Extensions of an Algorithm for the Analysis of Nongeneric Hopf Bifurcations, with Applications to Delay-Difference Equations

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ABSTRACT

A previously derived algorithm for the analysis of the Hopf bifurcation in functional differential equations is extended, allowing the elementary approximation of an existence and stability – determining scalar bifurcation function. With the assistance of the symbolic manipulation program MACSYMA [5], [9] this algorithm is used to implement the algorithm and to investigate the nature of nongeneric Hopf bifurcations in scalar delay – difference equations.

I. INTRODUCTION:

The practical application of the now well – understood theory of Hopf bifurcations in functional differential equations still poses many significant computational issues. The thorough analysis of the bifurcation structures (including questions of stability and direction of bifurcation) for specific applications often requires a sizeable amount of computation. Even when a computer - assisted analysis is considered adequate, the selection of the appropriate technique is an important consideration.

Over the last 15 years, many techniques have been developed to treat such problems [10]. Among them, three have been most extensively discussed in the liturature. Specifically, we refer to the method of averaging [3] [4], the use of the Poincare normal form [8], and the method of Liapunov-Schmidt [13]. Of course, each of these methods must ultimately produce the same result when applied to a specific equation. However, the ease of application of each of these methods can vary significantly.

Our purpose in this paper is to report on the use of symbolic manipulation software in the implementation of the third of these methods. This method differs from the other two in that it does not require the approximation of the center manifold existing near criticality at the equilibrium point under consideration. This is appears to have an advantage when hand calculations are attempted and, as we shall see, lends itself to a computationally efficient symbolic implementation, as well.

The specific technique to be considered here was introduced in [13]. A generalized algorithm appeared in [14], and a FORTRAN – based implementation was developed in [1], [11], [2]. We consider here the use of symbolic – manipulation software in the extension of the algorithm of [14], and the application of this algorithm to a class of scalar delay – difference equations. The material presented on these two topics is based on the results of [6], where additionally a MACSYMA [5], [9] - based symbolic manipulation package (BIPACK) was designed for analyzing generic and third-order nongeneric scalar FDE.

In the section to follow, the specific class of functional differential equations under consideration, and the technical assumptions required will be presented. Theorems 2.2 and 2.3 represent extensions of the results in [13] to the case of fifth order non-generic systems. The need for such results is illustrated in [12] where within the class of scalar integro-differential equations, elementary necessary and sufficient conditions are derived for third order degeneracy. A corollary addresses the important case of systems with odd nonlinearities. Section 3 is devoted to the application of these results to scalar delay – difference equations.

II. THE BIFURCATION FUNCTION:

In this section, we begin by making assumptions which remain throughout this paper. We define $C = C([-1,0] : \mathbb{R}^n)$, $L(\alpha) : C \to \mathbb{R}^n$, and $H(\alpha) : C \to \mathbb{R}^n$ and consider the system of equations

$$\dot{y}(t) = L(\alpha)y_t + H(\alpha; y_t) \tag{2.1}$$

where L and H are continuous, and α is a parameter in some (Euclidean) space. For fixed α, we assume $H(\alpha;\psi)$ can be expressed in the following expansion

$$H(\alpha;\psi) = \sum_{j=2}^{7} H_j(\psi^j) + \mathcal{O}(\|\psi\|^8), \tag{2.2}$$

where the H_j's, $j = 2, \ldots, 7$ are α-dependent, continuous, symmetric, j-linear forms taking values in $\mathbb{R}^n$. By the term symmetric, we mean that each H_j is invariant under a permutation of its j arguments. More precisely, we assume L and H are continuous in (α, ψ), and for fixed α, $H(\alpha;\psi)$ is at least 9 times continuously differentiable in ψ. As in [14], we assume that for $\psi \in C$ with derivatives $\psi^{(j)} \in C; j = 1, 2, \ldots, 7$, the functions $L(\alpha)\psi$, $H_j(\alpha;\psi)$, and $H(\alpha;\psi)$ are C^7 functions of α. Such assumptions are not uncommon to applications, where often derivatives of all orders are present.

Observe that $y \equiv 0$ defines a steady state for (2.1). The linearized equation

$$\dot{y}(t) = L(\alpha)y_t \tag{2.3}$$

has nontrivial solutions of the form $y(t) = \xi e^{\lambda t}$ with $\xi \in \mathbb{C}^n$ if and only there is a nontrivial ξ satisfying the characteristic system

$$0 = [\lambda I - L(\alpha)e^{\lambda\cdot}]\xi \equiv \Delta(\alpha;\lambda)\xi. \tag{2.4}$$

Assume for α near α_0 (2.4) possesses a nontrivial solution with $\lambda = \lambda(\alpha)$ such that $\lambda(\alpha_0) = i\omega, \omega \neq 0$. As usual, we assume that $\lambda = i\omega$ is a simple root of $det\Delta(\alpha_0;\lambda) = 0$ and all other roots (other than $\pm i\omega$) have negative real parts. Define $\xi^* = \xi^*(\alpha) \neq 0$ to be any solution of $\xi^*(\alpha)\Delta(\alpha;\lambda(\alpha)) = 0$ for α near α_0, and for λ near $\lambda(\alpha)$, let

$$\hat{\xi} = \hat{\xi}(\alpha;\lambda) \equiv \xi^*/[\xi^*\Delta'(\alpha;\lambda)\xi], \tag{2.5}$$

where $\Delta' = \partial\Delta/\partial\lambda$. See [13], [14] for details.

Our primary goal is to provide computational means of resolving the structure of Hopf bifurcations for (2.1) near criticality. The following proposition, proved in [13], asserts the existence of a scalar bifurcation function $g(\alpha, c)$ that facilitates such a study.

Proposition 2.1 *For ω in a neighborhood of ω_0 there exists a computable real-valued function g defined and C^8 in a neighborhood of $(\alpha_0, 0)$ whose zeros correspond in a 1-1 fashion with the small periodic solutions of (2.1) with period near $2\pi/\omega$. Under this correspondence, the periodic solution of (2.1) associated to a root c of $g(\alpha;\cdot)$ has the form*

$$y(t, \alpha; c, \nu) = 2Re\{\xi(\alpha)e^{\nu i t}\}c + \mathcal{O}(c^2), \tag{2.6}$$

(up to phase shift). Moreover, $y(t)$ is orbitally asymptotically stable (unstable) if and only if c is stable (unstable) when viewed as an equilibrium of the scalar equation $\dot{c} = g(\alpha; c)$.

Essential to the application of this result to specific equations is the effective approximation of the scalar bifurcation function g. This issue is considered in [14], where an inductive approximation algorithm is derived. It is shown in that reference that the small periodic solutions of (2.1) with periods $2\pi/\nu$ and α near α_0 coincide with those of the (complex) scalar bifurcation equation

$$\begin{aligned} 0 &= G(\alpha; \nu, c) && (2.7)\\ &= [\lambda(\alpha) - i\nu]c + \frac{\nu}{2\pi}\int_0^{2\pi/\nu} e^{-\nu i u}\hat{\xi}\cdot H(\alpha; y_u)du && (2.8)\\ &= (\lambda(\alpha) - i\nu)c + M_3(\alpha;\nu)c^3 + M_5(\omega;\nu)c^5 + M_7(\omega;\nu)c^7 + \cdots, && (2.9) \end{aligned}$$

where $y(t) = 2Re\{c\varphi(t)\} + \sum_{l=2}^{m} y^{(l)}(t)c^l + \ldots$ for $m < 8$, is defined inductively according to the following algorithm:

1. The expansion $y^{(l)}(t)$ has the form

$$y^{(l)}(t) = A_{l,l}e^{l\nu i t} + A_{l,l-2}e^{(l-2)\nu i t} + \cdots + A_{l,-l}e^{-l\nu i t},$$

where $\overline{A_{l,j}} = A_{l,-j}$.

2. $y^{(1)}(t) = 2Re\{\varphi(t)\} = A_{1,1}e^{\nu i t} + \bar{A}_{1,1}e^{-\nu i t}$, with $A_{1,1} = \xi(\alpha)$ and $\varphi(s) \equiv \xi(\alpha)e^{\nu i s}$,

3. Define $[\hat{\xi}\cdot\ ]$ to be the linear map from $\mathbb{C}^n$ to $\mathbb{C}$ given by

$$\hat{\xi}\cdot h = \sum_{j=1}^{n} \hat{\xi}_j \cdot h_j.$$

If, for $l \geq 2$, the coefficient of c^l in

$$\sum_{j=2}^{l} H_j(\alpha; [\sum_{m=1}^{l-1} y_t^{(m)} c^m]^j)$$

is $\sum_j B_{l,j}(\alpha;\nu)e^{j\nu i t}$, then

$$A_{l,j}(\alpha;\nu) = \begin{cases} \Delta^{-1}(\alpha; j\nu i)B_{l,j}(\alpha;\nu) & \text{for } j \neq \pm 1, \\ (\Delta^{-1}(\alpha;\nu i) - \frac{1}{\nu i - \lambda(\alpha)}\xi[\hat{\xi}\cdot\ ])B_{l,1}(\alpha;\nu) & \text{for } j = 1. \end{cases}$$

The singularity at $\lambda = \lambda(\alpha)$, in

$$\Delta^{-1}(\alpha;\lambda) - \frac{1}{\lambda - \lambda(\alpha)}\xi[\hat{\xi}\cdot]$$

is removable. In particular, for $h \in \mathbb{C}^n$, and λ near $\lambda(\alpha)$, we have the expansion

$$\begin{aligned} &\Delta^{-1}(\alpha;\lambda)h - \frac{1}{\lambda-\lambda(\alpha)}\xi[\hat{\xi}\cdot h] = \\ &\quad d - [\hat{\xi}\Delta'(\alpha;\lambda(\alpha))d]\xi - \frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi][\hat{\xi}\cdot h]\xi \\ &\quad + \Big[e - [\hat{\xi}\Delta'(\alpha;\lambda(\alpha))e]\xi + \frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi]\,[\hat{\xi}\Delta'(\alpha;\lambda(\alpha))d]\xi \\ &\quad - \frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))d]\xi + \Big\{(\frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi])^2 \\ &\quad - \frac{1}{6}[\hat{\xi}\Delta'''(\alpha;\lambda(\alpha))\xi]\Big\}\,[\hat{\xi}\cdot h]\xi\Big]\,(\lambda-\lambda(\alpha)) + O((\lambda-\lambda(\alpha))^2), \end{aligned}$$

where $d \in \mathbb{C}^n$ is any solution of

$$\Delta(\alpha; \lambda(\alpha))d = h - \Delta'(\alpha; \lambda(\alpha))\xi[\hat{\xi} \cdot h],$$

and $e \in \mathbb{C}^n$ is any solution of

$$\begin{aligned}\Delta(\alpha; \lambda(\alpha))e &= -\Delta'(\alpha; \lambda(\alpha))d + \Delta'(\alpha; \lambda(\alpha))\xi[\hat{\xi}\Delta'(\alpha; \lambda(\alpha))d] \\ &\quad + \left\{-\frac{1}{2}\Delta''(\alpha; \lambda(\alpha))\xi + \frac{1}{2}[\hat{\xi}\Delta''(\alpha; \lambda(\alpha))\xi]\Delta'(\alpha; \lambda(\alpha))\xi\right\}[\hat{\xi} \cdot h]\end{aligned}$$

For details, see [14], where this is derived through order 0.

Implimentation of this algorithm is obviously difficult to do by hand. We have choosen to perform the necessary details with the aid of the symbolic manipulation software MACSYMA [5], [9]; see [6] for complete details. As a result, we obtain the following the following theorem, which represents an extension of Theorem 2.1 [13], where the expansion though order 5 is presented.

Theorem 2.2 *Under the above hypotheses, there are $\varepsilon > 0$ and C^7 functions $G(\alpha; c, \nu)$ ($\mathbb{C}$ -valued), $y(t, \alpha; c, \nu)$ ($\mathbb{R}^n$-valued and $\frac{2\pi}{\nu}$ -periodic in t) defined for real c, $|c| < \varepsilon$, $|\nu - \omega| < \varepsilon$, $\|\alpha - \alpha_0\| < \varepsilon$, and $t \in \mathbb{R}$ such that (1.1) has a $2\pi/\nu$ -periodic solution $y(t)$ with $|y| < \varepsilon$, $|\nu - \omega| < \varepsilon$, and $\|\alpha - \alpha_0\| < \varepsilon$ if and only if $y(t) = y(t, \alpha; c, \nu)$ (up to phase shift) and (α, c, ν) solves the bifurcation equation: $G(\alpha; c, \nu) = 0$. Moreover, y satisfies (2.6), G is odd in c and*

$$G(\alpha; c, \nu) = [\lambda - i\nu]c + M_3(\alpha; \nu, \lambda)c^3 + M_5(\alpha; \nu, \lambda)c^5 + M_7(\alpha; \nu, \lambda)c^7 + O(c^9), \quad (2.10)$$

where $\lambda = \lambda(\alpha)$, $M_3(\alpha; \nu, \lambda) = \hat{\xi}(\alpha; \lambda) \cdot N_3(\alpha; \nu)$,

$$N_3(\alpha; \nu) \equiv 3H_3(\varphi^2, \bar{\varphi}) + 2H_2(\bar{\varphi}, A_{2,2}e^{2\nu i\cdot}) + 2H_2(\varphi, A_{2,0}), \quad (2.11)$$

with $\varphi(s) = \xi(\alpha)e^{i\nu s}$ for $s \leq 0$ and $A_{2,2}, A_{2,0}$ the unique solutions of

$$\begin{aligned}\Delta(\alpha; 2\nu i)A_{2,2} &= H_2(\varphi^2), \\ \Delta(\alpha; 0)A_{2,0} &= 2H_2(\varphi, \bar{\varphi}),\end{aligned}$$

respectively.

Similarly, $M_5(\alpha;\nu,\lambda) = \hat{\xi}(\alpha;\lambda)\cdot N_5(\alpha;\nu)$, *where*

$$
\begin{aligned}
N_5(\alpha;\nu) \;=\;& 2H_2(\varphi, A_{4,0}) + 2H_2(\bar{\varphi}, A_{4,2}e^{2\nu i\cdot}) + 2H_2(A_{2,2}e^{2\nu i\cdot}, \bar{A}_{3,1}e^{-\nu i\cdot}) \\
&+ 2H_2(\bar{A}_{2,2}e^{-2\nu i\cdot}, A_{3,3}e^{3\nu i\cdot}) + 2H_2(A_{2,0}, A_{3,1}e^{\nu i\cdot}) \\
&+ 3H_3(\varphi^2, \bar{A}_{3,1}e^{-\nu i\cdot}) + 6H_3(\varphi, \bar{\varphi}, A_{3,1}e^{\nu i\cdot}) \\
&+ 3H_3(\bar{\varphi}^2, A_{3,3}e^{3\nu i\cdot}) + 6H_3(\bar{\varphi}, A_{2,2}e^{2\nu i\cdot}, A_{2,0}) \\
&+ 6H_3(\varphi, A_{2,2}e^{2\nu i\cdot}, \bar{A}_{2,2}e^{-2\nu i\cdot}) + 3H_3(\varphi, (A_{2,0})^2) \\
&+ 12H_4(\varphi, \bar{\varphi}^2, A_{2,2}e^{2\nu i\cdot}) + 12H_4(\varphi^2, \bar{\varphi}, A_{2,0}) \\
&+ 4H_4(\varphi^3, \bar{A}_{2,2}e^{-2\nu i\cdot}) + 10H_5(\varphi^3, \bar{\varphi}^2),
\end{aligned}
$$

with $A_{3,3}, A_{3,1}, A_{4,2}, A_{4,0}$ *the unique solutions of*

$$\Delta(\alpha; 3\nu i)A_{3,3} = H_3(\varphi^3) + 2H_2(\varphi, A_{2,2}e^{2\nu i\cdot})$$

$$
\begin{aligned}
A_{3,1} \;=\;& d - [\hat{\xi}\Delta'(\alpha;\lambda(\alpha))d]\xi - \frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi]M_3\xi \\
&+ \Big[e - [\hat{\xi}\Delta'(\alpha;\lambda(\alpha))e]\xi + \frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi]\,[\hat{\xi}\Delta'(\alpha;\lambda(\alpha))d]\xi \\
&- \frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))d]\xi + \Big\{(\frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi])^2 \\
&- \frac{1}{6}[\hat{\xi}\Delta'''(\alpha;\lambda(\alpha))\xi]\Big\}\, M_3\xi\Big]\,(i\nu - \lambda(\alpha)),
\end{aligned}
$$

where d *is any solution of* $\Delta(\alpha;\lambda(\alpha))d = N_3 - (\Delta'\xi)M_3$, e *is any solution of*

$$
\begin{aligned}
\Delta(\alpha;\lambda(\alpha))e \;=\;& -\Delta'(\alpha;\lambda(\alpha))d + \Delta'(\alpha;\lambda(\alpha))\xi[\hat{\xi}\Delta'(\alpha;\lambda(\alpha))d] \\
&+ \Big\{-\frac{1}{2}\Delta''(\alpha;\lambda(\alpha))\xi + \frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi]\Delta'(\alpha;\lambda(\alpha))\xi\Big\}\, M_3,
\end{aligned}
$$

and

$\Delta^i \equiv (\partial^i\Delta/\partial\lambda^i)(\alpha;\lambda(\alpha)); i = 1, 2, \ldots$

$$
\begin{aligned}
\Delta(\alpha; 2\nu i)A_{4,2} \;=\;& 2H_2(\varphi, A_{3,1}e^{\nu i\cdot}) + 2H_2(\bar{\varphi}, A_{3,3}e^{3\nu i\cdot}) + 2H_2(A_{2,2}e^{2\nu i\cdot}, A_{2,0}) \\
&+ 6H_3(\varphi, \bar{\varphi}, A_{2,2}e^{2\nu i\cdot}) + 3H_3(\varphi^2, A_{2,0}) + 4H_4(\varphi^3, \bar{\varphi}),
\end{aligned}
$$

$$\Delta(\alpha;0)A_{4,0} = 2H_2(\varphi,\bar{A}_{3,1}e^{-\nu i\cdot}) + 2H_2(\bar{\varphi},A_{3,1}e^{\nu i\cdot}) + H_2((A_{2,0})^2) + 2H_2(A_{2,2}e^{2\nu i\cdot},\bar{A}_{2,2}e^{-2\nu i\cdot}) + 3H_3(\varphi^2,\bar{A}_{2,2}e^{-2\nu i\cdot}) + 3H_3(\bar{\varphi}^2,A_{2,2}e^{2\nu i\cdot}) + 6H_3(\varphi,\bar{\varphi},A_{2,0}) + 6H_4(\varphi^2,\bar{\varphi}^2).$$

Finally, $M_7(\alpha;\nu,\lambda) = \hat{\xi}(\alpha;\lambda)\cdot N_7(\alpha;\nu)$*, where at* $\alpha=\alpha_0$ *and* $\nu=\omega$

$$\begin{aligned} N_7(\alpha;\nu) = {} & 2H_2(\bar{\varphi},A_{6,2}e^{2\nu i\cdot}) + 2H_2(\varphi,A_{6,0}) + 2H_2(\bar{A}_{2,2}e^{-2\nu i\cdot},A_{5,3}e^{3\nu i\cdot}) \\ & + 2H_2(A_{2,0},A_{5,1}e^{\nu i\cdot}) + 2H_2(\bar{A}_{3,3}e^{-3\nu i\cdot},A_{4,4}e^{4\nu i\cdot}) \\ & + 2H_2(\bar{A}_{3,1}e^{-\nu i\cdot},A_{4,2}e^{2\nu i\cdot}) + 2H_2(A_{3,1}e^{\nu i\cdot},A_{4,0}) \\ & + 2H_2(\bar{A}_{4,2}e^{-2\nu i\cdot},A_{3,3}e^{3\nu i\cdot}) + 2H_2(\bar{A}_{5,1}e^{-\nu i\cdot},A_{2,2}e^{2\nu i\cdot}) \\ & + 3H_3(\bar{\varphi}^2,A_{5,3}e^{3\nu i\cdot}) + 6H_3(\bar{\varphi},\varphi,A_{5,1}e^{\nu i\cdot}) \\ & + 6H_3(\bar{\varphi},\bar{A}_{2,2}e^{-2\nu i\cdot},A_{4,4}e^{4\nu i\cdot}) + 6H_3(\bar{\varphi},\bar{A}_{2,0},A_{4,2}e^{2\nu i\cdot}) \\ & + 6H_3(\varphi,\bar{A}_{2,2}e^{-2\nu i\cdot},A_{4,2}e^{2\nu i\cdot}) + 6H_3(\bar{\varphi},A_{2,2}e^{2\nu i\cdot},A_{4,0}) \\ & + 6H_3(\varphi,A_{2,0},A_{4,0}) + 6H_3(\bar{A}_{2,2}e^{-2\nu i\cdot},A_{2,0},A_{3,3},e^{3\nu i\cdot}) \\ & + 6H_3(\bar{A}_{3,3}e^{-3\nu i\cdot},\varphi,A_{3,3},e^{3\nu i\cdot}) + 6H_3(\bar{\varphi},\bar{A}_{3,1}e^{-\nu i\cdot},A_{3,3},e^{3\nu i\cdot}) \\ & + 3H_3(\bar{\varphi},(A_{3,1}e^{\nu i\cdot})^2) + 6H_3(\bar{A}_{2,2}e^{-2\nu i\cdot},A_{2,2}e^{2\nu i\cdot},A_{3,1}e^{\nu i\cdot}) \\ & + 3H_3((A_{2,0})^2,A_{3,1}e^{\nu i\cdot}) + 6H_3(\bar{A}_{3,1}e^{-\nu i\cdot},\varphi,A_{3,1}e^{\nu i\cdot}) \\ & + 3H_3(\bar{A}_{3,3}e^{-3\nu i\cdot},(A_{2,2}e^{2\nu i\cdot})^2) + 6H_3(\bar{A}_{3,1}e^{-\nu i\cdot},A_{2,0},A_{2,2}e^{2\nu i\cdot}) \\ & + 6H_3(\bar{A}_{4,2}e^{-2\nu i\cdot},\varphi,A_{2,2}e^{2\nu i\cdot}) + 3H_3(\bar{A}_{5,1}e^{-\nu i\cdot},\varphi^2) \\ & + 4H_4((\bar{\varphi})^3,A_{4,4}e^{4\nu i\cdot}) + 12H_4((\bar{\varphi})^2,\varphi,A_{4,2}e^{2\nu i\cdot}) \\ & + 12H_4(\bar{\varphi},\varphi^2,A_{4,0}) + 12H_4((\bar{\varphi})^2,A_{2,0},A_{3,3}e^{3\nu i\cdot}) \\ & + 24H_4(\bar{\varphi},\varphi,\bar{A}_{2,2}e^{-2\nu i\cdot},A_{3,3}e^{3\nu i\cdot}) + 12H_4((\bar{\varphi})^2,A_{2,2}e^{2\nu i\cdot},A_{3,1}e^{\nu i\cdot}) \\ & + 24H_4(\bar{\varphi},\varphi,A_{2,0},A_{3,1}e^{\nu i\cdot}) + 12H_4(\bar{A}_{2,2}e^{2\nu i\cdot},\varphi^2,A_{3,1}e^{\nu i\cdot}) \\ & + 12H_4(\bar{\varphi},\bar{A}_{2,2}e^{-2\nu i\cdot},(A_{2,2}e^{2\nu i\cdot})^2) + 12H_4(\bar{\varphi},(A_{2,0})^2,A_{2,2}e^{2\nu i\cdot}) \\ & + 24H_4(\bar{A}_{2,2}e^{2\nu i\cdot},\varphi,A_{2,0},A_{2,2}e^{2\nu i\cdot}) + 12H_4(\bar{A}_{3,3}e^{3\nu i\cdot},\varphi^2,A_{2,2}e^{2\nu i\cdot}) \\ & + 24H_4(\bar{\varphi},\bar{A}_{3,1}e^{\nu i\cdot},\varphi,A_{2,2}e^{2\nu i\cdot}) + 4H_4(\varphi,(A_{2,0})^3) \end{aligned}$$

$$
\begin{aligned}
&+ 12H_4(\bar{A}_{3,1}e^{-\nu i\cdot},(\varphi)^2,A_{2,0}) + 4H_4(\bar{A}_{4,2}e^{-2\nu i\cdot},(\varphi)^3) \\
&+ 20H_5(\bar{\varphi}^3,\varphi,A_{3,3}) + 30H_5(\bar{\varphi}^2,\varphi^2,A_{3,1}e^{\nu i\cdot}) \\
&+ 10H_5(\bar{\varphi}^3,(A_{2,2}e^{2\nu i\cdot})^2) + 60H_5(\bar{\varphi}^2,\varphi,A_{2,0},A_{2,2}e^{2\nu i\cdot}) \\
&+ 60H_5(\bar{\varphi},\bar{A}_{2,2}e^{-2\nu i\cdot},\varphi^2,A_{2,2}e^{2\nu i\cdot}) + 30H_5(\bar{\varphi},\varphi^2,(A_{2,0})^2) \\
&+ 20H_5(\bar{A}_{2,2}e^{-2\nu i\cdot};\varphi^3,A_{2,0}) + 5H_5(\bar{A}_{3,3}e^{-3\nu i\cdot},\varphi^4) \\
&+ 20H_5(\bar{\varphi},\bar{A}_{3,1}e^{-\nu i\cdot},\varphi^3) + 60H_6(\bar{\varphi}^3,\varphi^2,A_{2,2}e^{2\nu i\cdot}) \\
&+ 60H_6(\bar{\varphi}^2,\varphi^3,A_{2,0}) + 60H_6(\bar{\varphi},\varphi^4,\bar{A}_{2,2}e^{-2\nu i\cdot}) + 35H_7(\bar{\varphi}^3,\varphi^4).
\end{aligned}
$$

In addition, $A_{4,4}$, $A_{5,1}$, $A_{5,3}$, $A_{6,0}$, and $A_{6,2}$ are unique solutions of

$$
\begin{aligned}
\Delta(\alpha;4\nu i)A_{4,4} &= 2H_2(\varphi,A_{3,3}e^{3\nu i\cdot}) + H_2((A_{2,2}e^{2\nu i\cdot})^2) \\
&\quad + H_3(\varphi^2,A_{2,2}e^{2\nu i\cdot}) + H_4(\varphi^4), \\
A_{5,1} &= f - [\hat{\xi}\Delta' f]\xi - \frac{1}{2}[\hat{\xi}\Delta''\xi]M_5\xi,
\end{aligned}
$$

where f is any solution of $\Delta(\alpha;\lambda(\alpha))f = N_5 - (\Delta'\xi)M_5$,

$$
\begin{aligned}
\Delta(\alpha;3\nu i)A_{5,3} &= 2H_2(\bar{\varphi},A_{4,4}e^{4\nu i\cdot}) + 2H_2(\varphi,A_{4,2}e^{2\nu i\cdot}) \\
&\quad + 2H_2(A_{2,0},A_{3,3}e^{3\nu i\cdot}) + 2H_2(A_{2,2}e^{2\nu i\cdot},A_{3,1}e^{\nu i\cdot}) \\
&\quad + 6H_3(\bar{\varphi},\varphi,A_{3,3}e^{3\nu i\cdot}) + 3H_3(\varphi^2,A_{3,1}e^{\nu i\cdot}) \\
&\quad + 3H_3(\bar{\varphi},(A_{2,2}e^{2\nu i\cdot})^2) + 6H_3(\varphi,A_{2,0},A_{2,2}e^{2\nu i\cdot}) \\
&\quad + 12H_4(\bar{\varphi},\varphi^2,A_{2,2}e^{2\nu i\cdot}) + 4H_4(\varphi^3,A_{2,0}) + 5H_5(\bar{\varphi},\varphi^4), \\
\Delta(\alpha;0)A_{6,0} &= 2H_2(\bar{\varphi},A_{5,1}e^{\nu i\cdot}) + 2H_2(\bar{A}_{2,2}e^{-2\nu i\cdot},A_{4,2}e^{2\nu i\cdot}) \\
&\quad + 2H_2(A_{2,0},A_{4,0}) + 2H_2(\bar{A}_{3,3}e^{-3\nu i\cdot},A_{3,3}e^{3\nu i\cdot}) \\
&\quad + 2H_2(\bar{A}_{3,1}e^{-\nu i\cdot},A_{3,1}e^{\nu i\cdot}) + 2H_2(\bar{A}_{4,2}e^{-2\nu i\cdot},A_{2,2}e^{2\nu i\cdot}) \\
&\quad + 2H_2(\bar{A}_{5,1}e^{-\nu i\cdot},\varphi) + 3H_3(\bar{\varphi}^2,A_{4,2}e^{2\nu i\cdot}) \\
&\quad + 6H_3(\bar{\varphi},\varphi,A_{4,0}) + 6H_3(\bar{\varphi},\bar{A}_{2,2}e^{-2\nu i\cdot},A_{3,3}e^{3\nu i\cdot}) \\
&\quad + 6H_3(\bar{\varphi},A_{2,0},A_{3,1}e^{\nu i\cdot}) + 6H_3(\bar{A}_{2,2}e^{-2\nu i\cdot},\varphi,A_{3,1}e^{\nu i\cdot}) \\
&\quad + 6H_3(\bar{A}_{2,2}e^{-2\nu i\cdot},A_{2,0},A_{2,2}e^{2\nu i\cdot}) + 6H_3(\bar{A}_{3,3}e^{-3\nu i\cdot},\varphi,A_{2,2}e^{2\nu i\cdot})
\end{aligned}
$$

$$
\begin{aligned}
&+ 6H_3(\bar{\varphi}, \bar{A}_{3,1}e^{-\nu i\cdot}, A_{2,2}e^{2\nu i\cdot}) + H_3((A_{2,0})^3) \\
&+ 6H_3(\bar{A}_{3,1}e^{-\nu i\cdot}, \varphi, A_{2,0}) + 3H_3(\bar{A}_{4,2}e^{-2\nu i\cdot}, \varphi^2) \\
&+ 4H_4(\bar{\varphi}^3, A_{3,3}e^{3\nu i\cdot}) + 12H_4(\bar{\varphi}^2, \varphi, A_{3,1}e^{\nu i\cdot}) \\
&+ 12H_4(\bar{\varphi}^2, A_{2,0}, A_{2,2}e^{2\nu i\cdot}) + 24H_4(\bar{\varphi}, \bar{A}_{2,2}e^{-2\nu i\cdot}, \varphi, A_{2,2}e^{2\nu i\cdot}) \\
&+ 12H_4(\bar{\varphi}, \varphi, (A_{2,0})^2) + 12H_4(\bar{A}_{2,2}e^{-2\nu i\cdot}, \varphi^2, A_{2,0}) \\
&+ 4H_4(\bar{A}_{3,3}e^{-3\nu i\cdot}, \varphi^3) + 12H_4(\bar{\varphi}, \bar{A}_{3,1}e^{-\nu i\cdot}, \varphi^2) \\
&+ 20H_5(\bar{\varphi}^3, \varphi, A_{2,2}e^{2\nu i\cdot}) + 30H_5(\bar{\varphi}^2, \varphi^2, A_{2,0}) \\
&+ 20H_5(\bar{\varphi}, \bar{A}_{2,2}e^{-2\nu i\cdot}, \varphi^3) + 20H_6(\bar{\varphi}^3, \varphi^3),
\end{aligned}
$$

$$
\begin{aligned}
\Delta(\alpha; 2\nu i)A_{6,2} \;=\;& 2H_2(\bar{\varphi}, A_{5,3}e^{3\nu i\cdot}) + 2H_2(\varphi, A_{5,1}e^{\nu i\cdot}) \\
&+ 2H_2(\bar{A}_{2,2}e^{-2\nu i\cdot}, A_{4,4}e^{4\nu i\cdot}) + 2H_2(A_{2,0}, A_{4,2}e^{2\nu i\cdot}) \\
&+ 2H_2(A_{2,2}e^{2\nu i\cdot}, A_{4,0}) + 2H_2(\bar{A}_{3,1}e^{-\nu i\cdot}, A_{3,3}e^{3\nu i\cdot}) \\
&+ H_2((A_{3,1}e^{\nu i\cdot})^2) + 3H_3(\bar{\varphi}^2, A_{4,4}e^{4\nu i\cdot}) \\
&+ 6H_3(\bar{\varphi}, \varphi, A_{4,2}e^{2\nu i\cdot},) + 3H_3(\varphi^2, \bar{A}_{4,0}) \\
&+ 6H_3(\bar{\varphi}, A_{2,0}, A_{3,3}e^{3\nu i\cdot}) + 6H_3(\bar{A}_{2,2}e^{-2\nu i\cdot}, \varphi, A_{3,3}e^{3\nu i\cdot}) \\
&+ 6H_3(\bar{\varphi}, A_{2,2}e^{2\nu i\cdot}, A_{3,1}e^{\nu i\cdot}) + 6H_3(\bar{A}_{2,0}, \varphi, A_{3,1}e^{\nu i\cdot}) \\
&+ 6H_3(\varphi, \bar{A}_{3,1}e^{-\nu i\cdot}, A_{2,2}e^{2\nu i\cdot}) + 3H_3((A_{2,0})^2, A_{2,2}e^{2\nu i\cdot}) \\
&+ 3H_3((A_{2,2}e^{2\nu i\cdot})^2, \bar{A}_{2,2}e^{-2\nu i\cdot}) + 12H_4(\bar{\varphi}^2, \varphi, A_{3,3}e^{3\nu i\cdot}) \\
&+ 12H_4(\bar{\varphi}, \varphi^2, A_{3,1}e^{\nu i\cdot}) + 6H_4(\bar{\varphi}^2, (A_{2,2}e^{2\nu i\cdot})^2) \\
&+ 24H_4(\bar{\varphi}, \varphi, A_{2,0}, A_{2,2}e^{2\nu i\cdot}) + 12H_4(\bar{A}_{2,2}e^{-2\nu i\cdot}, \varphi^2, A_{2,2}e^{2\nu i\cdot}) \\
&+ 6H_4(\varphi^2, (A_{2,0})^2) + 4H_4(\bar{A}_{3,1}e^{-\nu i\cdot}, \varphi^3) \\
&+ 20H_5(\bar{\varphi}, \varphi^3, A_{2,0}) + 30H_5(\bar{\varphi}^2, \varphi^2, A_{2,2}e^{2\nu i\cdot}) \\
&+ 5H_5(\bar{A}_{2,2}e^{-2\nu i\cdot}, \varphi^4) + 15H_6(\bar{\varphi}^2, \varphi^4).
\end{aligned}
$$

Assuming $\lambda(\alpha) = \mu(\alpha) + i\omega(\alpha)$, the real and imaginary parts of $G(\alpha; c, \nu) = 0$ become

$$
0 \;=\; \mu(\alpha)c + Re\{M_3(\alpha; \nu, \lambda)\}c^3 + Re\{M_5(\alpha; \nu, \lambda)\}c^5
$$

$$+ Re\{M_7(\alpha;\nu,\lambda)\}c^7 + \mathcal{O}(c^9), \tag{2.12}$$

$$\nu = \omega(\alpha) + Im\{M_3(\alpha;\nu,\lambda)\}c^2 + Im\{M_5(\alpha;\nu,\lambda)\}c^4$$
$$+ Im\{M_7(\alpha;\nu,\lambda)\}c^6 + \mathcal{O}(c^8), \tag{2.13}$$

for $c \neq 0$.

The following theorem (proved by iteration on equation (2.13) and elimination of variable ν) relates the real bifurcation function g of Proposition 2.1 to the complex bifurcation function G of the previous theorem.

Theorem 2.3 *The reduced bifurcation equation for higher order bifurcations is given by*

$$0 = g(\alpha;c) = \mu(\alpha)c + K_3(\alpha)c^3 + K_5(\alpha)c^5 + K_7(\alpha)c^7 + \mathcal{O}(c^9), \tag{2.14}$$

where

$$\begin{aligned}
K_3 &= Re\{M_3(\alpha;\omega(\alpha),\lambda(\alpha))\},\\
K_5 &= Re\{M_5(\alpha;\omega(\alpha),\lambda(\alpha))\} + Re\{\frac{\partial}{\partial\nu}(M_3(\alpha;\nu,\lambda(\alpha)))|_{\nu=\omega(\alpha)}\}\cdot w_2,\\
K_7 &= Re\{M_7(\alpha;\omega(\alpha),\lambda(\alpha))\} + Re\{\frac{\partial}{\partial\nu}(M_5(\alpha;\nu,\lambda(\alpha)))|_{\nu=\omega(\alpha)}\}\cdot w_2\\
&\quad + Re\{\frac{\partial}{\partial\nu}(M_3(\alpha;\nu,\lambda(\alpha)))|_{\nu=\omega(\alpha)}\}\cdot w_4\\
&\quad + \frac{1}{2}Re\{\frac{\partial^2}{\partial\nu^2}(M_3(\alpha;\nu,\lambda(\alpha)))|_{\nu=\omega(\alpha)}\}\cdot(w_2)^2,
\end{aligned}$$

and

$$\begin{aligned}
w_2 &= Im\{M_3(\alpha;\omega(\alpha),\lambda(\alpha))\},\\
w_4 &= Im\{M_5(\alpha;\omega(\alpha),\lambda(\alpha))\}\\
&\quad + Im\{\frac{\partial}{\partial\nu}(M_3(\alpha;\nu,\lambda(\alpha)))|_{\nu=\omega(\alpha)}\}\cdot Im\{M_3(\alpha;\omega(\alpha),\lambda(\alpha))\}.
\end{aligned}$$

The analysis of a particular equation then rests on identifying the critical parameter α_0 and the associated characteristic values and vectors, computing the terms in the expansion of the bifurcation function G in Theorem 2.2, then the evaluation of the expansion of g from the previous theorem. See [6] for a MACSYMA –

based implemetation of these formulas for scalar functional differential equations. A FORTRAN -based approach (numerical evaluation of K_3 and K_5) for systems is described in [2]. Only under very special circumstances can one hope to apply such a lengthy algorithm by hand calculation. However, in some important situations, many of the higher order terms H_j are identically zero causing significant simplifications. One such situation is that of equations with odd nonlinearities.

Corollary 2.4 *Under the above hypotheses, if H is odd there are $\varepsilon > 0$ and C^7 functions $G(\alpha; c, \nu)$ ($\mathbb{C}$ -valued), $y(t, \alpha; c, \nu)$ ($\mathbb{R}^n$-valued and $\frac{2\pi}{\nu}$ -periodic in t) defined for real c, $|c| < \varepsilon$, $|\nu - \omega| < \varepsilon$, $\|\alpha - \alpha_0\| < \varepsilon$, and $t \in \mathbb{R}$ such that (1.1) has a $2\pi/\nu$ -periodic solution $y(t)$ with $|y| < \varepsilon$, $|\nu - \omega| < \varepsilon$, and $\|\alpha - \alpha_0\| < \varepsilon$ if and only if $y(t) = y(t, \alpha; c, \nu)$ (up to phase shift) and (α, c, ν) solves the bifurcation equation: $G(\alpha; c, \nu) = 0$. Moreover, relation (2.6) holds, G is odd in c, and*

$$G(\alpha; c, \nu) = [\lambda - i\nu]c + M_3(\alpha; \nu, \lambda)c^3 + M_5(\alpha; \nu, \lambda)c^5 + M_7(\alpha; \nu, \lambda)c^7 + \mathcal{O}(c^9), \quad (2.15)$$

where $\lambda = \lambda(\alpha), M_3(\alpha; \nu, \lambda) = \hat{\xi}(\alpha; \lambda) \cdot N_3(\alpha; \nu)$,

$$N_3(\alpha; \nu) \equiv 3H_3(\varphi^2, \bar{\varphi}), \quad (2.16)$$

with $\varphi(s) = \xi(\alpha)e^{i\nu s}$ *for* $s \leq 0$. *Similarly,* $M_5(\alpha; \nu, \lambda) = \hat{\xi}(\alpha; \lambda) \cdot N_5(\alpha; \nu)$, *where*

$$\begin{aligned} N_5(\alpha; \nu) &= 3H_3(\varphi^2, \bar{A}_{3,1}e^{-\nu i \cdot}) + 6H_3(\varphi, \bar{\varphi}, A_{3,1}e^{\nu i \cdot}) \\ &\quad + 3H_3(\bar{\varphi}^2, A_{3,3}e^{3\nu i \cdot}) + 10H_5(\varphi^3, \bar{\varphi}^2), \end{aligned}$$

with $A_{3,3}, A_{3,1}$ *the unique solutions of*

$$\begin{aligned} \Delta(\alpha; 3\nu i)A_{3,3} &= H_3(\varphi^3) \\ A_{3,1} &= d - [\hat{\xi}\Delta'(\alpha; \lambda(\alpha))d]\xi - \frac{1}{2}[\hat{\xi}\Delta''(\alpha; \lambda(\alpha))\xi]M_3\xi \\ &\quad + \Big[e - [\hat{\xi}\Delta'(\alpha; \lambda(\alpha))e]\xi + \frac{1}{2}[\hat{\xi}\Delta''(\alpha; \lambda(\alpha))\xi]\,[\hat{\xi}\Delta'(\alpha; \lambda(\alpha))d]\xi \\ &\quad - \frac{1}{2}[\hat{\xi}\Delta''(\alpha; \lambda(\alpha))d]\xi + \Big\{(\frac{1}{2}[\hat{\xi}\Delta''(\alpha; \lambda(\alpha))\xi])^2 \\ &\quad - \frac{1}{6}[\hat{\xi}\Delta'''(\alpha; \lambda(\alpha))\xi]\Big\} M_3\xi\Big] (i\nu - \lambda(\alpha)), \end{aligned}$$

where d and e are any solutions of

$$\begin{aligned}\Delta(\alpha;\lambda(\alpha))d &= N_3-(\Delta'\xi)M_3,\\ \Delta(\alpha;\lambda(\alpha))e &= -\Delta'(\alpha;\lambda(\alpha))d+\Delta'(\alpha;\lambda(\alpha))\xi[\hat{\xi}\Delta'(\alpha;\lambda(\alpha))d]\\ &\quad+\left\{-\frac{1}{2}\Delta''(\alpha;\lambda(\alpha))\xi+\frac{1}{2}[\hat{\xi}\Delta''(\alpha;\lambda(\alpha))\xi]\Delta'(\alpha;\lambda(\alpha))\xi\right\}M_3\end{aligned}$$

and $\Delta^i \equiv (\partial^i\Delta/\partial\lambda^i)(\alpha;\lambda(\alpha))$; $i=1,2,3$. *Likewise,* $M_7(\alpha;\nu,\lambda)=\hat{\xi}(\alpha;\lambda)\cdot N_7(\alpha;\nu)$, *where at* $\alpha=\alpha_0$ *and* $\nu=\omega$

$$\begin{aligned}N_7(\alpha;\nu) &= 3H_3(\bar{\varphi}^2,A_{5,3}e^{3\nu i\cdot})+3H_3(\bar{\varphi},\varphi,A_{5,1}e^{\nu i\cdot})\\ &\quad+6H_3(\bar{A}_{3,3}e^{-3\nu i\cdot},\varphi,A_{3,3},e^{3\nu i\cdot})+6H_3(\bar{\varphi},\bar{A}_{3,1}e^{-\nu i\cdot},A_{3,3},e^{3\nu i\cdot})\\ &\quad+3H_3(\bar{\varphi},(A_{3,1}e^{\nu i\cdot})^2)+6H_3(\bar{A}_{3,1}e^{-\nu i\cdot},\varphi,A_{3,1}e^{\nu i\cdot})\\ &\quad+3H_3(\bar{A}_{5,1}e^{-\nu i\cdot},\varphi^2)+20H_5(\bar{\varphi}^3,\varphi,A_{3,3})\\ &\quad+30H_5(\bar{\varphi}^2,\varphi^2,A_{3,1}e^{\nu i\cdot})+5H_5(\bar{A}_{3,3}e^{-3\nu i\cdot},\varphi^4)\\ &\quad+20H_5(\bar{\varphi},\bar{A}_{3,1}e^{-\nu i\cdot},\varphi^3)+35H_7(\bar{\varphi}^3,\varphi^4),\end{aligned}$$

and

$$\begin{aligned}\Delta(\alpha;3\nu i)A_{5,3} &= 6H_3(\bar{\varphi},\varphi,A_{3,3}e^{3\nu i\cdot})\\ &\quad+3H_3(\varphi^2,A_{3,1}e^{\nu i\cdot})+5H_5(\bar{\varphi},\varphi^4),\\ A_{5,1} &= f-[\hat{\xi}\Delta'f]\xi-\frac{1}{2}[\hat{\xi}\Delta''\xi]M_5\xi,\end{aligned}$$

where f is any solution of $\Delta(\alpha;\lambda(\alpha))f=N_5-(\Delta'\xi)M_5$.

Example 2.5

The case of integrodifferential equations

$$\dot{y} = \alpha_1 y(t) + \alpha_2 \int_{-1}^{0} g(y_t(s))d\eta(s)$$

where $g(y) = y + h_2y^2 + h_3y^3 + \ldots$ illustrates the type of results obtainable, and their complexity. Examination of the previous results one sees that $K_3(\alpha;\omega) = c_1(\alpha,\omega)h_3 + c_2(\alpha,\omega)h_2^2$, with c_1 and c_2 computable functions of the bifurcation parameters $\alpha = (\alpha_1, \alpha_2)$ and frequency ω. (See [12] for an examination of the generic case in greater detail, and a derivation of conditions under which $K_3 \equiv 0$ for all choices of h_2 and h_3.) Similarly, one sees that K_5 will be a linear combination of the coefficient combinations h_5, h_2h_4, $h_3h_2^2$, h_3^2, and h_2^4, while K_7 will be a linear combination of the eleven terms h_7, h_6h_3, $h_5h_2^2$, $h_4h_2h_3$, $h_4h_2^3$, h_4^2, $h_3^2h_2^2$, h_3^3, $h_3h_2^4$, h_2^6 and h_2h_6. These reduce greatly in the case of odd nonlinearities since $h_2 = h_4 = h_6 = 0$.

III. SCALAR DELAY-DIFFERENCE EQUATIONS In this final section we will consider the scalar delay difference equation

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-1)) \\ &= \alpha x(t) + \beta x(t-1) + h(x(t), x(t-1)) \end{aligned} \tag{3.1}$$

where $h(x,y) = a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3 + \ldots$ is assumed to be smooth. Our goal is to illustrate the results of the previous section and provide insight into the nongeneric bifurcation structure for this important equation.

The analysis of the linearized equation $\dot{z}(t) = \alpha z(t) + \beta z(t-1)$ is found in [7]. With $\Delta(\alpha,\beta;\lambda) = \lambda - \alpha - \beta e^{-\lambda}$ one easily identifies the line $\alpha + \beta = 0$ to characterize those parameter values at which $\lambda = 0$ is a characteristic root. Similarly, substituting $\lambda = i\omega$ into the characteristic equation and separating the real and imaginary parts leads to the parametrization $\beta = \tilde{\beta}(\omega) \equiv -\omega/\sin(\omega)$; $\alpha = \tilde{\alpha}(\omega) \equiv -\tilde{\beta}(\omega)\cos(\omega)$ characterizing those parameter values along which there are (simple) imaginary root pairs $\lambda = \pm i\omega$; $\omega > 0$. The interval $0 < \omega < 2\pi$ generates the remaining boundary of the region Ω_- of parameter values at which all characteristic roots have negative real parts. This region contains the negative half-axis $\alpha < 0, \beta = 0$, and is pictured in Figure 3.1. See Section 2 of [12] for generalizations.

Along the imaginary root curve the usual transversality criteria are easy to verify, and at $(\tilde{\alpha}(\omega), \tilde{\beta}(\omega))$ all characteristic roots other that $\lambda = \pm i\omega$ have negative real parts. The representation of the higher order terms $h(x(t), x(t-1))$ in terms of symmetric, multilinear functionals is trivial, allowing one to apply Theorems 2.2 and 2.3 directly. The generic bifurcation constant $K_3 = K_3(\omega)$ with $\alpha = \tilde{\alpha}(\omega)$, $\beta = \tilde{\beta}(\omega)$ is seen to take the form

$$\begin{aligned} K_3(\omega) &= c_{a_2a_2}(\omega)a_2^2 + c_{a_2b_2}(\omega)a_2b_2 + c_{b_2b_2}(\omega)b_2^2 \\ &\quad + c_{a_2c_2}(\omega)a_2c_2 + c_{c_2c_2}(\omega)c_2^2 + c_{b_2c_2}(\omega)b_2c_2 \\ &\quad + c_{a_3}(\omega)a_3 + c_{b_3}(\omega)b_3 + c_{c_3}(\omega)c_3 + c_{d_3}(\omega)d_3 \end{aligned} \tag{3.2}$$

where by direct (but symbolically assisted) computation

$$
\begin{aligned}
c_{a_3}(\omega) &= 3\sin(\omega)(\sin(\omega)-\omega\cos(\omega))/D_1(\omega)\\
c_{b_3}(\omega) &= \sin(\omega)(3\cos(\omega)\sin(\omega)-2\omega\cos(\omega)^2-\omega)/D_1(\omega)\\
c_{c_3}(\omega) &= (-3\omega\cos(\omega)\sin(\omega)-2\cos(\omega)^4+\cos(\omega)^2+1)/D_1(\omega)\\
c_{d_3}(\omega) &= 3\sin(\omega)(\cos(\omega)\sin(\omega)-\omega)/D_1(\omega)\\
c_{a_2a_2}(\omega) &= 2(\cos(\omega)+1)[3(2\cos(\omega)+3)\sin(\omega)\\
&\quad -\omega(\cos(\omega)+2)(4\cos(\omega)+1)]/D_2(\omega)\\
c_{a_2b_2}(\omega) &= (\cos(\omega)+1)[3(2\cos(\omega)+3)(3\cos(\omega)+1)\sin(\omega)\\
&\quad -\omega(8\cos(\omega)^3+26\cos(\omega)^2+19\cos(\omega)+7)]/D_2(\omega)\\
c_{b_2b_2}(\omega) &= (\cos(\omega)+1)^2[(4\cos(\omega)^2+10\cos(\omega)+1)\sin(\omega)\\
&\quad -\omega(8\cos(\omega)^2+4\cos(\omega)+3)]/D_2(\omega)\\
c_{b_2c_2}(\omega) &= -(\cos(\omega)+1)[(8\cos(\omega)^4-8\cos(\omega)^3\\
&\quad -32\cos(\omega)^2-19\cos(\omega)-9)\sin(\omega)\\
&\quad -\omega(4\cos(\omega)^3-20\cos(\omega)^2-37\cos(\omega)-7)]/D_2(\omega)\\
c_{c_2c_2}(\omega) &= -2(\cos(\omega)+1)[(4\cos(\omega)^3-4\cos(\omega)^2-13\cos(\omega)-2)\sin(\omega)\\
&\quad -\omega(2\cos(\omega)^2-6\cos(\omega)-11)]/D_2(\omega)\\
c_{a_2c_2}(\omega) &= 2(\cos(\omega)+1)^2[3(2\cos(\omega)+3)\sin(\omega)-\omega(8\cos(\omega)+7)]/D_2(\omega)
\end{aligned}
$$

where

$$
\begin{aligned}
D_1(\omega) &= \sin(\omega)^2-2\omega\cos(\omega)\sin(\omega)+\omega^2\\
D_2(\omega) &= \omega(4\cos(\omega)+5)(\sin(\omega)^2-2\omega\cos(\omega)\sin(\omega)+\omega^2).
\end{aligned}
$$

Along the curve $0<\omega<2\pi$ the scalar equation $\dot{c}=\mu c+K_3(\omega)c^3$ completely characterizes the generic Hopf bifurcation structure of the equation (3.1). For example, as the coefficients $c_{a_3}(\omega)>0$ and $c_{d_3}(\omega)<0$ for ω in that interval, increases in the corresponding coefficients a_3 and d_3 are seen to have destabilizing and stabilizing effects, respectively, on the equilibrium $x = 0$ at criticality, as well as on nearby Hopf bifurcations.

The special case $\omega = \pi/2$ is of particular importance. With $\alpha = 0$ and $\beta = -\pi/2$ one computes

$$\begin{aligned} K_3(\pi/2) &= 2[2(c_3 + 3a_3) - \pi(b_3 + 3d_3)]/(\pi^2 + 4) \\ &\quad +4[4(9-\pi)a_2^2 + (2-3\pi)b_2^2 + 2(4-11\pi)c_2^2 \\ &\quad +(18-7\pi)(a_2b_2 + 2a_2c_2 + b_2c_2)]/(5\pi(\pi^2+4)) \end{aligned}$$

This extends Example 4.1 of [13]. Again the effects of the coefficients $a_2, b_2, \ldots, d_3$ on the stability of Hopf bifurcations can be easily deduced.

Where $K_3(\omega) = 0$ (a cone in (a_2, b_2, c_2) space), one must compute (at least) $K_5(\omega)$ to fully understand the bifurcation structure for (3.1). This can be accomplished symbolically/numerically without serious difficulty. We illustrate this point by considering the quadratic delay difference equation

$$\dot{x}(t) = \alpha x(t) + \beta x(t-1) + a_2x^2(t) + b_2x(t)x(t-1) + c_2x^2(t-1). \tag{3.3}$$

For such an equation, one might consider asking an analogue of Hilbert's 16^{th} Problem: How many simultaneous periodic orbits can this equation support? While this question is clearly difficult, our results of Section 2 shed light on the number of *small* periodic orbits that can be created via Hopf bifurcation at $x = 0$.

Using the quadratic nature of (3.3), we can normalize the coefficients of the higher order terms as $a_2 = \cos(\phi)$, $b_2 = \sin(\phi)\sin(\theta)$ and $c_2 = \sin(\phi)\cos(\theta)$, with $0 \le \phi \le \frac{\pi}{2}$; $0 \le \theta \le 2\pi$ now defining our parameter space (ω fixed). An examination of the results of the previous section show the K_5 and K_7 are homogeneous polynomials of degree 4 and 8, respectively, in the variables a_2, b_2, c_2, the coefficients of these polynomials again being functions of ω. As these polynomials and their coefficients are quite complicated we will restrict our attention to specific selections for ω, and identify the curves $K_5 = 0, K_7 = 0$ by numerical evaluation.

Figure 3.2 depicts the situation at $\omega = \pi/2$. Each of the coefficients K_3, K_5, K_7 are observed to be positive for $\phi = 0$ (corresponding to $a_2 = 1, b_2 = c_2 = 0$). A careful examination of these curves reveals that there are no simultaneous nontrivial solutions of $K_3 = K_5 = K_7 = 0$; thus $K_3 = K_5 = 0$ *implies* $K_7 \neq 0$. Consequently, at $\omega = \pi/2$ the complete Hopf bifurcation structure for (3.1) can be described by the normal equation $\dot{c} = \mu c + K_3 c^3 + K_5 c^5 + K_7 c^7$ with $\mu, K_3, K_5 \approx 0$; $K_7 \neq 0$. We conclude that the equation

$$\dot{x}(t) = \beta x(t-1) + a_2 x^2(t) + b_2 x(t) x(t-1) + c_2 x^2(t-1). \tag{3.4}$$

for $\beta \approx -\pi/2$ can support at most three small periodic solution families bifurcating from $x = 0$.

A similar numerical analysis at other selected values of ω suggests this behavior to be generic for (3.3). However, by an examination of the crossing orders of the curves $K_j = 0; j = 3, 5, 7$ and observing their apparent continuity in ω, we are lead to conclude the existence of at least one value of ω in the interval $(2\pi/3, 3\pi/4)$ at which $K_3 = K_5 = K_7 = 0$ nontrivially. At such a value, a complete resolution of the Hopf bifurcation structure for (3.3) would require (at least) the computation of K_9. Such a computation, while theoretically within the scope of the alogorithm of [14], would be a nontrivial task likely requiring careful partitioning of the calculations and hundreds of hours of cpu time on a current SUN or VAX-like workstation.

See [6] where Corollary 2.4 is used to derive analogous computations for K_3, K_5 and K_7 for (3.1) when $h(x(t), x(t-1))$ is assumed to be odd.

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Linear Stability Curve

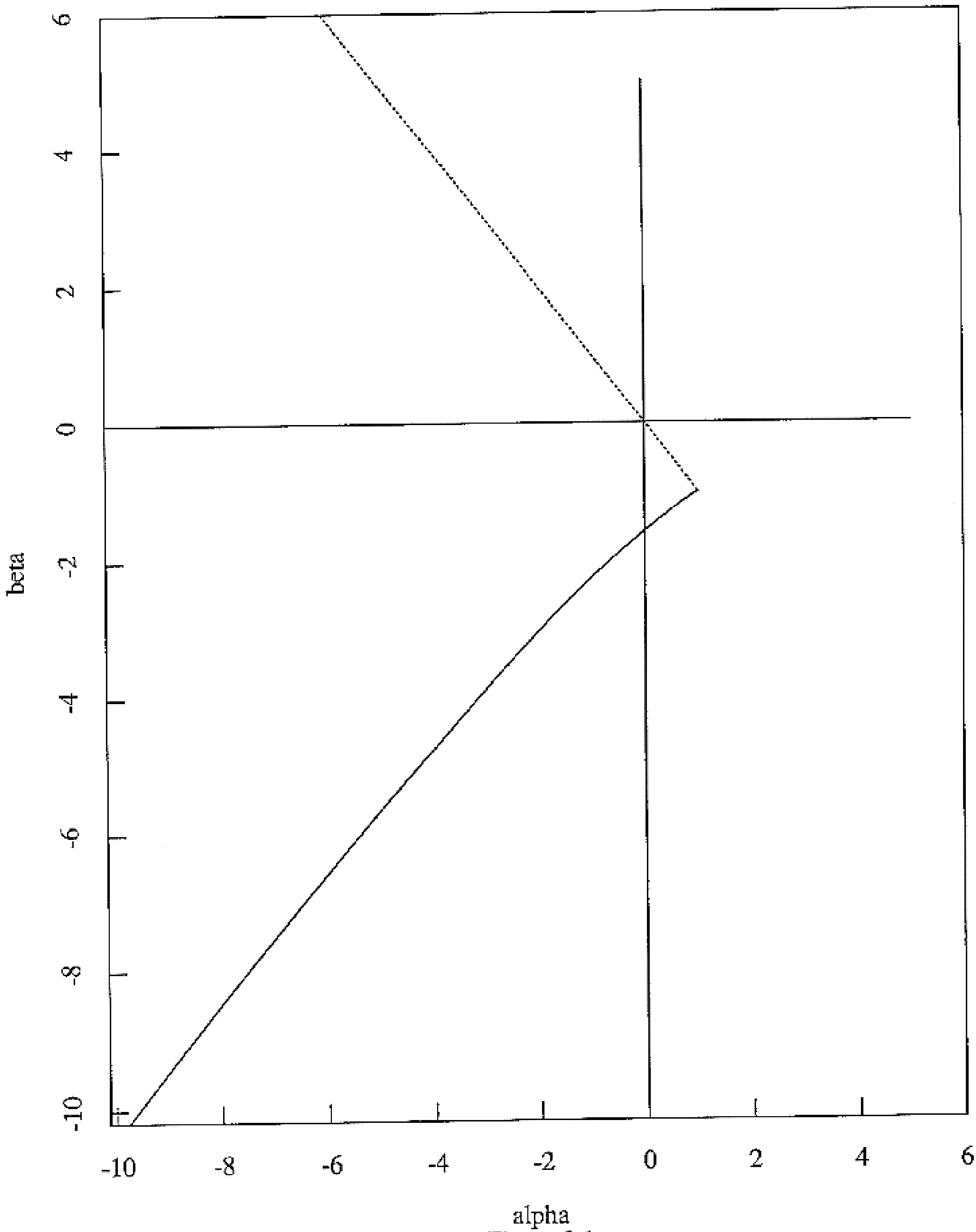

Figure 3.1

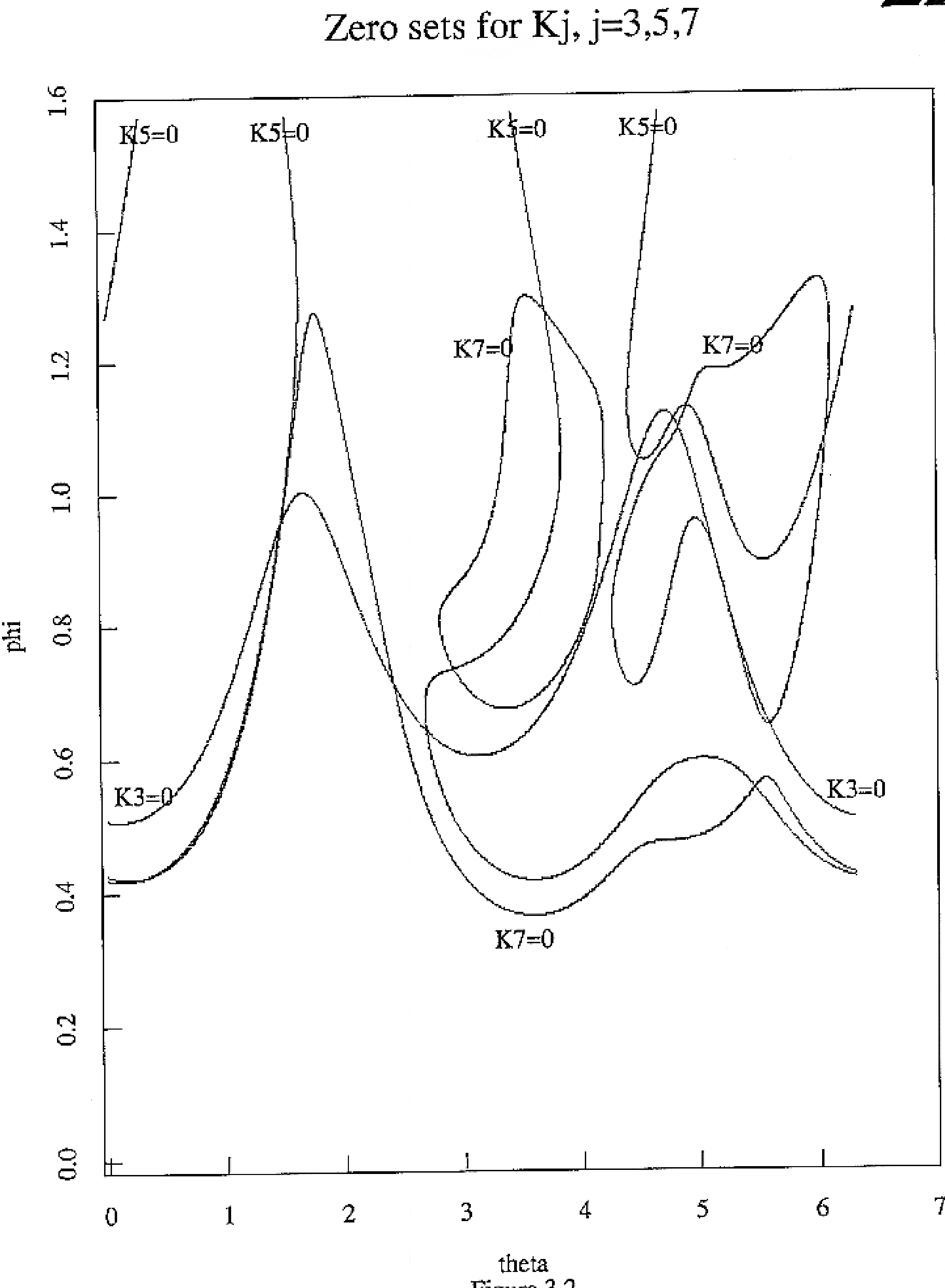
Zero sets for Kj, j=3,5,7
K5=0
K5=0
K5=0
K5=0
K7=0
K7=0
K3=0
K3=0
K7=0
phi
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
0
1
2
3
4
5
6
7
theta

Figure 3.2