ON THE GLOBAL REGULARITY PROBLEM
FOR 3-DIMENSIONAL NAVIER-STOKES EQUATIONS

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Abstract

For any 3-dimensional smooth bounded domain $\Omega \subset R^3$ and any viscosity $\nu > 0$, we describe an unbounded open set of initial data $u_0 \in D(A^{1/2})$ and forces $f \in L^\infty(0, \infty; D(A^0))$ generating globally regular solutions to 3-dimensional Navier-Stokes equations (here $A$ is the Stokes operator). In particular, it follows the existence of arbitrarily large initial data $u_0 \in H^1(\Omega)$ and forces in $L^\infty(0, \infty; L^2(\Omega))$ generating globally regular solutions to 3D NSE. This improves the classical small data regularity result known in the theory of 3D NSE for about 50 years.
The 3-dimensional Navier-Stokes equations (3D NSE)

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f,$$

$$\text{div } u = 0,$$

given on a smooth region $\Omega \subseteq \mathbb{R}^3$ are the basic equations of the Fluid Dynamics Theory (refs. 1, 2). They describe motion of a viscous incompressible fluid. The mathematical theory of these equations was begun about 60 years ago with the fundamental works of J. Leray (refs. 3, 4, 5) and later, E. Hopf [6]. However, one basic question about 3D NSE is not yet solved: it is still unknown whether all smooth initial data and forces generate globally regular solutions, i.e. solutions that remain smooth for all positive time. (Precise definitions are given below). Because of its importance, this global regularity problem has attracted considerable attention of many outstanding mathematicians; for a review of results, see, e.g., books 1, 2, 7, 8, 9 and papers [10]–[27]. Nevertheless, despite all efforts, the problem is still open and only partial results are known.

The difficulty of finding a solution which is generated by a smooth initial function and a smooth force, but is not globally regular, is addressed in the works of V. Scheffer [23] and L. Caffarelli, R. Kohn, and L. Nirenberg [10], who show that the Hausdorff measure of the set of possible time and time-space singularities is small. On the other hand, there are no theorems which guarantee that all smooth initial data and forces generate globally regular solutions. Most results dealing with globally regular solutions to 3D NSE are obtained under a priori assumptions on the flow, or the solution. For example, the classical Serrin result (see refs. 24, 25), which marks the beginning of the direction called partial regularity theorems, establishes global regularity assuming some partial regularity of the flow. Another classical result (ref. 7) establishes global regularity assuming the flow is symmetric. The result of ref. 28 implies the persistence of the global regularity property in a small neighborhood of the initial function which is assumed to generate globally regular flow $u(t)$ satisfying \( \int_0^\infty \| \nabla u(t) \|^4 \, dt < \infty \) under additional restriction that the force is either zero or, at least, belongs to $L^2(0, \infty; L^2(\Omega))$. 

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A noteworthy exception, which does not assume a priori hypotheses on the flow, is the classical small data regularity result stating that small initial data $u_0 \in V$ and small forces $f \in L^\infty(0, \infty; H)$ do generate globally regular solutions. (Here $H$ denotes the space of square-integrable divergence-free vector fields on $\Omega$ satisfying the corresponding boundary conditions and $V$ is the subspace consisting of vector fields having square-integrable spatial derivatives.) Also without a priori assumptions on the flow, G. Raugel and G. R. Sell (refs. 29, 30, 31, 32) established a large data regularity result for the case of a thin domain $\Omega = (0, l_1) \times (0, l_2) \times (0, \epsilon) \subset R^3$, where $\epsilon > 0$ is small. In particular, they show the existence of a large set of initial data $u_0 \in V$ and a large set of forces $f \in L^\infty(0, \infty; H)$ generating globally regular solutions. This set of data forms a large bounded open set in $V \times L^\infty(0, \infty; H)$. The radius of this set is proportional to $\epsilon^{-s}$, for some $s > 0$, see [32], which motivates the name large data regularity result.

In the present paper, we make no a priori hypotheses on the flow nor do we assume the thinness of the domain $\Omega \subset R^3$. We prove that there is an unbounded open set of initial data and forces in $V \times L^\infty(0, \infty; H)$ that generate globally regular solutions to 3D NSE.

We begin with the projection $L_r$ defined for any self-adjoint operator $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$ in a Hilbert space $X$:

$$L_r \overset{def}{=} \int_{\delta_r} dE_\lambda, \quad r > 0,$$  \hspace{1cm} (3)

where $\delta_r = [-r, r]$. The method is based on the following simple observation:

**Lemma.** For any self-adjoint operator $A$ and any $r > 0$, the following pair of inequalities holds:

$$\|w\|^2_\mathcal{X} \leq \frac{1}{r^2} \|Aw\|^2_\mathcal{X},$$  \hspace{1cm} (4)

for all $w \in D(A) \cap \mathcal{R}(I - L_r)$, and

$$\|Av\|^2_\mathcal{X} \leq r^2 \|v\|^2_\mathcal{X},$$  \hspace{1cm} (5)

for all $v \in \mathcal{R}(L_r)$, where $\|\cdot\|_\mathcal{X}$ denotes the norm in $X$, $D$ and $\mathcal{R}$ stand for the domain and range of an operator, respectively.
Note that in the case when $\mathcal{A}$ is the Laplacian operator $-\Delta$ on a bounded domain $\Omega$, the projection $L_r$ coincides with the projections $\mathcal{P}_r$, $G_{\leq r}$, and $P_{\leq r}$ used by J. Mallet-Paret and G. R. Sell [33], I. M. Glazman [34] (see also [35]), and B. Booß and K. P. Wojciechowski [36], respectively.

Recall (refs. 1, 2) that the Navier-Stokes equations with Dirichlet boundary conditions $u|_{\partial \Omega} = 0$ on a bounded smooth domain $\Omega \subset \mathbb{R}^3$ can be written in the form of the evolutionary equation in the space $H = Cl_{L^2(\Omega)}\{u \in C_0^\infty(\Omega) : \nabla \cdot u = 0\}$:

$$\frac{du}{dt} + \nu Au + B(u, u) = f, \quad u(0) = u_0, \quad (6)$$

where $A = -P\Delta$ is the Stokes operator, $P : L^2(\Omega) \to H$ is the orthogonal projection, and $B(u, v) = P(u \cdot \nabla)v$. Also recall the classical estimate for the trilinear term $b(u, v, w) := \langle B(u, v), w \rangle$:

$$|b(u, v, w)| \leq C_3 ||A^\frac{1}{2}u|| ||A^\frac{1}{2}v|| ||Av||^{\frac{1}{2}}||Aw||, \quad u \in V = D(A^\frac{1}{2}), \ v \in D(A), \ w \in H, \quad (8)$$

where $C_3 > 0$ is a constant independent of $u, v, w$. As in refs. 1, 2, by a globally regular solution of 3D NSE we understand a function $u = u(t)$ such that

$$u \in C([0, \infty); V) \cap L^\infty(0, \infty; V) \cap L^2_{loc}(0, \infty; D(A)),$$

$$u_t \in L^2_{loc}(0, \infty; H) \quad (9)$$

and $u$ satisfies (6) almost everywhere and (7). In the sequel we denote the norm in the space $H$ by $|| \cdot ||$ and the norm in $L^\infty(0, \infty; H)$ by $|| \cdot ||_\infty$. In particular, the norm $||u||_V$ in the space $V$ is equivalent to $||A^\frac{1}{2}u||$ and to $||u||_{H^1(\Omega)}$, the norm $||u||$ is equivalent to $||u||_{L^2(\Omega)}$.

Now we state our main theorem:

**Theorem A.** For any smooth bounded domain $\Omega \subset \mathbb{R}^3$ and any viscosity $\nu > 0$, there exist numbers $N_0 > 0$ and $N_1 > 0$ such that if $\alpha > 0$ and $\beta > 0$ satisfy the inequality $\alpha + \beta < N_0$ then there exists a number $r_0 = r_0(\nu, \Omega, \alpha, \beta) > 0$ such that the 3D NSE (6)-(7)
have a unique globally regular solution \( u = u(t) \) whenever \( u_0 \in V, f \in L^\infty(0, \infty; H) \) and \( r \geq r_0 \) satisfy

\[
|v_0|^2 < \alpha, \quad |L_r f|^2 < \beta, \tag{11}
\]

\[
|w_0|^2 < \ln r, \quad |(I - L_r)f|^2 < \ln r, \tag{12}
\]

where \( v_0 := L_r u_0, \ w_0 := (I - L_r)u_0 \). This solution satisfies the following global \( H^1 \)-norm estimates

\[
|v(t)|^2 \leq N_1(\alpha + \beta), \quad t \in [0, \infty), \tag{13}
\]

\[
|w(t)|^2 \leq \frac{3}{2} \ln r, \quad t \in [0, \infty), \tag{14}
\]

where \( v = L_r u, \ w = (I - L_r)u \). Moreover, there exists a time \( T_1 = T_1(r) = T_1(r; \alpha, \beta) \) independent of \( u_0 \) and \( f \) such that \( T_1 \to 0 \) as \( r \to \infty \) and

\[
|w(t)|^2 \leq 2D_2 r^{-2} \left[ \ln r + r(\alpha + \beta)^2 \right], \quad t \in [T_1, \infty),
\]

where \( D_2 \) is a constant depending only on \( \nu, \Omega \).

Taking the union over \( \{r \geq r_0\} \) of the sets (11)–(12), we obtain the next theorem:

**Theorem B.** For any 3-dimensional bounded smooth domain \( \Omega \subset \mathbb{R}^3 \) and any viscosity \( \nu > 0 \), there exists an unbounded open star-shaped set of initial data and forces in \( V \times L^\infty(0, \infty; H) \) generating globally regular solutions to Navier-Stokes equations (6)-(7). Moreover, there exists a number \( \kappa_1 > 0 \) such that, for any \( f \in L^\infty(0, \infty; H) \) with \( |f|_{L^\infty(0, \infty; H)} < \kappa_1 \), all \( u_0 \in V \) for which pairs \((u_0, f)\) belong to this set form an unbounded open star-shaped subset of \( V \). Similarly, there exists a number \( \kappa_2 > 0 \) such that, for any \( u_0 \in V \) with \( |u_0|_V < \kappa_2 \), all \( f \in L^\infty(0, \infty; H) \) for which pairs \((u_0, f)\) belong to this set form an unbounded open star-shaped subset of \( L^\infty(0, \infty; H) \).

Based on the Theorem A, it is possible to describe, under some conditions on the force \( f \), a local attractor in \( V \) with unbounded open basin of attraction of regular solutions of 3D NSE (6)-(7) (cf. with results of refs. [29]-[32] describing a local attractor in \( V \) with large bounded open basin of attraction of regular solutions in the case of a thin domain.
\[ \Omega = (0, l_1) \times (0, l_2) \times (0, \epsilon) \subset \mathbb{R}^3, \quad \epsilon > 0 \text{ small}. \] We will consider this issue in another publication.

**Proof of the Theorem A.** We set \( N_1 := \max\{(D_5 + \frac{\lambda_2}{\nu})^{\frac{3}{2}}, \frac{128}{\nu^2} + 1\} \) and \( N_0 := \min\{1, (13.5\nu^{-2}C_2^4N_1^3)^{-\frac{1}{2}}\} \), where \( D_5 = \max\{\frac{4}{\nu\lambda_1}, \frac{32}{\nu^2\lambda_1}, \frac{8}{\nu^3\lambda_1}\} \) and \( \lambda_1 > 0 \) is the first eigenvalue of the Stokes operator \( A \). Choose any \( r > \lambda_1, u_0 \in V \) and \( f \in L^\infty(0, \infty; H) \) satisfying (11), (12). According to the classical theory of 3D NSE (see, e. g., refs. 1, 2) there exists a time \( T_* = T_*(r), \quad 0 < T_* \leq \infty, \) such that (6)-(7) possess a unique solution \( u \) on \((0, T_*)\) and this solution satisfies

\[
\begin{align*}
    u & \in C([0, T_*); V) \cap L^\infty(0, T_*; V) \cap L^2_{\text{loc}}(0, T_*; D(A)) \\
    u_t & \in L^2_{\text{loc}}(0, T_*; H).
\end{align*}
\]

Since \( L_r \) is an orthogonal projection, we have

\[
\begin{align*}
    v & \in C([0, T_*); V) \cap L^\infty(0, T_*; V) \cap L^2_{\text{loc}}(0, T_*; D(A)) \\
    v_t & \in L^2_{\text{loc}}(0, T_*; H)
\end{align*}
\]

and

\[
\begin{align*}
    w & \in C([0, T_*); V) \cap L^\infty(0, T_*; V) \cap L^2_{\text{loc}}(0, T_*; D(A)) \\
    w_t & \in L^2_{\text{loc}}(0, T_*; H).
\end{align*}
\]

Let

\[
\begin{align*}
    R_1^2 & := \alpha + \beta, \\
    R_2^2 & := 2 \ln r.
\end{align*}
\]

Because of the continuity of \( ||v(t)||_V \) and \( ||w(t)||_V \) on \([0, T_*]\) and since \( ||v_0||_V^2 < N_1R_1^2 \) and \( ||w_0||_V^2 < R_2^2 \), there exists \( T^r = T(r), \quad 0 < T^r \leq T_* \), such that

\[
||v(t)||_V^2 < N_1R_1^2, \quad 0 \leq t < T^r,
\]

and

\[ 6 \]
\[ ||w(t)||_V^2 < R_2^2, \quad 0 \leq t < T' . \tag{18} \]

Without a loss of generality we assume that \([0, T']\) is the maximal time interval for which both (17) and (18) hold. Note that if \(T' < \infty\) then one has either

\[ ||v(T')||_V^2 = N_1 R_1^2 \tag{19} \]

or

\[ ||w(T')||_V^2 = R_2^2 . \tag{20} \]

We will next show that, for sufficiently large \(r\), neither (19) nor (20) hold, which then implies that \(T' = \infty\).

By applying the projection \(I - L_r\) to (6) and taking the scalar product of both sides of the obtained equation with \(A w\) we get the inequality for the \(w\)-term:

\[
\frac{1}{2} \frac{d}{dt} ||A^{\frac{1}{2}} w||^2 + \nu ||A w||^2 \leq |(f, A w)| + |b(w, w, A w)| + |b(v, w, A w)| + |b(v, v, A w)|, \tag{21}
\]

valid for all \(t \in [0, T']\). Using the Lemma and (8) we derive the following estimates:

\[ |b(w, w, A w)| \leq \frac{C_3}{r^\frac{1}{2}} ||A^{\frac{1}{2}} w|| ||A w||^2, \tag{22} \]

\[ |b(v, w, A w)| \leq \frac{C_3}{r^\frac{1}{2}} ||A^{\frac{1}{2}} v|| ||A w||^2, \tag{23} \]

\[ |b(v, v, A w)| \leq \frac{\nu}{4} ||A w||^2 + \frac{1}{\nu} C_3^2 r ||A^{\frac{1}{2}} v||^4. \tag{24} \]

Moreover, the Lemma and Sobolev embeddings yield the existence of a constant \(C > 0\) depending only on \(\nu\) and \(\Omega\) such that

\[ |b(w, v, A w)| \leq \frac{C}{r^\frac{1}{4}} ||A^{\frac{1}{2}} v|| ||A w||^2. \tag{25} \]

As a result of (15)-(18) and (21)-(25), we conclude that there exists \(r_1 = r_1(\nu, \Omega, \alpha, \beta) > \lambda_1\) independent of \(u_0, f, t\) such that for all \(r \geq r_1, \; t \in [0, T']\) the following holds:
\[
\frac{1}{2} \frac{d}{dt} ||A^{\frac{1}{2}} w||^2 + \frac{\nu}{32} ||Aw||^2 \leq \frac{1}{2\nu} ||(I - L_r) f||^2 \infty \\
+ D_1 r N_1^2 (\alpha + \beta)^2,
\]

where the constant \( D_1 = \frac{1}{\nu} C_3^2 \) depends only on \( \nu, \Omega \). From (26) and the Lemma we obtain, for all \( r \geq r_1 \) and \( t \in [0, T^r) \), that

\[
\frac{d}{dt} ||A^{\frac{1}{2}} w||^2 + \frac{\nu}{16} r^2 ||A^{\frac{1}{2}} w||^2 \leq \frac{1}{\nu} ||(I - L_r) f||^2 \infty \\
+ 2D_1 r N_1^2 (\alpha + \beta)^2,
\]

which implies by the Gronwall inequality and by (12) that

\[
||A^{\frac{1}{2}} w(t)||^2 \leq e^{-\frac{\nu}{2} r^2 t} \ln r \\
+ \frac{16}{\nu} \left( \frac{1}{r} \right)^2 \left[ \frac{1}{\nu} \ln r + 2D_1 r N_1^2 (\alpha + \beta)^2 \right]
\]

for all \( t \in [0, T^r) \), \( r > r_1(\nu, \Omega) \). This immediately yields that there exists \( r_2 = r_2(\nu, \Omega, \alpha, \beta) \geq r_1 \) such that, for all \( r \geq r_2 \), we have

\[
||A^{\frac{1}{2}} w(t)||^2 \leq \frac{3}{2} \ln r, \quad 0 \leq t < T^r.
\]

However, if \( T^r < \infty \) then (27) contradicts to (20). Thus, to show that \( T^r = \infty \) for all sufficiently large \( r \geq r_2 \) we need to find \( r_0 \geq r_2 \) such that, for all \( r \geq r_0 \), (19) can not be satisfied either. This is achieved by applying the projection \( L_r \) to (6) and estimating \( v(t) \) along the same lines as for \( w(t) \) above using the estimates

\[
|b(v, v, Av)| \leq \frac{\nu}{4} ||Av||^2 + \frac{27}{4\nu^3} C_3^4 N_1^3 (\alpha + \beta)^3,
\]

\[
|b(v, w, Av)| \leq \frac{\nu}{8} ||Av||^2 + \frac{2}{\nu} C_3^2 N_1 (\alpha + \beta) \frac{1}{r} ||Aw||^2,
\]

\[
|b(w, v, Av)| \leq \frac{\nu}{16} + \frac{432}{\nu^3} C_3^3 \ln(r) N_1 (\alpha + \beta) \frac{1}{r^2} ||Aw||^2,
\]

\[
|b(w, w, Aw)| \leq \frac{\nu}{32} C_3^2 \frac{3}{2} \ln(r) \frac{1}{r} ||Aw||^2,
\]

valid for all \( r \geq r_2 \), which are derived from the Lemma, (8), (27), (17), and (15). In particular, we obtain the inequality
\[
\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}}v\|^2 + \frac{\nu}{32} \|Av\|^2 \leq E_1 + E_2 \frac{\ln r}{r} \|Aw\|^2
\]

with some positive constants \(E_1, E_2\) depending only on \(\nu, \Omega, \alpha, \beta\). Using the Poincare and Gronwall inequalities and remembering (11), we obtain that

\[
\|A^{\frac{1}{2}}v(t)\|^2 \leq e^{-\frac{\nu t}{4K}} \alpha + E_1 t
\]

\[
+ E_2 \frac{\ln r}{r} \int_0^t \|Aw\|^2 \, ds
\]

(28)

By integrating (26), we obtain an estimate for \(\int_0^t \|Aw\|^2 \, ds\). This estimate and (28) imply that there exist \(T_2^r = T_2(r, \nu, \Omega, \alpha, \beta) > 0\) and \(r_3 = r_3(\nu, \Omega, \alpha, \beta) \geq r_2\) such that

\[
\|A^{\frac{1}{2}}v(t)\|^2 \leq \frac{3}{2} \alpha, \quad t \in [0, \min\{T_2^r, T^r\}].
\]

for all \(r \geq r_3\). Now we show that \(T_2^r < T^r\) for all \(r \geq r_3\). Indeed, otherwise we would get

\[
\|A^{\frac{1}{2}}v(T^r)\|^2 \leq \frac{3}{2} \alpha < N_1 R_1^2
\]

which together with (27) contradicts to (19)-(20). Continuing in this way to choose \(r\) large enough, we finally get \(r_0\) such that for all \(r \geq r_0\) we have

\[
\|A^{\frac{1}{2}}v(T^r)\|^2 < N_1 R_1^2
\]

which contradicts to (20) and, therefore, implies that \(T^r = \infty\). This finishes the proof.

\textbf{Remark.} Same results hold for periodic boundary conditions.

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