THE EXISTENCE OF VISCIOUS PROFILES
AND ADMISSIBILITY FOR TRANSONIC SHOCKS

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1. **Introduction**

   In this paper we describe an application of a particular shock admissibility criterion, the existence of travelling wave profiles under viscous perturbation, to an old problem in conservation laws: shock waves in steady inviscid transonic potential flow. As in some other well-known uses of this criterion, for example the isentropic unsteady gas dynamics equations, there is already a general understanding of what constitute the physically admissible shocks, and one purpose of the research reported here is to test the criterion itself. We are particularly interested in this test because of some new contexts in which conservation laws that change type (hyperbolic/elliptic, for example) may appear: models for steady visco-elastic flows and unsteady multiphase flows. In establishing mathematical properties of shocks in systems that change type, one sees immediately that a basic admissibility criterion for shocks, the so-called Lax criterion, cannot be directly applied if the equations are not hyperbolic at both end states of the shock. Thus we are motivated to look at a familiar model to see how effective are the tools being developed.

   This work is an outgrowth of research in the same spirit by Warnecke [WA] on viscous profiles for the transonic small disturbance equations, and is related to results by Keyfitz [KE1] on use of the travelling wave criterion for systems of two conservation laws in space and time which change type. Other authors have used viscous approximations to the inviscid transonic equations: we are particularly indebted to Morawetz's paper [MOR], and to conversations with her, for the formulation of the
isentropic equations for steady flow with and without viscosity. In particular, the existence theorem of Morawetz includes, implicitly, the case we consider here: the contribution of this paper is to make the connection explicit, as well as to consider the effects of different types of viscous perturbation. We mention also work of Feistauer and Necas [FN] and of Mock [MO].

A special feature of this steady flow problem is that the correct viscosities depend on the flow direction. This dependence also occurs in the so-called entropy inequality admissibility criterion, as we shall show below. That this happens is quite natural, since in steady flow one does not have time as a distinguished variable (direction). Admissibility criteria for shocks always involve a distinguished direction and in steady problems this appears explicitly in the formulation, as was observed by Warnecke [WA] for the transonic small disturbance problem. However, in the small disturbance equation one of the variables is distinguished by the approximation itself. In steady flows the streamlines replace time as the distinguished direction since physical particles move along them as time proceeds. In a fixed frame of reference one has only a local distinguished direction because the streamlines generally change direction.

The term "entropy inequality" takes its name from the context of non-isentropic flows. We shall use this name, because it is standard mathematical terminology, even though entropy is conserved, and therefore does not satisfy an inequality, in the problems discussed here. What we actually find is a momentum inequality. The model considered here conserves mass, energy, irrotationality and entropy across shocks.
The potential flow model is a physically reasonable approximation for aerodynamic flows in which the Mach number upstream of shocks is less than about 1.6. In that case the change in entropy across the shock is small: it is of third order in the shock strength, (see Landau and Lifschitz [LL], Section 83). Thus the assumption of isentropy across shocks introduces only a small error for weak shocks. It should also be noted that the change of type discussed in this paper is not particular to potential flow. In the 4 by 4 system of steady, two-dimensional Euler equations two of the four characteristic roots are not real for subsonic flow, i.e. this system is of composite type.

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2. The Basic Equations

We are interested in the system of two equations which describes steady, inviscid, adiabatic, irrotational, isentropic two dimensional flow of an ideal gas. The equations are

\[ f(\vec{q})_x + g(\vec{q})_y = \delta_x \left( \frac{\rho u}{v} \right) + \delta_y \left( \frac{\rho v}{-u} \right) = 0. \]  \hspace{1cm} (2.1)

Here \( \vec{q} = (u,v) \) represents the flow velocity at a point \((x,y)\)
of the plane. We shall speak of \( q = |\vec{q}| \) as the local speed of the flow.

The quantity \( \rho \), the local density of the gas, is a given function of \( q \) in the model. For an adiabatic, isentropic ideal gas, one has the pressure density relation \( p(\rho) = A\rho^\gamma \), where \( A \) can be normalized by the stagnation values of \( p \) and \( \rho \):

\[ A = \tilde{p} \tilde{\rho}^{-\gamma} \]

and \( \gamma \) is the ratio of specific heats (\( \gamma = 1.4 \) for dry air).

These equations, with \( \gamma = 2 \), also model steady surface waves in shallow water, as described in Courant and Friedrichs [CF], Section 19 or Stoker [ST], Section 2.3. This analogy has been exploited in 'water table' experiments, which may be used to visualize transonic flows. Some nice examples may be found in Stoker [ST], Section 10.12.

From Bernoulli's law (described below) we have

\[ \rho_b(q) = \tilde{\rho}(1 - q^2/\hat{q}^2)^{1/(\gamma-1)} \]  \hspace{1cm} (2.2)

where \( \hat{q} \) is the limit (cavitation) speed:

\[ \hat{q}^2 = \frac{2\gamma\tilde{p}}{(\gamma-1)\tilde{\rho}} \]  \hspace{1cm} (2.3)

The local sound speed, \( c(\rho) \), is defined by

\[ c^2(\rho) = \frac{dp}{d\rho} = \gamma A\rho^{\gamma-1} \]  \hspace{1cm} (2.4)

Bernoulli's law allows one to write \( c \) as a function of \( q \); for
a gamma-law gas, one has, explicitly,

$$c^2 = \frac{\gamma - 1}{2} (\hat{q}^2 - q^2) = \frac{1}{1-\mu^2} (c_\star^2 - \mu^2 q^2).$$  \hspace{1cm} (2.5)

We shall present our results for a gamma law gas, but we note that all that is essential here is that $c$ be a decreasing concave function of $q$ and that there be a unique value, $q = c_\star$, at which $c(q) = c_\star$. This value defines the sonic circle, $B$, in state space: its exterior, $\mathcal{E}$, is the supersonic region where (2.1) is a strictly hyperbolic system; in the interior, the subsonic region $\mathcal{E}$, (2.1) is elliptic.

This is seen by seeking characteristic surfaces (curves) $\phi(x,y) = 0$ in the equation: they satisfy

$$\det | \phi_x df(q) + \phi_y dg(q) | = 0.$$

(Note: we shall use the current terminology of linear theory, see F. John [JO] or Treves [TR], for definitions.) Linearizing about a constant state $\hat{q}$, we find the characteristic curves are straight lines, $x \cos \psi + y \sin \psi = \text{const}$. The angle $\psi$ determines a direction, $\mathbf{n}_\phi = (\cos \psi, \sin \psi)$ normal to the characteristic curve, called the characteristic conormal or characteristic covector. For the gamma-law gas model, we have

$$(c^2-u^2) \cos^2 \psi - 2uv \sin \psi \cos \psi + (c^2-u^2) \sin^2 \psi = 0. \hspace{1cm} (2.6)$$

This has real solutions if and only if $u^2 + v^2 \geq c^2(q)$; in this case, the solutions satisfy
\[ \tan \psi_{\pm} = \frac{uv \pm c(q) \sqrt{q^2 - c^2(q)}}{c^2(q) - v^2}. \] (2.7)

Note that the 4 by 4 system of steady, two-dimensional Euler equations also has the characteristic directions given by (2.7). In the steady Euler equations one also has \[ \tan \psi_{3,4} = \frac{-u}{v} : \] the streamlines are additional, doubly characteristic curves.

One sees by calculation that

\[ \psi_{\pm} = \theta \pm \alpha \]

(see Figure 1) where \( \theta = \tan^{-1}(v/u) \) is the angle of inclination of the flow \( \vec{q} \) to the horizontal x-axis and

\[ \tan \alpha = \frac{\sqrt{q^2 - c^2}}{c}. \] (2.8)

The quantity \( \pi/2 - \alpha \), the angle between \( \vec{q} \) and \( \phi \), is called the Mach angle [CF]. Thus for any supersonic state \( \vec{q} \) the two distinguished directions, \( n_{\phi_{\pm}} \), defined by \( (\cos \psi_{\pm}, \sin \psi_{\pm}) \), which are normal to the characteristic surfaces \( \phi_{\pm}(x,y) = 0 \), take the place of the characteristic speeds in time-dependent hyperbolic theory: they delineate the limits of the cone of timelike conormals and, strictly speaking, live in a dual space. Figure 1 illustrates the geometry for two cases: \( \vec{q} \) near the sonic line and \( q \) near the limit speed. For reference, \( N \) indicates the normal to the flow velocity \( \vec{q} \). We adopt the convention that \( \alpha \)
and the + quantities are measured counterclockwise from $\delta$.

**Shocks and the Shock Polar**

We consider, as usual, weak solutions of (2.1) consisting of constant states $\dot{q}_0$, $\dot{q}_1$ in half-planes separated by a shock line, $\gamma$:

$$x \cos \beta + y \sin \beta = 0$$  \hspace{1cm} (2.9)

To simplify the following discussion, we shall assume the **upstream flow**, $\dot{q}_0$, to be horizontal and directed to the right ($u_0 = q_0 > 0$, $\theta_0 = v_0/u_0 = 0$), and take $\beta = \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$. Then $\dot{q}_0$ occupies the half-plane $x \cos \beta + y \sin \beta < 0$; the **downstream flow**, $\dot{q}_1$, occupies the other half-plane. In linear systems, characteristic curves are the lines which can sustain discontinuities; in a genuinely nonlinear system, discontinuities in $\dot{x}$ are supported on curves transverse to the characteristics, but in the limit of weak discontinuities one has $\beta \to \alpha$, as we shall see. The notation reflects this; in fact, if $\dot{q}_0$ is not horizontal, $\beta$ should be replaced by $\beta + \theta_0$ in (2.9). As shown in Figure 2, $\beta$ is the angle of the shock normal, $n_\gamma$, to the x-axis (or to the upstream flow, $\dot{q}_0$).

The states $\dot{q}_0$, $\dot{q}_1$ and shock $\gamma$ form a weak solution to (2.1) if the **Rankine-Hugoniot** relation holds: $s[f] = [g]$, where $s = \frac{dy}{dx}$ is the slope of $\gamma$. Hence

$$(f_1 - f_0) \cos \beta + (g_1 - g_0) \sin \beta = 0$$  \hspace{1cm} (2.10)
where subscript $i$ means evaluation at state $i$. We note that the model considered by Courant and Friedrichs [CF] supplements (2.10) with a further relation, because entropy is not constant across a shock. Specifically, $\rho_B(q)$ is replaced by $\rho_B(S,q)$, a more general form of Bernoulli's law, in (2.10), and then momentum, in addition, is conserved by weak solutions: in the isentropic formulation used here and by others (see e.g. Jameson [JA1]), total momentum is not conserved. (Equation (2.10) is eq'ns (118.05) and (118.07) of [CF]; Bernoulli's law is (118.08) and without change of entropy (118.06) cannot hold simultaneously. It would be interesting to consider the more complete system.)

We solve (2.10) as follows. Writing

$$\rho_1 u_1 \cos \beta + \rho_1 v_1 \sin \beta = \rho_0 q_0 \cos \beta$$

(2.11)

$$v_1 \cos \beta - u_1 \sin \beta = -q_0 \sin \beta$$

we derive

$$u_1 = q_0 \left\{ 1 - \left( 1 - \frac{\rho_0}{\rho_1} \right) \cos^2 \beta \right\}$$

(2.12)

$$v_1 = -q_0 \left( 1 - \frac{\rho_0}{\rho_1} \right) \sin \beta \cos \beta.$$ 

Evaluating $q_1^2 = u_1^2 + v_1^2$, we find

$$2$$
\[
\cos^2 \beta = \frac{1 - (q_1/q_0)}{1 - (\rho_0/\rho_1)^2} 
\]  
(2.13)

For each \( q_1 < q_0 \) such that \((\rho q)_1 > (\rho q)_0\), there are two solutions, \( \beta_\pm \), to (2.13), and hence two states \((u_1, v_1)\) which can be recovered from (2.12) (these coalesce if \( \beta = 0 \)); for \( q_1 > q_0 \) the condition is \((\rho q)_1 < (\rho q)_0\).

Since

\[
(q \rho)' = \frac{\rho}{c^2} (c^2(q) - q^2) 
\]  
(2.14)

for a gamma-law gas, we see that \( q \rho(q) \) has a maximum at \( q = c_* \) and decreases monotonically to zero as \( q \) decreases to zero or increases to \( q^* \). Thus we have the following global result.

**Proposition 2.1**  
a) Let \( \vec{q}_0 \in \mathcal{B} \), then there exists a unique \( q^*(q_0) < c_* < q_0 \) such that \( \rho_B(q^*)q^* = \rho_B(q_0)q_0 \); the Hugoniot locus, \( \Gamma(\vec{q}_0) \), is restricted to values \( q_1 \geq q^* \). We have \( \beta = 0 \) if and only if \( q_1 = q^* \). Equations (2.12) and (2.13) give \( \vec{q}_1 \), parameterized by \( \beta \): at \( \vec{q}_1 = \vec{q}_0 \), \( \beta = \pm \alpha \) (as in (2.8)). For \( q_1 < q_0 \), the geometry of \( \Gamma \) is a closed loop; for \( q_0 < q_1 \leq q^* \), it consists of two finite curve segments.

b) If \( \vec{q}_0 \in \mathcal{B} \), the Hugoniot locus is restricted to \( q_1 \geq q_0 \); it consists of two finite branches that form a cusp at \( \vec{q}_0 \).

c) If \( \vec{q}_0 \in \mathcal{E} \), then there is a unique \( q^*(q_0) > c_* > q_0 \) at which \( \rho_B(q^*)q^* = \rho_B(q_0)q_0 \). The Hugoniot locus consists of the state \( \vec{q}_0 \) itself and a disconnected finite branch defined for
$q_1 \geq q^*(q_0)$.

In each case, the locus is star-shaped with respect to $\hat{q}_0$. The locus can be parameterized by $\beta$; except at the point $\hat{q}_1 = (q^*/q_0)\hat{q}_0$, it can be locally parameterized by $q_1$.

**Proof:** Most of the geometry is clear. To see that the curve is star-shaped, we note that, from (2.11),

$$v_1 = -(q_0 - u_1) \tan \beta;$$

(2.15)

that is (see Figure 3), the line segment joining $\hat{q}_0$ to $\hat{q}_1$ makes an angle $-\beta$ with the negative x-axis. Hence we wish to show that $\beta$ is a monotonic function of $q_1$. Differentiating the right side of (2.13) with respect to $q_1$ we obtain the expression (up to a positive factor)

$$I(q_1) = \rho_0^2 q_0^2 - \rho_0^2 q_1^2 + (\rho_0^2 - \rho_1^2)c^2(q_1).$$

(2.16)

Now if $q_1^2 \leq c^2(q_1)$, then (since $\rho_1 > \rho_0$), we have

$$I < (\rho q)_0^2 - (\rho_0 q_1)^2 + (\rho_0^2 - \rho_1^2)q_1^2 = (\rho q)_0^2 - (\rho q)_1^2 < 0.$$  

On the other hand, $I(q_0) = 0$, and

$$\frac{dI}{dq_1} = -2\rho_0^2 q_1 - 2\rho_1 q_1 c^2(q_1) + 2(\rho_0^2 - \rho_1^2)c(q_1)c'(q_1)$$

$$= 2(\rho_1^2 - \rho_0^2)q_1 - 2(\rho_1^2 - \rho_0^2)cc' > 0$$
since \( c' < 0 \) for a gamma-law gas.

Thus, if \( q_0 \in \mathcal{H} \), then for any \( q_1 < q_0 \), the right hand side of (2.13) is an increasing function of \( q_1 \); equation (2.15) now implies the star-shapedness of the loop. The remaining calculation is omitted. □

We can also relate \( \beta \) to the local Mach angle in the supersonic region:

**Proposition 2.2:** If \( q_0 > c_* \) and \( q_0 > q_1 \), then \( |\alpha_0| > |\beta| \), where \( \alpha_0 \) corresponds to \( q_0 \) in (2.8); if in addition \( q_1 > c_* \), then \( |\beta| > |\alpha_1| \).

**Proof:** We have seen that \( \cos^2 \beta \) is an increasing function of \( q_1 \). As \( q_1 \to q_0 \), we can evaluate, using l’Hôpital’s rule,

\[
\lim_{q_1 \to q_0} \cos^2 \beta = \left( -\frac{\rho_0^2}{q_0^2} \right) \frac{c^2(q_0)}{2 \rho_0 \rho'} = \frac{c^2(q_0)}{q_0^2}
\]

where we have used

\[
\rho' = \frac{d\rho}{dq} = -\frac{q^2}{c^2(q)}.
\]

Note that \( \frac{c^2(q_i)}{q_i^2} = \cos^2 \alpha_i \), from (2.8). Since \( \cos^2 \beta \) is increasing, it follows that \( \cos^2 \beta > \cos^2 \alpha_0 \) and hence \( |\beta| < |\alpha_0| \), for \( q_1 < q_0 \). Reversing the roles of \( q_0 \) and \( q_1 \), we
find the opposite inequality for $q_1$.

We note that precisely the states \( \{q_0', q_1', \beta \} \) with $q_1 < q_0$ and hence $\rho_1 > \rho_0$ are compressive. We shall refer to these as the physically admissible shocks, using the physical admissibility condition of the more general model given by the Euler equations. We want to compare this to the viscosity criterion, which we now derive.

The Viscous Isentropic Equations

We follow the deviation of Morawetz [MOR], except for the final form in which we shall use the equations. In particular, the standard equations for steady two-dimensional viscous isentropic flow are (Morawetz uses the derivation of Synge [SY]):

\[
\begin{align*}
(i) \quad & (\rho u)_x + (\rho v)_y = 0 \\
(ii) \quad & (\rho u^2)_x + (\rho uv)_y + p_x = \mu \Delta u \\
(iii) \quad & (\rho uv)_x + (\rho v^2)_y + p_y = \mu \Delta v.
\end{align*}
\]

(2.17)

Here $\rho, u$ and $v$ are functions of $x$ and $y$; $\mu$ is called the dynamic viscosity: $\mu = \nu \rho$ where $\nu$ is the kinematic viscosity.

Now (2.17 (ii) and (iii)) are equivalent to (using (i)):

\[
\begin{align*}
& uu_x + vu_y + i_x = \nu \Delta u \\
& uv_x + vv_y + i_y = \nu \Delta v.
\end{align*}
\]

(2.18)

where, for isentropic flow, $i$ is a function of $\rho$, namely
\[ i(\rho) = \int \frac{p'(\rho)}{\rho} \, d\rho = \frac{\lambda \gamma}{\gamma - 1} \rho^{\gamma - 1} \quad (2.19) \]

for a gamma-law gas.

Defining the vorticity,

\[ \omega = u_y - v_x, \quad (2.20) \]

one finds that (2.18) implies a diffusion equation for \( \omega \):

\[ \vec{q} \cdot \nabla \omega + \omega \text{ div } \vec{q} = \nabla \cdot (\omega \vec{q}) = \nu \Delta \omega. \quad (2.21) \]

Let us now assume \( \nu = \text{constant} \); this is convenient for integrating and not important in the limit \( \nu \to 0 \). The remaining information in (2.18) can be expressed as

\[ \nabla \left( \frac{1}{2}(u^2 + v^2) + i(\rho) - \nu(u_x + v_y) \right) = \omega \begin{pmatrix} -v \\ u \end{pmatrix} + \nu \begin{pmatrix} \omega_y \\ -\omega_x \end{pmatrix}, \quad (2.22) \]

which also implies, along streamlines

\[ \vec{q} \cdot \nabla \left( \frac{1}{2}(u^2 + v^2) + i(\rho) - \nu(u_x + v_y) \right) = \nu(u \omega_y - v \omega_x). \quad (2.23) \]

Now, since we are interested in flows where \( \vec{q} \) is constant upstream and hence \( \omega = 0 \) upstream, we see that compatibility with this condition implies, from (2.21)

\[ \omega = 0 \quad (2.24) \]

in \( \mathbb{R}^2 \), and hence, from (2.22)
\[ \frac{1}{2} (u^2 + v^2) + i(\rho) - \nu(u_x + v_y) = \text{const} = \frac{1}{2} q^2. \tag{2.25} \]

Now, since \( i' = p'/\rho \neq 0 \), we see that (2.25) in the case \( \nu = 0 \) is precisely Bernoulli's law. If \( \nu \neq 0 \), we write (from (2.19))

\[
\rho = \left( \frac{\gamma - 1}{\gamma A} \left( \frac{1}{2} q^2 - \frac{1}{2} q^2 + \nu \text{ div } \mathbf{q} \right) \right)^{1/(\gamma - 1)}
\]

\[
= \rho_B \left( \sqrt{q^2 - 2\nu \text{ div } \mathbf{q}} \right)
\]

\[
= \rho_B(q) + \nu F(\nu, q, \text{ div } \mathbf{q}). \tag{2.26}
\]

We thus conclude that the addition of a viscous term to (2.1) will, in the context of interest, maintain irrotationality and replace (2.2) by (2.26). Thus

\[
\left( \rho_B(q)u \right)_x + \left( \rho_B(q)v \right)_y = -\nu \left( (uF)_x + (vF)_y \right)
\]

\[
(v_x - u_y) = 0 \tag{2.27}
\]

Expanding the right side of (2.27), keeping only first-order terms in \( \nu \), gives

\[
-\nu \left\{ \left( \frac{\rho_B(q)}{c^2(q)} \text{ div } \mathbf{q} \right)_x + \left( \frac{\rho_B(q)}{c^2(q)} \text{ div } \mathbf{q} \right)_y \right\}
\]

for the viscous perturbation. Retaining only the terms with second-order derivatives gives
\[-\kappa \left( u(u_{xx} + v_{xy}) + v(u_{xy} + v_{yy}) \right) \]
\[= -\kappa \ (u \Delta u + v \Delta v) \]  

(2.28)

where \( \kappa > 0 \) and we have used (2.24). We note that near
\( q = \tilde{q}_0 = (u_0,0) \), with \( u_0 > 0 \), this gives a negative coefficient;
the elliptic regularization in (2.27) should perhaps be understood
in a more general context [KE2].

3. Travelling Wave Profiles

The system (2.27) or the equivalent system consisting of
(2.1), without the Bernoulli relation, supplemented by (2.25),
gives a natural viscous perturbation of (2.1). It is natural
because, for constant upstream flow, the irrotationality condition
is preserved and Bernoulli's relation replaced by (2.25); it may
not be completely physical because of the choice of constant
viscosity coefficient. (In addition, we are maintaining the
assumption that the flow is isentropic.) In this section we show
that the travelling wave criterion matches exactly the physical
admissibility criterion, when this natural viscosity is chosen.
Later we shall examine other choices.

Given a shock solution \( \{\tilde{q}_0, \tilde{q}_1, \beta\} \) of (2.1), satisfying
\[
\tilde{q}(x,y) = \begin{cases} 
\tilde{q}_0, & x \cos \beta + y \sin \beta < 0 \\
\tilde{q}_1, & x \cos \beta + y \sin \beta > 0.
\end{cases}
\]

where (2.11) holds, we seek a viscous profile or travelling wave
\[ \dot{q} = \dot{q}(\xi) = \dot{q}\left(\frac{x \cos \beta + y \sin \beta}{\nu}\right) \text{ satisfying (2.27) and the boundary conditions} \]

\[
\begin{align*}
\dot{q}(\xi) &\rightarrow \\
\dot{q}_0' &\quad \xi \rightarrow -\infty \\
\dot{q}_1' &\quad \xi \rightarrow +\infty.
\end{align*}
\tag{3.1}
\]

It is convenient to seek instead a solution \((q(\xi), \rho(\xi))\) to (2.1) and (2.5); again we do not assume the Bernoulli relation for \(\rho(\xi)\) but impose the boundary conditions

\[
\rho(\xi) \rightarrow \\
\begin{align*}
\rho_0 &= \rho_B(\dot{q}_0') &\xi \rightarrow -\infty \\
\rho_1 &= \rho_B(\dot{q}_1') &\xi \rightarrow +\infty
\end{align*}
\tag{3.2}
\]

(This corresponds exactly to (2.27) if (2.26) is used.)

Substituting \(q(\xi), \rho(\xi)\) in (2.1) and (2.5) yields

\[
\begin{pmatrix} \rho u \\ v \end{pmatrix}' \cos \beta + \begin{pmatrix} \rho v \\ -u \end{pmatrix}' \sin \beta = 0
\]

\[\tag{3.3}
\]

\[u' \cos \beta + v' \sin \beta = \frac{1}{2}(u^2 + v^2 - q^2) + i(\rho)
\]

Here \(\cdot = \frac{d}{d\xi}\). We can integrate the first two equations in (3.3) from \(\xi = -\infty\), using (3.1) and (3.2):

\[
(\rho u - \rho_0 q_0) \cos \beta + (\rho v) \sin \beta = 0
\]

\[\tag{3.4}
\]

\[v \cos \beta - (u - q_0) \sin \beta = 0\]
This implies that (3.3) reduces to a single first-order equation for \((\dot{q}(\xi), \rho(\xi))\) along the trajectories, which are conveniently described by \(u\), since \(v\) and \(\rho\) are given in terms of \(u\) by (3.4). In the \((u, v)\) plane, the trajectory lies on the line through \(\dot{q}_0\) with slope \(\tan \beta\):

\[v = (u - q_0) \tan \beta.\]  

(3.5)

Along the trajectory,

\[\rho(u - q_0 \sin^2 \beta) = \rho_0 q_0 \cos^2 \beta,\]  

(3.6)

and this admits a solution \(\rho(u)\) for \(u_1 \leq u \leq u_0\) since

\[u - q_0 \sin^2 \beta \geq u_1 - q_0 \sin^2 \beta = \frac{q_0 \rho_0}{\rho_1} \cos^2 \beta > 0,\]

from (2.12). At \(u = u_1\), (3.5) implies \(v = v_1\), by (2.15).

Furthermore, \(\rho = \rho_1 = \rho_b(q_1)\) satisfies (3.6) and by the star-shapedness of the shock polar (Proposition 2.1) it is the only solution.

Finally, using \(v' = u' \tan \beta\), we reduce the viscous profile equation to

\[u' \sec \beta = i(\rho) + \frac{1}{2} (u^2 + (u - q_0)^2 \tan^2 \beta - \hat{q}^2)\]  

(3.7)

**Proposition 3.1:** For each \((\dot{q}_0, \dot{q}_1, \beta)\) with \(q_1 < q_0\), equation
(3.7) has rest points at \( u_0 \) and \( u_1 \) and nowhere else in the interval \( (u_0, u_1) \); the first point is a repellor and the second an attractor.

**Corollary 3.2:** System (3.3) with boundary conditions (3.1), (3.2) admits solutions, unique up to a translation in \( \xi \); \( u, v \) and \( \rho \) are monotone along the trajectories.
Proof: Write (3.7) as \( u' \sec \beta = V(u) \). A calculation gives (using (3.6) for \( \rho'(u) \)):

\[
dV(u) = i'(\rho)\rho'(u) + u + (u-q_0) \tan^2 \beta \\
= \frac{p'(\rho)}{(u-q_0 \sin^2 \beta)} + u + (u-q_0) \tan^2 \beta.
\]

Now \( p'(\rho) = c^2(\rho) \) from (2.4); we use the fact that \( \rho \) satisfies Bernoulli's law at \( \hat{d}_0 \) and \( \hat{d}_1 \) to deduce

\[
dV(u_0) = -\frac{c^2(q_0)}{q_0 \cos^2 \beta} + q_0 = \frac{q_0}{\cos^2 \beta} (\cos^2 \beta - \cos^2 \alpha)
\]

where \( \alpha \) is the Mach angle defined in (2.8). From Proposition 2.2, \( dV(u_0) > 0 \): \( u_0 \) is a repellor for the one-dimensional vectorfield \( V \).

Similarly

\[
dV(u_1) = -\frac{c^2(q_1)}{(u_1 + v_1 \tan \beta) \cos^2 \beta} + u_1 + v_1 \tan \beta
\]

\[
= \frac{-c_1^2 + (u_1 \cos \beta + v_1 \sin \beta)^2}{(u_1 + v_1 \tan \beta) \cos^2 \beta}
\]

\[
= \frac{\rho_0^2 q_0^2 \cos^2 \beta / \rho_1^2 - c_1^2}{D}
\]

from (2.11); the numerator is (using (2.13))
\[
\frac{\rho_0^2 (q_0^2 - q_1^2)}{\rho_1^2 (1 - (\rho_0/\rho_1)^2)} - c_1^2 = \frac{\rho_0^2 (q_0^2 - q_1^2) - c_1^2 (\rho_1^2 - \rho_0^2)}{\rho_1^2 - \rho_0^2} \\
= \frac{I(q_1)}{\rho_1^2 - \rho_0^2}
\]

where \( I(q_1) \), defined by (2.16), is negative. Hence \( u_1 \) is an attractor for the one-dimensional flow. Because the Hugoniot locus is starshaped, \( V(u) < 0 \) in \((u_1, u_0)\), and hence a connecting orbit exists. The monotonicity of \( u \) and \( v \) follows from the fact that the trajectory is on the line (3.5); also from (3.6),

\[
\rho'(\xi) = \frac{d\rho}{du} u' = -\frac{u'(\xi)\rho}{u-q_0\sin^2\beta} > 0,
\]

so \( \rho \) is also monotonic.

4. Other Perturbation Matrices

We now look at viscous profiles and connecting orbits for (2.1) in a somewhat more abstract way, by looking at general perturbations

\[
f(\vec{q}) \xi + g(\vec{q})_y = cL(\vec{q}) \quad (4.1)
\]

Here \( L(\vec{q}) \) is a homogeneous second-order operator. If we restrict attention to shock polars \( H(\vec{q}_0) \) for \( \vec{q}_0 \in \mathcal{H} \) near the sonic line, then it is reasonable to consider \( L \) to be a constant coefficient operator.
In the context of evolution equations, where \( y \) represents time and \( g(\tilde{q}) = \tilde{q} \), it is natural to consider

\[
L(\tilde{q}) = D\tilde{q}_{xx},
\]

with \( D \) a positive definite matrix (more precisely, a stable matrix in the sense of Majda and Pego [MP]). The results of Warnecke [WA] on the transonic small disturbance equation show that this is not the correct framework for steady flow; indeed the perturbation considered in the last section does not have this character at all: as well as being singular, the associated matrix is negative. The local results of Keyfitz [KE1] on evolution equations that change type show that the structure of travelling wave profiles with perturbations of the type (4.2) and \( D \) near \( I \) is very different from what is expected in transonic flow. To unify the details, let us examine a travelling wave approximation to a shock \( (\tilde{q}_0, \tilde{q}_1, \beta) \) and replace (4.1) by the system

\[
P\tilde{q}'(\xi) = (f(\tilde{q}(\xi))-f_0)\cos \beta + (g(\tilde{q}(\xi))-g_0)\sin \beta. \quad (4.3)
\]

The matrix \( P \) will in general depend on \( \beta \). The physical viscosity of section 2 gives

\[
P = \begin{pmatrix} -\kappa u & -\kappa v \\ 0 & 0 \end{pmatrix} \quad (4.4)
\]

in equation (2.28).
If \( P \) were invertible, then (4.3) would represent a two dimensional vectorfield. Write it as

\[
P\dot{q}'(\xi) = V(q)
\]

or

\[
\dot{q}'(\xi) = P^{-1}V(q).
\] (4.5)

Now \( V(\dot{q}_0) = V(\dot{q}_1) = 0 \), and \( V \) has no other rest points near \( \dot{q}_0 \).

If we consider the special case \( \dot{q}_0 = \dot{q}_1 \in B \) (i.e. on the sonic line) we may ask: for what class of matrices \( P \) does the unfolding of (4.5) have the basic one-dimensional structure observed in the last section: all orbits connect \( \dot{q}_0 \) (at \( \xi = -\infty \)) to \( \dot{q}_1 \) (at \( \xi = +\infty \)), if \( q_1 < q_0 \). First we prove a preliminary result, which is independent of the perturbation \( P \) and shows that the vectorfield \( V(q) \) has a structure, near the sonic line, which is different from the transonic small disturbance equations [WA] or the normal form (consistent with the transonic small disturbance equations) found by Mock [MO].

**Proposition 4.1:** Consider the vectorfield

\[
V(q, \dot{q}_0, \beta) = \left(f(q) - f(\dot{q}_0)\right)\cos \beta + \left(g(q) - g(\dot{q}_0)\right)\sin \beta
\] (4.6)

for \( \dot{q}_0, \dot{q}_1 \) and \( \beta \) related by the Rankine-Hugoniot relation (2.10), and \( q_0 = |\dot{q}_0| = c_* \). Then the limit \( \dot{q}_1 \to \dot{q}_0 \).
\( \beta \to (\theta_0 \pm \alpha_0) \) defines a codimension one vectorfield which is equivalent (under the notion of vectorfield equivalence in Guckenheimer and Holmes [GH]) to the parameterized vectorfield

\[
X_\mu(x) = x(\mu - x). \tag{4.7}
\]

Furthermore, the limit \( \hat{q}_1 = \hat{q}_0 \) with \( q_0 = c_\ast \) is vectorfield equivalent to a codimension one vectorfield

\[
Y_\mu(x) = x(\mu^2 - x). \tag{4.8}
\]

**Proof:** We may consider (4.6) to be a parameterized family of vectorfields, parameterized by \( \beta \); for any \( \beta \), \( V(\hat{q}, \hat{q}_0, \beta) = 0 \), and at \( \beta = \alpha_0 = \cos^{-1} \frac{c_\ast}{q_0} \) there is joined a non-trivial branch of solutions to \( V(\hat{q}, \hat{q}_0, \beta) = 0 \), i.e. the solutions to the Rankine-Hugoniot relation, \( \hat{q} = q_1(\beta) \). Necessarily, then, \( \det dV(\hat{q}_0, \hat{q}_0, \alpha) = 0 \) and a calculation gives

\[
dV(\hat{q}_0, \hat{q}_0, \alpha) = \begin{bmatrix}
\rho_0 \left(1 - \frac{u_0^2}{c_0^2}\right) \cos \alpha & \rho_0 \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{bmatrix}
\]

Clearly \( \det dV = 0 \), and the eigenvalues are \( 0 \) and \( \cos \alpha (1 - \rho_0 \sin^2 \alpha) \); the corresponding eigenvectors are

\[
\hat{\nu}_0 = (\cos \alpha_0, \sin \alpha_0), \quad \hat{\nu}_1 = (\rho_0 \sin \alpha_0, 1).
\]
In the region of interest \((q_0 \text{ near } c_\ast)\), \(\beta\) is near zero, and the
eigenvalue of \(dV\) is near one; hence the rank of \(dV\) is always
greater than or equal to 1, and \(V\) is equivalent to a
one-dimensional problem. We introduce the coordinates \(x\)
obtained by projecting \(\vec{a} - \vec{a}_0\) onto \(\vec{v}_0\), and \(\mu = \beta - \alpha_0\). A
calculation shows \(d^2V(\vec{v}_0, \vec{v}_0)\) (the second directional derivative
of \(V\)) at \(\vec{a}_0\), and \(\frac{\partial}{\partial \beta} dV\) at \(\vec{a}_0\) are both non-zero if \(q_0 > c_\ast\).
Hence by Theorem 3.4.1 of Guckenheimer and Holmes [GH] (or rather,
by the theorem of Sotomayor [SO] quoted there) we obtain (4.7).
We note that this is called a codimension one vectorfield problem
because the parameterized family (4.7) is stable under
perturbations which preserve the trivial solution. In the context
of singularity theory for steady-state bifurcation with a
distinguished parameter, (4.7) has codimension zero under the
related notion of t-equivalence (Golubitsky and Schaeffer, [GS]
Vol I p 129).

If \(q_0 = c_\ast\), then \(dV\) assumes the simple form

\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]

and \(\vec{v}_0 = (1,0)\). Again, there is a one-dimensional null space and
projection onto \(\vec{v}_0\) is possible, using the center manifold
construction in Guckenheimer and Holmes [GH]. However, we now
find \(\frac{\partial}{\partial \beta} dV(\vec{v}_0) = 0\) but \(\frac{\partial^2}{\partial \beta^2} dV \neq 0\) (where it is necessary to
carry out one step of the center manifold reduction to calculate
the correct derivative here). This determines the singularity
(4.8); it has t-equivalence codimension 1, and an unfolding is given by

\[ W_\mu(x,a) = x(\mu^2 + a - x) \]  \hspace{1cm} (4.9)

We note that the difference between (4.7) and (4.8) results from a singularity in the bifurcation parameter, \( \mu \), rather than a degeneracy in higher-order terms as described in Guckenheimer and Holmes [GH Ch7 §1]. In particular, (4.7) is different from (4.8) only if there is a distinguished parameter present. This amounts (in our application) to examining the behavior of trajectories as \( \hat{q}_1 \) traverses the shock polar from \( \hat{q}_0 \) (distinguished parameter) as contrasted to examining the trajectories joining a pair of points \( \{\hat{q}_0, \hat{q}_1\} \), which are related by the Rankine-Hugoniot relation (but not distinguishing the role of \( \beta \)). In the context of vectorfield bifurcation with distinguished parameter, (4.8) has codimension one, while (4.7) is stable (codimension zero).

The equation \( \dot{x} = Y_\mu(x) \) exhibits the flow illustrated in Figure 4. The unfolding gives rise to Figures 5 (\( a > 0 \)) and 6 (\( a < 0 \)).

While we have not explicitly identified the unfolding parameter, \( a \), it is clear that in Figure 6 the line \( x = 0 \) represents a point \( \hat{q}_0 \) in the supersonic region; \( \hat{q}_1 \) is identified with the corresponding point on the parabola (which represents the shock polar parameterized by \( \mu \)); the intersections of the parabola and the \( \mu \)-axis define the loop in the shock polar and its extension beyond \( \hat{q}_0 \). In Figure 5, \( \hat{q}_0 \) is in the subsonic
zone and does not intersect its shock polar.

We now return to (4.3). Since the construction in Proposition 4.1 is valid only locally, we are limited to considering small amplitude shocks near the sonic line. Global extensions using other methods may be possible.

At \( q_0 = c_\ast \) and \( \beta = 0 \), the center manifold is horizontal (in the \( uv \) plane, it lies on the \( u \)-axis). Hence the physical viscosity of (4.4) has the effect of forcing the solution to lie on the one-dimensional center manifold, while the coefficient \( -\kappa u = -\kappa c_\ast \) determines the dynamics on the center manifold itself to be the opposite of Figures 4–6. In particular the finite trajectories always go from the higher speed state to the lower-speed one (we recover the physical condition). Because the nonzero eigenvalue of \( dV \) affects the vertical \( (v) \) direction and is positive, we see that a diagonal matrix \( P \) will preserve the dynamics if both diagonal entries are negative.

Let us consider a general perturbation

\[
P = \begin{pmatrix}
-1 & b \\
c & d
\end{pmatrix}
\]

Suppose \( \det P = -d - bc > 0 \), and \( d < 0 \), so \( P \) is negative definite (the case where \( P \) is singular can also be considered). We may understand the dynamics of (4.5) by examining the dynamics of

\[
\dot{\tilde{z}}' = P^{-1}W(\tilde{z})
\]   \hspace{1cm} (4.10)
where \( \hat{z} = (x,y) \) and \( W(\hat{z}) = \begin{pmatrix} x(\beta^2 + a - x) \\ y \end{pmatrix} \). We find that (4.10) has a saddle at \( \hat{z} = (0,0) \), while the other rest point is a sink as long as \( \beta^2 + a < 0 \) - i.e. as long as \( \hat{d}_1 \) is on the loop of the shock polar.

5. **The Momentum Inequality Criterion**

In the treatment of non-isentropic flows via conservation laws the change of entropy across shocks provides a natural admissibility criterion in the form of an inequality. The entropy must increase across a shock (see Landau and Lifschitz [LL]). For non-isentropic flows this implies \( \rho_1 > \rho_0 \); that is, shocks are compressive. The latter is the criterion commonly adopted for isentropic flows.

An important feature in the non-isentropic theory using the Euler equations is the fact that in the smooth parts of the flow the entropy is an additional conserved quantity. It is, therefore, natural to ask if such a situation arises in isentropic flows: is there an additional conserved quantity that provides an inequality type admissibility criterion at shocks?

Such a criterion is known in isentropic thermoelasticity in case the energy equation has a source term. Then an energy inequality provides a natural admissibility criterion (see Dafermos [DA]). But in the potential flow model discussed here the conserved quantities across a shock are mass, energy, entropy and rotationality. A natural candidate is, therefore, the momentum balance.

**Proposition 5.1:** The momentum equations
\[(\rho_B(q)u^2)_x + (\rho_B(q)uv)_y + p(\rho_B(q))_x = 0 \quad (5.1)\]

\[(\rho_B(q)u) + (\rho_B(q)v^2)_y + p(\rho_B(q))_y = 0\]

hold for smooth solutions to (2.1).

**Proof:** This can be seen by calculating \( p(\rho_B(q))_x \) and using \( u_y = v_x \), but there is an elegant way of deriving the equations (5.1) from variational principles. The irrotationality \( u_y - v_x = 0 \) implies locally the existence of a flow potential \( w : \nabla w = (u, v) \). This satisfies the second order partial differential equation

\[(\rho_B(|\nabla w|)w_x)_x + (\rho_B(|\nabla w|)w_y)_y = 0. \quad (5.2)\]

This equation is the Euler-Language equation for critical points of the functional

\[F(w) = \frac{1}{2} \int R(|\nabla w|)dx dy,\]

where \( R(q) = \int_0^q \rho(s)ds \). This fact is known as the Bateman variational principle. Now the integrand of \( F \) does not depend explicitly on \( x \) or \( y \). Hence, Noether's theorem gives two additional conservation laws satisfied by smooth solutions of (5.2), (or (2.1)). These are
\[
\left[ \frac{1}{2} R(q) - \rho_B(q) u^2 \right]_x - \left[ \rho_B(q) u v \right]_y = 0
\]
\[
\left[ -\rho_B(q) u v \right]_x + \left[ \frac{1}{2} R(q) - \rho_B(q) v^2 \right]_y = 0.
\]

From the definition of \( R \) one can show that \( \frac{1}{2} R = -p \), and we obtain (5.1). \( \square \)

Now let us project the left hand side of (5.1) onto an arbitrary unit vector \((\cos \varphi, \sin \varphi)\) and denote the resulting quantity by \( M(\varphi) \):

\[
M(\varphi) = \left[ (\rho_B u^2 + p) \cos \varphi + ((\rho_B u) v) \sin \varphi \right]_x
+ \left[ (\rho_B u v) \cos \varphi + ((\rho_B v^2 + p) \sin \varphi \right]_y.
\]

We claim that the inequality

\[ -M(\varphi) \leq 0, \quad (5.3) \]

understood in the sense of distributions, gives a local entropy inequality in the usual sense (cf. Smoller [SM] Sect. 20B.) for certain angles \( \varphi \) to be specified below. By local we mean that in different domains of a given flow one may have to use different angles \( \varphi \).

We have already seen that in smooth regions of a flow \( M(\varphi) = 0 \). Therefore, it remains to establish (5.3) at a shock. At any point of the shock curve, with normal \((\cos \beta, \sin \beta)\), (5.3) gives the inequality
\[
\left[ (\rho_B u^2 + p) \cos \varphi + \rho_B (uv) \sin \varphi \right] \cos \beta \\
+ \left[ (\rho_B uv) \cos \varphi + (\rho_B v^2 + p) \sin \varphi \right] \sin \beta \right]^0_1 \leq 0, \quad (5.4)
\]

where \([f]^0_1 = f_0 - f_1\) and the subscripts 0 and 1 denote the upstream and downstream values of a quantity, respectively.

Denote by \(U = u \cos \beta + v \sin \beta\) and \(L = -u \sin \beta + v \cos \beta\) the normal and tangential projections of the velocities, and set \(V(\varphi) = u \cos \varphi + v \sin \varphi\). Then we may rewrite (5.4) as

\[
\left[ \rho_B UV(\varphi) + p \cos (\beta - \varphi) \right]^0_1 \leq 0. \quad (5.5)
\]

Now suppose \(\beta - \varphi = \pm \frac{\pi}{2}\), then \(V(\varphi) = \pm L\) and (5.5) states \(\pm \left[ \rho_B UL \right]^0_1 \leq 0\). Actually, \(\rho_B(q_0)U_0L_0 = \rho_B(q_1)U_1L_1\) because the Rankine-Hugoniot conditions for (2.1) are

\[
\rho_B(q_0)U_0 = \rho_B(q_1)U_1 \quad \text{and} \quad L_1 = L_0. \quad (5.6)
\]

So for \(\varphi = \beta \pm \frac{\pi}{2}\) the inequality (5.5) poses no restriction on the upstream and downstream states.

Now let us consider \(\varphi = \beta\), then we have

\[
\left[ \rho_B u^2 + p \right]^0_1 \leq 0, \quad (5.7)
\]

or equivalently, the normal component of momentum increases through a shock.
**Lemma 5.1:** The inequality (5.7) is equivalent to the admissibility condition that physical shocks are compressive: 
\[
\begin{bmatrix}
\rho_b
\end{bmatrix}_1^0 \leq 0 \text{ or } \begin{bmatrix}
q
\end{bmatrix}_1^0 \geq 0.
\]

**Proof:** Since \( L_0 = L_1 \) note that for a given \( L = L_0 \) one has
\[
\frac{\partial}{\partial U} \left( \rho_b(U^2 + L^2)U^2 + p(U^2 + L^2) \right) = U \frac{\partial}{\partial U} \left( \rho_b(U^2 + L^2)U \right).
\]
So taking \( U^* \) to be the value where \( \rho_b(U^2 + L^2)U \) has its maximum one has
\[
\begin{bmatrix}
\rho_b(U^2 + L^2)U^2 + p(U^2 + L^2)
\end{bmatrix}_1^0 = \int_{U_0}^{U_1} \frac{\partial}{\partial \xi} (\rho_b(s^2 + L^2)s) ds
\]
\[
= \int_{U_0}^{U_1} (s - U^*) \frac{\partial}{\partial \xi} (\rho_b(s^2 + L^2)s) ds
\]
\[
+ U^* \int_{U_1}^{U_0} \frac{\partial}{\partial \xi} (\rho_b(s^2 + L^2)s) ds
\]
\[
= \int_{U_0}^{U_1} (s - U^*) \frac{\partial}{\partial \xi} (\rho_b(s^2 + L^2)s) ds
\]
\[
+ U^* \left[ \rho_b(U^2 + L^2)U \right]_{1}^0.
\]
Since \( \frac{\partial}{\partial s} (\rho B (s^2 + L^2) s) \) is positive for \( s < U^* \) and negative for \( s > U^* \) and \( \left[ \rho B (U^2 + L^2) U \right]_1^0 = 0 \) is one of the Rankine-Hugoniot conditions we have

\[
\left[ \rho B (U^2 + L^2) U^2 + p(U^2 + L^2) \right]_1^0 < 0 \quad (5.8)
\]

if and only if \( U_1 < U_0 \). If \( U_0 > U_1 \) the sign is reversed. \( \square \)

**Proposition 5.2:** Strict inequality in (5.5) holds for all \( \varphi \in (\beta - \frac{\pi}{2}, \beta + \frac{\pi}{2}) \) if and only if \( U_1 < U_0 \).

**Proof:** Differentiating the function

\[
f(\varphi) = \left[ \rho B (U^2 + L^2) UV(\varphi) + \cos (\beta - \varphi) p(U^2 + L^2) \right]_1^0
\]

one sees that it has only one extremum on the interval \( (\beta - \frac{\pi}{2}, \beta + \frac{\pi}{2}) \). Therefore we have \( f(\varphi) < 0 \) on this interval. Above we had seen that \( f \) vanishes at \( \varphi = \beta \pm \frac{\pi}{2} \) and one easily sees that \( f(\varphi) > 0 \) for the other values of \( \varphi \). Supposing now that (5.8) were valid for some \( U_1 > U_0 \) would lead to a contradiction since \( f(\beta) > 0 \) in that case. \( \square \)

The proposition shows that we may use the "entropy" inequality (5.3) for some fixed \( \varphi \) in any region of the flow.
where all shocks satisfy $\beta \in (\varphi - \frac{\pi}{2}, \varphi + \frac{\pi}{2})$. A restriction of possible values of $\beta$, if $\varphi$ is restricted to a certain range of angles and $q$ is bounded appropriately, is implied by Proposition 5.2. In certain relatively simple flows, such as flows through a laval nozzle or around an airfoil, the angles $\beta$ satisfy the restriction that they lie within a range of length $\pi$. In those cases one can work globally with a fixed $\varphi$.

A possible choice for $\varphi$ could be a typical local flow angle $\vartheta = \tan^{-1}\left(\frac{v}{u}\right)$. Since $\cos \vartheta = \frac{u}{q}$, $\sin \vartheta = \frac{v}{q}$ one could consider replacing (5.3) by

$$\frac{u}{q} \left[ (\rho_B u^2 + p)_x + (\rho_B uv)_y \right] + \frac{v}{q} \left[ (\rho_B uv)_x + (\rho_B v^2 + p)_y \right] \leq 0 \quad (5.9)$$

A variant of this was proposed by Morawetz [MOR]. The disadvantage of (5.9) is that it is not in divergence form and therefore an analogue of (5.4) is not defined.

It is interesting to note that the dependence of the momentum inequality on the flow direction corresponds to a property of the viscosity obtained in (3.29). The latter also depends on the direction of the flow.

Another entropy inequality was derived by Morice and Viviand [MV] and independently, using a different approach, by Osher, Hafez and Whitlow [OHW]. It uses polar coordinates $u = q \cos \theta$, $v = q \sin \theta$ and does not have a physical interpretation. We give it for the sake of completeness: it is

$$\left[ q(\theta \sin \theta + \cos \theta) - \left( \int_{\overline{q}}^{q} \frac{1}{\rho_B(s)} s \ ds \right) p_B(q) q \cos \theta \right]_x$$
\[ + \left( q(-\theta \cos \theta + \sin \theta) - \left( \int_{\tilde{q}}^{q} \frac{1}{\rho_B(s)s} \, ds \right) p_B(q) q \sin \theta \right)_{\bar{y} < 0} \]

where \( 0 < \bar{q} < \hat{q} \).

Another type of inequality that has proved useful, also for numerical purposes as in [BG], [BE], is obtained by choosing a simple convenient quantity and introducing a large constant to replace (5.3), dropping the requirement that a quantity be conserved. For example, one may require the divergence of the flow to be bounded in the sense of distributions, i.e. for some \( K > 0 \)

\[ u_x + v_y \leq K \quad (5.10) \]

should hold. This means that for any positive \( \varphi \in C^\infty_0(\Omega) \) we require

\[ -\int u \varphi_x + v \varphi_y \, dx dy \leq K \int \varphi \, dx dy \]

to hold. The constant \( K \) has to be chosen so large that (5.10) does not pose a restriction to the smooth parts of the flows to be considered. If we assume \( u_x \) and \( v_y \) to be measures, then (5.10) implies that the singular part of the measure \( u_x + v_y \) must be negative. This is nothing other than the inequality

\[ U_1 - U_0 \leq 0. \]

In a way, (5.10) is the simplest inequality one can pose.
However, it should be noted that if expansion wave fans are to be included in the solution, as in the Prandtl-Meyer flow, then (5.10) does not hold unless one replaces $K$ by $K/r$, where $r$ is the distance to the origin of the wave fan.

Bibliography


Figure 1

$q$ near $C_x$ ($\alpha \approx 0$)

Figure 2
Figure 3

Figure 4
Figure 5

Figure 6
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