we approximate this \( \sigma(u) \) by a sequence of positive functions \( \sigma_\varepsilon(u) \), we get a limiting pair \( (\varphi, u) \) with

\[
u \leq u^* \quad \text{a.e. ;}
\]

however it is not clear whether \( (\varphi, u) \) forms a “weak solution” of the problem (1.1) – (1.4) in the distribution sense defined in [2]. The main interest is actually to determine the set of infinite resistance; this is formally the set where \( \sigma(u) = 0 \) or, in view of (1.5), (1.6), the set

\[
S = \{(x, y) \in \mathbb{R}, \quad u(x, y) = u^*\}.
\]

For the \( \sigma \) and the boundary conditions considered in [2], \( S \) was found to be a level curve of a harmonic function; this curve connects one point on the boundary to another point on the boundary.

In this paper we adopt a more direct formulation of a solution to (1.1)–(1.4). We shall seek a solution for which \( S \) is either empty or an interval lying on the \( x \)-axis:

\[
S = \{(x, 0); \quad -a_* \leq x \leq a_*\}.
\]

Set

\[
S_0 = S \backslash \{(-a_*, 0), (a_*, 0)\}.
\]

The weak formulation of (1.1), (1.2) in a neighborhood of \( S \) formally implies that

\[
\begin{bmatrix}
\frac{\partial \varphi}{\partial n}
\end{bmatrix} = 0 \quad \text{on} \quad S_0 \equiv S \backslash \{(-a_*, 0), (a_*, 0)\},
\]

\[
\begin{bmatrix}
\frac{\partial u}{\partial n} + \varphi \frac{\partial \varphi}{\partial n}
\end{bmatrix} = 0 \quad \text{on} \quad S_0
\]

where \( n \) is the normal \( (0, 1) \) to \( S \) and \([\cdot\cdot\cdot]\) means the jump across \( S \). We shall require that

\[
u \text{ is continuous in } \overline{\mathbb{R}},
\]

\[
\varphi \text{ is continuous in } \overline{\mathbb{R}} \backslash S \text{ and uniformly continuous in } \mathbb{R} \cap \{y > 0\} \text{ and }
\]

\[
in \mathbb{R} \cap \{y < 0\},
\]

\[
3
\]
and that

\[(1.12) \quad \varphi \text{ and } u \text{ are uniformly } C^1 \text{ from each side of } S_0 ; \]

\(\varphi\) may be discontinuous across \(S_0\) (and in fact it will be).

Since \(u^* > 0\) and \(u\) is continuous in \(\overline{R}\) with \(u = 0\) on \(\partial R\), it is clear that \(a^* < a\). If \(A\) is small \(S\) will be empty, but if \(A\) is large then \(S\) will not be empty.

Notice in view of (1.12), the functions in (1.9) (1.10) are well defined.

Because the data for \(\varphi\) is symmetric in \(x\) and anti-symmetric in \(y\), and the data for \(u\) is symmetric in both \(x\) and \(y\), it is natural to look for a solution which satisfies:

\[(1.13) \quad \varphi(x,y) = -\varphi(x,-y), \quad \varphi(x,y) = \varphi(-x,y) \quad \text{in} \quad R \setminus S, \]
\[(1.14) \quad u(x,y) = u(x,-y) = u(-x,y) \quad \text{in} \quad R. \]

We add two more conditions, which are quite natural:

\[(1.15) \quad \varphi > 0 \quad \text{on} \quad R \cap \{y > 0\}; \]

further,

\[(1.16) \quad \frac{\partial \varphi(x,0^+)}{\partial y} \geq 0 \quad \text{on} \quad S_0. \]

**Definition 1.1.** A pair of functions \((\varphi, u)\) is called a solution of the thermistor problem (1.1)–(1.4) if it is a smooth solution of (1.1), (1.2) in \(R \setminus S\) satisfying (1.3), (1.4), and if (1.6)–(1.16) hold.

Observe that the relation (1.9) is a consequence of the first condition in (1.13). This condition also implies that

\[(1.17) \quad \varphi(x,0) = 0 \quad \text{off} \quad S. \]

**Theorem 1.1.** There exists a unique solution to the thermistor problem.

The proof is given in §§2–5.

In §6 we shall extend the concept of a solution, allowing \(S\) to have nonempty interior. We shall prove that the only possible solution is the one with \(S\) as in (1.8).

In §7 we shall extend all the results to the case where \(R\) is a general domain in \(\mathbb{R}^2\).
§2. Reduction of the problem. In this section we reduce the problem for \((\varphi, u)\) to a problem for \((\varphi, \psi)\) where

\begin{equation}
\psi = u + \frac{1}{2} \varphi^2 ;
\end{equation}

such a transformation was used already in [5] for other boundary conditions.

Clearly \(\psi\) is continuous in \(R\) and

\begin{equation}
\psi(x, y) = \psi(x, -y) = \psi(-x, y) \quad \text{in} \quad R .
\end{equation}

One can check that

\begin{equation}
\Delta \psi = 0 \quad \text{in} \quad R \setminus S .
\end{equation}

From (2.2) it follows that \(\psi_y(x, 0) = 0\) off \(S\) (since \(\psi \in C^1\) off \(S\)). The relation (1.10) gives \([\psi_y] = 0\) and, in view of the symmetry of \(\psi\) in \(y\), \(\psi_y = 0\) along \(S_0\). We conclude that

\begin{equation}
\psi_y(x, 0\pm) = 0 \quad \text{if} \quad -a < x < a , \quad x \neq \pm a_* .
\end{equation}

Finally,

\begin{equation}
\psi = \frac{A^2}{2} \quad \text{on} \quad y = \pm b ,
\end{equation}

\begin{equation}
\psi = \frac{\varphi^2}{2} \quad \text{on} \quad x = \pm a
\end{equation}

and

\begin{equation}
\psi < u^* + \frac{\varphi^2}{2} \quad \text{on} \quad \overline{R \setminus S} ,
\end{equation}

\begin{equation}
\psi = u^* + \frac{\varphi^2}{2} \quad \text{on} \quad S .
\end{equation}

The thermistor problem can now be reformulated in terms of \((\varphi, \psi)\):

(i) \(\varphi\) satisfies (1.1), (1.3), (1.13); it is continuously differentiable uniformly from each side of \(S\) except possibly at \((\pm a_*, 0)\), it is continuous in \(R \setminus S_0\), and it satisfies (1.15), (1.16);

(ii) \(\psi\) satisfies (2.1)–(2.8); it is continuously differentiable uniformly from each side of \(S\) except possible at \((\pm a_*, 0)\), and it is continuous in \(R\).
§3. Conformal mapping. It will suffice to construct \( \varphi, \psi \) in

\[
R^+ = R \cap \{ y > 0 \}.
\]

Set

\[
T = \{(x, 0); -a < x < -a_*, \text{ or } a_* < x < a\}.
\]

We shall use conformal mapping in order to transform the problem for \((\varphi, \psi)\) into a problem for \(\psi\) alone.

We introduce the harmonic conjugate \(-\bar{\varphi}\) of \(\varphi\); it is uniquely determined up to an additive constant. We choose \(\bar{\varphi}\) such that

\[
(3.1) \quad \bar{\varphi}(a, 0) = q, \quad q \text{ positive}.
\]

Set

\[
(3.2) \quad X = \bar{\varphi}, \ Y = \varphi
\]

and consider the conformal mapping

\[
(3.3) \quad (x, y) \rightarrow \bar{\varphi} + i\varphi = X + iY,
\]

or

\[
(3.4) \quad (x, y) \rightarrow (X(x, y), Y(x, y)).
\]

It maps \(\overline{R^+}\) into the rectangle \(R^*:\)

\[
(3.5) \quad R^* = \{(X, Y) ; \ -q < X \leq q, \ 0 < Y \leq A \}.
\]

here

\[
\partial R^+ \cap \{ y = b \} \quad \text{is mapped into} \quad \partial R^* \cap \{ Y = A \},
\]

\[
\partial R^+ \cap \{ x = \pm a \} \quad \text{is mapped into} \quad \partial R^* \cap \{ X = \pm q \}
\]

and

\[
\partial R^+ \cap T \quad \text{is mapped into segments}
\]

\[
- q \leq X \leq -q_1, \quad q_1 \leq X \leq q \quad \text{on} \quad Y = 0 \quad (0 < q_1 < q).
\]

The image of \(S\) is a curve \(S'\) in \(\overline{R^*}\) since \(\varphi(x, 0+) \geq 0\). From (1.16) we deduce that \(\bar{\varphi}_x(x, 0+) \geq 0\). This implies that \(S'\) must be a graph

\[
(3.6) \quad S' : Y = g(X), \quad 0 \leq g(X) < A
\]
We denote by $\Omega^*$ the region bounded by $S'$ and the $X$-axis, and set

$$R' = R^* \setminus \overline{\Omega^*}.$$  

**Lemma 3.1.** The mapping (3.3) is 1–1 from $\overline{R^+}$ onto $\overline{R'}$.

**Proof.** Using complex notation, let $\zeta_0$ be any point in $R'$. We shall prove that

$$\int_{\partial R'} \frac{dz}{f(z) - \zeta_0} = 2\pi i$$  

where $f(z) = \tilde{\varphi}(z) + i \varphi(z)$. As one travels along the edge $y = b$ of $R^+$ the argument of $f(z) - \zeta_0$ cannot change by any multiple of $2\pi i$, since $f(z)$ remains confined to the line $Y = A$. The same is true of the other edges of $R'$. Finally as one travels along $S'$, $X$ increases monotonically, by (3.6). These observations clearly yield the assertion (3.7).

Equality (3.7) implies that $f(z)$ takes the value $\zeta_0$ at precisely one point of $R'$. Thus $f$ is conformal mapping from $R^+$ into $R'$. Since both domains have piecewise $C^1$ boundary, we can appeal to a general result in conformal mappings [1; p. 369] to deduce that $f$ can be extended continuously as 1–1 mapping from $\overline{R^+}$ onto $\overline{R^-}$.

Set

$$\Psi(X, Y) = \psi(x, y).$$

Then $\Psi$ satisfies:

$$\Delta \Psi = 0 \quad \text{in} \quad R',$$  

$$\Psi = \frac{Y^2}{2} \quad \text{on} \quad X = \pm q,$$  

$$\Psi = \frac{A^2}{2} \quad \text{on} \quad Y = A,$$  

$$\Psi = 0 \quad \text{on} \quad T' = \{(X, 0), -q < X < -q_1, q_1 < X < q\},$$  

$$\frac{\partial \Psi}{\partial n} = 0 \quad \text{on} \quad S',$$  

and

$$\Psi = u^* + \frac{Y^2}{2} \quad \text{on} \quad S',$$  

$$\Psi < u^* + \frac{Y^2}{2} \quad \text{on} \quad R' \setminus S'.$$
We shall reduce the problem for \( \Psi \) into a variational inequality for a function \( W \) defined by

\[
W(X,Y) = \int_{g(X)}^{Y} \left[ u^* + \frac{Y^2}{2} - \Psi(X,Y) \right] dY \quad (\quad-q_1 < X < q_1)
\]

\[
= \int_{0}^{Y} \left[ u^* + \frac{Y^2}{2} - \Psi(x,Y) \right] dY \quad ((X,0) \in T')
\]

(3.15)

The condition \( \partial \Psi / \partial n = 0 \) on \( S' \) can be written in the form

\[
g'(X) \Psi_X + \Psi_Y = 0 \quad \text{on} \quad Y = g(X) .
\]

(3.16)

Using (3.16) one can verify that

\[ \Delta W = Y \quad \text{in} \quad R' . \]

Also, \( W > 0 \) in \( R' \), \( W = 0 \) on \( S' \) and

\[
\frac{\partial W}{\partial Y} = u^* + \frac{Y^2}{2} - \Psi = 1 \quad \text{on} \quad S' .
\]

This shows that if we extend \( W \) by 0 into \( \Omega^* \) then \( W \) satisfies a variational inequality:

**Lemma 3.2.** the function \( W \) defined by (3.15) and extended by 0 into \( \Omega^* \) satisfies the variational inequality

\[
-\Delta W \geq -Y , \quad W \geq 0 , \quad \Psi(-\Delta \Psi + Y) = 0 \quad \text{a.e. in} \quad R^* ,
\]

(3.17)

with boundary conditions

\[
\frac{\partial W}{\partial Y} = u^* \quad \text{on} \quad Y = A ,
\]

(3.18)

\[
W = u^* Y \quad \text{on} \quad X = \pm q ,
\]

\[
W = 0 \quad \text{on} \quad Y = 0 .
\]

§4. The variational inequality. It is well known (see, for instance, [6]) that the variational inequality (3.17), (3.18) has a unique solution. However, in order to go backward and recover the functions \( \psi \) and \( \varphi \), we must first show that

\[
(4.1) \quad \text{the free boundary is given by a curve} \quad S' : Y = g(X)
\]
and

\begin{equation}
\frac{\partial W}{\partial Y} > 0 \quad \text{in} \quad \{W > 0\},
\end{equation}

i.e., \(\psi < u^* + Y^2/2\) in \(R' \equiv \{W > 0\}\).

To do this we use the penalty method. We approximate \(W\) by the solution \(W_\varepsilon\) to

\begin{equation}
-\Delta W_\varepsilon + \beta_\varepsilon(W_\varepsilon) = -Y \quad \text{in} \quad R^* \quad (\varepsilon > 0)
\end{equation}

with the same boundary conditions (3.18) as for \(W\); here the \(\beta_\varepsilon(t)\) are \(C^\infty\) functions in \(t\) satisfying:

\begin{equation}
\begin{aligned}
\beta_\varepsilon(0) &= 0, \quad \beta_\varepsilon(t) = 0 \quad \text{if} \quad t > 0, \\
\beta_\varepsilon(t) &\to -\infty \quad \text{if} \quad t < 0, \quad \varepsilon \to 0, \\
\text{and} \quad \beta_\varepsilon'(t) &\geq 0, \quad \beta_\varepsilon''(t) \leq 0.
\end{aligned}
\end{equation}

We observe that the first compatibility condition holds at \((\pm q, 0)\) since, upon using the boundary conditions, we get

\[-\Delta W_\varepsilon + \beta_\varepsilon(W_\varepsilon) = \beta_\varepsilon(0) = 0 = Y \quad \text{at} \quad (\pm q, 0).\]

It follows that \(W_\varepsilon\) is \(C^2\) in a neighborhood of \((\pm q, 0)\). On the other hand, in a neighborhood of \((\pm q, A)\) the function \(\beta_\varepsilon(W)\) vanishes and (4.3) becomes \(\Delta W = Y\), so that \(\Delta(W_Y) = 1\). Since \(W_Y = u^*\) both on \(Y = A\) and on \(X = +q\), we can apply \(L^p\) boundary estimates to \(W_Y\) and conclude that, for any \(p > 1\),

\[W_Y \in W^{2,p}\] in a neighborhood \(N\) of \((\pm q, A)\).

This implies that \(W_{XX}, W_{XY}, W_{YY}\) and \(W_{YX}\) are in \(L^p(N)\) and, by Sobolev's imbedding, \(W_{XY}\) and \(W_{YY}\) are in \(C^\alpha(\overline{N})\). From (4.3) we deduce that also \(W_{XX} \in C^\alpha(\overline{N})\).

**Lemma 4.1.** The solution \(W\) satisfies:

\begin{equation}
W_{XX} \geq 0, \quad W_{YY} \geq 0 \quad \text{in} \quad R^*.
\end{equation}

**Proof.** Consider the function \(\zeta = \partial^2 W_\varepsilon / \partial Y^2\). Differentiating (4.3) twice with respect to \(Y\) we get

\[-\Delta \zeta + \beta_\varepsilon'(W_\varepsilon)\zeta = -\beta_\varepsilon''(W_\varepsilon) \left( \frac{\partial W_\varepsilon}{\partial Y} \right)^2 \geq 0.\]

On \(\{-q < X < q, Y = A\}\)

\[\zeta_Y = (W_\varepsilon)_{YY} = (-W_{\varepsilon,XX} + Y)_Y = 1.\]
On the sides $X = \pm q$, $\zeta = 0$, and on $Y = 0$
\[ \zeta = \beta_\varepsilon(0) - W_{\varepsilon,XX} = \beta_\varepsilon(0) = 0. \]

Since, as noted above, $\zeta$ is continuous in $\overline{R^*}$, the maximum principle yields $\zeta > 0$ in $R^*$. Taking $\varepsilon \to 0$ the assertion $W_{YY} \geq 0$ follows.

The proof of $W_{XX} \geq 0$ is similar. Here
\[ (W_{\varepsilon,XX})_Y = 0 \quad \text{on} \quad Y = A, \quad W_{\varepsilon,XX} = 0 \quad \text{on} \quad Y = 0, \]
and
\[ W_{\varepsilon,XX} = -W_{\varepsilon,YY} + \beta_\varepsilon(W_\varepsilon) + Y > 0 \quad \text{on} \quad X = \pm q. \]

**Corollary 4.2.** $W_Y(X,Y) \geq 0$.

Indeed, since $W(X,0) = 0$ and $W(X,Y) \geq 0$, we have $W_Y(X,0 \geq 0$. Since also $W_{YY} \geq 0$, the assertion follows.

From Corollary 4.2 we deduce that the free boundary for $W$ is given by (4.1) By the strong maximum principle (4.2) also holds.

By symmetry,

\[ W(X,Y) = W(-X,Y) \]

and by the maximum principle

\[ W_X(X,Y) \geq 0 \quad \text{if} \quad X \geq 0. \]

It follows that

\[ g(X) = g(-X), \]

and $g(X)$ is decreasing in $X$ for $X > 0$. We can further apply Theorems 6.1, 6.2 in [6; Chap. 2, Sec. 6] to deduce that

\[ g(X) \text{ is analytic for } -q_1 < X < q_1, \]
\[ g'(X) < 0 \quad \text{if} \quad 0 < X < q_1. \]

We also have

\[ g(q_1 - 0) = 0. \]

Indeed, if $g(q_1 - 0) > 0$ then along the segment $\ell = \{(q_1,Y); 0 < Y < g(q_1 - 0)\}$ the function $W_{YY}$ has zero Cauchy data. Since this function is harmonic in $R'$, it follows that $W_{YY} \equiv 0$, a contradiction.

We summarize:
Theorem 4.3. There exists a unique solution \( W \) to the variational inequality (3.17), (3.18), and it has the properties (4.1), (4.2), (4.5), (4.6); furthermore, the free boundary satisfies (4.7), (4.8) and (4.9).

§5. Proof of Theorem 1.1. In §4 we have constructed the function \( W(X,Y) \). We now define a function \( \Psi(X,Y) \) by

\[
\Psi = u^* + \frac{Y^2}{2} - W_Y.
\]

If \((\varphi, \psi)\) is a solution of the thermistor problem then, as shown in §3, \( \Psi(X,Y) \) must coincide with \( \psi(x,y) \). There remains the problem of reconstructing the functions \( \varphi \) and \( \psi \) from \( \Psi \). As we shall see, this can be done for one and only one choice of the parameter \( q \).

For clarity we shall write

\[
R^* = R_q^* = \{-q < X < q, \ 0 < Y < A\}
\]

and denote the corresponding \( W \) and \( \Psi \) by \( W_q \) and \( \Psi_q \). The free boundary will be denoted by

\[
Y = g_q(X) \quad -q_1 \leq X \leq q_1.
\]

We also set

\[
R'_q = R^*_q \setminus \{Y \leq g_q(X)\}.
\]

It is well known that there exists a unique conformal mapping \( h(Z) \) \( (Z = X + iY) \) of the closure of \( R'_q \) onto the rectangle in the \( z = x + iy \) plane:

\[
\overline{R^+} = \{-a \leq x \leq a, \ 0 \leq y \leq b\}
\]

such that

\[
\pm q + iA \quad \text{is mapped into} \quad \pm a + ib
\]

and \( ig(0) \) is mapped into \( 0 \).

From the uniqueness of \( h(Z) \) and the symmetry about the imaginary axis of \( R'_q \) and \( \overline{R^+} \) and of the points in (5.3) it follows that \( h(Z) \) is symmetric with respect to the imaginary axis; in particular,

\[
\text{if} \quad h(q) = a, \quad \text{then} \quad h(-q) = -a.
\]

If this is the case then

\[
h \text{ maps the edge } Y = A \text{ of } R'_q \text{ onto the edge } y = b \text{ of } R^+,
\]

(5.5) the edges \( X = \pm q \) of \( R'_q \) onto the edges \( x = \pm a \) of \( R^+ \),

and the remaining boundary of \( R'_q \) onto the edge \( y = 0 \) on \( R^+ \).
Denote the inverse of $h(Z)$ by $f$; i.e.,

\[ x + iy = h(X + iY) \quad \text{if and only if} \quad X + iY = f(x + iy). \]

Next define functions $\varphi, \tilde{\varphi}$ by

\[ \tilde{\varphi}(x, y) = X(x, y), \quad \varphi(x, y) = Y(x, y). \]

From (5.5) it follows that $\varphi$ satisfies all the boundary conditions in $R^+$ required in the definition of a solution to the thermistor problem (including (1.15), (1.16)) where

\[ \Delta V = Y \quad \text{in} \quad R^*_q \]

with the same boundary conditions as $W$, and set

\[ \tilde{V}_q(X, Y) = \frac{1}{q} V(qX, qY), \quad 0 < Y < \frac{A}{q}, \quad -1 < X < 1. \]

Then

\[ \Delta \tilde{V}_q = q^2 Y \to 0 \quad \text{as} \quad q \to 0 \]

so that $\tilde{V}_q \to V^*$ where $V^*$ is harmonic in $|X| < 1$, $0 < Y < \infty$ with boundary conditions

\[ V^*(X, 0) = 0, \quad V^*(\pm 1, Y) = u^* Y, \]

\[ V^* = O(Y) \quad \text{as} \quad Y \to \infty. \]
By a standard Phragmén–Lindelöf argument it follows that $V^* \equiv u^* Y$ and, therefore,
$$
\frac{\partial}{\partial Y} \tilde{V}_q(X, 0) \rightarrow \frac{\partial V^*}{\partial Y} = u^* \quad \text{as} \quad q \to 0 .
$$
The convergence is uniform, and it implies that
$$
V_Y(X, 0) \sim u^* \quad \text{uniformly in} \quad X ,
$$
provided $q$ is small.

Next we can apply the maximum principle to $V_{YY}$ (cf. the proof of Lemma 4.1) and deduce that $V_{YY} > 0$. It follows that $V_Y(X, Y) > 0$ if $q$ is small and therefore $V(X, Y) > 0$ in $R^*$. Hence $V$ is the solution $W_q$ of the variational inequality, and thus the coincidence set of $W_q$ is empty, i.e.,
$$
R'_q = \{-q < X < q, \ 0 < Y < A\} = R_q^* ,
$$
if $q$ is small.

Consider the function $x = x(X, Y)$, the real part of $h$ (cf. (5.6)). We need to show that
$$
(5.9) \quad x(q, 0) < q .
$$
Suppose (5.9) is not true. Then $x(q, 0) \geq q$ and there exists an $\alpha \in (0, q]$ such that $x(\alpha, 0) = q$.

We shall compare the harmonic function $x(X, Y)$ with the harmonic function $X$ in the rectangle $D = \{0 < X < q, \ 0 < Y < A\}$. On the top edge $Y = A$, $y \equiv A$ so that
$$
\frac{\partial x}{\partial Y} = -\frac{\partial y}{\partial X} = 0; \quad \text{also} \quad \frac{\partial X}{\partial Y} = 0 .
$$
On the edge $X = 0$, $x = 0$ (by symmetry of $h$). On the bottom edge $Y = 0$, if $0 \leq X \leq \alpha$ then $y(X, 0) = 0$ so that
$$
\frac{\partial x}{\partial Y} = -\frac{\partial y}{\partial X} = 0; \quad \text{also} \quad \frac{\partial X}{\partial Y} = 0 .
$$
Finally, on the remaining boundary of $D$ which consists of the segments from $(\alpha, 0)$ to $(q, 0)$ and from $(q, 0)$ to $(q, A)$, $x(X, Y) \geq q$ whereas $X \leq q$. It follows that
$$
x(X, Y) \geq X \quad \text{in} \quad D
$$
and, furthermore,
$$
\frac{\partial}{\partial X}(x - X)|_{X=0} > 0 .
$$
This implies that
$$
\frac{\partial}{\partial Y}(y - Y)|_{X=0} > 0
$$
and, by integration,
$$
0 = y(0, A) - A = \int_0^A \frac{\partial}{\partial Y}(y(0, Y) - Y) > 0 ,
$$
a contradiction.
Lemma 5.3. If \( q > a \) then \( y(q, 0) > 0 \).

Proof. It suffices to show that

\[
y(q, 0) > A \left( 1 - \frac{a}{q} \right).
\]

Suppose this is not true. Then there is an \( \alpha \in (0, A) \) such that

\[
y(q, \alpha) = A \left( 1 - \frac{a}{q} \right).
\]

We can now prove by comparison that

\[
y(X, Y) \leq A - \frac{a}{q} (A - Y) \quad \text{in} \quad R^*_q.
\]

Indeed, both sides of (5.12) are harmonic functions. On \( Y = A \) they coincide; on \( \{ X = 0 \} \) and on \( \{ X = q, \alpha < Y < A \} \)

\[
\frac{\partial y}{\partial X} = - \frac{\partial x}{\partial Y} = 0 \quad \text{and} \quad \frac{\partial}{\partial X} \left( A - \frac{a}{q} (A - Y) \right) = 0.
\]

Finally, by (5.11), on the remaining boundary which is contained in \( X > 0 \),

\[
y < y(q, \alpha) = A \left( 1 - \frac{a}{q} \right) \leq A - \frac{a}{q} (A - Y) \quad \text{since} \quad Y \geq 0.
\]

It follows that (5.12) holds and, furthermore,

\[
\left. \frac{\partial}{\partial Y} \left( y - A - \frac{a}{q} (a - Y) \right) \right|_{Y=1} > 0.
\]

Hence

\[
\frac{\partial x}{\partial X} = \frac{\partial y}{\partial Y} > \frac{a}{q} \quad \text{on} \quad Y = 1
\]

and, by integration,

\[
2a = x(q, 1) - x(-q, 1) = \int_{-q}^{q} \frac{\partial x}{\partial X} \, dX > \frac{a}{q} \cdot 2q = 2a
\]

a contradiction.
Denote the conformal mapping from $R'_q$ onto $\overline{R^+}$ by $h_q$. We have proved that
\[ h_q(q, 0) < q \quad \text{if} \quad q \quad \text{is small}, \]
\[ h_q(q, 0) = a + i\eta \quad \text{for some} \quad 0 < \eta < A, \quad \text{if} \quad q \quad \text{is large}. \]
By continuity of the function
\[ q \rightarrow h_q(q, 0) \]
It follows that there exists a $q_0 > 0$ such that
\[ h_{q_0}(0, 0) = q_0. \]
This proves the existence part of Lemma 5.1.

To prove that such a solution $q_0$ is unique we begin by examining more closely the dependence of $W_q$ and the free boundary
\[ Y = g_q(X), \quad -q_1 < X < q_1 \]
on $q$.

**Lemma 5.4.** If $q' > q$ then
\[ W_{q'} \left( \frac{q'}{q} X, Y \right) \leq W_q(X, Y) \quad \text{in} \quad R'_q \]
and
\[ g_{q'}(X) \geq g_q \left( \frac{q}{q'} X \right) \quad \text{for} \quad -\frac{q'}{q} q_1 < X < \frac{q'}{q} q_1. \]

**Proof.** Set
\[ \widetilde{W} = W_{q'} \left( \frac{q'}{q} X, Y \right). \]
Then, in $\{ \widetilde{W} > 0 \}$
\[ \Delta \widetilde{W} = \frac{q'^2}{q^2} \widetilde{W}_{XX} + \widetilde{W}_{YY} \geq \widetilde{W}_{XX} + \widetilde{W}_{YY} = Y \]
since $q'/q > 1$ and $\widetilde{W}_{XX} \geq 0$ (by Lemma 4.1). Since the boundary conditions for $\widetilde{W}$ majorize those for $W_q$, (5.13) follows by a comparison theorems for variational inequalities. The inequality (5.14) is a consequence of (5.13).

Consider the conformal mappings $h_q$ and $h_{q'}$ for $q < q'$, and suppose they satisfy:
\[ h_q(\pm q, 0) = h_{q'}(\pm q', 0) = \pm a. \]
We wish to show that this leads to a contradiction.

We begin by introducing the harmonic functions

\[ \eta_1(X, Y) = y_q(qX, A - q(A - Y)), \]
\[ \eta_2(X, Y) = y_{q'}(q'X, A - q'(A - Y)) \]

and the corresponding \( \xi_1(X, Y), \xi_2(X, Y) \) defined in the same way with \( y_q, y_{q'} \) replaced by \( x_q, x_{q'} \); here \( h_q = x_q + iy_q, h_{q'} = x_{q'} + iy_{q'} \).

Observe that \( \eta_1 \) is defined for \(-1 < X < 1\) and for \( Y \) which lies above the free boundary, i.e.,

\[ A - q(A - Y) \geq g_q(qX); \]

for simplicity we extend \( g_q(qX) \) by 0 to \(-1 \leq X < -q_1/q\) and to \( q_1/q < X \leq 1\), and similarly extend \( g_{q'}(q'X) \). Thus, for fixed \( X \), the \( Y \)-interval in the domain of definition of \( \eta_1 \) is \( \tilde{Y}_q \leq Y \leq A \) where

\[ A - q(A - \tilde{Y}_q) = g_q(qX). \]

The length of this interval is \( A - \tilde{Y}_q \).

Similarly, for fixed \( X \) the \( Y \)-interval in the domain of definition of \( \eta_2 \) is \( \tilde{Y}_{q'} \leq Y \leq A \) where

\[ A - q'(A - \tilde{Y}_{q'}) = g_{q'}(q'X) \]

and its length is \( A - \tilde{Y}_{q'} \). Since \( g_{q'}(q'X) \geq g_q(qX) \), we get

\[ q'(A - \tilde{Y}_{q'}) < q(A - \tilde{Y}_q) \]

and, since \( q' > q \),

\[ A - \tilde{Y}_{q'} < A - \tilde{Y}_q. \]

This means that the domain of definition \( I_q \) of \( \eta_1 \) contains the domain of definition \( I_{q'} \) of \( \eta_2 \).

We now compare \( \eta_1, \eta_2 \) in the domain \( I_{q'} \). Clearly \( \eta_1 \geq 0 = \eta_2 \) on the free boundary. From (5.15) we deduce that the horizontal (vertical) sides of the domains of definitions of \( \eta_1 \) and \( \eta_2 \) are mapped into the same horizontal (vertical) sides of \( R^+ \). Therefore

\[ \eta_1 = \eta_2 \text{ on } Y = A \]

and

\[ \frac{\partial \eta_1}{\partial X} = \frac{\partial \xi_1}{\partial Y} = 0 \text{ on the vertical sides}. \]

Applying the maximum principle we conclude that \( \eta_1 > \eta_2 \) in \( I_{q'} \) and, furthermore,

\[ \frac{\partial(\eta_1 - \eta_2)}{\partial Y} \bigg|_{Y=A} < 0. \]
It follows that
\[ \frac{\partial}{\partial X} (\xi_1 - \xi_2)\big|_{Y=A} < 0 \]
and, by integration,
\[ 0 = (\xi_1 - \xi_2)(1, A) - (\xi_1 - \xi_2)(-1, A) = \int_{-1}^{1} \frac{\partial}{\partial X} (\xi_1 - \xi_2)(X, A)dX < 0 , \]
a contradiction.

We have thus proved that there cannot be more than one solution \( q \) to the equation \( h_q(q, 0) = q \). This completes the proof of Theorem 1.1.

§6. A more general uniqueness theorem. Definition 1.1 of a solution \((\varphi, u)\) presupposes that \( S \) consists of an interval lying on the \( x \)-axis. From the physical background of the problem one might equally well look for a solution where \( S \) has the form
\[ (6.1) \quad S = \{(x, y); |y| \leq k(x), \ -\gamma \leq x \leq \gamma \} \]
where
\[ k(-x) = k(x) , \]
\[ k(x) > 0 \quad \text{if} \quad 0 \leq x < \gamma_1 , \]
\[ k(x) = 0 \quad \text{if} \quad \gamma_1 \leq x \leq \gamma . \]

Let
\[ (6.2) \]
\[ S_0 = \{(x, 0), \quad k(x) = 0 \} , \]
\[ S_1 = S \setminus S_0 . \]

Formally, the weak formulation of (1.1) gives
\[ (6.4) \quad \frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \partial S_1 \]
where \( n \) is the outward normal to \( S_1 \). Since the function \( \psi \) introduced in §2 satisfies (see [5] [6])
\[ (6.5) \quad \nabla (\sigma(u)\nabla \psi) = 0 , \]
the weak formulation of (1.2), written in terms of \( \psi \), gives
\[ (6.6) \quad \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \partial S_1 . \]

We also need to replace (1.13), (1.14) by
\[ (6.7) \quad \psi > 0 \quad \text{in} \quad (R \setminus S) \cap \{y > 0 \} , \]
\[ (6.8) \quad \frac{\partial \varphi}{\partial n} \geq 0 \quad \text{on} \quad (S_0 \cup \partial S_1) \cap \{y \geq 0 \} , \quad \text{except at} \quad (\pm \gamma, 0) . \]

Finally we assume that (1.13), (1.14) hold in \( R \setminus S \), \( u \) is continuous in \( R \setminus (\text{int } S_1) \), and continuously differentiable in \( (R \setminus \text{int } S_1) \cap \{y \geq 0 \} \), and \( \varphi \) has the same smoothness properties except for jump discontinuity across the interior of the interval \( S_0 \).
**Definition 6.1.** A pair \((\varphi, u)\) satisfying all the above properties as well as (1.1)–(1.4) in \(R \setminus S\) is called a solution to the thermistor problem.

**Theorem 6.1.** If \((\varphi, u)\) is a solution to the thermistor problem (according to Definition 6.1) then \(S_1\) is empty so that \((\varphi, u)\) coincides with the solution asserted in Theorem 1.1.

**Proof.** Suppose \((\varphi, u)\) is a solution and consider the mapping \(\tilde{\varphi} + i\varphi\) introduced in §3. It maps \(S_1\) onto an interval \(l\) along which \(\tilde{\varphi} = 0\). Thus, the image of \(R\) is \(R' \setminus \bar{l}\) where \(R'\) is defined as in §3 and \(\bar{l}\) is the closure of an interval

\[
l = \{(0, Y) ; g(0) < Y < \delta\}, \quad \delta \geq g(0) .
\]

By properties of the solution, the function \(\Psi(X, Y) = \psi(x, y)\) satisfies:

\[
\Psi = u^* - \frac{Y^2}{2} \text{ on } l
\]

if \(l\) is nonempty, and (cf. (6.6))

\[
\frac{\partial \Psi}{\partial X} = 0 \text{ from both sides of } l .
\]

Hence \(\Psi\) is harmonic across \(l\). By uniqueness to the Cauchy problem,

\[
\Psi(X, Y) \equiv u^* + \frac{X^2}{2} - \frac{Y^2}{2} ,
\]

a contradiction. This proves that \(l\) is empty and then so is the set \(S_1\), and the theorem follows.

**§7. General domains.** Let \(\Omega\) be a general domain with piecewise \(C^1\) boundary \(\partial \Omega\) and choose four points \(A, B, C, D\) on \(\partial \Omega\), arranged clockwise along \(\partial \Omega\). Let \(R\) be a rectangle

\[
R_p : -p < x < p, \quad -b < y < b
\]

and set

\[
A' = (-p, b), \quad B' = (p, b), \quad C' = (p, -b), \quad D' = (-p, -b) .
\]

Denote by \(F_p\) the conformal mapping which maps \(\Omega\) onto \(R_p\) such that

\[
F_pA = A', \quad F_pB = B', \quad F_pD = D' .
\]

**Theorem 7.1.** There exists a unique value \(p > 0\) such that \(F_p(C) = C\).
Once this theorem is proved, we can immediately solve the thermistor problem which consists of (1.1), (1.2) in $\Omega$ with

\[ \varphi = A \quad \text{on the arc } \widehat{AB} , \]
\[ \varphi = -A \quad \text{on the arc } \widehat{CD} , \]
\[ \frac{\partial \varphi}{\partial n} = 0 \quad \text{on the arcs } \widehat{BC} \text{ and } \widehat{DA} , \]
\[ u = 0 \quad \text{on } \partial \Omega , \]

by first solving it in the rectangle $R_p$ with a pair $(\varphi, u)$, and then taking $(\varphi \circ F_p^{-1}, u \circ F_p^{-1})$ to be the solution in $\Omega$ (cf. [7]).

**Proof of Theorem 7.1.** Denote by $G_p$ the point on the arc $DCB$ whose image under $F_p$ is $C'$. We claim that

\[ (7.1) \quad \text{if } p' > p \text{ then } G_{p'} \text{ lies between } G_p \text{ and } B . \]

Suppose this is not true and consider the imaginary parts $Y_p$ and $Y_{p'}$ of $F_p$ and $F_{p'}$. On $\widehat{AB}$ both are equal to $b$. On $\widehat{DA}$ and the arc $G_pCB$ both have zero normal derivatives. On $\widehat{DG_{p'}}$ both $Y_p$ and $Y_{p'}$ are equal to $-b$. Finally, on the arc $G_{p'}G_p$, $Y_{p'} = p'$, $Y_p \leq p$ so that $Y_{p'} \geq Y_p$. By the maximum principle it follows that $Y_{p'} \geq Y_p$ in $\Omega$ and

\[ \frac{\partial}{\partial n} (Y_{p'} - Y_p) > 0 \quad \text{on } \widehat{AB} , \]

so that

\[ \frac{\partial}{\partial s} (X_{p'} - X_p) > 0 \quad \text{on } \widehat{AB} \]

where $\partial/\partial s$ denote the tangential derivative. Integrating on $s$ we get

\[ 2p = X_p(B) - X_p(A) > X_{p'}(B) - X_{p'}(A) = 2p' \]

a contradiction.

The assertion (7.1) implies that there exists at most one value $p$ for which $F_p(C) = C'$

Observe next that the function $p \rightarrow F_p(C)$ is continuous. Therefore, in order to complete the proof of Theorem 7.1 it suffices to show:

\[ (7.2) \quad \text{if } p \rightarrow 0 \text{ then } G_p \rightarrow D , \]

\[ (7.3) \quad \text{if } p \rightarrow \infty \text{ then } G_p \rightarrow B . \]

To prove (7.2) suppose the assertion is not true. Then

\[ \text{as } p \rightarrow 0 , \quad G_p \rightarrow C_* , \quad C_* \neq D . \]
The harmonic function $X_p$ satisfies

$$|X_p| \leq p \quad \text{in} \quad \Omega$$

and therefore

$$X_p \to 0 \quad Y_p \to \text{const.}.$$ 

uniformly in $\Omega$, as $p \to 0$. This is a contradiction since $Y \equiv \lim Y_p$ is harmonic in $\Omega$ and

$$Y = b \quad \text{on} \quad \widehat{AB}, \quad Y = -b \quad \text{on} \quad \widehat{DC}.$$ 

To prove (7.3) we work with $\frac{1}{p} X_p$, $\frac{1}{p} Y_p$. The image of $\Omega$ under $\frac{1}{p} (X_p + iY_p)$ is a rectangle with one side of length 1 and the other side of length $b/p$ which goes to zero as $p \to \infty$. Thus we can apply the proof of (7.2) to deduce the assertion (7.3).

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