LAGRANGIAN BLOCK DIAGONALIZATION

By

Debra Lewis

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LAGRANGIAN BLOCK_DIAGONALIZATION

DEBRA LEWIS
Institute for Mathematics and its Applications
University of Minnesota, Mpls. MN 55455

CONTENTS

1 INTRODUCTION ................................................................................................................. 2

2 RELATIVE EQUILIBRIA ................................................................................................. 4
  2.a Definitions ................................................................................................................. 4
  2.b Relative equilibria as critical points .......................................................................... 6

3 FORMAL STABILITY OF RELATIVE EQUILIBRIA ...................................................... 9
  3.a The energy-momentum method and the configuration–velocity split .................... 9
  3.b The amended energy ............................................................................................... 13
  3.c Block diagonalization and formal stability conditions ............................................ 14
  3.d Decomposition of configuration variations ............................................................. 15
  3.d.i Group variations ................................................................................................. 16
  3.d.ii Internal variations ............................................................................................. 18

4 APPLICATION OF BLOCK_DIAGONALIZATION ....................................................... 20
  4.a Orbit regular equilibria ............................................................................................ 20
  4.b Orbital stability .................................................................................................... 22
  4.c One dimensional momentum isotropy subgroups .................................................... 25

5 EXAMPLES ..................................................................................................................... 26
  5.a Natural mechanical systems ................................................................................... 26
  5.a.i Definitions .......................................................................................................... 26
  5.a.ii Formal stability .................................................................................................. 27
  5.a.iii A particle in an electromagnetic field .................................................................. 28
  5.b The symmetric rigid body ....................................................................................... 30
  5.c $SO(3)$ invariant Lagrangians on $T\mathbb{R}^3$ .......................................................... 32

REFERENCES ................................................................................................................... 35

APPENDIX: SUMMARY OF THE METHOD ................................................................. 36
1 Introduction

Consider a Lagrangian system with symmetry group $G$. Given some configuration $q_e \in Q$ and some element $\xi$ of the Lie algebra $\mathcal{G}$ of $G$, we wish to determine conditions on $q_e$ and $\xi$ under which the configuration $q_e$ will undergo a steady group motion with group velocity $\dot{\xi}$, i.e., that the evolution of $q_e$ according to the dynamics of the Lagrangian system will be given by

$$q_t = \exp(t \dot{\xi}) \cdot q_e,$$  \hspace{1cm} (1.1)

with velocity

$$\dot{q}_t = \xi Q(q_t) := \frac{\partial}{\partial t} \exp(t \dot{\xi}) \cdot q_e.$$ \hspace{1cm} (1.2)

For example, if $G = SO(3)$, the group of rigid rotations in $\mathbb{R}^3$, then an element $\xi \in \mathcal{G}$ can be identified with a vector in $\mathbb{R}^3$ and the motion $q_t$ consists of a steady rotation about the axis $\xi$. Once we have identified a pair $(q_e, \xi)$ satisfying (1.1), we wish to consider the stability of this motion. Generally, such motions cannot be strictly stable, since the motion is insensitive to perturbations in particular directions along the group orbit; however, by suitably relaxing the criteria for stability, one can obtain a physically meaningful and verifiable standard of stability for such motions. The two key ingredients of this analysis are the augmented Lagrangian $L_\xi : Q \rightarrow \mathbb{R}$ and the generalized inertia mapping $\Pi : Q \times \mathcal{G} \rightarrow \mathcal{G}^*$. The augmented Lagrangian $L_\xi$ is constructed by identifying each configuration $q \in Q$ with the velocity vector $\xi Q(q) \in TQ$ obtained by letting $q$ move with group velocity $\dot{\xi}$. The value of $L_\xi$ at $q$ is simply the value of the Lagrangian $L$ at $\xi Q(q)$. We shall show that the augmented Lagrangian can be used to determine equilibria of the form $v_e = \xi Q(q_e)$. The generalized inertia mapping expresses the momentum associated to group motions; $\Pi(q, \xi)$ is the group momentum of the vector $\xi Q(q)$. This construction makes possible a simple characterization of the level sets of the momenta which is central to the stability analysis.

The variational structure of Hamiltonian and Lagrangian dynamics is the key to energy stability analyses. Given a relative equilibrium, there exists a conserved quantity, the energy-momentum functional, for which the equilibrium is a critical point. If the energy-momentum functional can be modified so as to produce a conserved quantity for which the relative equilibrium is a local minimum modulo symmetries, then this new conserved quantity can be used, much like a Liapunov function, to demonstrate nonlinear stability of the equilibrium modulo symmetries. The Lagrangian energy-momentum method presented here is a direct extension of the energy-momentum and reduced energy-momentum (REM) methods described in Simo et al. [1989a] and Simo et al. [1989b]. The energy-momentum methods are applicable to simple mechanical systems, i.e. Hamiltonian systems on a cotangent bundle $T^*Q$ with Hamiltonian

$$H(z) = \frac{1}{2} |z|^2_{g^{-1}} + V(q),$$ \hspace{1cm} (1.3)

where $g$ is a group invariant metric on $Q$ and the group invariant potential $V$ depends only on the basepoint $q$ of $z$. These methods specify a quotient manifold $P_\mu = J^{-1}(\mu)/G_\mu$ on which the induced energy-momentum functional may have a local extremum at $[v_e]$. Conservation
of energy and momentum implies that if \([v_e]\) is a local extremum of the energy-momentum functional, then it remains a local extremum under the induced dynamics; hence the energy-momentum functional can be used as a Liapunov function. If certain regularity conditions on the equilibrium momentum are satisfied, then equivariance of the dynamics imply that the dynamics on \(TQ\) are accurately modeled (modulo \(G_\mu\)) by the induced dynamics on \(P_\mu\).

The REM method is an improved version of the energy-momentum method; the two methods provide identical stability conditions, but the REM method is simpler and more efficient to apply. The REM method provides sufficient conditions for (formal) orbital stability of a relative equilibrium with configuration \(q_e\) and group momentum \(\mu\) as follows: By identifying the configuration manifold \(Q\) with a subset of the level set of the group momenta, it is possible to construct the amended potential \(V_\mu\), which is due to Smale [1970a,b]; if certain non-degeneracy assumptions are satisfied, definiteness of the second variation of \(V_\mu\) modulo momentum conserving group motions, i.e. modulo the orbit of the isotropy subgroup \(G_\mu\), implies orbital stability of the relative equilibrium. The energy-momentum method and REM method have been applied to coupled rigid bodies and rods (Simo et al. [1989a]), pseudo-rigid bodies (Lewis and Simo [1989a]), and multiple coupled rigid bodies (Patrick [1990]), among other examples; a trio of simple applications, intended to illustrate the REM method, can be found in Lewis and Simo [1989b]. A related energy method is the energy-Casimir method, in which the quotient \(J^{-1}(\mu)/G_\mu\) is replaced by the quotient \(T^*Q/G\). See Holm et al. [1985] for a detailed discussion of the energy-Casimir method and numerous applications.

We shall show that the block diagonalization results obtained for simple mechanical systems are valid for a much larger class of systems, including all natural Lagrangian systems, as characterized by Gantmacher [1975], §11. It is not necessary for stability that the quotient \(P_\mu\) or the level set \(J^{-1}(\mu)\) be manifolds. In the current treatment, we test the restriction of the second variation, \(D^2\mathcal{L}(v_e)\), of the Lagrangian energy-momentum function to \(\ker DJ(v_e)\) for definiteness modulo variations tangent to the orbit of the isotropy subgroup \(G_\mu\). (If \(P_\mu\) is a manifold, this is equivalent to testing the second variation of the induced functional for definiteness on \(T[v_e]P_\mu\).) We then use a straightforward extension of a result due to Patrick [1990] to show that, for finite dimensional systems satisfying appropriate conditions on the isotropy subgroups \(G_\mu\) and \(G_{v_e}\), definiteness of the restricted second variation implies nonlinear stability modulo \(G_\mu\). In the infinite dimensional case, additional analysis is required to establish nonlinear stability.

The most notable change in the derivation of the block diagonalization constructions is the use of the Lagrangian \(L\) in place of the Hamiltonian \(H = \text{'kinetic' + 'potential'}.\) The key to the extension of the reduced energy-momentum method is the observation that the appropriate generalizations of a \(G\) invariant metric are the Legendre transformation and linearized Legendre transformation. The existence of an energy dependent pairing between variations (in particular, between group variations) is central to the construction of the augmented Lagrangian and stability form; we show that a metric is not necessary for these constructions – it is sufficient that the linearized Legendre transformation satisfy a non-degeneracy condition on the space of group variations. An additional, relatively
minor generalization of the method is the observation that an arbitrary complement to the
tangent to the orbit of the isotropy subgroup may be chosen, rather than the orthogonal
complement specified in the original treatment of the reduced energy momentum method.
The stability conditions obtained are independent of the choice of complement; hence the
most convenient choice for the application at hand may be used.

The motivation for the present treatment is two-fold: first, the desire to encompass as
large a class of practical applications as possible; second, the theoretical consideration of
the assumptions necessary for block diagonalization. In the interest of the latter, we have
attempted to minimize the assumptions made in the general treatment of block diagonaliza-
tion. (For example, we do not assume that relative equilibria are orbit regular, as defined in
§4.a, even though we do not presently know of a physically reasonable example which fails
to be orbit regular.) We note, however, that many of the central constructions and proofs
are actually simplified by a more general treatment. We shall attempt where possible to
indicate the simplest and most common cases; the case of a simple mechanical system with a
free left $SO(3)$ action (i.e., material frame indifference) may be kept in mind as a prototype.

2 Relative equilibria

2.a Definitions

We now briefly describe some of the geometric constructions to be used in the derivation of
the equilibrium and stability conditions. For a more detailed description of Hamiltonian and
Lagrangian mechanics in the presence of symmetry, see, e.g., Abraham and Marsden [1978].
Our configurations are elements of a manifold $Q$; our velocities, or configuration-velocity
pairs, are elements of the tangent bundle $TQ$, while momenta, or configuration-momentum
pairs, are elements of the cotangent bundle $T^*Q$. We assume that a Lie group $G$ acts on $Q$
on the left; this action is denoted by

$$g \cdot q = \Psi_g(q)$$

(2.1)
for $g \in G$ and $q \in Q$. The induced action on $TQ$ is given by

$$g \cdot v := T_q \Psi_g \cdot v$$

(2.2)
for $v \in T_q Q$. Let $\pi : TQ \to Q$ denote the canonical fiber bundle projection; in coordinates,
$\pi(q, v) = q$. We require a notation for variation within a fixed tangent space fiber, i.e.
a variation of the velocity $v$ which leaves the configuration $q$ fixed. Given $v \in T_q Q$, let
vert$_v : T_q Q \to T_v TQ$ denote the vertical lift

$$\text{vert}_v(w) := \frac{d}{d\epsilon}\big|_{\epsilon=0} v + \epsilon w \quad \forall \; w \in T_q Q;$$

(2.3)
in coordinates: $\text{vert}_v(w) = (0, w) \in T_{(q, v)} TQ$. 

Let $L : TQ \to \mathbb{R}$ be a Lagrangian which is invariant under the action of $G$ on $Q$, i.e., which satisfies $L(g \cdot v) = L(v)$ for all $g \in G$ and $v \in TQ$. Let $\text{FL} : TQ \to T^*Q$ denote the Legendre transformation (fiber derivative) of $L$:

$$\text{FL}(v) \cdot w := DL(v) \cdot \text{vert}_v(w) \quad \forall \ w \in T_{\pi(v)}Q.$$  \hfill (2.4)

Let $E : TQ \to \mathbb{R}$ denote the energy

$$E(v) := \text{FL}(v) \cdot v - L(v)$$  \hfill (2.5)

associated to the Lagrangian $L$. The following equivariance results are well-known; we include the proofs both for the sake of completeness and to illustrate in a simple setting the techniques that will be used throughout the paper.

**Proposition 2.1** The Legendre transformation $\text{FL}$ is $G$ equivariant and the energy $E$ is $G$ invariant, i.e.

$$\text{FL}(T_q \Psi_g \cdot v) = T^*_q \Psi_{g^{-1}} \cdot \text{FL}(v) \quad \text{and} \quad E(g \cdot v) = E(v)$$  \hfill (2.6)

for all $g \in G$ and $v \in TQ$.

**Proof:** Given $v$ and $w \in T_{qQ}$, let $v_\epsilon$ be a curve in $T_qQ$ such that $w = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} v_\epsilon$. $G$ invariance of $L$ implies that

$$\left( T_q \Psi_g^* \cdot DL(T_q \Psi_g \cdot v) \right) \cdot w = DL(T_q \Psi_g \cdot v) \cdot (T_q \Psi_g \cdot w)$$  \hfill (2.7)

$$= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(T_q \Psi_g \cdot v_\epsilon)$$

$$= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(v_\epsilon)$$

$$= DL(v) \cdot w;$$

hence

$$\text{FL}(T_q \Psi_g \cdot v) = T^*_q \Psi_{g^{-1}} \cdot \text{FL}(v).$$  \hfill (2.9)

It follows from the equivariance of $\text{FL}$ that

$$\text{FL}(g \cdot v) \cdot (g \cdot v) = \left( T^*_q \Psi_{g^{-1}} \cdot \text{FL}(v) \right) \cdot (T_q \Psi_g \cdot (v))$$  \hfill (2.10)

$$= \text{FL}(v) \cdot v;$$

hence, since $L$ is $G$ invariant, $E$ is $G$ invariant. \hfill \!

We now give a brief summary of the relationship between the dynamics and the Lagrangian $L$. (For a more detailed description, see Abraham and Marsden [1978].) The Lagrange two-form $\omega_L$ is given by

$$\omega_L := \text{FL}^* \omega,$$  \hfill (2.11)

where $\omega$ denotes the canonical symplectic form on the cotangent bundle $T^*Q$. A Lagrangian vector field is a vector field $X_E$ on $TQ$ satisfying

$$\iota_{X_E} \omega_L = DE,$$  \hfill (2.12)

5
i.e.
\[ \omega_L(X_E(v), \delta v) = DE(v) \cdot \delta v \]  \hspace{1cm} (2.13)
for all \( v \in TQ \) and \( \delta v \in T_vTQ \). If \( L \) is regular, then \( X_E \) exists and is second order, i.e. \( X_E \) satisfies \( T\pi \circ X_E = \text{Id} \). We shall assume throughout that a second order Lagrangian vector field \( X_E \) exists and that the dynamics of the system are determined by the relation
\[ \dot{v} = X_E(v). \]  \hspace{1cm} (2.14)
If \( Q \) is finite dimensional and \( X_E \) is second order, then (2.14) is equivalent to the Euler-Lagrange equation
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}. \]  \hspace{1cm} (2.15)

Given a configuration \( q_e \) undergoing a steady group motion with total velocity \( \xi \), i.e. satisfying (1.1), we say that the associated configuration-velocity pair \( v_e \in TQ \) is a relative equilibrium. In this case, the evolution of \( v_e \) under the Lagrangian dynamics \( \dot{v} = X_E(v) \) is
\[ v_t = \exp(t \xi) \cdot v_e. \]  \hspace{1cm} (2.16)
Hence \( v_e \), as well as \( q_e \), evolves by a steady group motion. We define the infinitesimal generator \( \xi_Q : Q \rightarrow TQ \) associated to an element \( \xi \) of the Lie algebra \( \mathcal{G} \) by
\[ \xi_Q(q) := \left. \frac{d}{dt} \right|_{t=0} \exp(t \xi) \cdot q. \]  \hspace{1cm} (2.17)
The infinitesimal generator \( \xi_{TQ} : TQ \rightarrow T(TQ) \) is defined analogously. Thus, if \( q_t \) satisfies (1.1), then \( v_e := \xi_Q(q_e) \) and
\[ \dot{v}_t = X_E(v_t) = \xi_{TQ}(v_t). \]  \hspace{1cm} (2.18)

2.b Relative equilibria as critical points

It is clear from (2.12) that equilibria of the system determined by the Lagrangian vector field \( X_E \) must be critical points of the energy \( E \). (If \( L \) is regular, then critical points and equilibria are in exact correspondence.) Relative equilibria are also critical points of an appropriate functional. This functional is obtained from the energy by adding a component of the total momentum to the energy \( E \).

Define the momentum map \( J : TQ \rightarrow \mathcal{G}^* \) by
\[ J(v) \cdot \eta := FL(v) \cdot \eta_Q(\pi(v)). \]  \hspace{1cm} (2.19)
Given \( \xi \in \mathcal{G} \), define the momentum map \( J_\xi : TQ \rightarrow \mathbb{R} \) in the direction of \( \xi \) by
\[ J_\xi(v) := J(v) \cdot \xi. \]  \hspace{1cm} (2.20)
The momentum map satisfies
\[ \iota_{\xi_Q} \omega_L = DJ_\xi \]  \hspace{1cm} (2.21)
for all \( \xi \in \mathcal{G} \). Noether's theorem states that \( J \) is an integral of the Lagrangian dynamics. (See, e.g., Abraham and Marsden [1978].)
The relative equilibrium theorem for Hamiltonian systems states that \( z_e \in T^*Q \) is a relative equilibrium if and only if there exists \( \xi \in \mathcal{G} \) such that \( z_e \) is a critical point of the Hamiltonian energy-momentum function \( H_\xi : T^*Q \to \mathbb{R} \) given by

\[
H_\xi(z) := H(z) - (J(z) - \mu_e) \cdot \xi,
\]

where \( J \) denotes, for the moment, the momentum map on the cotangent bundle \( T^*Q \) and \( \mu_e := J(z_e) \). An analogous result holds for Lagrangian systems as well. Define the Lagrangian energy-momentum function \( E_\xi : TQ \to \mathbb{R} \) by

\[
E_\xi(v) := E(v) - J_\xi(v).
\]

Proposition 2.2 If \( v_e \in TQ \) is a relative equilibrium, i.e. if \( v_e \) satisfies (2.18) for some element \( \xi \in \mathcal{G} \), then \( v_e \) is a critical point of the energy-momentum functional \( E_\xi \). If the Lagrangian \( L \) is regular, then \( v_e \) is a relative equilibrium if and only if \( v_e \) is a critical point of \( E_\xi \) for some \( \xi \).

Proof: The first claim of the theorem follows from (2.21) and (2.12), since

\[
\iota_{X_E} \omega_L = DE \quad \text{and} \quad \iota_{\xi_{TQ}} \omega_L = DJ_\xi
\]

imply

\[
0 = \iota_{(X_E(v_e) - \xi_{TQ}(v_e))} \omega_L(v_e) = \iota_{X_E} \omega_L(v_e) - \iota_{\xi_{TQ}} \omega_L(v_e) = DE(v_e) - DJ_\xi(v_e) = DE_\xi(v_e).
\]

If \( L \) is regular, then (2.18) holds if and only if \( DE_\xi(v_e) = 0 \), since nondegeneracy of \( \omega_L \) implies that

\[
\iota_{(X_E(v_e) - \xi_{TQ}(v_e))} \omega_L(v_e) = 0 \quad \iff \quad X_E(v_e) = \xi_{TQ}(v_e).
\]

The relative equilibrium theorem can be phrased in the following fashion: Given a velocity \( v_e \), does there exist an algebra element \( \xi \) such that \( v_e \) is a critical point of \( E_\xi \)? In many contexts, a more natural question is the following: Given a specific algebra element \( \xi \in \mathcal{G} \), for what \( q_e \in Q \) is \( v_e := \xi_Q(q_e) \) a relative equilibrium? This question can be partially answered by determining the set of critical points of \( E_\xi \) of the form \( v_e = \xi_Q(q_e) \); the set of relative equilibria with total velocity \( \xi \) is contained within this set. (If \( L \) is regular, then these two sets coincide.) In fact, relative equilibria with total velocity \( \xi \) can be identified with critical points of a simpler function, which depends only on the configuration \( q \). Given \( \xi \in \mathcal{G} \), define the augmented Lagrangian \( L_\xi : Q \to \mathbb{R} \) by

\[
L_\xi(q) := L(\xi_Q(q)).
\]
We shall show below that \( v_e := \xi_Q(q_e) \) is a critical point of \( E_\xi \) if and only if \( q_e \) is a critical point of \( L_\xi \). (If \( q_e \) has non-trivial symmetry, the algebra element \( \xi \) need not be uniquely determined.)

We now discuss some of the decompositions of variations which are central to the block diagonalization results. A crucial construction is the decomposition of a general variation in \( T(TQ) \) into the image of a variation in \( TQ \) under a vector field and the remaining ‘vertical’ component.

**Proposition 2.3**

(i) Given a vector field \( v : Q \to TQ \) and a variation \( \delta v \in T_{v(q)} TQ \), there exists \( \Delta v \in T_{v(q)} Q \) such that

\[
DE_\xi(v(q)) \cdot \delta v = D(E_\xi \circ v)(q) \cdot (T_{v(q)} \pi \cdot \delta v) + D^2_v L(v(q)) \cdot (\Delta v, v(q) - \xi_Q(q)),
\]

where \( D_v \) denotes differentiation within the fiber \( T_{v(q)} Q \). In particular,

\[
DE_\xi(v(q)) \cdot \operatorname{vert}_{v(q)}(v) = D^2_v L(v(q))(v, v(q) - \xi_Q(q))
\]

for all \( v \in T_{v(q)} Q \).

(ii) If \( v_e \in T_{v_e} Q \) is a relative equilibrium, then there exists \( \xi \in \mathcal{G} \) such that \( v_e = \xi_Q(q_e) \) and \( q_e \) is a critical point of \( L_\xi \).

(iii) If \( L \) is regular at \( v_e \), then \( v_e \) is a relative equilibrium if and only if there exists \( \xi \in \mathcal{G} \) such that \( v_e = \xi_Q(q_e) \) and \( q_e \) is a critical point of \( L_\xi \).

**Proof:** Let \( v := v(q) \). Consider a variation \( \delta \varepsilon \in T_{v(q)} TQ \) and define \( \delta q := T_{v(q)} \pi \cdot \delta v \);

\[
(T_{v(q)} \pi \cdot (\delta \varepsilon - T_{v(q)} \varepsilon \cdot \delta q)) = 0
\]

implies there exists \( \Delta v \in T_{v(q)} Q \) such that

\[
\delta v = T_{v(q)} \varepsilon \cdot \delta q + \operatorname{vert}_v(\Delta v).
\]

Thus

\[
DE_\xi(v) \cdot \delta v = DE_\xi(v) \cdot (T_{v(q)} \varepsilon \cdot \delta q) + DE_\xi(v) \cdot \operatorname{vert}_v(\Delta v)
\]

\[
= D(E_\xi \circ v)(q) \cdot \delta q + (D_v L_v(v) \cdot \Delta v) \cdot [v(q) - \xi_Q(q)].
\]

This proves the first claim.

Proposition 2.2 states that if \( v_e \) is a relative equilibrium, then there exists \( \xi \in \mathcal{G} \) such that \( v_e \) is a critical point of \( E_\xi \) (if \( L \) is regular at \( v_e \), then \( v_e \) is a relative equilibrium if and only if such a \( \xi \) exists). It follows from (2.32) that \( DE_\xi(v_e) = 0 \) implies \( v_e = \xi_Q(q_e) \), where \( q_e := \pi(v_e) \). Hence, to establish the second and third claims, we need to show that \( v_e \) is a critical point of \( E_\xi \) if and only if \( q_e \) is a critical point of \( L_\xi \). We note that

\[
E_\xi(v) = FL(v) \cdot v - L(v) - J_\xi(v)
\]

\[
= FL(v) \cdot (v - \xi_Q(q)) - L(v)
\]
implies

\[ E_\xi(\xi_Q(q)) = -L_\xi(q). \]  \hspace{1cm} (2.34)

Thus, applying the previous result,

\[ DE_\xi(\xi_Q(q)) \cdot \delta v = D(E_\xi \circ \xi_Q)(q) \cdot \delta q = -DL_\xi(q) \cdot \delta q, \]  \hspace{1cm} (2.35)

so \( DE_\xi(v_e) = 0 \) if and only if \( DL_\xi(q_e) = 0 \). √

3 Formal Stability of Relative Equilibria

3.1 The energy-momentum method and the configuration-velocity split

Throughout this section, we shall assume that \( v_e \) is a relative equilibrium, i.e. a critical point of \( E_\xi \). The argument presented in the preceding section shows that the augmented Lagrangian can be used to identify relative equilibria with a specified velocity; however, the augmented Lagrangian does not provide reliable stability information. In the construction of the augmented Lagrangian, a configuration \( q \) is identified with the velocity \( \xi_Q(q) \), which need not lie on the momentum level set \( J^{-1}(\mu) \). This failure to satisfy the momentum constraint can lead to artificially restrictive stability conditions. In the case of a simple mechanical system, the augmented Lagrangian equals the augmented potential \( V_\xi \); the second variation of \( V_\xi \) is known to yield overly conservative stability conditions. (See Simo et al. [1989b].) As in the simple mechanical case, we must appropriately incorporate the momentum constraints to obtain realistic stability conditions. In the asymmetric simple mechanical case, it was shown that associated to each configuration \( q \) and total momentum \( \mu \in \mathcal{G}_* \) there is a unique algebra element \( \eta \in \mathcal{G} \) such that the configuration \( q \) moving with total velocity \( \eta \) has total momentum \( \mu \). This fact was used to identify the configuration manifold \( Q \) with a subset of \( J^{-1}(\mu) \) consisting solely of rigid (group) motions. In the current work, we wish to relax the preceding assumptions, while preserving the essential features of the construction.

The central construction to be used in identifying \( Q \) (or a subset of \( Q \)) with a subset of \( J^{-1}(\mu) \) is the generalized inertia mapping. This mapping assigns to a pair consisting of a configuration \( q \) and an element of the Lie algebra \( \mathcal{G} \) the total momentum of the configuration \( q \) moving with total velocity \( \xi \). We shall show that for a very general class of Lagrangian systems, the inertia mapping is invertible, or nearly invertible, with respect to the algebra element. This allows us to associate to each configuration a ‘rigid motion’ with total momentum (nearly) equal to \( \mu \).

We now introduce some of the terminology associated to the action of the symmetry group \( G \) on the manifold \( Q \). The group orbit \( G \cdot q_e = \{ g \cdot q_e : g \in G \} \) has tangent space

\[ \mathcal{G} \cdot q_e := \{ \eta_q(q_e) : \eta \in \mathcal{G} \}. \]  \hspace{1cm} (3.1)

Let \( G_{q_e} \) denote the isotropy subgroup (symmetry group) of \( q_e \):

\[ G_{q_e} := \{ g \in G : g \cdot q_e = q_e \}. \]  \hspace{1cm} (3.2)
The algebra $\mathcal{G}_{q_e}$ of $G_{q_e}$ satisfies
\begin{equation}
\mathcal{G}_{q_e} = \{ \eta \in \mathcal{G} : \eta_Q(q_e) = 0 \};
\end{equation}
hence $\mathcal{G}_{q_e}$ is the kernel of the mapping $\eta \mapsto \eta_Q(q_e)$ taking algebra elements to infinitesimal group motions of $q_e$.

Define the generalized inertia mapping $\Pi : Q \times \mathcal{G} \rightarrow \mathcal{G}^*$ by
\begin{equation}
\Pi(q, \eta) := J(\eta_Q(q)),
\end{equation}
i.e.
\begin{equation}
\Pi(q, \eta) \cdot \zeta = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L((\eta + \epsilon \zeta)_Q(q)).
\end{equation}
The Lagrangian block diagonalization procedure can be carried out if there exists a neighborhood $U$ of $q_e$ and a map $\Xi : U \rightarrow \mathcal{G}$ such that $\Xi(q_e) = \xi$ and the map $\mu : Q \rightarrow \mathcal{G}^*$ defined by
\begin{equation}
\mu(q) := \Pi(q, \Xi(q))
\end{equation}
satisfies
\begin{equation}
(D\mu(q_e) \cdot \delta q) \cdot \eta \neq 0 \implies \eta \in \mathcal{G}_{q_e}
\end{equation}
for all $\delta q \in T_{q_e}Q$. It is often possible to construct a map $\Xi$ which exactly satisfies the condition
\begin{equation}
\Pi(q, \Xi(q)) = \mu
\end{equation}
for some specified value of $\mu$. (We shall always consider the equilibrium momentum $\mu := J(v_e)$.) In this case, the map $\Xi$ can be used to identify $Q$ with a subset of $J^{-1}(\mu)$, as discussed below. This identification is, conceptually, the goal of the following constructions. Condition (3.8) can be satisfied, for example, for simple mechanical systems on which the symmetry group $G$ acts freely. However, (3.8) is substantially more restrictive than (3.7) and is, in fact, more restrictive than is necessary. We shall show below that condition (3.7) is sufficient to enable us to identify a subspace of $T_{q_e}Q$ with a subspace of $T_{v_e}J^{-1}(\mu)$. This more general identification is sufficient for application of the energy-momentum method. We describe in Section 4.a a fairly general condition under which a map $\Xi$ satisfying (3.7) exists. Given such maps $\Xi$ and $\mu$, the analysis proceeds as follows.

The space ker $DJ(v_e)$ can be decomposed into two subspaces, one modeled on configuration variations, the other modeled on velocity variations. We first construct the model spaces $Q$ and $(G \cdot q_e)^{\perp}$; both are subspaces of $T_{q_e}Q$, but elements of $Q$ are to be interpreted as configuration variations, while elements of $(G \cdot q_e)^{\perp}$ are viewed as velocities. Define the locked momentum map $\Pi_{\xi} : Q \rightarrow \mathcal{G}^*$ by
\begin{equation}
\Pi_{\xi}(q) := \Pi(q, \xi)
\end{equation}
and define
\begin{equation}
Q := \left\{ \delta q \in T_{q_e}Q : D\Pi_{\xi}(q_e) \cdot \delta q \in \mathcal{G}_{q_e}^A \right\},
\end{equation}

10
where $G_{q_e}^4 \subset G^*$ denotes the annihilator of the isotropy algebra of $q_e$. Let $\langle \langle , \rangle \rangle_e : T_{q_e}Q \times T_{q_e}Q \to \mathbb{R}$ denote the bilinear form induced by the linearized fiber derivative:

$$\langle \langle v, w \rangle \rangle_e := \left. \frac{d}{d\tau} \right|_{\tau=0} \frac{d}{d\epsilon} \bigg|_{\epsilon=0} L(\xi Q(q_e) + \tau v + \epsilon w)$$

for all $v, w \in T_{q_e}TQ$ and let $(G \cdot q_e)^\perp$ denote the $\langle \langle , \rangle \rangle_e$ orthogonal complement to $G \cdot q_e$:

$$(G \cdot q_e)^\perp := \{ v \in T_{q_e}Q : \langle \langle v, \eta Q(q_e) \rangle \rangle_e = 0 \ \forall \ \eta \in G \}.$$  

(3.12)

The variations associated to $Q$ capture the configuration-velocity coupling induced by the momentum constraint; the elements of $(G \cdot q_e)^\perp$ are associated to the remaining ‘uncoupled’ velocity variations. The subspace $Q$ is embedded in $\ker DJ(v_e)$ by means of the the locked total momentum vector field $v_\mu : U \to TQ$, given by

$$v_\mu(q) := \left[ \Xi(q) \right]_Q(q).$$

(3.13)

The designation ‘locked momentum’ is motivated by the fact that

$$J(v_\mu(q)) = \mu(q)$$

(3.14)

for all $q \in U$, which follows from (3.4) and (3.6). The space $(G \cdot q_e)^\perp$ is simply vertically lifted to $\ker DJ(v_e)$.

**Proposition 3.1** Define

$$\mathcal{K}_q := \{ T_{q_e}v_\mu \cdot \delta q : \delta q \in \mathcal{Q} \}$$

(3.15)

and

$$\mathcal{K}_v := \left\{ \text{vert}_{v_e}(v) : v \in (G \cdot q_e)^\perp \right\}. $$

(3.16)

Then

$$\ker DJ(v_e) = \mathcal{K}_q \oplus \mathcal{K}_v.$$  

(3.17)

**Proof:** Given an element $\delta v \in \ker DJ(v_e)$, define $\delta q := T_{q_e} \pi \cdot \delta v$,

$$\delta v_q := T_{q_e}v_\mu \cdot \delta q \quad \text{and} \quad \delta v_v := \delta v - \delta v_q.$$  

(3.18)

By construction, $\delta v_v = \text{vert}_{v_e}(v)$ for some $v \in T_{q_e}Q$. In addition, $\delta v \in \ker DJ(v_e)$ and (3.14) imply

$$\langle \langle v, \eta Q(q_e) \rangle \rangle_e = (DJ(v_e) \cdot \delta v_v) \cdot \eta$$

$$= (DJ(v_e) \cdot \delta v - DJ(v_e) \cdot \delta v_q) \cdot \eta$$

$$= -(D\mu(q_e) \cdot \delta q) \cdot \eta.$$  

(3.19)

By assumption, $\eta \notin G_{q_e}$ implies $(D\mu(q_e) \cdot \delta q) \cdot \eta = 0$ and hence $\langle \langle v, \eta Q(q_e) \rangle \rangle_e = 0$. On the other hand, $\eta \in G_{q_e}$ if and only if $\eta Q(q_e) = 0$; hence $\eta \in G_{q_e}$ implies

$$0 = \langle \langle v, \eta Q(q_e) \rangle \rangle_e = -(D\mu(q_e) \cdot \delta q) \cdot \eta.$$  

(3.20)

Thus $\delta v \in \ker DJ(v_e)$ implies $\delta v_v \in \mathcal{K}_v$ and $D\mu(q_e) \cdot \delta q = 0$.  

11
Define the linearized inertia tensor \( \mathbb{II}_e : \mathcal{G} \rightarrow \mathcal{G}^* \) by

\[
\mathbb{II}_e \eta := D_\xi \mathbb{II}(q_e, \xi) \cdot \eta.
\]  
(3.21)

We can use (3.5) to derive the following expressions for \( \mathbb{II}_e \): Given \( \eta \) and \( \zeta \in \mathcal{G} \),

\[
\zeta \cdot \mathbb{II}_e \eta = \left. \frac{d}{d\tau} \right|_{\tau=0} \left. \frac{d}{dq_e} \right|_{q_e=0} \left[ \mathbb{II}(\xi + \tau \eta + \epsilon \zeta)(q_e) \right] = \langle \eta Q(q_e), \zeta Q(q_e) \rangle_e.
\]  
(3.22)

Next, note that application of the chain rule to \( \mu(q) = \mathbb{II}(q, \Xi(q)) \) yields

\[
D\mu(q_e) \cdot \delta q = D\mathbb{II}(q_e, \xi) \cdot \delta q + D\zeta \mathbb{II}(q_e, \xi) \cdot (D\Xi(q_e) \cdot \delta q) = D\mathbb{II}_e(q_e) \cdot \delta q + \mathbb{II}_e(D\Xi(q_e) \cdot \delta q).
\]  
(3.23)

Hence \( D\mu(q_e) \cdot \delta q = 0 \) implies

\[
D\mathbb{II}_e(q_e) \cdot \delta q = -\mathbb{II}_e(D\Xi(q_e) \cdot \delta q).
\]  
(3.24)

Since (3.22) implies \( \mathbb{II}_e|_{q_e=0} = 0 \),

\[
(D\mu(q_e) \cdot \delta q)|_{q_e=0} = (D\mathbb{II}_e(q_e) \cdot \delta q)|_{q_e=0}.
\]  
(3.25)

Thus \( \delta v \in \ker DJ(v_e) \) implies \( \delta q = T_{v_e} \pi \cdot \delta v \in \mathcal{Q} \) and, hence, \( \delta v_e \in K_q \). It follows that \( \ker DJ(v_e) = K_q \oplus K_v \). √

**Lemma 3.1**

\[
(D\mathbb{II}_e(q_e) \cdot \delta q) \cdot \zeta = D^2 L(v_e)(\text{vert}_{v_e}(\zeta Q(q_e)), T_{q_e} \zeta Q \cdot \delta q) + D L(v_e) \cdot (T_{q_e} \zeta Q \cdot \delta q).
\]  
(3.26)

**Proof:** Let \( q_e \in U \) be a curve tangent to \( \delta q \) at \( q_e \). Then

\[
(D\mathbb{II}_e(q_e) \cdot \delta q) \cdot \zeta = \left. \frac{d}{d\tau} \right|_{\tau=0} \left. \frac{d}{dq_e} \right|_{q_e=0} L((\xi + \tau \zeta)(q_e)) = \left. \frac{d^2}{d\tau^2} \right|_{\tau=0} D L((\xi + \tau \zeta)(q_e)) \cdot (T_{q_e} (\xi + \tau \zeta)(q_e) \cdot \delta q) = D^2 L(v_e) \left( \text{vert}_{\zeta Q(q_e)}(\zeta Q(q_e)), T_{q_e} \zeta Q \cdot \delta q \right) + D L(v_e) \cdot (T_{q_e} \zeta Q \cdot \delta q).
\]  
(3.27)

√

Lemma 3.1 implies that

\[
\mathcal{Q} = \{ \delta q \in T_{q_e} \mathcal{Q} : D L(v_e) \cdot (T_{q_e} \zeta Q \cdot \delta q) = 0 \quad \forall \ z \in \mathcal{G}_{q_e} \}.
\]  
(3.28)

In particular, \( \mathcal{Q} \) contains all variations which preserve the symmetry of the equilibrium configuration.
3.b The amended energy

We can identify the space $\mathcal{K}_q$ of configuration-velocity variations with its model space $Q$ of configuration variations by pulling back the function $E_\xi$ from $TQ$ to $Q$. Consideration of variations of the function $E_\xi$ with respect to vectors of the form $\delta v = T_q e \cdot \delta q$ naturally suggests the function $E_\xi \circ v_\mu$. We define the amended energy $E_\mu : Q \to \mathbb{R}$ by

$$E_\mu(q) := E_\xi(v_\mu(q)) = \text{FL}(v_\mu(q)) \cdot v_\mu(q) - L(v_\mu(q)) - J(v_\mu(q)) \cdot \xi = \mu(q) \cdot (\Xi(q) - \xi) - L(v_\mu(q)).$$

Proposition 3.2

(i)

$$DE_\mu(q) \cdot \delta q = [D\mu(q) \cdot \delta q] \cdot (\dot{\xi} - \xi) - D(L \circ \dot{\xi}_Q)(q) \cdot \delta q,$$

where $\dot{\xi} := \Xi(q)$, for all $q \in U$ and $\delta q \in T_q Q$. In particular,

$$DE_\mu(q_e) = 0.$$  \hspace{1cm} (3.30)

(ii)

$$D^2 E_\mu(q_e)(\delta q, \Delta q) = -D^2 L_\xi(q_e)(\delta q, \Delta q) - (D\Xi(q_e) \cdot \Delta q)(D\Xi(q_e) \cdot \delta q).$$  \hspace{1cm} (3.31)

Proof:

$$DE_\mu(q) \cdot \delta q = (D\mu(q) \cdot \delta q) \cdot (\dot{\xi} - \xi) + \mu(q) \cdot (D\Xi(q) \cdot \delta q)$$

$$- DL(v_\mu(q)) \cdot \left[ \text{vert}_{\dot{\xi}_Q(q)} \left( \left[ D\Xi(q) \cdot \delta q \right]_{Q} (q) \right) \right]$$

$$= (D\mu(q) \cdot \delta q) \cdot (\dot{\xi} - \xi) + \mu(q) \cdot (D\Xi(q) \cdot \delta q)$$

$$- DL(v_\mu(q)) \cdot [D\Xi(q) \cdot \delta q]_{Q} (q) - DL(\dot{\xi}_Q(q)) \cdot \left( T_q \dot{\xi}_Q(q) \cdot \delta q \right)$$

$$= (D\mu(q) \cdot \delta q) \cdot (\dot{\xi} - \xi) - D(L \circ \dot{\xi}_Q)(q) \cdot \delta q,$$

since $v_\mu(q) = \dot{\xi}_Q(q)$ and

$$\text{FL}(v_\mu(q)) \cdot \eta_Q(q) = J \left( \left[ \Xi(q) \right]_{Q} (q) \right) \cdot \eta$$

$$= \Pi(q, \Xi(q)) \cdot \eta$$

$$= \mu(q) \cdot \eta$$  \hspace{1cm} (3.33)

for all $\eta \in \mathcal{G}$.

To prove the second claim, let $\tilde{\eta} : U \to TQ$ denote an arbitrary vector field extension of the variation $\delta q$, let $q_e$ be a curve tangent to $\Delta q$, and define $\xi^e := \Xi(q_e)$; using (3.24), (3.27), and (3.29), we derive the relationship

$$D^2 E_\mu(q_e)(\delta q, \Delta q)$$

$$= \left. \frac{d}{de} \right|_{e=0} \left[ DE_\mu(q_e) \cdot \tilde{\eta}(q_e) \right]$$

13
\[ \begin{align*}
&= \frac{d}{dx}\bigg|_{x=0} \left[ (D\mu(q_e) \cdot \tilde{\delta}q(q_e)) \cdot (\xi^e - \xi) - DL(\xi^e Q(q_e)) \cdot (T_{q_e} \xi^e Q \cdot \tilde{\delta}q(q_e)) \right] \\
&= -D^2 L(\xi Q(q_e))(T_{q_e} \xi Q \cdot \delta q, T_{q_e} \xi Q \cdot \Delta q) - D^2 L(\xi Q(q_e)) \left( \text{vert}_{\xi Q(q_e)} \left( [D\Xi(q_e) \cdot \Delta q]_Q (q_e) \right) , T_{q_e} \xi Q \cdot \delta q \right) \\
&\quad - DL(\xi Q(q_e)) \left( T_{q_e} [D\Xi(q_e) \cdot \Delta q]_Q \cdot \delta q \right) \\
&= -D^2 L(\xi Q(q_e))(\delta q, \Delta q) - (D^2 L(\xi Q(q_e))(\delta q) \cdot \Delta q) \cdot \Delta q).
\end{align*} \]

3.3 Block diagonalization and formal stability conditions

**Proposition 3.3**

(i) The second variation $D^2 E_\xi(v_e)|_{\ker D^2 v_e}$ block diagonalizes with respect to $K_q$ and $K_v$, i.e.

\[ D^2 E_\xi(v_e)(\delta v_q, \delta v_v) = 0 \quad \forall (\delta v_q, \delta v_v) \in K_q \times K_v. \]  \hspace{1cm} (3.35)

(ii) The second variation of $E_\xi$ restricted to $K_q$ takes the form

\[ D^2 E_\xi(v_e)(T_{q_e} \nu_m \cdot \delta q, T_{q_e} \nu_m \cdot \Delta q) = B_e(\delta q, \Delta q), \]  \hspace{1cm} (3.36)

where the stability two form $B_e : Q \times Q$ is given by

\[ B_e(\delta q, \Delta q) := (D\Xi(q_e) \cdot \Delta q) \cdot I_e (D\Xi(q_e) \cdot \delta q) - D^2 L_\xi(q_e)(\delta q, \Delta q). \]  \hspace{1cm} (3.37)

(iii) The second variation of $E_\xi$ restricted to $K_v$ takes the form

\[ D^2 E_\xi(v_e)(\text{vert}_{v_e}(v_1), \text{vert}_{v_e}(v_2)) = \langle v_1, v_2 \rangle_e. \]  \hspace{1cm} (3.38)

**Proof:** Given $\delta v_q \in K_q$ and $\delta v_v \in K_v$, let $\delta q \in Q$ and $v \in (G \cdot q_e)^\perp$ denote the associated configuration variation and velocity. Let $v : U \rightarrow T^*Q$ denote a vector field such that $v(q_e) = v$ and let $q_e$ be a curve tangent to $\delta q$. Then, using (2.29) and the fact that $v_e$ is a critical point of $E_\xi$, we see that

\[ D^2 E_\xi(v_e)(\delta v_q, \delta v_v) = \frac{d}{dx}\bigg|_{x=0} \left[ D(E_\xi(v_m(q_e)) \cdot \text{vert}_{v_m(q_e)}(v(q_e))) \right] \\
= \frac{d}{dx}\bigg|_{x=0} \left[ D\nu \cdot FL(v_m(q_e)) \cdot v(q_e) \right] \cdot \delta q(q_e) - \xi Q(q_e) \right) \\
= \langle \langle v, [D\Xi(q_e) \cdot \delta q]_Q (q_e) \rangle, v \rangle_e \\
= 0. \]  \hspace{1cm} (3.39)

Proposition 3.2 and (3.24) imply

\[ \begin{align*}
D^2 E_\xi(v_e)(T_{q_e} \nu_m \cdot \delta q, T_{q_e} \nu_m \cdot \Delta q) \\
&= D^2 E_\mu(q_e)(\delta q, \Delta q) \\
&= -D^2 L_\xi(q_e)(\delta q, \Delta q) - (D\Pi_\xi(q_e) \cdot \Delta q)(D\Xi(q_e) \cdot \delta q) \\
&= -D^2 L_\xi(q_e)(\delta q, \Delta q) + (D\Xi(q_e) \cdot \delta q) \cdot I_e (D\Xi(q_e) \cdot \Delta q) \\
&= B_e(\delta q, \Delta q).
\end{align*} \]  \hspace{1cm} (3.40)
for $\delta q, \Delta q \in Q$, which proves the second claim.

The proof of the third claim is straightforward:

$$D^2E_\xi(v_e)(\text{vert}_{v_e}(v), \text{vert}_{v_e}(\hat{v})) = D^2L(v_e)(v, \hat{v}) = \langle v, \hat{v} \rangle_{c^*}.$$ (3.41)

\[ \checkmark \]

The $G_\mu$ invariance of $E_\xi$ implies that

$$D^2E_\xi(v_e)(\zeta_Q(q_e), \delta v) = 0$$ (3.42)

for all $\zeta \in G_\mu$ and $\delta v \in T_{v_e}(TQ)$. (See Proposition 3.5 for a direct proof of this fact.) Hence, $D^2E_\xi(v_e)$ can never be definite; instead, we consider definiteness modulo the tangent to the $G_\mu$ orbit. This can be accomplished either by constructing an explicit complement $S$ to the tangent space $G_\mu \cdot v_e \subset \ker DJ(v_e)$ or by replacing the usual condition that the second variation be definite with the condition that the second variation be semi-definite, with kernel $G_\mu \cdot v_e$. We briefly describe both approaches.

Let $\tilde{V}$ be a complement to the linearized isotropy orbit $G_\mu \cdot q_e$ in $T_{q_e}Q$ and define

$$V := \tilde{V} \cap Q$$ (3.43)

and

$$S := \{ \delta v \in \ker DJ(v_e) : T_{q_e} \pi \cdot \delta v \in V \}.$$ (3.44)

Clearly, $K_v \subset S$, since $T_{q_e} \pi \cdot \text{vert}_{v_e}(v) = 0$ for all $v \in T_{q_e}Q$. The complement condition does, however, restrict the configuration variations:

$$T_{q_e}v_\mu \cdot \delta q \in S \iff \delta q \in V.$$ (3.45)

Thus, the stability conditions may be summarized as follows: The relative equilibrium $v_e$ is formally stable, in the sense that the second variation $D^2E_\xi(v_e)|_S$ is positive (negative) definite, if

(i) $\langle \ , \ \rangle |_{(G_\mu \cdot q_e)^\perp}$ is positive (negative) definite

(ii) $B_e|_V$ is positive (negative) definite.

The stability conditions are independent of the choice of complement. Hence the second condition may be replaced by the following:

(ii)' $B_e$ is positive (negative) semi-definite, with kernel $G_\mu \cdot q_e$.

3.4 Decomposition of configuration variations

There is a further decomposition available which can be used to simplify the determination of the definiteness of the stability form $B_e$. The configuration variations in the space $V$ can be decomposed into ‘rigid variations’ and ‘internal variations’. This further decomposition not only reduces the size of the blocks to be checked for definiteness; it also exploits the group invariance of the Lagrangian to obtain a relatively simple expression for the form $B_e$ on the subspace of group variations.
3.6.1 Group variations

While the original Lagrangian $L$ is $G$ invariant, and hence variations of $L$ with respect to group motions are trivial, the modified functions $L_\xi$, $E_\xi$, and $E_\mu$ are not $G$ invariant, since the introduction of the total velocity $\xi$ and total momentum $\mu$ results in a 'preferred direction' which breaks the symmetry of the original system. However, the variations of the augmented Lagrangian $L_\xi$ and the inertia tensor $\mathbb{I}$ with respect to variations along the group orbit $G \cdot q_e$ take a particularly simple form which depends only on the fiber derivatives of the Lagrangian.

**Proposition 3.4**

(i) \[
D(L \circ \zeta Q)(q) \cdot \eta Q(q) = \text{ad}^*_\xi \mathbb{II}(q, \zeta) \cdot \eta \tag{3.46}
\]

for all $\eta$ and $\zeta \in G$ and $q \in Q$.

(ii) $v_e = \xi Q(q_e)$ implies $\xi \in G_\mu$.

(iii) \[
DE_\mu(q) \cdot \eta Q(q) = -\text{ad}^*_\xi \mu(q) \cdot \eta + [D \mu(q) \cdot \eta Q(q)] \cdot (\Xi(q) - \xi) \tag{3.47}
\]

for all $\eta \in G$ and $q \in U$.

(iv) \[
\mathbb{II}_\xi(q_e) \cdot \eta Q(q_e) = -\left(\mathbb{II}(\text{ad}_\eta \xi) + \text{ad}^*_\mu \mu\right). \tag{3.48}
\]

**Proof:** The following equivariance property is the key to the proposition: $G$ invariance of $L$ implies

\[
L(\zeta Q(g \cdot q)) = L\left(g \cdot \left([\text{Ad}_{g^{-1}} \zeta]_Q(q)\right)\right)
= L\left([\text{Ad}_{g^{-1}} \zeta]_Q(q)\right) \tag{3.49}
\]

for all $\zeta \in G$, $g \in G$, and $q \in Q$.

(3.49) implies

\[
D(L \circ \zeta Q)(q) \cdot \eta Q(q) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(\zeta Q(\exp(\epsilon \eta) \cdot q))
= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L([\text{Ad}_{\exp(-\epsilon \eta)} \zeta]_Q(q))
= -\mathbb{F}L(\zeta Q(q)) \cdot [\text{ad}_\eta \zeta]_Q(q)
= -\mathbb{II}(q, \zeta) \cdot \text{ad}_\eta \zeta
= \text{ad}^*_\zeta \mathbb{II}(q, \zeta) \cdot \eta. \tag{3.50}
\]

In particular,

\[
0 = DL_\xi(q_e) \cdot \eta Q(q_e) = \text{ad}^*_\xi \mu \cdot \eta \tag{3.51}
\]

for all $\eta \in G$ implies $\xi \in G_\mu$. Combining the first result with Proposition 3.2 and $\mu(q) = \mathbb{II}(q, \Xi(q))$ yields

\[
DE_\mu(q) \cdot \eta Q(q) = [D \mu(q) \cdot \eta Q(q)] \cdot (\Xi(q) - \xi) - \text{ad}^*_\Xi(q) \mu(q) \cdot \eta, \tag{3.52}
\]
which proves the third claim.

Given \( \eta \in \mathcal{G} \), we derive the expression for \( D \Pi_\xi(q_e) \cdot \eta_Q(q_e) \) as follows: (3.5) and (3.49) imply

\[
\Pi(g \cdot q, \zeta) \cdot \omega = \left. \frac{d}{dt} \right|_{t=0} \Pi(\zeta + t\omega, q(g \cdot q)) \\
= \left. \frac{d}{dt} \right|_{t=0} L \left( \left[ \text{Ad}_{g^{-1}}(\zeta + t\omega) \right] (q) \right) \\
= \mathcal{F}L \left( \left[ \text{Ad}_{g^{-1}} \zeta \right] (q) \right) \cdot \left[ \text{Ad}_{g^{-1}} \omega \right] (q) \\
= \Pi(q, \text{Ad}_{g^{-1}} \zeta) \cdot \text{Ad}_{g^{-1}} \omega. 
\]

for all \( q \in Q \), \( g \in G \), and \( \zeta \) and \( \omega \in \mathcal{G} \). Application of this result to the curve \( g_e := \exp(\epsilon \eta) \) in \( G \) yields

\[
(D \Pi(q, \zeta) \cdot \eta_Q(q)) \cdot \omega = \left. \frac{d}{dt} \right|_{t=0} \Pi(\exp(\epsilon \eta), q, \zeta) \cdot \omega \\
= \left. \frac{d}{dt} \right|_{t=0} \left[ \Pi(q, \text{Ad}_{\exp(-\epsilon \eta)} \zeta) \cdot \text{Ad}_{\exp(-\epsilon \eta)} \omega \right] \\
= \left( D \zeta \Pi(q, \zeta) \cdot (-\text{ad}_\eta \zeta) \right) \cdot \omega + \Pi(q, \zeta) \cdot (-\text{ad}_\eta \omega) \\
= -\left[ D \zeta \Pi(q, \zeta) \cdot (\text{ad}_\eta \zeta + \text{ad}_\eta^* \Pi(q, \zeta)) \right] \cdot \omega. 
\]

Hence (3.48) holds. \( \sqrt{\text{ }} \)

We now characterize the ‘rigid variations’, which are a subspace of the group orbit \( \mathcal{G} \cdot q_e \).

Define the space

\[ Q_{\text{rig}} := \mathcal{G} \cdot q_e \cap Q \quad (3.55) \]

Our immediate goal is the specification of a subspace \( \mathcal{G}_Q \subseteq \mathcal{G} \) such that \( Q_{\text{rig}} = \mathcal{G}_Q \cdot q_e \).

Since \( \mathcal{G}_{q_e} \subseteq \ker \Pi_e \), it follows from Proposition 3.4.i that \( D \Pi_\xi(q_e) \cdot \eta_Q(q) \in \mathcal{G}_{q_e}^A \), and hence \( \eta_Q(q_e) \in Q \), if and only if \( \text{ad}_\eta^* \mu \in \mathcal{G}_{q_e}^A \). Hence, if we define the subspace \( \mathcal{G}_Q \subseteq \mathcal{G} \) by

\[ \mathcal{G}_Q := \left\{ \eta \in \mathcal{G} : \text{ad}_\eta^* \mu \in \mathcal{G}_{q_e}^A \right\}, \]

then

\[ Q_{\text{rig}} = \mathcal{G}_Q \cdot q_e. \quad (3.57) \]

Note that \( \mathcal{G}_Q \supseteq \mathcal{G}_\mu \cup \mathcal{G}_{q_e} \), since \( \mathcal{G}_{q_e} \) is a subalgebra and \( \mu \in \mathcal{G}_{q_e}^A \).

We now show that the stability two form \( B_e \) takes a particularly simple form on group variations. Define the generalized Arnold form \( A : \mathcal{G}_Q \times \mathcal{G}_Q \to \mathbb{R} \) by

\[ A(\zeta, \eta) := \text{ad}_\eta^* \mu \cdot (D \Xi(q_e) \cdot \eta_Q(q_e)). \quad (3.58) \]

**Proposition 3.5**

\[ D^2 E_\mu(q_e)(\eta_Q(q_e), \delta q) = \text{ad}_\eta^* \mu \cdot (D \Xi(q_e) \cdot \delta q) \quad (3.59) \]

for all \( \eta \in \mathcal{G} \) and \( \delta q \in Q \). In particular,

\[ D^2 E_\mu(q_e)(\eta_Q(q_e), \zeta Q(q_e)) = A(\eta, \zeta) \quad \forall \ \eta, \ \zeta \in \mathcal{G}_Q. \quad (3.60) \]
Proof: Consider \( \eta \in \mathcal{G} \) and \( \delta q \in \mathcal{Q} \); let \( q_e \) be a curve tangent to \( \delta q \). Proposition 3.4.iii implies

\[
D^2 E_\mu(q_e)(\eta\mathcal{Q}(q_e), \delta q) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} D E_\mu(q_e) \cdot \eta\mathcal{Q}(q_e) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left[ -\text{ad}_\xi^* \mu(q_e) \cdot \eta + [D\mu(q_e) \cdot \eta\mathcal{Q}(q_e)] \cdot (\Xi(q_e) - \xi) \right] = -\text{ad}_{D\Xi(q_e) \cdot \delta q}^* \mu \cdot \eta = \text{ad}_\eta^* \mu \cdot (D\Xi(q_e) \cdot \delta q),
\]

since \( D\mu(q_e) \cdot \delta q = 0 \) for all \( \delta q \in \mathcal{Q} \). \( \sqrt{\big} \)

3.d.ii Internal variations

The next step in the decomposition is the definition of a space \( \mathcal{Q}_{\text{int}} \) of 'internal variations' such that the stability form \( \mathcal{B}_e \) block diagonalizes with respect to \( \mathcal{Q}_{\text{rig}} \) and \( \mathcal{Q}_{\text{int}} \). The space \( \mathcal{Q}_{\text{int}} \) is simply the orthogonal complement to \( \mathcal{Q}_{\text{rig}} \) with respect to the bilinear form \( \mathcal{B}_e \); this apparent tautology is justified by the fact that the simple form (3.59) taken by \( \mathcal{B}_e |_{\mathcal{Q}_{\text{rig}} \times \mathcal{Q}} \) makes the determination of \( \mathcal{Q}_{\text{int}} \) relatively simple. For a large class of systems, there is a simple physical interpretation of the space \( \mathcal{Q}_{\text{int}} \).

Proposition 3.4.iv and (3.57) imply that

\[
D^2 E_\mu(q_e)(\Delta q, \delta q) = 0 \quad \forall \Delta q \in \mathcal{Q}_{\text{rig}}
\]

if and only if

\[
\text{ad}_{D\Xi(q_e) \cdot \delta q}^* \mu \cdot \eta = 0 \quad \forall \eta \in \mathcal{G}_\mathcal{Q}.
\]

Hence, we define the subspace \( \mathcal{Q}_{\text{int}} \subset \mathcal{Q} \) by

\[
\mathcal{Q}_{\text{int}} := \{ \delta q \in \mathcal{Q} : \text{ad}_{D\Xi(q_e) \cdot \delta q}^* \mu \in \mathcal{G}_\mathcal{Q} \}.
\]

By construction, \( D^2 E_\mu(q_e)|_{\mathcal{Q}_{\text{rig}} \times \mathcal{Q}_{\text{int}}} = 0 \).

Remark: For a large class of relative equilibria, including asymmetric equilibria, the definitions of the spaces \( \mathcal{Q}_{\text{rig}} \) and \( \mathcal{Q}_{\text{int}} \) take a somewhat simpler form. Specifically, if \( G_\mu \supseteq G_{q_e} \), then \( \mathcal{G}_\mathcal{Q} = \mathcal{G} \); hence

\[
\mathcal{Q}_{\text{rig}} = \mathcal{G} \cdot q_e \quad \text{and} \quad \mathcal{Q}_{\text{int}} = \{ \delta q \in \mathcal{Q} : D\Xi(q_e) \cdot \delta q \in \mathcal{G}_\mu \}.
\]

Define the subspaces \( \mathcal{V}_{\text{rig}} \) and \( \mathcal{V}_{\text{int}} \) of \( \mathcal{V} \) by

\[
\mathcal{V}_{\text{rig}} := \mathcal{Q}_{\text{rig}} \cap \mathcal{V} \quad \text{and} \quad \mathcal{V}_{\text{int}} := \mathcal{Q}_{\text{int}} \cap \mathcal{V}.
\]

Proposition 3.6 If \( G \) is finite dimensional and \( A(\zeta, \cdot) = 0 \) implies \( \zeta \in \mathcal{G}_\mu \cup \mathcal{G}_{q_e} \), e.g. if \( D^2 E_\mu(q_e)|_{\mathcal{V}_{\text{rig}} \times \mathcal{V}_{\text{rig}}} \) is definite, then

\[
\mathcal{Q} = \mathcal{Q}_{\text{rig}} + \mathcal{Q}_{\text{int}}, \quad \mathcal{Q}_{\text{rig}} \cap \mathcal{Q}_{\text{int}} = \mathcal{G}_\mu \cdot q_e.
\]
and hence
\[ V = \mathcal{V}_\text{rig} \oplus \mathcal{V}_\text{int}. \]  
(3.68)

*If G is infinite dimensional, we must additionally assume that* \[ \text{Range } \mathcal{A} + \mathcal{G}_Q^A = (\mathcal{G}_\mu \cup \mathcal{G}_{q_e})^A, \]  
(3.69)

where \( \mathcal{A} \) is viewed as a linear map from \( \mathcal{G}_Q \) to \( \mathcal{G}^* \), to guarantee that the decompositions are valid.

**Proof:** To simplify the notation, let \( \mathcal{G}_t := \mathcal{G}_\mu \cup \mathcal{G}_{q_e} \subseteq \mathcal{G}_Q \). Equations (3.59) and (3.60) imply that \( \ker \mathcal{A} \supseteq \mathcal{G}_t \) and, since \( \mathcal{A} \) is symmetric,
\[ \text{Range } \mathcal{A} + \mathcal{G}_Q^A \subseteq \mathcal{G}_t^A. \]  
(3.70)

It follows from (3.60) and (3.64) that \( \zeta_Q(q_e) \in \mathcal{Q}_\text{int} \) implies \( \mathcal{A}(\zeta, \cdot) = 0 \) and, since we assume that \( \mathcal{A} \) is nondegenerate on the complement in \( \mathcal{G}_Q \) to \( \mathcal{G}_t \),
\[ \mathcal{Q}_\text{rig} \cap \mathcal{Q}_\text{int} = \mathcal{G}_t \cdot q_e = \mathcal{G}_\mu \cdot q_e. \]  
(3.71)

Nondegeneracy of \( \mathcal{A} \) modulo \( \mathcal{G}_t \) implies that \( \text{Range } \mathcal{A} \cap \mathcal{G}_Q^A = 0 \). Hence, if \( G \) is finite dimensional, \( \text{Range } \mathcal{A} + \mathcal{G}_Q^A = \mathcal{G}_t^A \), since
\[ \text{codim}_{\mathcal{G}_t^A} \mathcal{G}_Q^A \leq \dim \mathcal{G}_Q - \dim \mathcal{G}_t \]
\[ = \dim \text{Range } \mathcal{A} \]  
(3.72)

implies
\[ \text{codim}_{\mathcal{G}_t^A} \left( \text{Range } \mathcal{A} + \mathcal{G}_Q^A \right) = \dim \text{Range } \mathcal{A} - \text{codim}_{\mathcal{G}_t^A} \mathcal{G}_Q^A - \dim \left( \text{Range } \mathcal{A} \cap \mathcal{G}_Q^A \right) \]
\[ = 0. \]  
(3.73)

Let \( \delta \in \mathcal{Q} \). Proposition (3.5) implies
\[ \text{ad}^*_\mathcal{D}(\mathcal{Q}_\text{int} \cdot \delta) \in \mathcal{G}_t^A; \]  
(3.74)

hence there exists \( \eta \in \mathcal{G}_Q \) such that \( \tilde{\delta} := \delta \cdot \eta \) satisfies
\[ \text{ad}^*_\mathcal{D}(\mathcal{Q}_\text{int} \cdot \delta) \in \mathcal{G}_Q^A; \]  
(3.75)

Thus \( \tilde{\delta} \in \mathcal{Q}_\text{rig} \) and, since \( \eta_Q(q_e) \in \mathcal{Q}_\text{rig}, \mathcal{Q} = \mathcal{Q}_\text{rig} + \mathcal{Q}_\text{int}. \) Finally, \( \mathcal{V} \cap (\mathcal{G}_\mu \cdot q_e) = 0 \) implies \( \mathcal{V} = \mathcal{V}_\text{rig} \oplus \mathcal{V}_\text{int}. \)

Let \( \mathcal{G}_Q \subseteq \mathcal{G}_Q \) be a subspace satisfying the conditions
\[ \mathcal{V}_\text{rig} = \mathcal{G}_Q \cdot q_e \quad \text{and} \quad \mathcal{G}_Q \cap \mathcal{G}_{q_e} = 0. \]  
(3.76)

The relative equilibrium \( \nu_e \) is formally stable, in the sense that the second variation \( D^2E_\xi(\nu_e)|_S \) is positive (negative) definite, if
(i) $\langle , \rangle_e|_{G \cdot q_e}^\perp$ is positive (negative) definite

(ii) $A|_{\tilde{G}_q}$ is positive (negative) definite

(iii) $B_e|_{\mathcal{V}_{int}}$ is positive (negative) definite.

(iv) (3.69) holds; e.g., $G$ is finite dimensional.

The stability criteria can be expressed without making a specific choice of the complement to $G_\mu \cdot q_e$. If we define $B_{int} := B_e|_{\mathcal{Q}_{int}}$, then conditions (ii) and (iii) may be replaced by:

(ii)′ $A$ is positive (negative) semi-definite, with kernel $G_\mu \cup G_{\gamma e}$

(iii)′ $B_{int}$ is positive (negative) semi-definite, with kernel $G_\mu \cdot q_e$.

4 APPLICATION OF BLOCK DIAGONALIZATION

In the preceding section, our goal was the derivation of the block diagonalization algorithm in as general a context as possible. In the present section we shall address the following practical considerations. First, are there simple, readily verifiable conditions which guarantee that the block diagonalization procedure can be carried out? Second, under what conditions does formal stability imply nonlinear orbital stability? We first describe a sufficiency condition, referred to as orbit regularity, for applicability of block diagonalization. It is straightforward to test for this condition if the group $G$ is finite dimensional; if $G$ is infinite, verification may be more delicate. We then discuss the conditions under which formal stability implies orbital stability with respect to an appropriate symmetry group. If the configuration manifold $Q$ is finite dimensional, we can apply an extension of a result due to Patrick [1990]; if appropriate conditions on the isotropy subgroups $G_\mu$ and $G_{\gamma e}$ are satisfied, then formal stability implies orbital stability with respect to $G_\mu$. While the spirit of these results carries over to the infinite dimensional case, analytic difficulties can arise if the topology of $TQ$ is poorly matched to the Lagrangian. Hence, we make no general claims regarding the relationship between formal and orbital stability in the infinite dimensional case.

We note that if the relative equilibrium $v_e$ has nontrivial isotropy, then the equilibrium group velocity is not uniquely defined. In this case, it is necessary to consider all possible choices of group velocity to obtain optimal stability conditions.

4.1 Orbit regular equilibria

In this section, we describe a class of relative equilibria for which the block diagonalization procedure can be carried out, i.e., for which a map $\Xi : Q \to \mathcal{G}$ satisfying (3.7) can be found. We shall say that a triplet $(L, \xi, q_e)$ consisting of a $C^1$ $G$ invariant Lagrangian $L : Q \to \mathbb{R}$, an element $\xi$ of the Lie algebra $\mathcal{G}$, and a point $q_e$ is orbit regular if there exists a complement $\tilde{G}$ to $G_{\gamma e}$ in $\mathcal{G}$ such that $\xi \in \tilde{G}$ and $\Pi_e|_\tilde{G}$ is an isomorphism between $\tilde{G}$ and $\mathcal{G}_{\gamma e}$, where $\mathcal{G}_{\gamma e}$ denotes the annihilator of $G_{\gamma e}$. Since

$$\eta \cdot \Pi_e \xi = \langle \eta \mathcal{Q}(q_e), \xi \mathcal{Q}(q_e) \rangle_e = 0 \quad (4.1)$$

20
for all $\zeta \in G_{q_e}$ and $\eta \in G$, $\ker \Pi_e \supseteq G_{q_e}$; hence orbit regularity can be interpreted as the condition that $\Pi_e$ have no degeneracies other than those due to the symmetry of the equilibrium configuration.

If $G$ is finite dimensional, then $(L, \xi, q_e)$ is orbit regular if

$$\ker \Pi_e = G_{q_e}, \quad (4.2)$$

since

$$\dim G^A_{q_e} = \dim G - \dim G_{q_e} = \dim \tilde{G} \quad (4.3)$$

implies $\Pi_e$ is an isomorphism if it is one to one. If $G$ is finite dimensional and $(\langle , \rangle_e \mid G_{q_e})$ is definite, then $(L, \xi, q_e)$ is orbit regular, since $(\langle , \rangle_e \mid G_{q_e})$ definite implies $\zeta \in \ker \Pi_e$ if and only if $\zeta q_e = 0$, i.e. if $\zeta \in G_{q_e}$. In particular, simple mechanical systems with finite dimensional symmetry groups are orbit regular at all points. (If the group $G$ is infinite dimensional, orbit regularity may be influenced by the choice of the dual space $G^*$.)

**Proposition 4.1** If $(L, \xi, q_e)$ is orbit regular, then there exists a neighborhood $U \subset Q$ of $q_e$ and a $C^1$ map $\Xi : Q \to G$ such that $\Xi(q_e) = \xi$ and the mapping $\mu(q) := \Pi(q, \Xi(q))$ satisfies

$$(D\mu(q_e) \cdot \delta q) \cdot \zeta \neq 0 \quad \implies \quad \zeta \in G_{q_e} \quad (4.4)$$

for all $\delta q \in T_{q_e}Q$. If $q_e$ has nontrivial isotropy subalgebra $G_{q_e}$, then for each choice of a complement $\tilde{G}$ to $G_{q_e}$, isomorphic to $G^A_{q_e}$ and containing $\xi$ there is an associated mapping $\Xi$. If $q_e$ is a point without isotropy, then there is a unique map $\Xi$ such that $\mu(q) \equiv \mu$.

**Proof:** If $q_e$ has trivial isotropy, then the first claim of the proposition is a straightforward application of the implicit function theorem to $\Pi$. If $q_e$ has nontrivial isotropy, then $\Pi_e : G \to G^A_{q_e} \neq G^*$; hence $\Pi$ must be modified before the implicit function theorem can be applied. Let a complement $\tilde{G}$ to $G_{q_e}$ be given and let $\tilde{P}$ denote the projection $\tilde{P} : G \to \tilde{G}$. Define the restricted inertia mapping $\tilde{\Pi} : Q \times \tilde{G} \to G^A_{q_e}$

$$\tilde{\Pi}(q, \eta) := \Pi(q, \eta) \circ \tilde{P} \quad (4.5)$$

$\tilde{\Pi}(q_e, \eta) = \Pi(q_e, \eta)$ for all $\eta \in G$; in particular, $\mu \in G^A_{q_e}$. Hence we can apply the implicit function theorem to $\tilde{\Pi}$ to find a curve $\Xi : U \to \tilde{G}$ such that

$$\tilde{\Pi}(q, \Xi(q)) \equiv \mu \quad (4.6)$$

It follows that $\mu(q) := \Pi(q, \Xi(q))$ satisfies $(D\mu(q_e) \cdot \delta q) \cdot \eta = 0$ for all $\eta \in \tilde{G}$ and $\delta q \in T_{q_e}Q$.

If the triplet $(L, \xi, q_e)$ is orbit regular, then not only do we know that a curve $\Xi$ exists; we also have a very simple expression for its first variation at $q_e$ in terms of the linearized inertia tensor and the first variation of the locked momentum mapping. This provides expressions for the stability two form and the Arnold form which are straightforward generalizations of the forms derived in the simple mechanical case.

**Proposition 4.2** If $(L, \xi, q_e)$ is orbit regular, then
(i) Let $\overline{\Pi}_e : \overline{\mathcal{G}} \to (\mathcal{G}_{q_e})^A$ denote the restriction of $\Pi_e$ to $\overline{\mathcal{G}}$. $\overline{\Pi}_e$ is invertible and

$$D\Xi(q_e) \cdot \delta q = \overline{\Pi}_e^{-1} (D\Pi_e(q_e) \cdot \delta q).$$

(4.7)

(ii) The stability two form $B_e : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ takes the form

$$B_e(\delta q, \Delta q) = (D\Pi_e(q_e) \cdot \delta q) \cdot \overline{\Pi}_e^{-1} (D\Pi_e(q_e) \cdot \Delta q) - D^2L_e(q_e)(\delta q, \Delta q).$$

(4.8)

(iii) The generalized Arnold form $A : \mathcal{G}_\mathcal{Q} \times \mathcal{G}_\mathcal{Q} \to \mathbb{R}$ takes the form

$$A(\zeta, \eta) = \text{ad}^*_\xi \mu \cdot \left( \text{ad}_\eta \xi + \overline{\Pi}_e^{-1} \text{ad}^*_\eta \mu \right).$$

(4.9)

The forms $A$ and $B_e$ do not depend on the choice of the subspace $\overline{\mathcal{G}}$.

**Proof:** The first claim follows directly from (3.24). By substituting (4.7) into (3.37), we obtain (4.8). To establish the third claim, we note that (3.48) and (4.7) imply

$$D\Xi(q_e) \cdot \eta Q(q_e) = \overline{\Pi}_e^{-1} \left( \Pi_e \text{ad}_\eta \xi + \text{ad}^*_\eta \mu \right)$$

$$= \text{P} \text{ad}_\eta \xi + \overline{\Pi}_e^{-1} \text{ad}^*_\eta \mu.$$  

(4.10)

Inserting (4.10) into (3.58) yields

$$A(\zeta, \eta) = \text{ad}^*_\xi \mu \cdot \left( \text{ad}_\eta \xi + \overline{\Pi}_e^{-1} \text{ad}^*_\eta \mu \right).$$

(4.11)

The independence of the definitions from the choice of the complement follows from the condition that $D\Pi_e(q_e) \cdot \delta q \in \mathcal{G}^A_{q_e}$ for all $\delta q \in \mathcal{Q}$.

4.b Orbital stability

Once formal stability of a relative equilibrium has been established, we can attempt to demonstrate actual orbital stability of the relative equilibrium. Orbital stability will typically be taken with respect to the isotropy subgroup $G_{\mu}$, although in cases in which the relative equilibrium possesses nontrivial isotropy, orbital stability may be determined with respect to a subgroup of $G_{\mu}$. We discuss here methods of proving orbital stability by means of a Liapunov function. Hence, we need to determine conditions under which formal stability of a relative equilibrium $v_e$ implies that $v_e$ is, in some appropriate sense, a local minimum of a conserved functional. For infinite dimensional systems, definiteness of the second variation of a function at a critical point need not imply that the critical point is a local extremum. Such results are highly dependent on the underlying function spaces; a point which is a local minimum with respect to a specific topology may fail to be a minimum with respect to other topologies. These issues are discussed, for example, in Ball [1978]. To simplify matters, we shall, for the remainder of this section, consider only finite dimensional systems.

Given a local extremum of a constrained variational problem, one can typically construct an augmented functional for which the extremum becomes a local minimum of an
unconstrained variational problem. The augmented functional is obtained by adding an appropriate positive function of the constraints to the original functional; see Hestenes [1975]. The systems we consider cannot be directly treated by such methods. Due to the $G_\mu$ invariance of the energy-momentum functional, we do not have true extrema even for the constrained system; the energy-momentum functional attains its (locally) optimal value along the entire orbit $G_\mu \cdot v_e$. To obtain an isolated extremum, we must, in some fashion, 'factor out' the $G_\mu$ symmetry of the system.

If the quotient $T^*Q/G_\mu$ is a manifold in a neighborhood of the equivalence class $[v_e]$, then we can apply the usual augmentation techniques to the induced energy-momentum function on the quotient manifold to show that $[v_e]$ is a local extremum of some conserved functional. If the quotient is not a manifold, then an alternative approach is to construct a slice with respect to $G_\mu$ and demonstrate the existence of a local minimum on the slice. Recent works which apply slice techniques to the analysis of the stability and bifurcation of relative equilibria include Field [1980], Krupa [1990], and Patrick [1990]. Patrick addresses the orbital stability of relative equilibria of Hamiltonian systems, combining augmentation and slice techniques. We quote without proof (adapting notation) the following result from Patrick [1990], where it appears as Theorem 1:

Let $(P, \omega, H, G, \mathcal{J})$ be a Hamiltonian system with symmetry. Suppose $z_e$ is a regular relative equilibrium with evolution $t \mapsto \exp(\xi t) \cdot z_e$, $\mathcal{J}(z_e) = \mu$, the action of $G_\mu$ on $P$ is proper, and $G$ admits an inner product invariant under the Ad action of $G_\mu$. Then $D(H - J_\xi)(z_e) = 0$, and $z_e$ is stable if it is formally stable; that is, if $D^2(H - J_\xi)(z_e)|_{T_{z_e}J^{-1}(\mu)}$ is positive or negative definite on some (and hence any) complement to $G_\mu \cdot z_e$ in $T_{z_e}J^{-1}(\mu)$.

Here stability is taken modulo $G_\mu$.

The result as stated is not directly applicable to relative equilibria of Lagrangian systems which fail to be regular either in the sense that the Lagrangian itself fails to be regular, and hence cannot be transformed to an equivalent Hamiltonian system, or in the sense that the relative equilibrium is a singular point of the momentum map, i.e. that the relative equilibrium has nontrivial isotropy. (See Arms et al. [1981] for a discussion of the relationship between isotropy and singularities of the level sets of the momentum map.) The following two observations indicate that the theorem can be easily extended to a very general class of Lagrangian relative equilibria, including some equilibria with nontrivial isotropy. First, we note that the result relies on the conservation of energy and momentum, not on the existence of a symplectic structure. Hence the theorem is applicable to Lagrangian systems, with the energy replacing the Hamiltonian. Second, we observe that regularity of the relative equilibrium, in the sense that the level set $J^{-1}(\mu)$ is a manifold, is used only to guarantee the existence of a smooth map $\tilde{\Psi} : G_\mu \cdot z_e \to \mathcal{G}$ which yields the velocities of the relative equilibria along the orbit $G_\mu \cdot z_e$. This assumption can be relaxed as follows: Let $S$ and $\pi : G_\mu \cdot S \to G_\mu \cdot z_e$ (or, in the Lagrangian case, $v_e$), denote, respectively, the slice and the orbit translation map defined in Patrick [1990]. Obviously, $G_\mu$ acts transitively on $G_\mu \cdot z_e$; hence the isotropy subgroup $G_{\mu \cdot z_e} := G_\mu \cap G_{z_e}$ is a closed Lie subgroup of
\( G_{\mu}, G_{\mu}/G_{\mu, z_e} \) is a smooth manifold diffeomorphic to \( G_{\mu} \cdot z_e \), and every element of \( G_{\mu} \) lies in the image of a local section of \( G_{\mu}/G_{\mu, z_e} \) into \( G_{\mu} \). (See, for example, Boothby [1975], §IV.9 for proofs of these claims.) It follows that \( G_{\mu} \rightarrow G_{\mu}/G_{\mu, z_e} \) is a principal bundle with fiber group \( G_{\mu, z_e} \). If there exists a smooth global section \( \sigma : G_{\mu}/G_{\mu, z_e} \rightarrow G_{\mu} \), i.e., if the bundle \( G_{\mu} \rightarrow G_{\mu}/G_{\mu, z_e} \) is trivial, then the conclusions of Patrick’s theorem hold even in the absence of regularity. This can be seen as follows: Let \( \phi : G_{\mu} \cdot v_e \rightarrow G_{\mu}/G_{\mu, z_e} \) denote the diffeomorphism between the orbit and the quotient manifold and define \( \gamma := \sigma \circ \phi \circ \pi \). The smooth map \( \hat{\Psi} : G_{\mu} \cdot S \rightarrow G \) given by

\[
\hat{\Psi}(z) := \text{Ad}_{\gamma(z)} \xi
\]

(4.12)

replaces Patrick’s map \( \Psi \) in the proof. This minor modification allows us to take advantage of the power of slice techniques, by allowing us to consider relative equilibrium with nontrivial isotropy. (Since we assume that \( G_{\mu} \) acts properly, it is only in the neighborhood of a symmetric point that the quotient \( T^*Q/G_{\mu} \) can fail to be a manifold.)

Thus, we can state the stability result as follows:

**Corollary 4.1** Let \( v_e \in TQ \) be a formally stable relative equilibrium with group momentum \( \mu \) of a \( G \) equivariant finite dimensional Lagrangian system. If

(i) the Lie algebra \( \mathcal{G} \) possesses an \( \text{Ad} \, G_{\mu} \) invariant inner product

(ii) \( G_{\mu} \) acts properly on \( TQ \)

(iii) the principal bundle \( G_{\mu} \rightarrow G_{\mu}/(G_{\mu} \cap G_{v_e}) \) is trivial,

then \( v_e \) is orbitally stable with respect to \( G_{\mu} \).

Clearly, this statement also holds for a relative equilibrium \( z_e \in T^*Q \) of an equivariant Hamiltonian system.

As a simple example, consider a trio of Lie groups

\[
G_1 \supseteq G_2 \supseteq H
\]

(4.13)

and a formally stable relative equilibrium \( v_e \) for which \( G_{\mu} = G_1 \times G_2 \) and \( G_{\mu} \cap G_{v_e} \) is the diagonal of \( H \times H \). This situation arises for mechanical systems possessing both spatial and body symmetries; e.g., if we consider isotropic homogeneous elasticity with respect to a reference configuration possessing an axis of symmetry, then \( G_1 = G_2 = H = S^1 \). The cosets of \( G_{\mu}/(G_{\mu} \cap G_{v_e}) \) are determined by the relation

\[
(g_1, g_2) \approx (g_1, g_2) \quad \iff \quad (g_1, g_2) = (hg_1, hg_2) \quad \text{for some } h \in H.
\]

(4.14)

Hence a global section is given by

\[
\sigma \left( (g_1, g_2) \right) := \left( g_2^{-1} g_1, \text{Id} \right),
\]

(4.15)

where \( \text{Id} \) denotes the identity element of \( G_1 \). If \( \mathcal{G} \) possesses an invariant inner product, then we can apply Corollary 4.1 to show that \( v_e \) is orbitally stable.
\textbf{4.c One dimensional momentum isotropy subgroups}

If the isotropy subgroup $G_\mu$ of the equilibrium momentum $\mu$ is one dimensional, several convenient simplifications take place. This situation arises in the case of an $SO(3)$ or $S^1$ action, e.g. in the case of a mechanical system possessing no symmetries other than material frame indifference.

We first note that one of the assumptions on $G_\mu$ given in Corollary 4.1 is automatically satisfied in this case. The subgroup $G_\mu \cap G_{\nu_e}$ is either zero dimensional or equal to $G_\mu$; hence the bundle $G_\mu \rightarrow G_\mu/(G_\mu \cap G_{\nu_e})$ is trivial. Thus, if $Q$ is finite dimensional and $G_\mu$ acts properly on $TQ$, and $G$ possesses a $G_\mu$ invariant inner product, formal stability implies nonlinear orbital stability.

If $G_\mu$ is one dimensional and $G_\mu \supseteq G_{\nu_e}$, then the determination of the formal stability conditions associated to internal variations is substantially simplified. By combining the equilibrium conditions and the defining conditions of the space $\mathcal{Q}_{\text{int}}$ of internal variations, we shall show that $B_{\text{int}}$ can be expressed in terms of variations of scalar functions. Specifically, define $f_\xi : Q \rightarrow \mathbb{R}$ by

$$f_\xi(q) := \Pi_\xi(q) \cdot \xi$$ \hspace{1cm} (4.16)

and set

$$\kappa_\xi := \xi \cdot \Pi_\xi \xi.$$ \hspace{1cm} (4.17)

Note that, for simple mechanical systems, $f_\xi = -2C_\xi$ and $\kappa_\xi = -2C_\xi(q_e)$, where $C_\xi$ denotes the centrifugal potential $C_\xi(q) := -\frac{1}{2} \left| \xi_Q(q) \right|^2_g$. If $\kappa_\xi = 0$, then $B_{\text{int}} = D^2L_\xi |_{\mathcal{Q}_{\text{int}}}$; otherwise

$$B_{\text{int}}(\delta q, \Delta q) = \frac{1}{\kappa_\xi} (Df_\xi(q_e) \cdot \delta q) (Df_\xi(q_e) \cdot \Delta q) - D^2L_\xi(q_e)(\delta q, \Delta q).$$ \hspace{1cm} (4.18)

This can be seen as follows: The equilibrium conditions imply that $\xi \in G_\mu$; hence, if $G_\mu$ is one dimensional, then $\xi$ spans $G_\mu$. Thus, using (3.65), we see that

$$\delta q \in \mathcal{Q}_{\text{int}} \iff D\Xi(q_e) \cdot \delta q = \lambda_\delta q \xi \quad \text{for some } \lambda_\delta q \in \mathbb{R}$$ \hspace{1cm} (4.19)

and

$$(D\Xi(q_e) \cdot \delta q) \cdot \Pi_\xi (D\Xi(q_e) \cdot \Delta q) = \lambda_\delta q \lambda_{\Delta q} \kappa_\xi.$$ \hspace{1cm} (4.20)

The identity (3.24) implies

$$Df_\xi(q_e) \cdot \delta q = \xi \cdot (D\Pi_\xi(q_e) \cdot \delta q)$$

$$= -\xi \cdot \Pi_\xi (D\Xi(q_e) \cdot \delta q)$$

$$= -\lambda_\delta q \kappa_\xi$$ \hspace{1cm} (4.21)

Hence, if $\kappa_\xi \neq 0$, then

$$\lambda_\delta q = \frac{-Df_\xi(q_e) \cdot \delta q}{\kappa_\xi}$$ \hspace{1cm} (4.22)

and, using (3.37),

$$B_{\text{int}}(\delta q, \Delta q) = (D\Xi(q_e) \cdot \delta q) \cdot \Pi_\xi (D\Xi(q_e) \cdot \Delta q) - D^2L_\xi(q_e)(\delta q, \Delta q)$$

$$= \frac{1}{\kappa_\xi} (Df_\xi(q_e) \cdot \delta q) (Df_\xi(q_e) \cdot \Delta q) - D^2L_\xi(q_e)(\delta q, \Delta q).$$ \hspace{1cm} (4.23)
5 Examples

5.1 Natural mechanical systems

Gantmacher [1975], §11, defines a natural mechanical system as a Lagrangian system with Lagrangian of the form

\[ L(v) := \frac{1}{2} |v|^2 + \langle v, Y(q) \rangle_g - V(q). \]  

(5.1)

where \( q := \pi(v) \) is the basepoint of the tangent vector \( v \). We first treat the general case of a natural mechanical system, showing, in particular, that all relative equilibria of a natural system with a finite dimensional symmetry group are orbit regular. We then apply the results obtained for the general case to the analysis of a particle moving in an electromagnetic field.

5.1.1 Definitions

Let \( Q \) be a manifold acted upon by a Lie group \( G \). Let the Lagrangian \( L \) be of the form (5.1), where \( g \) is a \( G \) invariant metric on \( Q \), \( Y : Q \to TQ \) is a \( G \) equivariant vector field, and \( V : Q \to \mathbb{R} \) is a \( G \) invariant function. The Legendre transformation \( FL : TQ \to T^*Q \) associated to \( L \) is determined by

\[ FL(v) \cdot w = \langle v + Y(q), w \rangle_g \]  

(5.2)

for all \( v \in TQ \) and \( w \in T_{\pi(v)}Q \). The energy associated to \( L \) is

\[
E(v) = FL(v) \cdot v - L(v) \\
= \langle v + Y(q), v \rangle_g - \frac{1}{2} |v|^2 - \langle v, Y(q) \rangle_g + V(q) \\
= \frac{1}{2} |v|^2 + V(q).
\]

(5.3)

Note that \( E \) is independent of \( Y \).

Let \( J_g : TQ \to G^* \) denote the map determined by

\[ J_g(v) \cdot \eta := \langle v, \eta_Q(q) \rangle_g. \]  

(5.4)

The augmented Lagrangian \( L_\xi : Q \to \mathbb{R} \) is given by

\[
L_\xi(q) := L(\xi_Q(q)) \\
= \frac{1}{2} \xi \cdot \Pi_g(q) \xi + \beta(q) \cdot \xi - V(q),
\]

(5.5)

where \( \beta := J_g \circ Y \) maps \( Q \to G^* \) and \( \Pi_g \) denotes the inertia tensor associated to \( g \), i.e.

\[ \Pi_g(q) \xi := J_g(\xi_Q(q)). \]  

(5.6)

The inertia tensor \( \Pi : Q \times \mathcal{G} \to G^* \) is affine with respect to \( \mathcal{G} \):

\[
\Pi(q, \xi) := J_g(\xi_Q(q) + Y(q)) \\
= J_g(\xi_Q(q)) + J_g(Y(q)) \\
= \Pi_g(q) \xi + \beta(q).
\]

(5.7)
It follows that the linearized inertia tensor $\Pi_e$ is simply the metric inertia tensor at equilibrium, i.e.

$$\Pi_e = \Pi_g(q_e).$$  \hfill (5.8)

If $G$ is finite dimensional, then the triplet $(L, \xi, q_e)$ is orbit regular for all $q_e \in Q$ and all $\xi \in G$, since $\zeta \in \ker \Pi_e$ implies

$$|\zeta Q(q_e)|^2 = \zeta \cdot \Pi_g(q_e) \zeta = 0$$  \hfill (5.9)

and hence $\zeta \in G_{q_e}$.

5.a.ii  Formal stability

Let $v_e = \xi Q(q_e)$ be a relative equilibrium. The analysis of the ‘vertical’ block associated to $(G \cdot q_e)^\perp$ is straightforward, since

$$\langle [v, v]_e \rangle = |v|^2 > 0$$  \hfill (5.10)

for all $v \neq 0$. Assume $(L, \xi, q_e)$ is orbit regular.

The stability two form $B_e$ is given by

$$B_e(\delta q, \Delta q) = (D(\Pi(q_e) \cdot \delta q) \xi) \cdot \Pi^{-1}_e (D(\Pi(q_e) \cdot \Delta q) \xi) + D^2V(q_e)(\delta q, \Delta q) - \frac{1}{2} \xi \cdot (D^2 \Pi(q_e)(\delta q, \Delta q) \xi)$$  \hfill (5.11)

$$= [D(\Pi_g(q_e) \cdot \delta q) \xi + D\beta(q_e) \cdot \delta q] \cdot \Pi^{-1}_e \left[D(\Pi_g(q_e) \cdot \Delta q) \xi + D\beta(q_e) \cdot \Delta q\right] + D^2V(q_e)(\delta q, \Delta q) - \frac{1}{2} \xi \cdot \left[D^2 \Pi_g(q_e)(\delta q, \Delta q) \xi - D^2 \beta(q_e)(\delta q, \Delta q)\right].$$

If we define the set $Q_\alpha$ of asymmetric points by

$$Q_\alpha := \{q \in Q : \eta q(q) \neq 0 \quad \forall \eta \in G\},$$  \hfill (5.12)

then the restriction of $\Pi$ to $Q_\alpha \times G$ is invertible with respect to $G$, with inverse $\Pi^{-1} : Q_\alpha \times G^* \to G$ given by

$$\Pi^{-1}(q, \mu) = \Pi_g^{-1}(q)(\mu - \beta(q)).$$  \hfill (5.13)

The constant momentum vector field $v_\mu : Q_\alpha \to TQ$ is given by

$$v_\mu(q) := \left(\Pi^{-1}(q, \mu)\right)_Q(q)$$

$$= \left[\Pi_g^{-1}(q)(\mu - \beta(q))\right]_Q(q).$$  \hfill (5.14)

Hence, the restriction of the amended energy $E_\mu$ to $Q_\alpha$ has the form

$$E_\mu(q) := E_{\xi}(v_\mu(q))$$

$$= \frac{1}{2} |v_\mu(q)|^2_g + V(q) - \mu \cdot \xi$$

$$= \frac{1}{2} \left[\Pi_g^{-1}(q)(\mu - \beta(q))\right]_Q^2(q) + V(q) - \mu \cdot \xi$$

$$= \frac{1}{2}(\mu - \beta(q))\Pi_g^{-1}(q)(\mu - \beta(q)) + V(q) - \mu \cdot \xi.$$  \hfill (5.15)
Remark: Note that if $\beta$ is constant, then the equilibrium configurations and their stabilities are identical to those of the simple mechanical system with metric $g$ and potential $V$. For example, if the vector field $Y$ is horizontal, i.e. if $Y(q)$ lies within the horizontal subspace associated to the simple mechanical connection $\alpha$, then $\beta \equiv 0$. If $\beta$ is constant to first order at $q_e$, then the equilibrium conditions are unaffected by $Y$ and the stability conditions differ from those of the associated simple mechanical system by the term $\frac{1}{2} D^2 \beta(q_e)(\dot{q}, \Delta q) \cdot \xi$.

5.a.iii A particle in an electromagnetic field

As an example of a natural mechanical system, we now consider a point mass moving in a electromagnetic field. In this case, the configuration space is $Q = \mathbb{R}^3$ and the tangent bundle $TQ$ is identified with $\mathbb{R}^3 \times \mathbb{R}^3$. We assume that the fields in question are cylindrically symmetric, i.e. there exists an axis (e.g., the $\hat{z}$ axis) such that the fields are invariant under rotations about the axis and translations along the axis. The symmetry group is then $G = S^1 \times \mathbb{R}$, with group action

$$(\theta, \zeta) \cdot q := R_\theta q + \zeta \hat{z}, \quad (5.16)$$

where $R_\theta \in SO(3)$ is the matrix associated to rotation through the angle $\theta$ about the axis $\hat{z}$.

The Lagrangian is given by

$$L(v) = \frac{m}{2} |v|^2 + \varepsilon m \langle v, A(q) \rangle - e\varphi(q), \quad (5.17)$$

where

(i) $m$ is the mass of the particle

(ii) $e$ is the electric charge of the particle

(iii) $\varepsilon := \frac{\varepsilon}{c}$, where $c$ is the speed of light

(iv) $A$ is the magnetic vector potential

(v) $\varphi$ is the electric scalar potential

(vi) $\langle , \rangle$ is the Euclidean inner product on $\mathbb{R}^3$.

Thus $\langle , \rangle_g = m \langle , \rangle$, $Y(q) := \varepsilon A(q)$ and $V(q) := e\varphi(q)$. $G$ equivariance implies the vector potential $A$ is of the form

$$A(q) = \nu(\rho)pq + \sigma(\rho)(\hat{z} \times q) + \tau(\rho)\hat{z}, \quad (5.18)$$

where $pq := (\hat{z} \times q) \times \hat{z}$ denotes the projection of $q$ onto the $x$-$y$ plane and $\rho := |\hat{z} \times q|^2$, and the scalar potential $\varphi$ is a function of $\rho$. The magnetic field is given by

$$\nabla \times A(q) = (2\rho\sigma'(\rho) + \sigma(\rho))\hat{z} - 2\tau'(\rho)(\hat{z} \times q). \quad (5.19)$$
The Lie group $G$ has Lie algebra $\mathcal{G} \approx \mathbb{R}^2$, with infinitesimal generator
\[ (\xi, \zeta)_{Q}(q) = (q, \xi \dot{q} + \zeta \dot{q}). \] (5.20)

The momentum map associated to the action of $G$ on $TQ$ and the metric $g$ is
\[ J_{g}(q, v) = m(v \cdot (\dot{q} 	imes q), v \cdot \dot{q}); \] (5.21)

hence the metric inertia tensor $I_{g}: G \to G^*$ is given by
\[ I_{g}(q, (\xi, \zeta)) := J_{g}((\xi, \zeta)_{Q}(q)) \\
= J_{g}(q, \xi \dot{q} \times q + \zeta \dot{q}) \\
= m(\rho \xi, \zeta). \] (5.22)

Using (5.21), we see that the map $\beta := J_{g} \circ Y$ is given by
\[ \beta(q) = \frac{m}{\epsilon} (\rho \sigma(\rho), \tau(\rho)). \] (5.23)

Hence (5.7) implies
\[ I_{\xi, \zeta}(q) = m(\rho (\xi + \epsilon \sigma(\rho), \zeta + \epsilon \tau(\rho)). \] (5.24)

The augmented Lagrangian is given by
\[ L_{\xi, \zeta}(q) = \frac{m}{2} (\xi^2 \rho + \zeta^2) + \epsilon m (\xi \rho \sigma(\rho) + \zeta \tau(\rho)) - \epsilon \varphi(\rho). \] (5.25)

The group $G$ is abelian, hence $G_{\mu} = G$. We consider a point $q_{e}$ such that $\rho_{e} \neq 0$; $q_{e}$ has trivial isotropy. The tangent to the orbit, $G \cdot q_{e}$, is spanned by the vectors $\dot{z} \times q_{e}$ and $\dot{z}$, thus the complement $V$ to the orbit can be chosen as the span of $q_{e}$.

The augmented Lagrangian has first variation
\[ DL_{\xi, \zeta}(q_{e}) = 2 \left[ m \xi \left( \frac{\xi}{2\epsilon} + \epsilon (\sigma(\rho_{e}) + \rho_{e} \sigma(\rho_{e})) \right) - \epsilon m \zeta \tau(\rho_{e}) - \epsilon \varphi(\rho_{e}) \right] q_{e}, \] (5.26)

where ' denotes variation with respect to $\rho$.

Hence $q_{e}$ is a critical point of $L_{\xi, \zeta}$ if
\[ \varphi'(\rho_{e}) = \epsilon m \left[ \xi \left( \frac{\xi}{2\epsilon} + \sigma(\rho_{e}) + \rho_{e} \sigma(\rho_{e}) \right) - \zeta \tau(\rho_{e}) \right]. \] (5.27)

The augmented Lagrangian has second variation
\[ D^2 L_{\xi, \zeta}(q_{e}) = 4 (\epsilon m [\xi (2\sigma'(\rho_{e}) + \rho_{e} \sigma''(\rho_{e})) + \zeta \tau''(\rho_{e})] - \epsilon \varphi''(\rho_{e})) q_{e} \otimes q_{e}. \] (5.28)

The first variation of $I_{\xi, \zeta}(q)$ at $q_{e}$ is given by
\[ D I_{\xi, \zeta}(q_{e}) = 2 \epsilon m \left( \xi^{-1} + \sigma(\rho_{e}) + \rho_{e} \sigma'(\rho_{e}), \tau'(\rho_{e}) \right) q_{e}. \] (5.29)

The linearized inertia tensor is given by $I_{e} = \text{diag}[m \rho_{e}, m]$; hence
\[ B_{e}(q_{e}^2) = 4 \rho_{e}^2 \left[ \frac{1}{\rho_{e}} (\epsilon^{-1} \xi + \sigma(\rho_{e}) + \rho_{e} \sigma(\rho_{e}))^2 + (\tau'(\rho_{e}))^2 \right] \\
- \epsilon m [\xi (2\sigma'(\rho_{e}) + \rho_{e} \sigma''(\rho_{e})) + \zeta \tau''(\rho_{e})] + \epsilon \varphi''(\rho_{e}). \] (5.30)
Thus \( v_e = (q_e, \alpha \hat{\alpha} + q_e + \alpha \hat{\alpha}) \) is a formally stable relative equilibrium if (5.27) holds and (5.30) is positive.

Since \( G \) is abelian, the coadjoint action is trivial and the Euclidean inner product on \( \mathcal{G} \cong \mathbb{R}^2 \) is trivially \( G_\mu \) invariant. Since \( v_e \) has trivial isotropy, Corollary 4.1 implies that a formally stable relative equilibrium is nonlinearly orbitally stable.

5.6 The symmetric rigid body

As a simple example of a symmetric configuration, we consider a rigid body with one axis of symmetry. The symmetry group is \( SO(3) \times S^1 \), with action

\[
(\Lambda, \theta) \cdot Q = \Lambda QR_\theta^T,
\]

(5.31)

where \( R_\theta \in SO(3) \) is the matrix associated to rotation through the angle \( \theta \) about the axis of symmetry. If the body is symmetric about its axis of rotation, then it is sufficient to apply the energy-momentum method using the subgroup \( SO(3) \) with the usual left action. If the body is symmetric about an axis \( \sigma \) in the plane of rotation, then the stability two form associated to the pure left action is indefinite. In this case, it is necessary to consider the full group \( G \) to obtain definite stability conditions.

The configuration space of the rigid body is the rotation group \( SO(3) \). The Lagrangian is given by

\[
L(\hat{\alpha}) := \frac{1}{2} \left\langle \hat{\alpha}, \hat{\alpha} \right\rangle_B,
\]

(5.32)

where

\[
\left\langle U, V \right\rangle_B := \int_B \rho_{\text{ref}} \left\langle Ux, Vx \right\rangle d^3x.
\]

(5.33)

\( G \) has Lie algebra \( \mathcal{G} \cong \mathbb{R}^3 \times \mathbb{R} \) with infinitesimal generator

\[
(\xi, \alpha)Q(\Lambda) = \hat{\xi} \Lambda + \alpha (\text{Id} - \Lambda) \sigma.
\]

(5.34)

Let \( \Pi_L(\Lambda) \) denote the usual inertia diadic associated to the left action of \( SO(3) \). We assume that the coordinate axes are chosen such that \( \Pi_L \) is diagonal; the symmetry assumption implies that \( \Pi_L = \text{diag}[I_1, I_2, I_1] \) for some constants \( I_1 \neq I_2 \in \mathbb{R} \). Additionally, we assume that the body is in steady rotation with angular velocity \( \xi \) such that \( \xi \) is not parallel to \( \sigma \). Given \( (\xi, 0) \in \mathcal{G} \),

\[
L_{\xi,0}(\Lambda) = \frac{1}{2} \left\langle (\xi, 0)Q(\Lambda), (\xi, 0)Q(\Lambda) \right\rangle_B = \frac{1}{2} \left\langle (\xi, 0), \Pi(\Lambda)(\xi, 0) \right\rangle_B = \frac{1}{2} \xi \cdot \Pi_L(\Lambda) \xi.
\]

(5.35)

(5.36)

If we identify \( \delta \alpha \in T_\Lambda SO(3) \) with \( \delta \theta \in \mathbb{R}^3 \) as usual, then \( L_{\xi,0} \) has first variation

\[
DL_{\xi,0}(\Lambda_e) \cdot \delta \theta = \frac{1}{2} \xi \cdot [D\Pi_L(\Lambda_e) \cdot \delta \theta] \xi.
\]

(5.37)

Hence the usual rigid body equilibrium conditions imply that \( \xi \) is parallel to a principal axis of \( B \).
The ‘locked momentum map’ is given by

$$\Pi_\xi(\Lambda) \cdot (\eta, \alpha) = \xi \cdot \Pi_L(\Lambda)(\eta + \alpha(\text{Id} - \Lambda)\sigma).$$  \hfill (5.38)

The equilibrium momentum is given by

$$\Pi_\xi(L_e)(\xi, 0) = (\mu, 0),$$  \hfill (5.39)

where

$$\mu := \Pi_L(L_e) = I_1 \xi. $$  \hfill (5.40)

The locked momentum map has first variation

$$D\Pi_\xi(L_e) \cdot \delta \theta = (\delta \theta \times \mu - \Pi_L(L_e)(\delta \theta \times \xi), \mu \cdot (\sigma \times \delta \theta))$$  \hfill (5.41)

at $L_e$.

The relevant isotropy subalgebras are

(i) $G_{\Lambda_e} = \text{span} \{(0,1)\}$

(ii) $G_{\nu_e} = (0,0)$

(iii) $G_\mu = \text{span} \{(\xi,0), (0,1)\}$.

Hence

$$Q = \left\{ \delta \theta \in \mathbb{R}^3 : \delta \theta \cdot (\xi \times \sigma) = 0 \right\} = \text{span} \{\sigma, \xi\}, $$  \hfill (5.42)

$$G_{\mathcal{Q}} = \left\{ (\eta, \alpha) \in \mathbb{R}^3 \times \mathbb{R} : (\xi \times \eta) \cdot \sigma = 0 \right\}$$

$$= \text{span} \{(\xi,0), (0,1), (\sigma,0)\}$$

$$= G_\mu \cup G_{\Lambda_e} + \text{span} \{(\sigma,0)\}. $$ \hfill (5.43)

and

$$Q = G_{\mathcal{Q}} \cdot \Lambda_e. $$ \hfill (5.44)

$G \cdot \Lambda_e = T_{\Lambda_e}SO(3)$ implies $(G \cdot \Lambda_e)^\perp = 0$. Thus $\mathcal{A}$ can be either positive or negative semi-definite and

$$\mathcal{A}((\sigma,0),(\sigma,0)) = \frac{I_1 - I_2}{I_1 I_2} |\mu \times \sigma|^2 \neq 0 $$ \hfill (5.45)

implies the equilibrium is formally stable. The Euclidean inner product on $G \approx \mathbb{R}^3 \times \mathbb{R}$ is $G_\mu$ invariant. If the equilibrium $\nu_e$ is not in rotation about the axis of symmetry, then $\nu_e$ has trivial isotropy, so we can apply Corollary 4.1. If $\nu_e$ is in rotation about the axis of symmetry, then, as stated earlier, $\nu_e$ is formally stable if we replace $G$ with $SO(3)$, which acts freely. Hence we can again apply the corollary and we have shown that any relative equilibrium is orbitally stable.
5.3 \textit{SO}(3) invariant Lagrangians on $T \mathbb{R}^3$

As our final example, we consider a general $\text{SO}(3)$ equivariant Lagrangian system on $T \mathbb{R}^3$. $\text{SO}(3)$ invariance implies that the Lagrangian $L$ takes the form

$$L(q, \dot{q}) = f(\rho, \chi, \kappa),$$

(5.46)

where

$$\rho := |q|^2 \quad \chi := \langle q, \dot{q} \rangle \quad \kappa := \frac{1}{2} |\dot{q}|^2$$

(5.47)

for some function $f : \mathbb{R}^3 \to \mathbb{R}$. $L$ has partial derivatives

$$L_\dot{q} = f_\chi q + f_\kappa \dot{q}$$
$$L_{\dot{q}\dot{q}} = f_{\chi\chi} q \otimes q + f_{\chi\kappa} (q \otimes \dot{q} + \dot{q} \otimes q) + f_{\kappa \chi} \text{Id} + f_{\kappa \kappa} \dot{q} \otimes \dot{q}$$
$$L_q = 2 f_\rho q + f_\chi \dot{q}$$
$$L_{qq} = 2 f_{\rho \rho} q \otimes q + 4 f_{\rho \chi} q \otimes \dot{q} + 2 f_{\chi \chi} \dot{q} \otimes \dot{q}$$
$$L_{\dot{q}q} = 2 f_{\rho \kappa} \dot{q} \otimes q + f_{\chi \chi} \dot{q} \otimes \dot{q} + f_{\chi \kappa} \dot{q} \otimes \dot{q} + f_{\kappa \chi} \dot{q} \otimes \dot{q},$$

(5.48)

where all derivatives of $L$ are evaluated at the point $(q, \dot{q})$ and all derivatives of $f$ are evaluated at the corresponding triplet $(\rho, (q, \dot{q}), \frac{1}{2} |\dot{q}|^2)$.

We consider now a candidate relative equilibrium $v_e = (q_e, \xi \times q_e)$ for some $\xi \in \mathbb{R}^3$ such that $\xi \times q_e \neq 0$. Henceforth, all derivatives of $f$ are to be evaluated at the point $(\rho, 0, \frac{1}{2} \xi^2 \rho)$. The augmented Lagrangian has first variation

$$D L_{\xi}(q_e) \cdot \delta q = \langle L_{\dot{q}}, \delta q \rangle + \langle L_{\dot{q}q}, \xi \times \delta q \rangle$$
$$= \langle 2 f_\rho q - f_\kappa \xi \times (\xi \times q_e), \delta q \rangle$$
$$= \left( \left( 2 f_\rho + |\xi|^2 f_\kappa \right) q_e - f_\kappa (\xi, q_e) \xi \delta q \right).$$

(5.49)

Hence $v_e = (q_e, \xi \times q_e)$ is a relative equilibrium if

$$f_\kappa = f_\chi = 0 \quad \text{or} \quad \rho_e f_\rho + \kappa_e f_\kappa = \langle \xi, q_e \rangle = 0.$$

(5.50)

We now compute the linearized Legendre transformation and inertia tensor. $\langle \cdot, \cdot \rangle_e = L_{\dot{q}\dot{q}}(q_e, \xi \times q_e)$; hence, if we take $\{q_e/\sqrt{\rho_e}, (\xi \times q_e)/\sqrt{2\kappa_e}, \xi/|\xi|\}$ as a basis of $T_{q_e}Q$, then

$$\langle \cdot, \cdot \rangle_e = \begin{pmatrix}
    f_\kappa + \rho_e f_{\chi\chi} & \sqrt{2} \rho_e \kappa_e f_{\chi\kappa} & 0 \\
    \sqrt{2} \rho_e \kappa_e f_{\chi\kappa} & f_\kappa + 2 \kappa_e f_{\kappa\kappa} & 0 \\
    0 & 0 & f_\kappa
\end{pmatrix}.$$

(5.51)

Define $\Pi_\eta : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Pi_\eta(q) := |q|^2 \text{Id} - q \otimes q.$$ (5.52)

The linearized inertia tensor is determined by

$$\eta \cdot \Pi_\eta \zeta = \langle \eta \times q_e, \zeta \times q_e \rangle_e.$$ (5.53)

32
hence, if we take \( \{ q_e \sqrt{\rho_e}, (\xi \times q_e) / \sqrt{2\kappa_e}, \xi / |\xi| \} \) as a basis of of \( \mathcal{G} \), then (5.51) implies
\[
\Pi_e = \text{diag} \left[ 0, \rho_e^2 f_\kappa, \rho_e (f_\kappa + 2\kappa_e f_{\kappa\kappa}) \right]. \tag{5.54}
\]

\((L, \xi, q_e)\) is orbit regular if \( \ker \Pi_e = \mathcal{G}_{q_e} = \text{span} \{ q_e \} \); hence \((L, \xi, q_e)\) is orbit regular if
\[
-2\kappa e f_{\kappa\kappa} \neq f_\kappa \neq 0. \tag{5.55}
\]

We now consider the locked momentum map \( \Pi_\xi \), which takes the form
\[
\Pi_\xi(q) = f_\kappa(\rho, 0, \frac{1}{2} \xi^2 \rho) \Pi_g(q) \xi. \tag{5.56}
\]

The equilibrium momentum \( \mu \) is
\[
\mu = \Pi_\xi(q_e) = \rho_e f_\kappa \xi, \tag{5.57}
\]
with isotropy subalgebra \( \mathcal{G}_\mu = \text{span} \{ \xi \} \). At \( q_e \), \( \Pi_\xi(q) \) has first variation
\[
D\Pi_\xi(q_e) \cdot \delta q = f_\kappa(D\Pi_g(q_e) \cdot \delta g) + 2f_{\rho\kappa} \Pi_g(q_e) \xi \langle q_e, \delta q \rangle
+ f_{\kappa\kappa} \left( \frac{1}{2} \xi \cdot (D\Pi_g(q_e) \cdot \delta g) \right) \Pi_g(q_e) \xi
= 2 (f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa}) \langle q_e, \delta q \rangle \xi - f_\kappa \langle \xi, \delta q \rangle q_e. \tag{5.58}
\]

The space \( \mathcal{Q} \) of admissible variations is computed as follows: \( \mathcal{G}_{q_e} = \text{span} \{ q_e \} \) implies
\[
\mathcal{Q} := \left\{ \delta q \in T_{q_e} \mathcal{Q} : D\Pi_\xi(q_e) \cdot \delta q \in \mathcal{G}_{q_e}^A \right\}
= \left\{ \delta q \in T_{q_e} \mathcal{Q} : q_e \cdot (D\Pi_\xi(q_e) \cdot \delta q) = 0 \right\}
= \left\{ \delta q \in T_{q_e} \mathcal{Q} : \rho_e f_\kappa \langle \xi, \delta q \rangle = 0 \right\}
= \text{span} \{ q_e, \xi \times q_e \}. \tag{5.59}
\]

If we choose \( \tilde{\mathcal{V}} = \text{span} \{ q_e, \xi \} \) as a complement to \( \mathcal{G}_\mu \cdot q_e = \text{span} \{ \xi \times q_e \} \), then \( \mathcal{V} = \text{span} \{ q_e \} \). We now proceed to the computation of the stability tensor \( \mathcal{B}_e \).
\[
D\Pi_\xi(q_e) \cdot q_e = 2\rho_e (f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa}) \xi. \tag{5.60}
\]

For convenience, we choose \( \tilde{\mathcal{G}} = \text{span} \{ \xi, \xi \times q_e \} \); (5.54) implies
\[
\xi \cdot \tilde{\Pi}_e^{-1} \xi = \frac{|\xi|^2}{\rho_e (f_\kappa + 2\kappa_e f_{\kappa\kappa})}
= \frac{\rho_e^2 (f_\kappa + 2\kappa_e f_{\kappa\kappa})}{2\kappa_e}
= \rho_e^2 \left( \frac{f_\kappa}{2\kappa_e} + f_{\kappa\kappa} \right)^{-1} \tag{5.61}
\]
and hence
\[
(D\Pi_\xi(q_e) \cdot q_e) \cdot \tilde{\Pi}_e^{-1} (D\Pi_\xi(q_e) \cdot q_e) = 4 \left( \frac{f_\kappa}{2\kappa_e} + f_{\kappa\kappa} \right)^{-1} (f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa})^2. \tag{5.62}
\]
The second variation of the augmented Lagrangian is

\[ D^2 L_{\xi}(q_e) \left( q_e^2 \right) = \left( f_{\rho \rho} + \frac{1}{2} |\xi|^2 f_{\rho \kappa} + \frac{1}{4} f_{\kappa \kappa} \right) \left( 2 |q_e|^2 \right)^2 = 4 \left( \rho_e^2 f_{\rho \rho} + \rho_e \kappa_e f_{\rho \kappa} + \kappa_e^2 f_{\kappa \kappa} \right). \]  

(5.63)

Hence

\[ B_e \left( q_e^2 \right) = 4 \left( \frac{f_k}{2\kappa_e} + f_{\kappa \kappa} \right)^{-1} \left( f_k + \rho_e f_{\rho \kappa} + \kappa_e f_{\kappa \kappa} \right)^2 - \left( \rho_e^2 f_{\rho \rho} + \rho_e \kappa_e f_{\rho \kappa} + \kappa_e^2 f_{\kappa \kappa} \right). \]  

(5.64)

To obtain the final stability conditions, it is necessary to determine the definiteness of \( \langle \cdot, \cdot \rangle_e \) on \( \mathcal{G} \cdot q_e \perp = \text{span} \{ q_e \} \).

\[ \langle q_e, q_e \rangle_e = \rho_e (f_k + \rho_e f_{\kappa \kappa}); \]  

(5.65)

hence \( \langle \cdot, \cdot \rangle_e \mid_{\mathcal{G} \cdot q_e \perp} \) is positive (negative) definite if \( f_k + \rho_e f_{\kappa \kappa} > ( < ) 0 \).

To show that formal stability implies orbital stability, we note that \( \mathcal{G} \approx \mathbb{R}^3 \) possesses an invariant inner product. \( \xi \times q_e \neq 0 \) implies \( q_e \) and \( \xi \times q_e \) are not parallel; hence the relative equilibrium has trivial isotropy and we can apply Corollary 4.1.

To summarize, if \( q_e \) and \( \xi \) satisfy \( \xi \times q_e \neq 0 \) and

(i) \( \langle \xi, q_e \rangle = \rho_e f_{\rho \rho} + \kappa_e f_{\kappa \kappa} = 0 \) (equilibrium)

(ii) \( -2\kappa_e f_{\kappa \kappa} \neq f_k \neq 0 \) (orbit regularity)

(iii) \( f_k + \rho_e f_{\kappa \kappa} > ( < ) 0 \) and

\[ \left( \frac{f_k}{2\kappa_e} + f_{\kappa \kappa} \right)^{-1} \left( f_k + \rho_e f_{\rho \kappa} + \kappa_e f_{\kappa \kappa} \right)^2 - \rho_e^2 f_{\rho \rho} + \rho_e \kappa_e f_{\rho \kappa} + \kappa_e^2 f_{\kappa \kappa} > ( < ). \]  

(5.66)

(5.66)

(5.66)

(5.66)

then \( q_e \) and \( \xi \) determine an orbitally stable relative equilibrium.

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REFERENCES


35
APPENDIX: SUMMARY OF THE METHOD

For the convenience of the reader, we present here a brief summary of the method in the case that $Q$ and $G$ are both finite dimensional and the equilibrium is orbit regular. For the appropriate stability conditions when $Q$ or $G$ is infinite dimensional or the equilibrium is not known to be orbit regular, refer to the body of the text.

Let $L : TQ \to \mathbb{R}$ be a Lagrangian which is invariant under the tangent lift of the action of a Lie group $G$ on $Q$. Let there be a second order Lagrangian vector field $X_E$ associated to $L$; e.g., let $L$ be regular. Given $\xi \in \mathcal{G}$, define the infinitesimal generator $\xi_Q : Q \to TQ$ by

$$\xi_Q(q) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \exp(\epsilon \xi) \cdot q$$  \hspace{1cm} (A.1)

and the augmented Lagrangian $L_\xi : Q \to \mathbb{R}$ by

$$L_\xi(q) := L(\xi_Q(q)).$$  \hspace{1cm} (A.2)

$v_e \in T_{q_e}Q$ is a relative equilibrium if and only if there exists $\xi \in \mathcal{G}$ such that $v_e := \xi_Q(q_e)$ and $q_e$ is a critical point of $L_\xi$.

The stability conditions can be specified as follows: Let $v_e$ be a relative equilibrium with associated total velocity $\xi$. Define

(i) the Legendre transformation $F_L : TQ \to T^*Q$

$$F_L(v) \cdot w := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(v + \epsilon w)$$  \hspace{1cm} (A.3)

(ii) the locked momentum mapping $\Pi_\xi : Q \to \mathcal{G}^*$

$$\Pi_\xi(q) \cdot \eta := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L((\xi + \epsilon \eta)_Q(q)) \quad \forall \eta \in g$$  \hspace{1cm} (A.4)

(iii) the equilibrium momentum $\mu := \Pi_\xi(q_e)$

(iv) the bilinear form $\langle \cdot, \cdot \rangle_e$ on $T_{q_e}Q$ induced by the linearized Legendre transformation

$$\langle v, w \rangle_e := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left. \frac{d}{d\tau} \right|_{\tau=0} L(\xi_Q(q_e) + \epsilon v + \tau w)$$  \hspace{1cm} (A.5)

(v) the linearized inertia tensor $\Pi_e : \mathcal{G} \to \mathcal{G}^*$

$$\zeta \cdot \Pi_e \eta := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left. \frac{d}{d\tau} \right|_{\tau=0} L((\xi + \epsilon \eta + \tau \zeta)_Q(q_e))$$  \hspace{1cm} (A.6)
(vi) the $\langle , \rangle_e$-orthogonal complement $(G \cdot q_e)^\perp$ to the tangent space $G \cdot q_e$ of the group orbit

$$(G \cdot q_e)^\perp := \{ v \in T_{q_e}Q : \langle v, \eta Q(q_e) \rangle_e = 0 \ \forall \ \eta \in G \} \quad (A.7)$$

(vii) the isotropy subgroup $G_\mu$ of $\mu$ with respect to the coadjoint action and its algebra $G_\mu$

$$G_\mu := \{ g \in G : \text{Ad}^*_g \mu = \mu \} \quad \text{and} \quad G_\mu := \{ \eta \in G : \text{ad}^*_\eta \mu = 0 \} \quad (A.8)$$

(viii) the orbit $G_\mu \cdot q_e$ of the subalgebra $G_\mu$

$$G_\mu \cdot q_e := \{ \eta q(q_e) : \eta \in G_\mu \} \quad (A.9)$$

(ix) the isotropy subgroup $G_{q_e}$ of the equilibrium configuration $q_e$ and its subalgebra $G_{q_e}$

$$G_{q_e} := \{ g \in G : g \cdot q_e = q_e \} \quad \text{and} \quad G_{q_e} := \{ \eta \in G : \eta q(q_e) = 0 \} \quad (A.10)$$

(x) the isotropy subgroup $G_{v_e}$ of the equilibrium velocity $v_e$

$$G_{v_e} := \{ g \in G : g \cdot v_e = v_e \} \quad (A.11)$$

(xi) the subspace $G_Q \subset G$ of admissible group motions

$$G_Q := \{ \eta \in G : \text{ad}^*_\eta \mu \in G_{q_e}^A \} \quad (A.12)$$

If the group $G$ is finite dimensional, then the assumption that the triplet $(L, \xi, q_e)$ is orbit regular is equivalent to the assumption that $\ker \Omega_e = G_{q_e}$. Let $\bar{G}$ be a complement of $G_{q_e}$ and let $\bar{\Omega}_e$ denote the restriction of $\Omega_e$ to $\bar{G}$. (If $q_e$ has trivial isotropy, then $\bar{\Omega}_e = \Omega_e$.) Define

(i) the generalized Arnold form $\mathcal{A} : G_Q \times G_Q \to \mathbb{R}$

$$\mathcal{A}(\zeta, \eta) := \text{ad}^*_\xi \mu \cdot \left( \text{ad}^*_\eta \xi + \bar{\Omega}_e^{-1} \text{ad}^*_\eta \mu \right) \quad (A.13)$$

(ii) the space $Q_{\text{int}}$ of ‘internal’ variations

$$Q_{\text{int}} = \{ \delta q \in T_{q_e}Q : D\Omega_e(q_e) \cdot \delta q \in G_{q_e}^A \} \quad (A.14)$$

where

$$G_{\text{int}} := G_{q_e} \cup \{ \bar{\Omega}_e^{-1} (\text{ad}^*_\eta \mu) : \eta \in G_Q \} \quad (A.15)$$

(iii) the ‘internal’ stability two form $B_{\text{int}} : Q_{\text{int}} \times Q_{\text{int}} \to \mathbb{R}$

$$B_{\text{int}}(\delta q, \Delta q) := (D\Omega_e(q_e) \cdot \delta q) \cdot \bar{\Omega}_e^{-1} (D\Omega_e(q_e) \cdot \Delta q) - D^2L_\xi(q_e)(\delta q, \Delta q) \quad (A.16)$$

A relative equilibrium $v_e$ is formally stable if
(i) $\langle \cdot , \cdot \rangle_c$ is positive (negative) definite on $(G \cdot q_c)^\perp$

(ii) $A$ is positive (negative) semi-definite, with kernel $G_\mu \cup G_{q_c}$

(iii) $B_{\text{int}}$ is positive (negative) semi-definite, with kernel $G_\mu \cdot q_c$.

$v_\varepsilon$ is nonlinearly stable modulo $G_\mu$ if

(i) $v_\varepsilon$ is formally stable

(ii) there is a $G_\mu$ invariant inner product on $G$

(iii) there is a smooth global section $\sigma : G_\mu/(G_\mu \cap G_{v_\varepsilon}) \rightarrow G_\mu$,
    i.e. the principal bundle $G_\mu \rightarrow G_\mu/(G_\mu \cap G_{v_\varepsilon})$ is trivial.

Remarks: $A$, $Q_{\text{int}}$, and $B_{\text{int}}$ do not depend on the choice of $\tilde{G}$. Note that if $G_{q_\varepsilon} \subset G_\mu$, then $G_Q = G$ and

$$Q_{\text{int}} = \left\{ \delta q \in T_{q_\varepsilon}Q : \tilde{\Pi}_c^{-1}(D\tilde{\Pi}_\xi(q_\varepsilon) \cdot \delta q) \in G_\mu \right\}.$$  \hspace{1cm} (A.17)
<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>569</td>
<td>Li Ta-Tsien (Li Da-qian) and Zhao Yan-Chun</td>
<td>Global Existence of Classical Solutions to the Typical Free Boundary Problem for General Quasilinear Hyperbolic Systems and its Applications</td>
</tr>
<tr>
<td>570</td>
<td>Thierry Cazenave and Fred B. Weissler</td>
<td>The Structure of Solutions to the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation</td>
</tr>
<tr>
<td>571</td>
<td>Marshall Slemrod and Athanasiou E. Tsavaras</td>
<td>A Limiting Viscosity Approach for the Riemann Problem in Isentropic Gas Dynamics</td>
</tr>
<tr>
<td>573</td>
<td>P.J. Vassiliev</td>
<td>On the Geometry of Semi-Linear Hyperbolic Partial Differential Equations in the Plane Integrable by the Method of Darboux</td>
</tr>
<tr>
<td>574</td>
<td>Jerome V. Moloney and Alan C. Newell</td>
<td>Nonlinear Optics</td>
</tr>
<tr>
<td>575</td>
<td>Keti Tenenblat</td>
<td>A Note on Solutions for the Intrinsic Generalized Wave and Sine-Gordon Equations</td>
</tr>
<tr>
<td>576</td>
<td>P. Szmolyan</td>
<td>Heteroclinic Orbits in Singularly Perturbed Differential Equations</td>
</tr>
<tr>
<td>577</td>
<td>Wenxiong Liu</td>
<td>A Parabolic System Arising In Film Development</td>
</tr>
<tr>
<td>578</td>
<td>Daniel B. Dix</td>
<td>Temporal Asymptotic Behavior of Solutions of the Benjamin-Ono-Burgers Equation</td>
</tr>
<tr>
<td>579</td>
<td>Michael Renardy and Yuriko Renardy</td>
<td>On the nature of boundary conditions for flows with moving free surfaces</td>
</tr>
<tr>
<td>580</td>
<td>Werner A. Stahel</td>
<td>Robust Statistics: From an Intellectual Game to a Consumer Product</td>
</tr>
<tr>
<td>581</td>
<td>Avner Friedman and Fernando Reitich</td>
<td>The Stefan Problem with Small Surface Tension</td>
</tr>
<tr>
<td>582</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Separation of Variables Methods for Systems of Differential Equations in Mathematical Physics</td>
</tr>
<tr>
<td>583</td>
<td>Mitchell Luskin and George R. Sell</td>
<td>The Construction of Inertial Manifolds for Reaction-Diffusion Equations by Elliptic Regularization</td>
</tr>
<tr>
<td>584</td>
<td>Konstantin Mischaikow</td>
<td>Dynamic Phase Transitions: A Connection Matrix Approach</td>
</tr>
<tr>
<td>585</td>
<td>Philippe Le Floch and Li Tatsien</td>
<td>A Global Asymptotic Expansion for the Solution to the Generalized Riemann Problem</td>
</tr>
<tr>
<td>586</td>
<td>Matthew Witten, Ph.D.</td>
<td>Computational Biology: An Overview</td>
</tr>
<tr>
<td>587</td>
<td>Matthew Witten, Ph.D.</td>
<td>Peering Inside Living Systems: Physiology in a Supercomputer</td>
</tr>
<tr>
<td>588</td>
<td>Michael Renardy</td>
<td>An existence theorem for model equations resulting from kinetic theories of polymer solutions</td>
</tr>
<tr>
<td>590</td>
<td>Luigi Preziosi</td>
<td>An Invariance Property for the Propagation of Heat and Shear Waves</td>
</tr>
<tr>
<td>591</td>
<td>Gregory M. Constantine and John Bryant</td>
<td>Sequencing of Experiments for Linear and Quadratic Time Effects</td>
</tr>
<tr>
<td>592</td>
<td>Prabir Daripa</td>
<td>On the Computation of the Beltrami Equation in the Complex Plane</td>
</tr>
<tr>
<td>593</td>
<td>Philippe Le Floch</td>
<td>Shock Waves for Nonlinear Hyperbolic Systems in Nonconservative Form</td>
</tr>
<tr>
<td>595</td>
<td>Mark J. Friedman and Eusebius J. Doedel</td>
<td>Numerical computation and continuation of invariant manifolds connecting fixed points</td>
</tr>
<tr>
<td>596</td>
<td>Scott J. Spector</td>
<td>Linear Deformations as Global Minimizers in Nonlinear Elasticity</td>
</tr>
<tr>
<td>597</td>
<td>Denis Serre</td>
<td>Richness and the classification of quasilinear hyperbolic systems</td>
</tr>
<tr>
<td>598</td>
<td>L. Preziosi and F. Rosso</td>
<td>On the stability of the shearing flow between pipes</td>
</tr>
<tr>
<td>599</td>
<td>Avner Friedman and Wenxiong Liu</td>
<td>A system of partial differential equations arising in electrophotography</td>
</tr>
<tr>
<td>600</td>
<td>Jonathan Bell, Avner Friedman, and Andrew A. Lacey</td>
<td>On solutions to a quasilinear diffusion problem from the study of soft tissue</td>
</tr>
<tr>
<td>601</td>
<td>David G. Schaeffer and Michael Shearer</td>
<td>Loss of hyperbolicity in yield vertex plasticity models under nonproportional loading</td>
</tr>
<tr>
<td>602</td>
<td>Herbert C. Krazer and Barbara Lee Keyfitz</td>
<td>A strictly hyperbolic system of conservation laws admitting singular shocks</td>
</tr>
<tr>
<td>603</td>
<td>S. Laederich and M. Levi</td>
<td>Qualitative dynamics of planar chains</td>
</tr>
<tr>
<td>604</td>
<td>Milan Miklavčič</td>
<td>A sharp condition for existence of an inertial manifold</td>
</tr>
<tr>
<td>605</td>
<td>Charles Collins, David Kinderlehrer, and Mitchell Luskin</td>
<td>Numerical approximation of the solution of a variational problem with a double well potential</td>
</tr>
<tr>
<td>606</td>
<td>Todd Arbogast</td>
<td>Two-phase incompressible flow in a porous medium with various nonhomogeneous boundary conditions</td>
</tr>
<tr>
<td>#</td>
<td>Author/s</td>
<td>Title</td>
</tr>
<tr>
<td>----</td>
<td>----------------------------------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>607</td>
<td>Peter Poláčik</td>
<td>Complicated dynamics in scalar semilinear parabolic equations in higher space dimension</td>
</tr>
<tr>
<td>608</td>
<td>Bei Hu</td>
<td>Diffusion of penetrant in a polymer: a free boundary problem</td>
</tr>
<tr>
<td>609</td>
<td>Mohamed Sami ElBialy</td>
<td>On the smoothness of the linearization of vector fields near resonant hyperbolic rest points</td>
</tr>
<tr>
<td>610</td>
<td>Max Jodeit, Jr. and Peter J. Olver</td>
<td>On the equation $\text{grad } f = M \text{ grad } g$</td>
</tr>
<tr>
<td>611</td>
<td>Shui-Nee Chow, Kening Lu, and Yun-Qiu Shen</td>
<td>Normal form and linearization for quasiperiodic systems</td>
</tr>
<tr>
<td>612</td>
<td>Prabir Daripa</td>
<td>Theory of one dimensional adaptive grid generation</td>
</tr>
<tr>
<td>613</td>
<td>Michael C. Mackey and John G. Milton</td>
<td>Feedback, delays and the origin of blood cell dynamics</td>
</tr>
<tr>
<td>614</td>
<td>D.G. Aronson and S. Kamin</td>
<td>Disappearance of phase in the Stefan problem: one space dimension</td>
</tr>
<tr>
<td>615</td>
<td>Martin Krupa</td>
<td>Bifurcations of relative equilibria</td>
</tr>
<tr>
<td>616</td>
<td>D.D. Joseph, P. Singh, and K. Chen</td>
<td>Couette flows, rollers, emulsions, tall Taylor cells, phase separation and inversion, and a chaotic bubble in Taylor-Couette flow of two immiscible liquids</td>
</tr>
<tr>
<td>617</td>
<td>Artemio González-López, Niky Kamran, and Peter J. Olver</td>
<td>Lie algebras of differential operators in two complex variables</td>
</tr>
<tr>
<td>618</td>
<td>L.E. Fraenkel</td>
<td>On a linear, partly hyperbolic model of viscoelastic flow past a plate</td>
</tr>
<tr>
<td>619</td>
<td>Stephen Schechter and Michael Shearer</td>
<td>Undercompressive shocks for nonstrictly hyperbolic conservation laws</td>
</tr>
<tr>
<td>620</td>
<td>Xinfu Chen</td>
<td>Axially symmetric jets of compressible fluid</td>
</tr>
<tr>
<td>621</td>
<td>J. David Logan</td>
<td>Wave propagation in a qualitative model of combustion under equilibrium conditions</td>
</tr>
<tr>
<td>622</td>
<td>M.L. Zeeman</td>
<td>Hopf bifurcations in competitive three-dimensional Lotka-Volterra Systems</td>
</tr>
<tr>
<td>623</td>
<td>Allan P. Forsdy</td>
<td>Isospectral flows: their Hamiltonian structures, Miura maps and master symmetries</td>
</tr>
<tr>
<td>624</td>
<td>Daniel D. Joseph, John Nelson, Michael Renardy, and Yuriko Renardy</td>
<td>Two-Dimensional cusped interfaces</td>
</tr>
<tr>
<td>625</td>
<td>Avner Friedman and Bei Hu</td>
<td>A free boundary problem arising in electrophotography</td>
</tr>
<tr>
<td>626</td>
<td>Hamid Bellout, Avner Friedman and Victor Isakov</td>
<td>Stability for an inverse problem in potential theory</td>
</tr>
<tr>
<td>627</td>
<td>Barbara Lee Keyfitz</td>
<td>Shocks near the sonic line: A comparison between steady and unsteady models for change of type</td>
</tr>
<tr>
<td>628</td>
<td>Barbara Lee Keyfitz and Gerald G. Warnecke</td>
<td>The existence of viscous profiles and admmissibility for transonic shocks</td>
</tr>
<tr>
<td>629</td>
<td>P. Szmolyan</td>
<td>Transversal heteroclinic and homoclinic orbits in singular perturbation problems</td>
</tr>
<tr>
<td>630</td>
<td>Philip Boyland</td>
<td>Rotation sets and monotone periodic orbits for annulus homeomorphisms</td>
</tr>
<tr>
<td>631</td>
<td>Kenneth R. Meyer</td>
<td>Apollonius coordinates, the N-body problem and continuation of periodic solutions</td>
</tr>
<tr>
<td>632</td>
<td>Chjan C. Lim</td>
<td>On the Poincare–Whitney circuitspace and other properties of an incidence matrix for binary trees</td>
</tr>
<tr>
<td>634</td>
<td>Stanley Minkowitz and Matthew Witten</td>
<td>Periodicity in cell proliferation using an asynchronous cell population</td>
</tr>
<tr>
<td>635</td>
<td>M. Chipot and G. Dal Maso</td>
<td>Relaxed shape optimization: The case of nonnegative data for the Dirichlet problem</td>
</tr>
<tr>
<td>636</td>
<td>Jeffery M. Franke and Harlan W. Stech</td>
<td>Extensions of an algorithm for the analysis of nongeneric Hopf bifurcations, with applications to delay-difference equations</td>
</tr>
<tr>
<td>637</td>
<td>Xinfu Chen</td>
<td>Generation and propagation of the interface for reaction–diffusion equations</td>
</tr>
<tr>
<td>638</td>
<td>Philip Korman</td>
<td>Dynamics of the Lotka–Volterra systems with diffusion</td>
</tr>
<tr>
<td>639</td>
<td>Harlan W. Stech</td>
<td>Generic Hopf bifurcation in a class of integro-differential equations</td>
</tr>
<tr>
<td>640</td>
<td>Stephane Laederich</td>
<td>Periodic solutions of non linear differential difference equations</td>
</tr>
<tr>
<td>641</td>
<td>Peter J. Olver</td>
<td>Canonical Forms and Integrability of BiHamiltonian Systems</td>
</tr>
<tr>
<td>642</td>
<td>S.A. van Gils, M.P. Krupa and W.F. Langford</td>
<td>Hopf bifurcation with nonsemisimple 1:1 Resonance</td>
</tr>
<tr>
<td>643</td>
<td>R.D. James and D. Kinderlehrer</td>
<td>Frustration in ferromagnetic materials</td>
</tr>
<tr>
<td>644</td>
<td>Carlos Rocha</td>
<td>Properties of the attractor of a scalar parabolic P.D.E.</td>
</tr>
<tr>
<td>645</td>
<td>Debra Lewis</td>
<td>Lagrangian block diagonalization</td>
</tr>
</tbody>
</table>