AN INVERSE PROBLEM FOR HARMONIC ACOUSTICS IN STRATIFIED OCEANS

By

Robert P. Gilbert

and

Yongzhi Xu

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Robert P. Gilbert $^1$ and Yongzhi Xu $^2$

Abstract. In this paper we study the problem of determining the refraction index $n(z)$, as well as the propagating sound field, produced by a line source in a stratified ocean.

Key words. Inverse problem, Helmholtz equation, eigenvalue problem, transmutation, nonlinear Volterra integral equation.

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1 Introduction

In this paper we study the problem of determining the refraction index $n(z)$, as well as the propagating sound field, produced by a line source in a stratified ocean. This problem is motivated from a current method of measuring the noise of ships or other vessels in the ocean. The usual arrangement for measuring vessels' noise involves using an array of hydrophones, strung either in a line along the bottom in shallow water, or vertically in deeper water, as illustrated in Figure 10.2 of [12]. A similar arrangement allows us to measure the sound pressure, $u$, which is produced by a known source, at the bottom of the ocean or at a proper distance from the source. We are motivated by this to consider the problem of determining the refraction index from known data at either the bottom or at a chosen distance. More precisely, we seek to determine $n^2(z) < 1$, where the sound pressure satisfies:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k^2 n^2(z)u = -\frac{\delta(r)}{2\pi r} f(z), \ 0 \leq z \leq h, \ 0 \leq r < \infty, \quad (1.1)$$

$$u(r, 0) = 0, \ 0 \leq r < \infty, \quad (1.2)$$

$$\frac{\partial u}{\partial z}(r, h) = 0, \ 0 \leq r < \infty, \quad (1.3)$$

$$\frac{\partial u_n}{\partial r} - ika_n u_n = o\left(\frac{1}{r^{1/2}}\right), \ as \ r \to \infty, \quad (1.4)$$

$$u(r, h) = g(r), \ 0 \leq r < \infty, \quad (1.5)$$

or

$$u(r_0, z) = l(z), \ 0 \leq z \leq h. \quad (1.6)$$

$^1$Department of Mathematical Sciences, University of Delaware, Newark, DE 19716

$^2$Institute for Mathematics and its Applications, University of Minnesota, 514 Vincent Hall, 206 Church Street SE, Minneapolis, MN 55455
Here $n^2(z) < 1$ is unknown, the $n^{th}$ normal mode $u_n$ is defined by

$$u_n(r) = \int_0^h u(r, z) \phi_n(z) dz,$$

(1.7)

where $\phi_n(z)$, $a_n$ are the $n^{th}$ normalized eigenfunction and eigenvalue for the eigenvalue problem

$$\phi_n''(z) + k^2(n^2(z) - a_n^2)\phi_n(z) = 0,$$

(1.8)

$$\phi_n'(h) = 0,$$

(1.9)

and

$$\phi_n(0) = 0.$$

(1.10)

The eigenvalue problem (1.8)-(1.10) is obtained from (1.1)-(1.3) by separation of variables (cf.[1]).

The functions $f(z)$, $g(r)$, $l(z)$ are given functions whose properties will be specified later. We call (1.1)-(1.5) the problem (B) and call (1.1)-(1.4), (1.6) the problem (V).

Similar to Rudell [10], we define the pair of functions $(n(z), u(r, z))$ as a solution of the problem (B) (or the problem (V)) provided that

1. $n(z) \in C[0, h]$, $u \in C^2((0, h) \times (0, \infty)) \cap C([0, h] \times [0, \infty))$;
2. $\lim_{z \to 0} u_z(r, z)$ and $\lim_{z \to h} u_z(r, z)$ exist;
3. (1.1)-(1.5) are satisfied, (or (1.1)-(1.4), (1.6) are satisfied).

In this paper we consider only the problem (B). In section 2, we obtain an a priori estimate to the transmutation kernel. In section 3, we reduce the inverse problem to an equivalent nonlinear Volterra integral equation. In section 4, we prove that there exists a unique solution to the Volterra equation, which implies problem (B) is uniquely solvable.

2 A priori estimate of the transmutation kernel

Let $\phi_n(z)$ be an eigenfunction of the problem (1.8)-(1.10), and let us represent $u(r, z)$ in the form

$$u(r, z) = \sum_{n=0}^{\infty} u_n(r) \phi_n(z),$$

where the $u_n(r)$ satisfy the radiating condition (1.4).

Following Gelfand and Levetan [3] and others (see, for example, [4],[10]), we represent the eigenfunction $\phi_n(x)$ in transmutation form as

$$\phi_n(x) = \cos \sqrt{\lambda_n(h - z)} + \int_1^z K(z, s) \cos \sqrt{\lambda_n(h - s)} ds,$$

(2.1)

where $K(z, s)$ is determined uniquely by the Goursat problem:

$$\frac{\partial^2 K}{\partial z^2} - \frac{\partial^2 K}{\partial s^2} + k^2[n^2(z) - 1]K = 0, \quad 0 \leq z \leq h,$$

(2.2)
\[
\frac{\partial K}{\partial z}(z, z) + k^2[n^2(z) - 1]K = 0, \quad 0 \leq z \leq h, \quad (2.3)
\]
\[
2\frac{\partial K}{\partial z}(z, h) = 0, \quad 0 \leq z \leq h, \quad K(h, h) = 0, \quad (2.4)
\]

and where \( n^2(z) \in C[0, h] \) such that \( n^2(z) < 1 \). If we extended \( K(z, s) \) as an even function about \( s = h \) into \( s \in [h, 2h] \), then (2.4) can be replaced by
\[
2\frac{\partial K}{\partial z}(z, 2h - z) + k^2[n^2(z) - 1]K = 0, \quad 0 \leq z \leq h. \quad (2.5)
\]

By introducing the new unknown
\[
M(\xi, \eta) = K(h - \xi - \eta, h - \xi + \eta),
\]
the above problem is equivalent to
\[
\frac{\partial^2 M}{\partial \xi \partial \eta} + k^2[n^2(h - \xi - \eta) - 1]M = 0, \quad 0 \leq \xi + \eta \leq h, \quad (2.6)
\]
\[
M(\xi, 0) = \frac{1}{2} k^2 \int_0^\xi [n^2(h - t) - 1]dt, \quad 0 \leq \xi \leq h, \quad (2.7)
\]
\[
M(0, \eta) = \frac{1}{2} k^2 \int_0^\eta [n^2(h - t) - 1]dt, \quad 0 \leq \eta \leq h, \quad (2.8)
\]
which is uniquely solvable.

We now give some a priori estimates for the transmutation kernel \( K(z, s) \).

**Lemma 2.1:** If \( n^2(z) \leq 1 \) and \( n(z) \in C[0, h] \), then
\[
M(\xi, \eta) \leq 0 \text{ for } 0 \leq \xi + \eta \leq h.
\]

**Proof:** Denote
\[
a(h - \xi - \eta) = k^2[1 - n^2(h - \xi - \eta)],
\]
then \( a(h - \xi - \eta) \geq 0 \) for \( 0 \leq \xi + \eta \leq h \). Hence, \( M(\xi, \eta) \) can be written as
\[
M(\xi, \eta) = -\frac{1}{2} \int_0^\xi a(h - t)dt - \frac{1}{2} \int_0^n a(h - t)dt + \int_0^\xi \int_0^\eta a(h - \xi - \eta)M(\xi, \eta)d\xi d\eta
\]
\[
= b(\xi, \eta) + \int_0^\eta \int_0^\xi a(h - \xi - \eta)M(\xi, \eta)d\xi d\eta, \quad (2.9)
\]
where
\[
b(\xi, \eta) = -\frac{1}{2} \int_0^\xi a(h - t)dt - \frac{1}{2} \int_0^n a(h - t)dt \leq 0.
\]
We look for a solution of (2.8) in the form of
\[
M(\xi, \eta) = M_0(\xi, \eta) + \sum_{n=1}^{\infty} [M_n(\xi, \eta) - M_{n-1}(\xi, \eta)], \quad (2.10)
\]
where
\[ M_0(\xi, \eta) = b(\xi, \eta), \]
\[ M_n(\xi, \eta) = b(\xi, \eta) + \int_0^\xi \int_0^\eta a(h - \xi - \eta)M_{n-1}(\xi, \eta)d\xi d\eta. \]
Moreover, we define
\[ \tilde{M}_n(\xi, \eta) := M_n(\xi, \eta) - M_{n-1}(\xi, \eta) \]
\[ = \int_0^\xi \int_0^\eta a(h - \xi - \eta)\tilde{M}_{n-1}(\xi, \eta)d\xi d\eta. \]
Since \( a \in C[0, h] \) and, hence is bounded, say, \( \max_{t \in [0, h]} |a(t)| = A \), we have
\[ |M_0(\xi, \eta)| \leq \frac{A}{2}(\xi + \eta), \]
and
\[ |\tilde{M}_n(\xi, \eta)| \leq A \int_0^\xi \int_0^\eta a(h - \xi - \eta)\tilde{M}_{n-1}(\xi, \eta)d\xi d\eta. \]
By induction it follows that
\[ |\tilde{M}_n(\xi, \eta)| \leq \frac{A(\xi + \eta)}{2} \frac{(A \xi \eta)^n}{(n + 1)!n!} \]
for \( 0 \leq \xi + \eta \leq h, \ n \geq 1. \)
This implies that the series (2.9) is uniformly convergent and defines a solution of the integral equation (2.8). But from \( M_0(\xi, \eta) = b(\xi, \eta) \leq 0 \) and \( a(h - \xi - \eta) \geq 0 \) it follows by induction that \( M_n(\xi, \eta) \leq 0 \) for \( n = 1, 2, \cdots \). Hence,
\[ M(\xi, \eta) = \lim_{n \to \infty} M_n(\xi, \eta) \leq 0. \]
We use the following notation for the sup norm
\[ ||u||_h = \sup_{0 \leq z \leq h} |u(z)|. \]
Let \( K(z, s; a_1(z)), \ K(z, s; a_2(z)) \) be the solutions of (2.2), (2.3) and (2.4) with undetermined \( a(z) = k^2[1 - n^2(z)] \) and boundary data determined by \( a_1(z) \) and \( a_2(z) \) respectively, and let
\[ \Gamma(z) := \sup_{z \leq \xi \leq 2h - z} |K(z, s, a(z))|. \]
It may be seen from (2.9), \( K(z, s, n(z)) \) satisfies
\[ K(z, s; a(z)) = b(h - (z + s)/2, (s - z)/2) + \int_{D(z, s)} a(\tau)K(\tau, t, a(\tau))d\tau dt, \]
(2.11)
where \( D(z, s) \) is the parallelogram bounded by the characteristic lines of slope \( \pm 1 \) through \((h, h)\) and \((z, s)\) respectively. Since \( D(z, s) \subseteq \{(\tau, t) : \tau \leq t \leq 2h - \tau, z \leq \tau \leq h\} \), it follows from (2.11) that for \( z \in [0, h] \),

\[
\Gamma(z) \leq 2h\|a\|_h + 2h\|a\|_h \int_z^h \Gamma(\tau) d\tau.
\]  

(2.12)

Gronwall’s inequality now implies for \( 0 \leq z \leq h \), that

\[
\Gamma(z) \leq 2h\|a\|_h e^{2zh\|a\|_h}.
\]  

(2.13)

**Lemma 2.2**: Let \( a_1(z), a_2(z) \in C[0, h] \) and \( a_1(z) \leq C_1, a_2(z) \leq C_1 \) for some fixed constant \( C_1 \). If \( K(z, s; a_1(z)), K(z, s; a_2(z)) \) are the solutions of (2.2), (2.3) and (2.4) with undetermined coefficient \( a(z) \) and boundary conditions determined by \( a_1(z) \) and \( a_2(z) \) respectively, then for a constant \( C = C(h, C_1) \),

\[
\sup_{z \leq s \leq 2h - z} |K(z, s; a_1(z)) - K(z, s; a_2(z))| \leq C \sup_{z \leq \tau \leq h} |a_1(\tau) - a_2(\tau)|.
\]

**Proof**: From (2.11) we obtain

\[
K(z, s; a_1(z)) - K(z, s; a_2(z))
\]

\[
= -\frac{1}{2} \int_0^{h-(z+s)/2} [a_1(h - t) - a_2(h - t)] dt - \frac{1}{2} \int_0^{(s-z)/2} [a_1(h - t) - a_2(h - t)] dt
\]

\[
+ \int_{D(z, s)} [a_1(\tau)K(\tau, t; a_1(\tau)) - a_2(\tau)K(\tau, t; a_2(\tau))] d\tau dt
\]

\[
= -\frac{1}{2} \int_0^{h-(z+s)/2} [a_1(h - t) - a_2(h - t)] dt - \frac{1}{2} \int_0^{(s-z)/2} [a_1(h - t) - a_2(h - t)] dt
\]

\[
+ \int_{D(z, s)} [a_1(\tau) - a_2(\tau)] K(\tau, t; a_1(\tau)) d\tau dt
\]

\[
+ \int_{D(z, s)} a_2(\tau) [K(\tau, t; a_1(\tau)) - K(\tau, t; a_2(\tau))] d\tau dt.
\]

It follows

\[
|K(z, s; a_1(z)) - K(z, s; a_2(z))|
\]

\[
\leq h\|a_1 - a_2\|_h + \int_{D(z, s)} \|a_1 - a_2\|_h |K(\tau, t; a_1(\tau))| d\tau dt
\]

\[
+ \int_{D(z, s)} \|a_2\|_h |K(\tau, t; a_1(\tau)) - K(\tau, t; a_2(\tau))| d\tau dt.
\]

Setting

\[
\Lambda(\tau) := \sup_{\tau \leq \tau \leq 2h - \tau} |K(\tau, t, a_1(\tau)) - K(\tau, t, a_2(\tau))|
\]

we have

\[
|K(z, s; a_1(z)) - K(z, s; a_2(z))|
\]
\[ \leq (h + \int_{D(z,s)} |K(\tau, t; a_1(\tau))|d\tau dt)\|a_1 - a_2\|_h + 2C_1h \int_{z}^{h} \Lambda(\tau)d\tau. \]

Using (2.13), we can conclude that for \(0 \leq z \leq h\),

\[ \Lambda(z) \leq C_2 h \|a_1 - a_2\|_h + C_3 \int_{z}^{h} \Lambda(\tau)d\tau, \tag{2.14} \]

where

\[ C_2 = 1 + C_1 h^2 e^{2C_1 h^2}, \]

and \(C_3 = 2C_1 h\). Using Gronwall's inequality we have

\[ \Lambda(z) \leq C_2 h \|a_1 - a_2\|_h e^{C_3 z}, \]

and by setting \(C = C_2 he^{C_3 h}\), we have proved the lemma.

### 3 A nonlinear Volterra integral equation deduced from the inverse problem

In this section we reduce the inverse problem to a nonlinear Volterra integral equation. We will assume that the given functions \(f(z)\), \(g(r)\) satisfy the following conditions.

(C1) The function \(f(z) \in C^2[0, h]\) is positive for all \(z \in [0, h]\) and satisfies \(f(0) = f'(h) = 0\). Also \(f'(0) \neq 0\), \(f(h) \neq 0\) and \(f''(z)/f(z)\) is bounded for \(z \in [0, h]\).

(C2) The function \(g(r) \in C^1[0, \infty)\) and is analytic in the half plane \(Re r > 0\), having the series expansion \(g(r) = \sum_{n=0}^{\infty} \alpha_n H_0^{(1)}(ka_nr)\) where \(\{a_n\}_{n=0}^{\infty}\) is a negative decreasing sequence of real numbers satisfying the asymptotic formula

\[ \sqrt{\lambda_n} h = kh \sqrt{1 - a^2} = (n + \frac{1}{2})\pi + O(\frac{1}{n}). \tag{3.1} \]

Let \(\{a_n\}_{n=0}^{\infty}\) be the sequence generated by \(g(r)\), and

\[ \mathcal{N} = \{n(z) \in C[0, 1] | \phi_n''(z) + k^2(n^2(z) - a_n^2)\phi_n(z) = 0, \phi_n(0) = 0, \]

\[ \phi_n'(h) = 0 \text{ for some set of eigenfunctions } \{\phi_n(z)\}_{n=1}^{\infty}, \]

then \(\mathcal{N}\) consists of those continuous potentials appearing in the Sturm-Liouville problem (1.8) with boundary conditions (1.9) and (1.10) that have the set \(\{\lambda_n\}_{n=0}^{\infty}\) as spectrum.

For any \(n(z) \in \mathcal{N}\), if the corresponding eigenfunction basis is \(\{\phi_n(z)\}_{n=0}^{\infty}\) given by

\[ \phi_n(z) = \cos\sqrt{\lambda_n} (h - z) + \int_{h}^{z} K(z, s)\cos\sqrt{\lambda_n} (h - s)ds \tag{3.2} \]

where \(K(z, s)\) is determined uniquely by the Goursat problem (2.2),(2.3) and (2.4), then the function

\[ u(r, z) = \sum_{n=0}^{\infty} \alpha_n \phi_n(z) H_0^{(1)}(ka_nr) \tag{3.3} \]
satisfies (1.1)-(1.4). Moreover,

\[ u_{rr} + \frac{1}{r} u_r + u_{zz} + k^2 n^2(z) u \]

\[ = \sum_{n=0}^{\infty} \alpha_n \left[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 a_n^2 \right) H_0^{(1)}(ka_n r) \phi_n(z) \right. \]

\[ + \left( \frac{\partial^2}{\partial z^2} + k^2 (n^2(z) - a_n^2) \right) \phi_n(z) H_0^{(1)}(ka_n r) \right] \]

\[ = \sum_{n=0}^{\infty} \alpha_n \phi_n(z) \delta(r), \]

so

\[ f(z) = \sum_{n=0}^{\infty} \alpha_n \phi_n(z). \]

Now we consider the representation (2.1). If we multiply both sides by \( \alpha_n \) and sum from \( n = 1 \) to infinity, then, in view of that \( \sum_{n=0}^{\infty} \lambda_n \alpha_n < \infty \) due to \( f''(h) \) existing, we can interchange the order of summation and integration, and obtain

\[ \sum_{n=0}^{\infty} \alpha_n \phi_n(z) = \sum_{n=0}^{\infty} \alpha_n \cos(\sqrt{\lambda_n}(h - z)) + \int_{h}^{z} K(z, s) \sum_{n=0}^{\infty} \alpha_n \cos(\sqrt{\lambda_n}(h - s)) ds. \]

If we define a function \( h(z) \) by

\[ h(z) = \sum_{n=0}^{\infty} \alpha_n \cos(\sqrt{\lambda_n}(h - z)), \quad (3.4) \]

then

\[ f(z) = h(z) + \int_{h}^{z} K(z, s, a(z)) h(s) ds. \quad (3.5) \]

Here in order to emphasize the dependence of \( K(z, s) \) on the \( a(z) \),

\[ a(z) := k^2[1 - n^2(z)], \]

we denote \( K(z, s) \) as \( K(z, s, a(z)) \).

We want to convert (3.5) to a Volterra equation of the second kind for an unknown \( a(z) \). Differentiating (3.5) twice, we have

\[ f''(z) = h''(z) + \frac{\partial}{\partial z} [K(z, z, a(z)) h(z)] + K_1(z, z, a(z)) h(z) \]

\[ + \int_{h}^{z} K_{11}(z, s, a(z)) h(s) ds. \]

But

\[ \int_{h}^{z} K_{11}(z, s, a(z)) h(s) ds \]
\[
= \int_h^z [K_{22}(z, s, a(z)) - k^2(n^2(z) - 1)K(z, s, a(z))] h(s) ds
\]
\[
= K_2(z, z, a(z))h(z) - K_2(z, h, a(z))h(h) - K(z, z, a(z))h'(z)
+ K(z, h, a(z))h'(h) + \int_h^z K(z, s, a(z))h''(s) ds
+ \int_h^z [k^2(n^2(z) - 1)]K(z, s, a(z)) h(s) ds.
\]

It follows then that
\[
f''(z) = h''(z) + 2 \left[ \frac{\partial}{\partial z} K(z, z, a(z)) \right] h(z) + k^2(n^2(z) - 1) \int_h^z K(z, s, a(z)) h(s) ds + \int_h^z K(z, s, a(z)) h''(s) ds
\]
\[
= h''(z) + k^2(n^2(z) - 1) h(z) + k^2(n^2(z) - 1) \int_h^z K(z, s, a(z)) h(s) ds
+ \int_h^z K(z, s, a(z)) h''(s) ds
\]
\[
= h''(z) + k^2(n^2(z) - 1) f(z) + \int_h^z K(z, s, a(z)) h''(s) ds.
\]

Hence, the inverse problem is equivalent to the following nonlinear Volterra equation
\[
f(z) a(z) + \int_h^z K(z, s, a(z)) h''(s) ds = - f''(z) - h''(z).
\] (3.6)

Now we need to show that \( h(z) \) can be determined by \( g(r) \) in problem (B). Let
\[
v(r, z) = \sum_{n=0}^{\infty} \alpha_n \cos[\sqrt{\lambda_n}(h - z)] H_{0}^{(1)}(ka_n r),
\] (3.7)
then \( v(r, z) \) satisfies
\[
v_{rr} + \frac{1}{r} v_r + v_{zz} + k^2 v = \frac{1}{2\pi r} \delta(r) h(z),
\] (3.8)
\[
v(r, h) = \sum_{n=0}^{\infty} \alpha_n H_{0}^{(1)}(ka_n r) = g(r),
\] (3.9)
\[
v_z(r, h) = 0,
\] (3.10)
and \( v \) satisfies an outgoing radiating condition as \( r \to \infty \) and \( v \) is bounded as \( z \to \infty \).

Notice that (3.9) and (3.10) give the Cauchy data for the homogeneous Helmholtz equation when \( r \neq 0 \), so \( g(r) = 0 \) follows \( v(r, z) = 0 \) for \( 0 < r < \infty \), \( z \in \mathbb{R}^1 \). It then forces \( h(z) = 0 \) for \( z \in \mathbb{R}^1 \), i.e. \( h(z) \) is determined uniquely by \( g(r) \). Furthermore, by separation of variables in (3.8) and using (3.10) and the radiating condition as \( r \to \infty \), we can express \( v(r, z) \) in the form
\[
v(r, z) = \int_{-\infty}^{1} C(t^2) \cos[k \sqrt{1 - t^2} (h - z)] H_{0}^{(1)}(ktr) dt^2,
\] (3.11)
where $C(t^2)$ is an arbitrary constant depending on $t^2$. From direct calculation it follows that

$$v_{rr} + \frac{1}{r}v_r + v_{zz} + k^2v$$

$$= \int_{-\infty}^{1} C(t^2) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) H_0^{(1)}(ktr) \cos[k\sqrt{1 - t^2}(h - z)]dt^2$$

$$+ \int_{-\infty}^{1} C(t^2)H_0^{(1)}(ktr) \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \cos[k\sqrt{1 - t^2}(h - z)]dt^2$$

$$= \int_{-\infty}^{1} C(t^2) \left( -k^2t^2 H_0^{(1)}(ktr) + \frac{\delta(r)}{2\pi r} \right) \cos[k\sqrt{1 - t^2}(h - z)]dt^2$$

$$+ \int_{-\infty}^{1} C(t^2)k^2t^2 H_0^{(1)}(ktr) \cos[k\sqrt{1 - t^2}(h - z)]dt^2$$

$$= \frac{\delta(r)}{2\pi r} \int_{-\infty}^{1} C(t^2) \cos[k\sqrt{1 - t^2}(h - z)]dt^2$$

$$= \frac{\delta(r)}{2\pi r} h(z).$$

Therefore,

$$\int_{-\infty}^{1} C(t^2) \cos[k\sqrt{1 - t^2}(h - z)]dt^2 = h(z), \quad (3.12)$$

and

$$\int_{-\infty}^{1} C(t^2)H_0^{(1)}(kar)dt^2 = g(r) \quad (3.13)$$

due to (3.9) and (3.11). But (3.13) means that $g(r)$ is some kind of $H$-transform of $C(t^2)$, hence, $C(t^2)$ can be represented as some $Y$-transform of $g(r)$, (ref. p.191 and p.221 of [9]), and $h(z)$ can be constructed.

4 Solution to the nonlinear Volterra integral equation

Since $f(z) > 0$ for $z \in (0, h]$ and $f'(0) \neq 0$, $z = 0$ is the simple zero point of $f(z)$. $\sum_{n=0}^{\infty} \alpha_n \lambda_n < \infty$ insures

$$h''(z) + \int_{h}^{z} K(z, s)h''(s)ds$$

$$= -\sum_{n=0}^{\infty} \alpha_n \lambda_n [\cos \sqrt{\lambda_n}(h - z) + \int_{h}^{z} K(z, s) \cos \sqrt{\lambda_n}(h - s)ds]$$

$$= -\sum_{n=0}^{\infty} \alpha_n \lambda_n \phi_n(z).$$

Hence,

$$h''(0) + \int_{h}^{0} K(0, s)h''(s)ds = 0.$$
Therefore,
\[
a(z) = \frac{f''(z)}{f(z)} - \frac{1}{f(z)} \left[ h''(z) + \int_{h}^{z} K(z, s) h''(s) ds \right]
\]
is bounded at \( z = 0 \), and the integral equation (3.6) is regular on \([0, h] \).

The following lemmas for the nonlinear Volterra equation can be proved in a standard way, we omit the detail here (cf. [8]).

Consider a Volterra integral equation of the form
\[
x(t) = f(t) + \int_{0}^{t} g(t, s, x(t)) ds, \quad (t \geq 0).
\]

Let \( T > 0 \) and suppose \( f \) and \( g \) in equation (4.1) satisfy the following hypotheses:

(H1) \( f \) is defined and continuous for all \( t \) in \([0, T] \),

(H2) \( g \) is measurable in \((t, s, x)\) for \( 0 \leq s \leq t < T \), \( x \in \mathbb{R} \), \( g(t, s, x) \) is continuous in \( x \) for each fixed pair \((t, s)\) and \( g(t, s, x) = 0 \) if \( s > t \).

(H3) For each real number \( T \geq K > 0 \) and each bounded subset \( B \) of \( \mathbb{R}^1 \) there exists a measurable function \( m \) such that
\[
|g(t, s, x)| \leq m(t, s), \quad (0 \leq s \leq t \leq K, \ x \in B)
\]

and
\[
\sup \left\{ \int_{0}^{t} m(t, s) ds : 0 \leq t \leq K \right\} < \infty.
\]

(H4) For each compact interval \( I \subset [0, T] \), each continuous function \( \phi : I \rightarrow \mathbb{R}^1 \) and each \( t_0 \) in \([0, T] \)
\[
\lim_{t \to t_0} \int_{I} \{ g(t, s, \phi(t)) - g(t_0, s, \phi(t_0)) \} ds = 0.
\]

(H5) For each constant \( T \geq K > 0 \) and each bounded \( B \subset \mathbb{R}^1 \) there exists a measurable function \( k(t, s) \) such that
\[
|g(t, s, x) - g(t, s, y)| \leq k(t, s)|x - y|
\]
whenever \( 0 \leq s \leq t \leq K \) and both \( x \) and \( y \) are in \( B \). For each \( t \in [0, K] \) the function \( k(t, s) \) is in \( L^1(0, t) \) as a function of \( s \) and
\[
\lim_{h \to 0} \int_{t}^{t+h} k(t + h, s) ds = 0.
\]

We have the following Lemmas:

**Lemma 4.1:** Let the functions \( f \) and \( g \) satisfy hypotheses (H1)-(H5). If \( \int_{0}^{t} m(t, s) ds \to 0 \) as \( t \to 0^+ \) for each \( m \) in (H3), then there exists a constant \( \beta > 0 \) such that equation (1.11) has a unique continuous solution \( x(t) \) on the interval \([0, \beta] \).

**Lemma 4.2:** Let \( f \) and \( g \) satisfy (H1)-(H3) and \( g \) satisfies (H6):
(H6) For each compact subinterval $J$ of $T$, each bounded set $B \in R$ and each $t_0$ in $T$

$$\sup \{ \int_J |g(t, s, \phi(t)) - g(t_0, s, \phi(t_0))| ds : \phi \in C(J : B) \}$$

tend to zero as $t \to t_0$.

If $x(t)$ is a bounded continuous solution of (1.11) on an interval $[0, \alpha)$, then $x(t)$ can be extended as a continuous solution of (1.11) to an interval $0 \leq t \leq \alpha_0$ where $\alpha_0 > \alpha$.

By virtue of the above lemmas, we can prove the following theorems:

**Theorem 4.1:** If assumptions (C1) and (C2) are satisfied, then there is a constant $\beta < h$ and the linear Volterra integral equation (3.6) has a unique solution $n(z) \in C[\beta, h]$.

**Proof:** Rewrite (3.6) as

$$a(z) + \int_h^z K(z, s, a(z)) \frac{h''(s)}{f(s)} ds = \frac{f''(z) - h''(z)}{f(z)}.$$  \hspace{1cm} (4.2)

If (C1) and (C2) hold, then the hypotheses (H1) is satisfied by the free term of (4.2). For any $n(z) \in C[0, h]$, the transmutation kernel $K(z, s)$, as a solution to the Goursat problem for hyperbolic equation, is in $C^1$. If $a_0(z), a_n(z) \in C[0, h]$ and $a_n(z) \to a_0(z)$ in the supremum norm, then $K(z, s; a_n(z)) \to K(z, s; a_0(z))$ in the supremum norm by Lemma 2.2. Hence, (H2) holds. Also (H3) holds from (2.14) for $m(z, s) = 2h\hat{C} e^{2\hat{C}z}$ where

$$\hat{C} = \max \{ ||n||_h : n \in B \},$$

and $B$ is a bounded subset of $R$. (H4) is implied by Lemma 2.2. Now let

$$k(z, s) = (1 + \hat{C} h^2 e^{2\hat{C}z}) e^{\hat{C}h},$$

then

$$\sup_{z \leq s \leq 2h - z} |K(z, s; a_1(z) - K(z, s; a_2(z))| \leq k(z, s)||a_1 - a_2||_h$$

due to Lemma 2.2. Moreover,

$$\lim_{\epsilon \to 0} \int_{z+\epsilon}^{z+\epsilon} k(z + \epsilon, s) ds = 0,$$

hence (H5) is satisfied.

Furthermore,

$$\lim_{z \to 0^+} \int_0^z m(z, s) ds = \lim_{z \to 0^+} \int_0^z 2h\hat{C} e^{2\hat{C}z} ds = 0,$$

for each $\hat{C}$. By Lemma 4.1, there exists a constant $\beta < h$ such that the equation (4.2) has a unique continuous solution $a(z)$ on the interval $[\beta, h]$.

**Theorem 4.2:** If assumptions (C1) and (C2) are satisfied, then Volterra integral equation (3.6) has a solution $a(z) \in C[0, h]$.
**Proof:** We first prove that if (C1) (C2) are satisfied, then $K(z, s, a(z))$ satisfies (H6). For our problem, it is sufficient to consider the compact set $J \subset [0, h]$ and for each $z_0 \in [0, h]$. If $B$ is a bounded set of $R$, and $a \in C(J, B)$, then

$$\sup\{|a(z)|; a \in C(J, B)\} \leq M \quad \text{for some constant } M.$$  \hspace{1cm} (4.3)

For any $\epsilon > 0$ and $z_0 + \epsilon \in [0, h]$, using (2.11) we obtain for any $a \in C(J, B)$

$$\left| K(z + \epsilon, s, a(z + \epsilon)) - K(z, s, a(z)) \right|$$

$$\leq \frac{1}{2} \left[ \int_{0}^{h-(z+\epsilon)/2} a(h - t)dt - \int_{0}^{h-(z+\epsilon)/2} a(h - t)dt \right]$$

$$+ \frac{1}{2} \left[ \int_{0}^{s-(z+\epsilon)/2} a(h - t)dt - \int_{0}^{s-(z+\epsilon)/2} a(h - t)dt \right]$$

$$\leq M \left( \frac{\epsilon}{4} + \frac{\epsilon}{4} + 2hC\epsilon \right) \to 0, \quad \text{as } \epsilon \to 0.$$

Here $C = 2hMe^{2Mh^2}$ is the bound of $|K(z, s, a(z))|$ due to (2.13) and (4.3). From Lemma 4.2, $a(z)$ can be extended as a continuous function of (4.2) to an interval $\beta_0 \leq z \leq h$ where $\beta_0 < \beta$. Repeating this argument a finite number of times we come to the conclusion that there exists a continuous solution $a(z)$ on $[0, h]$.

Now we show the uniqueness of the solution to equation (4.1).

**Theorem 4.3:** If assumptions (C1) and (C2) are satisfied, then the solution of the Volterra integral equation (3.6) in $[0, h]$ is unique.

**Proof:** For simplicity we make a change of coordinates: $z' = h - z$, $s' = h - s$ and rename these once more as $z$, $s$. Also we denote the corresponding function as $K(z, s, a(z))$ instead of $K(h - z, h - s, a(h - z))$. We first show that if $a_1, a_2 \in C([0, h])$, then for any positive parameters $\lambda$ and $\epsilon$ it holds that

$$|K(z, s, a_1(z)) - K(z, s, a_2(z))| \leq \frac{C}{\lambda} \|a_1 - a_2\|_{s, \epsilon} e^{\lambda z},$$

where

$$\|a_1 - a_2\|_{s, \epsilon} = \sup_{0 \leq z \leq h - \epsilon} \left\{|a_1(z) - a_2(z)|e^{-\lambda z} \right\}, \quad \epsilon > 0.$$
In the new coordinates, (2.11) follows

\[|K(z, s, a_1(z)) - K(z, s, a_2(z))|\]

\[\leq \frac{1}{2} \int_0^{(z-s)/2} |a_1(t) - a_2(t)| dt + \frac{1}{2} \int_0^{(z+s)/2} |a_1(t) - a_2(t)| dt\]

\[+ \int_0^z \int_{-t}^t |K(t, s, a_1(t)) - K(t, s, a_2(t))||a_1(t) - a_2(t)||dsdt\]

\[+ \int_0^z \int_{-t}^t |K(t, s, a_2(t))||a_1(t) - a_2(t)||dsdt\]

\[\leq \left( \frac{1}{2} + \frac{1}{2} + 2hC_1 \right) \int_0^z |a_1(t) - a_2(t)| dt\]

\[+ 2Mh \int_0^z \sup_{-t \leq s \leq t} |K(t, s, a_1(t)) - K(t, s, a_2(t))||dsdt.\]

Here we have noticed the fact that in the new coordinates \(z \geq s, \text{ and } z \geq -s\) and used \(C_1 = \sup_{0 \leq s \leq h-\epsilon} \{K(z, s, a_2(z))\}\), \(M = \sup_{0 \leq s \leq h-\epsilon} \{a_1(z)\}\). Using Gronwall's inequality, the above inequality implies

\[\sup_{-z \leq s \leq z} |K(z, s, a_1(z)) - K(z, s, a_2(z))|\]

\[\leq (1 + 2hC_1) \int_0^z |a_1(t) - a_2(t)|e^{-\lambda t} dt e^{2Mhz}\]

\[\leq C_2 ||a_1 - a_2||_{\lambda, z} e^{\lambda z},\]

where \(C_2 = (1 + 2hC_1)e^{2Mh^2}\).

If \(a_1(z)\) and \(a_2(z)\) are solutions to the Volterra integral equation (4.2), then, in the new coordinates,

\[a_1(z) - a_2(z) = \int_0^{h-z} [K(z, s, a_1(z)) - K(z, s, a_2(z))] \rho(s) ds\]

where \(\rho(z, s) = |h''(s)/f(z)| \leq C_\epsilon\) for some constant \(C_\epsilon\) in \([0, h - \epsilon]\). It follows that

\[|a_1(z) - a_2(z)| \leq \int_0^z |K(z, s, a_1(z)) - K(z, s, a_2(z))||\rho(s)| ds\]

\[\leq C \int_0^{h-\epsilon} |K(z, s, a_1(z)) - K(z, s, a_2(z))| ds\]

\[\leq C \int_0^{h-\epsilon} \frac{C_2}{\lambda} ||a_1 - a_2||_{\lambda, z} e^{\lambda z} ds, \text{ for } 0 \leq z \leq h - \epsilon.\]

Still using \(C\) to denote the constant \(CC_2h\), we obtain for any \(z \in [0, h - \epsilon]\)

\[|a_1(z) - a_2(z)|e^{-\lambda z} \leq \frac{C}{\lambda} ||a_1 - a_2||_{\lambda, \epsilon}.\]

Therefore, for any \(\lambda > 0\),

\[||a_1 - a_2||_{\lambda, \epsilon} \leq \frac{C}{\lambda} ||a_1 - a_2||_{\lambda, \epsilon}.\]
This implies that \( \|a_1 - a_2\|_{\lambda, \epsilon} = 0 \); hence, \( a_1(z) \equiv a_2(z) \) for \( z \in [0, h - \epsilon] \), and letting \( \epsilon \to 0 \), we have \( a_1(z) \equiv a_2(z) \) for \( z \in [0, h) \). Now from the continuity and boundness of \( a_1(z) - a_2(z) \) at \( z = h \), it follows that it is also true at \( z = h \).

From Theorem 4.2 and Theorem 4.3 we have immediately

**Corollary 4.1**: If assumptions (C1) and (C2) are satisfied, then the problem (B) has unique solution pair \((n(z), u(z))\) for \( z \in [0, h] \).

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