DYNAMICS OF THE LOTKA–VOLTERRA SYSTEMS WITH DIFFUSION

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1. Introduction.

We study the steady states and the long term behavior for the following system describing interaction of two species in the Lotka-Volterra model with diffusion,

\[ u_t = \Delta u + u(a - bu + cv) \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \]

\[ (1.1) \]

\[ v_t = \Delta v + v(d + eu - fv) \text{ in } \Omega, \ v = 0 \text{ on } \partial \Omega. \]

Here \( a, d, b, \) and \( f \) are positive constants; the constants \( c \) and \( e \) can be of either sign, and those signs determine the type of interaction. Throughout the paper \( \Omega \) denotes a smooth domain in \( \mathbb{R}^n \), and we are looking for the classical solutions \( u(x,t) \) and \( v(x,t) \), \( x \in \Omega \) and \( t > 0 \). Also, we shall always assume that \( b = f = 1 \), which can be achieved by rescaling. The problem \((1.1)\), particularly its steady states, have been studied in a number of papers, see e.g., [3,4,7,8,11] and the references therein. Local existence and uniqueness for \((1.1)\) follow from [1].

Two things determine the global behavior of solutions of \((1.1)\). One is existence or non-existence of steady states for \((1.1)\). For example, in the cooperating species case (i.e. \( c > 0, \ e > 0 \)), we show that condition \( ec > 1 \) implies blow up in finite time of any nontrivial non-negative solution of \((1.1)\). This is because the corresponding steady state problem has no positive solution, as was shown in [8]. The other thing determining the global behavior is stability or instability of trivial solutions of the steady state problem for \((1.1)\), i.e. of \((0,0)\), \((u_a,0)\) and \((0,u_d)\). Here and throughout we

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denote by $u_a$ the positive in $\Omega$ solution of

$$
\Delta u + u(a-u) = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega,
$$

which exists for $a > \lambda_1$, where $\lambda_1$ and $\varphi_1(x) > 0$ denote the principal eigenpair of $-\Delta$ on $\Omega$.

The importance of the steady states for global behavior can be seen even in the scalar case. Consider, for example

$$(1.2) \quad u_t = \Delta u + u^3 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega.$$

If we define the "energy" $J(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{4} u^4 \right) dx$, we easily see that for any nontrivial steady state $\bar{u}(x)$, $J(\bar{u}) > 0$. Also it is easy to check that $J(u(x,t)) \leq J(u(x,0))$. Assume now that

$$(1.3) \quad J(u(x,0)) \leq 0.$$

If the solution $u(x,t)$ was bounded, then by H. Matano [10] its $\omega$-limit set in $C^1(\Omega)$ would be non-empty and consist of steady-state solutions. This would contradict the above considerations, so that $u(x,t)$ must become unbounded, and it is natural to expect that this would happen in finite time. Here is a rigorous proof that (1.3) implies non-existence of a global solution (see also [9]). Multiply (1.2) by $u$, and denote $G = \int_{\Omega} u^2 dx$. Then

$$
\frac{1}{2} G' = -2J(u) + \frac{1}{4} \int_{\Omega} u^4 dx \geq c_1 G^2 \text{ for some } c_1 > 0.
$$

Some remarks on notation. By $u(x,t;u_0,v_0)$ and $v(x,t;u_0,v_0)$ we denote the solution of (1.1) depending on the data $u(x,0) = u_0(x), \ v(x,0) = v_0(x)$. For scalar equations (like (1.2)) the corresponding notation is $u(x,t;u_0)$. We abbreviate $\int_{\Omega} u = \int_{\Omega} u(x) dx$. By $\lambda_1(\Delta + a(x))$ be denote the principal eigenvalue of the operator $\Delta + a(x)$.

2. Blow up for parabolic equations.

We consider the problem
(2.1) \( u_t = \Delta u + f(u) \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \) \( u(x,0) = u_0(x) \),

although our result can be stated more generally, similarly to H. Levine [9].

In particular we can cover second order systems and higher order equations
like \( u_t = -\Delta^2 u + f(u) \).

Define \( G = \int u^2 \), and the "energy" \( J(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - F(u) \),

\( F\)-antiderivative of \( f \). Notice that formally

(2.2) \[
\frac{d}{dt} J(u) = -\int u_t^2 \leq 0.
\]

**Theorem 2.1.** Assume that for some \( c_1 > 0, \beta > 0 \)

(1) \[
\int f(u) u - 2F(u) \geq c_1 G^{1+\beta} \text{ along the solution of (2.1)},
\]

(11) \[
c_1 G^{1+\beta}(u_0) - 2J(u_0) > 0.
\]

Then the problem (2.1) cannot have a global smooth solution (smooth enough to
justify (2.2)).

**Proof.** Multiply (2.1) by \( u \) and integrate

\[
\frac{1}{2} G' = -2J(u) + \int uf(u) - 2F(u) \geq -2J(u_0) + c_1 G^{1+\beta}.
\]

This differential inequality in view of (ii) cannot have global solutions.

**Example 1.** \( f(u) = u^p \), \( p \geq 3 \) odd integer. Then

\[
\int uf(u) - F(u) \geq \frac{p}{p+1} |\Omega| \frac{-p-1}{2} \frac{p+1}{2} G^2(u),
\]

and so the condition for blow up is

\[
\frac{p}{p+1} |\Omega| \frac{-p-1}{2} \frac{p+1}{2} G^2(u_0) - 2J(u_0) > 0.
\]

In particular, \( J(u_0) \leq 0 \) and \( u_0 \equiv 0 \) imply the blow up.

**Example 2.** \( f(u) = u^p \), \( p \geq 2 \) even. This time we need to assume
additionally that \( u_0 \equiv 0 \), so that by the maximum principle \( u(x,t) \equiv 0 \) for all \( x \) and \( t \), and the rest is as above.

**Example 3.** The problem \( (k = \text{const}) \)

(2.3) \[
\frac{1}{t} \frac{\partial}{\partial t} u = \Delta u + u^3 + k\int u dx \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega
\]

belongs to a class recently studied by B. Fiedler and P. Polačik [5]. They
showed that (2.3) has a Lyapunov functional

\[
J(u) = \int \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{4} u^4 \right) - \frac{k}{2} \int u^2.
\]

As in the theorem 2.1 one shows that the condition \( J(u(x,0)) \leq 0 \) implies blow up (notice that \( u(x,0) \) does not have to be "large" for that).

3. Dynamics of a logistic equation with diffusion.

Before considering interactions of several species, we study dynamics of a single specie with population density \( u(x,t) \) satisfying

\[
(3.1) \quad u_t = \Delta u + u(a(x) - u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]

\[
u(x,0) = u_0(x) \quad (u_0(x) \in C^2(\Omega) \text{ and } u_0 = 0 \text{ on } \partial \Omega).
\]

Here \( \Omega \) is a smooth domain in \( \mathbb{R}^n \), \( a(x) \in C^\alpha(\bar{\Omega}) \) for some \( 0 < \alpha < 1 \).

**Lemma 3.1.** Assume that \( u(x) > 0 \) in \( \Omega \) is a solution of

\[
(3.2) \quad \Delta u + u(a(x) - u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]

with \( a(x) \geq 0 \) and \( a(x) \neq 0 \) in \( \Omega \). Then \( u(x) \) is the only nontrivial non-negative solution of (3.2).

**Proof.** Since \( \varphi = \max_{\Omega} a(x) \) is a supersolution, and \( \psi = 0 \) a subsolution, it follows that (3.2) has a maximal solution. It has to be \( u(x) \), since (3.2) cannot have more than one positive solution (the proof is similar to the one below, but easier). Let \( v(x) \) be another nontrivial solution with

\[
0 \leq v(x) \leq u(x).
\]

Since

\[
\int_{\Omega} uv(u-v) dx = 0,
\]

\( u(x) = v(x) \) except possibly when \( v(x) = 0 \). Let \( \overline{x} \in \Omega \) be such that \( v(\overline{x}) = 0 \) but \( v(x_n) > 0 \) for a sequence \( \{x_n\} \to \overline{x} \). Then by the continuity of \( u \) and \( v \)

\[
u(\overline{x}) = \lim_{n \to \infty} u(x_n) = \lim_{n \to \infty} v(x_n) = v(\overline{x}) = 0,
\]

which contradicts \( u > 0 \).

**Lemma 3.2.** The problem (3.2) has a positive solution if and only if

\[
\lambda = \lambda_1(\Delta + a(x)) > 0.
\]
Proof. Sufficiency. Large constants are supersolutions of (3.2). Let $\bar{u} > 0$ be a principal eigenfunction of $\Delta + a(x)$, i.e.

\begin{equation}
\Delta \bar{u} + a(x)\bar{u} = \lambda \bar{u}.
\end{equation}

Then one easily checks that $\epsilon \bar{u}$ is a subsolution of (3.2), provided $\epsilon$ is sufficiently small.

Necessity. We assume that $\lambda_1(\Delta + a(x)) \leq 0$. This implies that

$$
\int_\Omega (\Delta u + a(x)u) u dx \leq 0 \text{ for any } u \in C^2(\Omega).
$$

Multiplying (3.2) by $u$ and integrating, we see it can have no nontrivial solution.

We denote by $u_a$ the positive solution of (3.2). By $u(x, t; u_0)$ we denote the solution of (3.1).

**Theorem 3.1.** Assume that (3.3) holds and $u_0 \geq 0$ in $\Omega$, $u_0 \neq 0$. Then

\begin{equation}
\lim_{t \to \infty} u(x, t; u_0) = u_a \quad \text{in } C^1(\overline{\Omega}).
\end{equation}

If the condition (3.3) fails then $\lim_{t \to \infty} u(x, t; u_0) = 0$.

Proof. Suppose (3.3) holds, and assume first that $\epsilon \varphi_1(x) \leq u_0(x)$ for all $x \in \Omega$, if $\epsilon$ is small enough. By the maximum principle

\begin{equation}
u(x, t; \epsilon \varphi_1) \leq u(x, t; u_0) \leq u(x, t; M).
\end{equation}

By [12] it follows that $u(x, t; \epsilon \varphi_1)$ and $u(x, t; M)$ tend respectively to the minimal and maximal solutions of (3.2). In view of lemma 3.1 they both tend to $u_a$, and the theorem follows by (3.6).

For the general $u_0$ the theorem 2.8 in Matano [10] implies that the $\omega$-limit set of $u_0$ in $C^1(\overline{\Omega})$ is non-empty, and consists of non-negative solutions of (3.2), which by lemma 3.1 are zero and $u_a$. If $u_a \in \omega(u_0)$ then at some $T$, $u(x, T) > \epsilon \varphi_1(x)$ for some small $\epsilon$, and then taking $T$ as a new initial time we conclude (3.5).

It remains to exclude the possibility that $\omega(u_0) = \{0\}$ in $C^1(\overline{\Omega})$. If that was the case then $u(x, t) \to 0$ uniformly in $x$, i.e. $u(x, t) < \epsilon$ for $t \geq T$.
Setting $H = \int_{\Omega} u(x,t) \tilde{u}(x) \, dx$ ($\tilde{u}$ as defined by (3.4)) we obtain from (3.1) for $t \geq T$

$$H' \geq \tilde{\lambda} H - cH \geq cH \quad (\text{for some } c > 0)$$

with $H(T) > 0$. This shows that $H(t)$ cannot tend to zero as $t \to \infty$, a contradiction. ($H(t)$ cannot become zero at a finite time $T$, for otherwise step back a little, and repeat the above argument).

If (3.3) fails, then in view of lemma 3.2, $u(x,t; M)$ must tend to zero, and so does $u(x,t; u_0)$, which finishes the proof.

Assuming (3.3), denote by $u_{a+\epsilon}$, $\epsilon > 0$, the positive solution of

$$\Delta u + u(a(x)+\epsilon - u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \delta \Omega.$$  \tag{3.7}

Then $u_{a+\epsilon} > u_a$ in $\Omega$ (notice that $u_a$ is a subsolution of (3.7), and then the inequality follows as in [11]).

We shall need the following lemmas.

\textbf{Lemma 3.3.} \[ \lim_{\epsilon \to 0} \| u_{a+\epsilon} - u_a \|_{L^\infty(\Omega)} = 0. \]

\textbf{Proof.} Let $\epsilon \leq 1$. By the maximum principle

$$\| u_{a+\epsilon} \|_{L^\infty(\Omega)} \leq a+1.$$ 

By the usual boot-strap (for any $0 < \alpha < 1$)

$$\| u_{a+\epsilon} \|_{C^{2+\alpha}(\Omega)} \leq c \quad \text{independent of } \epsilon \leq 1.$$ 

Let $\{ \epsilon_k \}$ be an arbitrary sequence such that $\epsilon_1 > \epsilon_2 > \ldots > 0$. Then

$$u_{a+\epsilon_1} > u_{a+\epsilon_2} > \ldots > u_a > 0 \quad \text{in } \Omega.$$ 

Since $C^{2+\alpha}(\Omega)$ imbeds compactly into $C^2(\Omega)$, a subsequence $u_{a+\epsilon_{k_j}}$ converges in $C^2(\Omega)$ to some $w(x) > 0$, which is then a solution of (3.2). By uniqueness $w = u_a$, and by monotonicity the entire sequence $\{ u_{a+\epsilon_k} \}$ converges to $u_a$.

\textbf{Lemma 3.4.} For the problem

$$v_t = \Delta v + a(x,t)v \text{ in } \Omega, \quad v = 0 \text{ on } \delta \Omega,$$

assume that $v(x,0) \geq 0$, $v(x,0) \not\equiv 0$, and $|a(x,t)| \leq c$ uniformly in $x \in \Omega$ and
t > 0. Then v(x,t) cannot go to zero in finite time.

**Proof.** Set $H = \int_{\Omega} v(x,t) \varphi_1(x) dx$. Then $H(0) > 0$ and $H(t) \geq H(0)e^{-(\lambda_1+c)t}$.


We consider a system ($\Omega$ a smooth domain in $\mathbb{R}^n$)

\[
\begin{align*}
    u_t &= \Delta u + u(a - u + cv) \text{ in } \Omega, \quad u = 0 \text{ on } \delta\Omega \\
    v_t &= \Delta v + v(d + eu - v) \text{ in } \Omega, \quad v = 0 \text{ on } \delta\Omega \\
    u(x,0) &= u_0(x), \quad v(x,0) = v_0(x),
\end{align*}
\]

(4.1) describing cooperative interaction of two species with population densities $u(x,t)$ and $v(x,t)$. We assume that $a, c, d, e$ are positive constants, although we can admit for $a$ and $d$ to be functions of $x$. Corresponding steady state system

\[
\Delta u + u(a - u + cv) = 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \delta\Omega
\]

(4.2)

\[
\Delta v + v(d + eu - v) = 0 \quad \text{in } \Omega, \quad v = 0 \text{ on } \delta\Omega,
\]

was analyzed in Korman-Leung [8], where the following theorem was proved.

**Theorem 4.1.** Assume that $a > \lambda_1, d > \lambda_1$. Then (4.2) has a positive solution (i.e. $u > 0$ and $v > 0$ in $\Omega$) if and only if $ec < 1$.

If $ec > 1$ then it is easy to see that both $u(x,t)$ and $v(x,t)$ become unbounded as $t$ increases, if

(4.3) $u_0 \geq 0, \quad v_0 \geq 0 \text{ with } u_0 \neq 0, \quad v_0 \neq 0 \text{ in } \Omega$.

The following theorem shows more.

**Theorem 4.2.** Assume that $a > \lambda_1, d > \lambda_1$ and the initial data satisfies (4.3). If $ec > 1$, then solution of (4.1) blows up in finite time. If $ec = 1$ then solution of (4.1) exists for all time, and as $t \to +\infty$,

(4.4) $\|u(x,t)\|_{L^\infty(\Omega)} \to \infty, \quad \|v(x,t)\|_{L^\infty(\Omega)} \to \infty$.

If $ec < 1$ then solution exists for all time and is bounded in $L^\infty(\Omega)$.

We prove this theorem at the end of this section after presenting some results on which it depends, which are also of independent interest. We start with a corresponding ODE.
Theorem 4.3. Let $u(t)$ and $v(t)$ be solutions of

\[ \begin{align*}
\dot{u} &= u(a - u + cv), \quad u(0) = u_0 > 0 \\
\dot{v} &= v(d + eu - v), \quad v(0) = v_0 > 0.
\end{align*} \tag{4.5} \]

Here $a,c,d,e$ are positive constants.

(i). If $ec < 1$ then

\[ \lim_{t \to \infty} u(t) = \frac{a + cd}{1 - ec}, \quad \lim_{t \to \infty} v(t) = \frac{ae + d}{1 - ec}. \]

(ii). If $ec = 1$ then solution exists for all $t > 0$, and $\lim_{t \to +\infty} u(t) = +\infty$.

\[ \lim_{t \to +\infty} v(t) = +\infty. \]

(iii). If $ec > 1$ then both $u(t)$ and $v(t)$ go to $+\infty$ in finite time.

Proof. The claim (i) is standard. The limits in (i) are coordinates of the point of intersection of the lines $\ell_1: a - u + cv = 0$ and $\ell_2: d + eu - v = 0$ in the $(u,v)$ plane. In cases (ii) and (iii) the lines $\ell_1$ and $\ell_2$ do not intersect and hence all solutions must tend to $\infty$. This is because all trajectories eventually enter and stay in the region

\[ A = \{(u,v) \in \mathbb{R}^2_+ | 0 < v < d+eu \text{ for } 0 < u \leq a, \quad \frac{1}{c}u - \frac{1}{c}a < v < d+eu \text{ for } u > a\}. \]

To prove (iii) decompose $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{(u,v) \in A | d+eu - v \leq cv\}$, $A_2 = \{(u,v) \in A | a - u + cv \leq cu\}$, where $c > 0$ is fixed small enough so that the remaining set $A_3$ extends to $\infty$ (i.e. $A_1$ and $A_2$ are disjoint). If a trajectory in $A$ stays outside $A_1$ then $v$ blows up, if it stays outside $A_2$ then $u$ blows up, and in either case both $u$ and $v$ blow up in finite time. It remains to notice that a trajectory in $A$ cannot go between $A_1$ and $A_2$ more than once. This is because in $A$

\[ \frac{dv}{du} = \frac{v(d+eu-v)}{u(a-u+cv)} > 0. \]

Turning to the final case (ii), set $\mu_1(t) = e^{-at}$, $\mu_2(t) = e^{-dt}$. Rewrite (4.5) as ($e = \frac{1}{c}$)

\[ \frac{d}{dt}(\mu_1 u) = \mu_1 u(-u + cv) \]

\[ \tag{4.6} \]
\[
\frac{d}{dt}(\mu_2 v) = \mu_2 (\frac{1}{c}u - v).
\]

Denoting further \(U = \mu_1 u, V = \mu_2 v\) and dividing the first equation in (4.6) by the second one,
\[
\frac{dU}{dV} = -c \frac{U}{V}.
\]

Integrating and returning to the original \(u\) and \(v\),
\[
\mu_1 u = \frac{c_1}{\mu_2 v} \quad \text{(} c_1 \text{ is a constant of integration)}.
\]

This implies that \(v\) cannot go to \(\infty\) in finite time, since otherwise \(u\) would have to go to 0 in finite time. Since we already know that \(u(t)\) and \(v(t)\) tend to \(\infty\) as \(t \to \infty\), the theorem is proved.

**Lemma 4.1.** For the problem (4.1) assume that \(u_0 \geq 0, u_0 \neq 0, v_0 \geq 0, v_0 \neq 0\) in \(\Omega\), and
\[
-\Delta u_0 \leq u_0 (a - u_0 + cv_0) \text{ in } \Omega, \quad u_0 = 0 \text{ on } \partial \Omega
\]
(4.7)
\[-\Delta v_0 \leq v_0 (d + eu_0 - v_0) \text{ in } \Omega, \quad v_0 = 0 \text{ on } \partial \Omega.
\]

Let \((u^0, v^0)\) satisfy inequalities like (4.7) but with the signs reversed and \(u^0 \geq 0\) and \(v^0 \geq 0\) on \(\partial \Omega\). Assume moreover that \(u^0 \geq u_0\) and \(v^0 \geq v_0\) in \(\Omega\). Then
\[
u_0 \leq u(x, t; u^0, v^0) \leq u^0
\]
(4.8)
\[

\]

and
\[
(4.9) \quad u_t(x, t; u^0, v^0) \geq 0, \quad v_t(x, t; u^0, v^0) \geq 0 \text{ in } \Omega \times R_+.
\]

**Proof.** Set \(w_1 = u-u^0, w_2 = v-v^0\). Then
\[
w_{1t} - \Delta w_1 \geq w_1 (a - u + cv - u_0) + cu_0 w_2
\]
\[
w_{2t} - \Delta w_2 \geq ev_0 w_1 + w_2 (d + eu - v + v_0)
\]
\[
w_1(x, 0) = 0 \text{ on } \partial \Omega, \quad w_i = 0 \text{ on } \partial \Omega, \quad i = 1, 2.
\]

By the maximum principle for weakly coupled systems, \(u \geq u_0\) and \(v \geq v_0\) in \(\Omega \times R_+\). To prove the other side of (4.8), define \(w_1 = u^0 - u, w_2 = v^0 - v\). Then
\[ w_{1t} - \Delta w_1 \geq w_1(a - u^0 - u + c\nu^0) + cuw_2 \]
\[ w_{2t} - \Delta w_2 \geq evw_1 + w_2(d + eu^0 - v + \nu^0) \]
\[ w_i(x, 0) \geq 0, \quad i = 1, 2. \]

Notice that by the previous part \( u \geq 0, \nu \geq 0 \) in \( \Omega \). Applying the maximum principle again, we conclude (4.8).

Next, define \( u_h = \frac{u(x, t+h; u_0, \nu^0)}{h} - u(x, t; u_0, \nu^0) \)
\[ v_h = \frac{v(x, t+h; u_0, \nu^0)}{h} - v(x, t; u_0, \nu^0). \]

Since by (4.8) \( u_h(x, 0) \geq 0 \) and \( v_h(x, 0) \geq 0 \), an argument similar to the one above shows that \( u_h(x, t) \geq 0 \) and \( v_h(x, t) \geq 0 \) for all \( x \in \Omega, \quad t > 0 \). Letting \( h \to 0 \), we conclude (4.9).

**Remark.** By a similar argument one sees that \( u(x, t; u^0, \nu^0) \) and \( v(x, t; u^0, \nu^0) \) satisfy (4.8), and \( u_t(x, t; u^0, \nu^0) \leq 0, \quad v_t(x, t; u^0, \nu^0) \leq 0 \) in \( \Omega \times R_+ \).

**Theorem 4.4.** For the problem (4.1) assume existence of two pairs of functions \( (u_0, \nu^0) \) and \( (u^0, \nu^0) \) is in lemma 4.1. Then the following limits exist for all \( x \in \Omega \):
\[ \lim_{t \to \infty} u(x, t; u_0, \nu_0) = u(x), \quad \lim_{t \to \infty} v(x, t; u_0, \nu_0) = v(x), \]
\[ \lim_{t \to \infty} u(x, t; u^0, \nu^0) = U(x), \quad \lim_{t \to \infty} v(x, t; u^0, \nu^0) = V(x), \]
where \( (u, v) \) and \( (U, V) \) are positive solutions of (4.2). The \( \omega \)-limit set of (4.1) (with respect to data in \( [u_0, u^0] \times [\nu_0, \nu^0] \) is then contained in the order rectangle \( [u, U] \times [v, V] \).

**Proof.** For \( h > 0 \) define \( u_m(x, t) = u(x, t+mh; u^0, \nu^0), \quad v_m(x, t) = v(x, t+mh; u^0, \nu^0), \) \( m \) a positive integer. By lemma 4.1 both sequences are decreasing in \( m \) and the limits \( \lim_{m \to \infty} u_m(x, t) = U(x, t) \) and \( \lim_{m \to \infty} v_m(x, t) = V(x, t) \) exist, and \( U(x, t) = V(x, t) = 0 \) for \( x \in \delta \Omega \). By the interior Schauder's estimates (see e.g., [10])
\[ |u_m|_{2+\gamma, [\delta_1, \delta_2], K} + |v_m|_{2+\gamma, [\delta_1, \delta_2], K} \leq c \]
for any \( 0 < \gamma < 1, \delta_2 > \delta_1 > 0 \) and \( K \subset \subset \Omega \). Since \( C^{2+\gamma, 1+\gamma/2}(K \times [\delta_1, \delta_2]) \) is
compactly imbeddend in $C^{2,1}(K \times [\delta_1, \delta_2])$ it follows that for a subsequence

$$u_{m_k} \to U(x, t), \quad v_{m_k} \to V(x, t) \text{ in } C^{2,1}(K \times [\delta_1, \delta_2]).$$

By monotonicity the entire sequences $\{u_m\}$ and $\{v_m\}$ converge in $C^{2,1}(K \times [\delta_1, \delta_2])$. Since $(u_m, v_m)$ are solutions of (4.1), we can pass to the limit in the equations and conclude that $(U(x, t), V(x, t))$ is a solution of (4.1) for $x \in \Omega$, $t > 0$.

Finally, we claim that $U_t = V_t = 0$ on $\Omega \times R_+$. Let $K \subseteq \Omega$, and $k = \text{integer} > 1$. Consider $\bar{u}_m(x, t) = u(x, t + m, 0, 0, v_0)$,

$$\bar{v}_m(x, t) = v(x, t + m, 0, 0, v_0).$$

The new sequences $\{\bar{u}_m\}$ and $\{\bar{v}_m\}$ have the old ones as subsequences. Since as above $\bar{u}_m$ and $\bar{v}_m$ converge, their limits are again $U(x, t)$ and $V(x, t)$ respectively, i.e.

$$\bar{u}_m \to U(x, t), \quad \bar{v}_m \to V(x, t) \text{ in } C^{2,1}(K \times [\delta_1, \delta_2]).$$

Set $\tau = \frac{h}{k}$, and assume $\tau$ is small (i.e. $k$ is large). Let $x_0 \in K$, $t_0 \in (\delta_1 + \tau, \delta_2 - \tau)$, with $\tau$ so small that $\delta_2 - \tau > \delta_1 + \tau$. Then

$$U(x_0, t_0 + \tau) - U(x_0, t_0) = \lim_{m \to \infty} u(x_0, t_0 + \tau + m\tau; \theta^0, v^0) - \lim_{m \to \infty} u(x_0, t_0 + m\tau) = 0.$$ 

Since $U$ is differentiable it follows that

$$U_t(x_0, t_0) = \lim_{k \to \infty} \frac{U(x_0, t_0 + \frac{h}{k}) - U(x_0, t_0)}{h/k} = 0.$$

Since $x_0, t_0, \delta_1, \delta_2$ and $K$ were arbitrary, the claim follows, and the theorem is proved (the final statement follows easily by the maximum principle).

**Corollary 1.** Assume that $a > \lambda_1, d > \lambda_1, ec < 1$. Then the problem (4.2) has minimal and maximal positive solutions $(u, v)$ and $(\bar{u}, \bar{v})$ respectively, and the $\omega$-limit set of (4.1) with respect to data satisfying (4.3) is contained in the order rectangle $[u, \bar{u}] \times [v, \bar{v}]$.

**Proof:** Existence of the steady states was proved in [7]. In that work
subsolutions were \( u_0 = \epsilon \varphi_1, v_0 = \epsilon \varphi_2 \) with sufficiently small \( \epsilon \). It remains to notice that in view of the theorem 3.1 we may assume without loss of generality that the initial data is above \( \varphi_1 \) for \( \epsilon \) small (just shift the original time if necessary).

**Corollary 2.** Assume that \( a \geq d > \lambda_1, \epsilon c < \delta^2 \), where \( \delta = \inf_{\Omega} \frac{u_d}{u_a} \). Then (4.2) has a unique positive solution, which attracts all solutions of (4.1), with data satisfying (4.3).

**Proof.** Uniqueness of the steady state was proved in [8], the rest is as above.

For cooperating species conditions \( a > \lambda_1, d > \lambda_1 \) are not necessary for existence of positive steady states as the following result shows.

**Theorem 4.5.** Assume that \( a > \lambda_1, \epsilon c < 1 \) and \( \lambda_1 (\Delta + d + \epsilon u_a) > 0 \). Then the problem (4.2) has a positive solution.

**Proof.** Fix constants \( M, N > 0 \) such that
\[
a - M + cN < 0, \quad d + eM - N < 0.
\]
By lemma 4.1, \( u(x,t) \equiv u(x,t;M,N) \) and \( v(x,t) \equiv v(x,t;M,N) \) are monotone decreasing in \( t \), and by the argument of the theorem 4.4 they converge to a non-negative solution \( (u(x),v(x)) \) of (4.2). Clearly \( u(x) \geq u_a(x) > 0 \), since \( u(x,t) \) lies above the solution \( u(x,t;M) \) of (3.1), which tends to the \( u_a > 0 \) from above. Since \( \lambda_1 (\Delta + d + \epsilon u(x,t)) \geq \lambda_1 (\Delta + d + \epsilon u_a) > 0 \), it follows by lemma 3.2 that the second equation in (4.2) has a positive solution, and then by lemma 3.1, \( v(x) \) is either that positive solution or zero. It remains to exclude the latter possibility. Indeed if \( v \equiv 0 \) then by the Vitali's theorem \( v(x,t) \to 0 \) as \( t \to \infty \) uniformly in \( x \in \Omega \). We claim that then \( u(x,t) \to u_a \) as \( t \to \infty \) uniformly in \( x \in \Omega \). Indeed for any \( \epsilon > 0 \) one can find \( T > 0 \), such that \( cv(x,t) < \epsilon \) for \( t > T \) and \( x \in \Omega \). Hence for \( t > T \)
\[
u_a \leq u(x,t) \leq u_{a+\epsilon'}.
\]
and the claim follows by lemma 3.3.

Rewrite the second equation in (4.1)

\[(4.10) \quad v_t = \Delta v + v(d+eu_a) - v^2 + v(eu-eu_a).\]

Denote by $\bar{v} > 0$ the principal eigenfunction of $\Delta + d + eu_a$ corresponding to $\bar{\lambda} = \lambda_1(\Delta+d+eu_a)$. Multiply both sides of (4.10) by $\bar{v}$ and integrate, setting

\[H = \int_\Omega \bar{v} \, dx. \quad \text{Then for } t > T, \text{ with } T \text{ large enough (using lemma 3.4)}
\]

\[(4.11) \quad H' \geq \frac{\bar{\lambda}}{2} H, \quad H(T) > 0.\]

But $H(t) \to 0$ as $t \to \infty$, which contradicts (4.11).

We need two more comparison results whose proofs easily follow from the maximum principle for weakly coupled systems.

**Lemma 4.2.** Let $\bar{u}(t), \bar{v}(t)$ be solutions of

\[
\begin{align*}
\bar{u}' &= \bar{u}(a-\bar{u}+c\bar{v}), \quad \bar{u}(0) = \max_\Omega u_0(x), \\
\bar{v}' &= \bar{v}(d+e\bar{u}-\bar{v}), \quad \bar{v}(0) = \max_\Omega v_0(x).
\end{align*}
\]

Then for all $x \in \Omega$ and $t > 0$,

\[u(x,t;u_0,v_0) \leq \bar{u}(t), \quad v(x,t;u_0,v_0) \leq \bar{v}(t).\]

**Lemma 4.3.** Let $U(x,t)$ and $V(x,t)$ be solutions of

\[
\begin{align*}
U_t &= \Delta U + U(A-U+cV) \text{ in } \Omega, \quad U = 0 \text{ on } \partial \Omega, \\
V_t &= \Delta V + V(D+eU-V) \text{ in } \Omega, \quad V = 0 \text{ on } \partial \Omega, \\
U(x,0) &= U_0(x), \quad V(x,0) = V_0(x),
\end{align*}
\]

with $A \geq a, D \geq d, U_0(x) \geq u_0(x) \geq 0$ and $V_0(x) \geq v_0(x) \geq 0$ for all $x \in \Omega$.

Then $u(x,t) \leq U(x,t), \quad v(x,t) \leq V(x,t)$ for all $x \in \Omega, \ t > 0$.

**Proof of the theorem 4.2.**

(i) Case $c \varepsilon > 1$. Assume first that $d = a$, and $u_0(x) = pv_0(x)$, where $p = \frac{c+1}{d+1}$. Then solution of (4.1) can be found in the form $u = pv$, with

\[(4.12) \quad v_t = \Delta v + v(a+\alpha v),\]

where $\alpha = c-p > 0$ (by [1] the solution is unique). Any solution of (4.12) with data satisfying (4.3) blows up in finite time. Indeed if
\[ H(t) = \int_{\Omega} v(x, t) \varphi_1(x) \, dx, \text{ then } H(0) > 0 \text{ and (assuming } \int_{\Omega} \varphi_1(x) \, dx = 1) \]

\[ H' \geq -\lambda_1 H + aH + \alpha H^2 \geq aH^2. \]

For the general case with say \( a \geq d \), we can assume in view of the theorem 3.1 that \( v_0(x) \equiv g(x), u_0(x) \equiv pg(x) \) for some \( g(x) > 0 \) in \( \Omega \). Then by lemma 4.3, the solution of (4.1) lies above the solution of the same system with \( d \) in place of \( a \), and \( pg(x) \) and \( g(x) \) in place of \( u_0(x) \) and \( v_0(x) \) respectively. By the above this implies the blow up.

(ii) Case \( ce = 1 \). By the theorem 3.1 we may assume that \( u_0(x) \equiv c \varphi_1, v_0(x) \equiv c \varphi_1 \) for some \( c > 0 \). By taking \( c \) sufficiently small we may assume that \( \overline{u}_0 \equiv c \varphi_1 \) and \( \overline{v}_0 \equiv c \varphi_1 \) satisfy (4.7), and hence by lemma 4.1,

\[ (4.13) \quad u_t(x, t; \overline{u}_0, \overline{v}_0) \geq 0, \quad v_t(x, t; \overline{u}_0, \overline{v}_0) \geq 0 \quad \text{in } \Omega \times \mathbb{R}_+. \]

Notice that either both \( u(x, t; \overline{u}_0, \overline{v}_0) \) and \( v(x, t; \overline{u}_0, \overline{v}_0) \) are bounded in \( L^\infty(\Omega) \) or both are unbounded in \( L^\infty(\Omega) \) (if one quantity is bounded then an easy comparison argument for (4.1) shows that the other one is bounded too). If both of the above quantities were bounded, then (4.13) would imply that \( u(x, t; \overline{u}_0, \overline{v}_0) \) and \( v(x, t; \overline{u}_0, \overline{v}_0) \) converge, and by the theorem 4.4 the limiting functions would give a positive solution of (4.2). But by the theorem 3.1 in [8], the problem (4.2) has no positive solutions if \( ce = 1 \). Hence in view of lemma 4.3 as time increases

\[ \| u(x, t; u_0, v_0) \|_{L^\infty(\Omega)} \to \infty, \quad \| v(x, t; u_0, v_0) \|_{L^\infty(\Omega)} \to \infty. \]

Finally, using the lemma 4.2 and the theorem 4.2, we conclude that blow up cannot occur in finite time.

5. Competing species.

The results of this section are similar to those of the preceding one, and so we omit the proofs.

We consider a system (\( \Omega \) a smooth domain in \( \mathbb{R}^n \))
\[ u_t = \Delta u + u(a-u-cv) \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \]
\[ (5.1) \]
\[ v_t = \Delta v + v(d-eu-v) \text{ in } \Omega, \ v = 0 \text{ on } \partial \Omega \]
\[ u(x,0) = \bar{u}_0(x), \ v(x,0) = \bar{v}_0(x) \ (a,c,d,e \text{ - positive constants}), \]
and the corresponding steady state system
\[ (5.2) \]
\[ \Delta u + u(a-u-cv) = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \]
\[ \Delta v + v(d-eu-v) = 0 \text{ in } \Omega, \ v = 0 \text{ on } \partial \Omega. \]

**Lemma 5.1.** Assume there are two pairs of positive in \( \Omega \) functions \((u_0^0, v_0^0)\) and \((u_0^1, v_0^1)\), with \( u_0 = v_0 = 0 \) on \( \partial \Omega \) and \( u_0^0 \geq 0, \ v_0^0 \geq 0 \) on \( \partial \Omega \), which satisfy
\[ -\Delta u_0 \leq u_0(a-u_0-cv_0^0) \]
\[ (5.3) \]
\[ -\Delta v_0 \geq v_0^0(d-eu_0-v_0^0) \text{ in } \Omega, \]
and similar inequalities for \((u_0^1, v_0^1)\). Moreover, assume that \( u_0 \leq u_0^0 \) and \( v_0 \leq v_0^0 \) in \( \Omega \). Then
\[ u_0 \leq u(x,t;u_0^0,v_0^0) \leq u_0^0 \]
\[ (5.4) \]
\[ v_0 \leq v(x,t;u_0^0,v_0^0) \leq v_0^0 \text{ in } \Omega \times \mathbb{R}^+. \]
and
\[ (5.5) \]
\[ u_t(x,t;u_0^0,v_0^0) \geq 0, \ v_t(x,t;u_0^0,v_0^0) \leq 0 \text{ in } \Omega \times \mathbb{R}^+, \]
and similar inequalities hold for \( u(x,t;u_0^0,v_0^0) \) and \( v(x,t;u_0^0,v_0^0) \).

**Theorem 5.1.** Assume existence of \((u_0^0, v_0^0)\) and \((u_0^1, v_0^1)\) as above. Then the following limits exist for all \( x \in \Omega \):
\[ \lim_{t \to \infty} u(x,t;u_0^0,v_0^0) = U(x), \]
\[ \lim_{t \to \infty} v(x,t;u_0^0,v_0^0) = V(x), \]
\[ \lim_{t \to \infty} u(x,t;u_0^1,v_0^0) = U(x), \]
\[ \lim_{t \to \infty} v(x,t;u_0^1,v_0^0) = V(x). \]

The pairs \((u(x), V(x))\) and \((U(x), V(x))\) are positive solutions of \((5.2)\).

The \( \omega \)-limit set of \((5.2)\) with respect to data in \([u_0^0,u_0^0] \times [v_0^0,v_0^0]\) is contained in \([u,U] \times [v,V] \).

**Corollary.** Assume that
\[ \lambda_1(\Delta+a-cu_d) > 0; \ \lambda_1(\Delta+d-eu_a) > 0. \]
Then the problem \((5.2)\) has positive solutions \((u(x), V(x))\) and \((U(x), V(x))\), and the \( \omega \)-limit set of \((5.1)\) with respect to the strictly positive data (i.e.
satisfying (5.6) below) is \([u,U] \times [v,V]\).

**Proof.** Let \(\tilde{u}(x)\) and \(\tilde{v}(x)\) be the principal eigenfunctions of \(\Delta + a - cu_d\)
and \(\Delta + d - eu_a\) respectively. Then we can take \(u^0 = u_a\), \(v^0 = u_d\), \(u_0 = e\tilde{u}\), \(v_0 = e\tilde{v}\) with \(e\) sufficiently small. Hence we can apply the preceding theorem if
the data satisfies

\[(5.6) \quad \tilde{u}_0(x) \geq e\tilde{u}, \quad \tilde{v}_0(x) \geq e\tilde{v} \quad \text{for } e \text{ sufficiently small.}\]

Finally we mention that in [4] E.N. Dancer proved that (5.1) can have
multiple positive solutions.


We study the systems

\[
\begin{align*}
  u_t &= \Delta u + u(a-u-cv) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega \\
  v_t &= \Delta v + v(-d+eu-v) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega \\
  u(x,0) &= u_0(x), \quad v(x,0) = v_0(x).
\end{align*}
\]

(6.1)

Here \(u(x,t)\) denotes the population density of a prey, and \(v(x,t)\) that of a
predator; \(a,c,d,e\), are positive constants. We assume that the data satisfies
(4.3), which implies by the maximum principle that \(u(x,t)\) and \(v(x,t)\) are
non-negative for all \(x\) and \(t\).

Unlike the previous cases the system (6.1) has no monotonicity
properties. Its behavior is determined by stability of the trivial solutions
\((0,0)\) and \((u^*_a,0)\). If the solution approaches \((u^*_a,0)\) then \(v(x,t)\) tends to 0.
However the lack of monotonicity does not allow us to use the Vitali's theorem
as in the previous sections to conclude that \(v(x,t)\) tends to zero uniformly in
\(x\). In case \(n = 1\) we are able to conclude the uniform convergence, by using
the following global a priori bound.

**Lemma 6.1.** Consider the problem \((\Omega \subset \mathbb{R}^n, n \geq 1)\)

\[
\begin{align*}
  u_t &= \Delta u + f(x,t) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega \\
  u(x,0) &= u_0(x).
\end{align*}
\]

(6.2)
Assume that

(1) \( \| \mathbf{f} \|_{L^2(\Omega)} \leq c_1 \),

(2) \( \| u(x,t) \|_{L^2} \leq c_2 \),

with constants \( c_1, c_2 \) independent of \( t \). Then

(6.3) \( \| u \|_{1} \leq c, \) \( c \) is independent of \( t \).

Here \( \| \cdot \|_{1} \) denotes the norm in the Sobolev's space \( H^1(\Omega) \); \( c \) depends on \( c_1, c_2 \) and \( \| u_0 \|_{1} \).

**Proof.** Our proof follows Babin and Vishik [2], and consists of several steps.

**Step 1.** Multiply (6.2) by \( u \) and integrate over \( \Omega \), and then in \( t \) from 0 to \( T \). Obtain

(6.4) \( \int_0^T \| u(x,\tau) \|_{1}^2 d\tau \leq c(c_1, c_2, T, \| u_0 \|_{1}) \).

**Step 2.** Multiply (6.2) by \( \Delta u \) and integrate over \( \Omega \). After the usual manipulations,

(6.5) \( \| u(x,t) \|_{1}^2 \leq \| u_0(x) \|_{1}^2 + c_1^2 T. \)

**Step 3.** Multiply (6.2) by \( t \Delta u \) and integrate,

(6.6) \( D_t \left( \frac{1}{2} t \| u \|_{1}^2 \right) + t \int (\Delta u)^2 = \frac{1}{2} \| u \|_{1}^2 - t \int f \Delta u \)

\[ \leq \frac{1}{2} \| u \|_{1}^2 + \frac{t}{2} \int (\Delta u)^2 + \frac{t}{2} \int f^2. \]

Integrating from 0 to \( t \) and using (6.4),

\[ \frac{1}{2} t \| u \|_{1}^2 \leq \frac{1}{2} \int_0^t \| u \|_{1}^2 d\tau + c_1 \frac{t^2}{4} \leq c. \]

Let now \( 0 < \delta \leq t \leq T \). Then

(6.7) \( \| u \|_{1} \leq c(c_1, c_2, T, \delta). \)

**Step 4.** Fix say \( \delta = 1 \) and \( T = 10 \). Then for \( 0 < t \leq 1 \) the estimate (6.3) follows from (6.5), while for \( 1 \leq t \leq 10 \) it follows from (6.7). Take \( t = 1 \) as the new initial time. Then \( \| u \|_{1} \) is bounded by the same quantity as in (6.7) for \( 2 \leq t \leq 11 \), and so on.

**Lemma 6.2.** Assume that \( \lim_{t \to \infty} v(x,t) = 0 \) for each \( x \in \overline{\Omega} \subset \mathbb{R}^1 \), and
\|v\|_1 \leq c, \text{ uniformly in } t.

Then \( v(x,t) \) tends to zero uniformly in \( x \in \Omega \).

**Proof.** Take an arbitrary sequence \( \{t_m\} \to \infty \). Suffices to show that \( u_m(x) = v(x,t_m) \) tends to zero uniformly in \( x \), as \( m \to \infty \). Since \( n = 1 \),
\[
\|u_m\|_{C^\alpha} \leq c \|v\|_1 \leq c, \text{ uniformly in } t, \text{ for any } 0 < \alpha < \frac{1}{2}.
\]
Since \( C^\alpha(\Omega) \) is compactly imbedded in \( C^0(\Omega) \), it follows that a subsequence of \( u_m(x) \) converges to zero uniformly in \( x \). Remove this subsequence, and repeat the procedure until the entire sequence is exhausted.

**Definition.** We say that a specie \( u(x,t) \) dies out, if \( \lim_{t \to \infty} u(x,t) = 0 \), otherwise we say that \( u(x,t) \) persists.

**Theorem 6.1.** For the problem (6.1) assume that \( n = 1 \) and the data satisfies (4.3). Then the conditions

(i) \( a > \lambda_1 \)

(ii) \( \lambda_1 (\Delta + eu_a - d) > 0 \)

are necessary and sufficient for the persistence of both species.

**Proof.** Necessity follows by the theorem 3.1.

Sufficiency. If \( v(x,t) \to 0 \) as \( t \to \infty \), then by lemma 6.2 the convergence is uniform in \( x \) (notice that both \( u \) and \( v \) are bounded in \( L^\infty(\Omega) \)). Then as in the proof of the theorem 4.5, \( u \to u_a \) as \( t \to \infty \) uniformly in \( x \in \Omega \). If we now denote by \( \bar{v} \) the principal eigenfunction of \( \Delta + eu_a - d \), and by \( H = \int_\Omega \bar{v} v \mathrm{d}x \), then from the second equation in (6.1) we obtain a contradiction as before. If \( u(x,t) \to 0 \) as \( t \to \infty \), then again the convergence is uniform in \( x \). From the second equation in (6.1) it follows that \( v(x,t) \to 0 \) as \( t \to \infty \) uniformly in \( x \) (compare \( v(x,t) \) with the solution of \( V'(t) = -\frac{d}{2} V(t) \), for \( t \) large). But then from the first equation in (6.1), \( u(x,t) \to u_a \), a contradiction.

In [3] under conditions similar to ours, E.N. Dancer has proved existence of positive steady state for (6.1).
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