A FREE BOUNDARY PROBLEM ARISING FROM
SWELLING-CONTROLLED RELEASE PROCESSES

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Abstract. In this paper, we study a mathematical model describing the diffusion of a penetrant solvent in a polymer which arises from swelling-controlled release processes. The model is formulated as a one-dimensional free boundary problem. Global existence, uniqueness and the regularity of free boundaries are established, as well as asymptotic behavior.

1. Introduction. In this paper, we study a model arising from swelling–controlled release processes. A slab of polymer matrix (which contains drug) is contacted with a slab of penetrant solvent. The matrix starts to swell immediately at the penetrant surface. There are two moving fronts (free boundaries). One, $X = L(t)$, separates the pure solvent from the swollen polymer and is caused to move towards pure solvent region by volume expansion of the polymer due to gradual swelling; and the other, $X = R(T)$, separates the the solvent–free polymer from the swollen polymer and is driven towards matrix region by the excess $C(X, T) - C^*$ at a finite rate, where $C(X, T)$ is the concentration of the solvent and $C^*$ corresponds to its equilibrium or threshold value in the polymer. The release rate of drug is thus determined by the rate of diffusion of the solvent (according to Fick’s law) in the polymer as well as the rate of change of two moving fronts. In this paper we shall investigate the diffusion of the solvent and the free boundaries. We assume that initially the penetrant solvent occupies the region \( \{ X < 0 \} \) with the uniform concentration \( C_0 \) \( (C_0 > C^*) \) while the matrix occupies the region \( \{ X > 0 \} \). Introduce dimensionless variables $u = (C - C^*)/(C_0 - C^*)$ and $l, r, x, t$ as in [3], the problem can be formulated as follows [3]:

**Problem 1.** Find a triple \( \{ u(x, t), l(t), r(t) \} \) such that for any $0 < T < \infty$, $l(t), r(t) \in C^1[0, T], u(x, t) \in C^{2,1}(D_T) \cap C(D_T), D_T = \{(x, t) : l(t) < x < r(t), 0 < t < T\}, u_x$ is continuous up to $x = l(t)$ and $x = r(t)$, and such that

\[
\begin{align*}
(1.1) & \quad \varepsilon u_t = u_{xx} \text{ for } (x, t) \in D_T, \\
(1.2) & \quad u = 1 \text{ for } x = l(t), 0 < t < T, \\
(1.3) & \quad -u_x = (1 + \varepsilon u)\dot{r} \text{ for } x = r(t), 0 < t < T, \\
(1.4) & \quad \dot{r} = u^\mu \text{ for } x = r(t), 0 < t < T, \\
(1.5) & \quad -l(t) = \lambda \int_{l(t)}^{r(t)} (\mu + \varepsilon u)dx \text{ for } 0 < t < T,
\end{align*}
\]

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Here $\varepsilon = (C_0 - C^*)/(C^* + K)$, $K = k_2/k_1$, $k_1$ and $k_2$ are two positive proportionality constants such that $\dot{R} = k_1(C - C^*)^n$ and $-C_2 - C \dot{R} = k_2(C - C^*)^n$ on $X = R(T)$, respectively, while $\mu = C^*/(C^* + K)$, and $\lambda = \overline{v} \cdot (C^* + K)$, where $\overline{v}$ is the molar volume of the swelling agent (the solvent).

Equation (1.3) is a mass balance equation at $x = r(t)$, and equation (1.4) is $n$-th order law describing the swelling kinetics as an interface reaction. Typically, $n$ takes value varying from $10^{-2}$ to $10^2$. Equation (1.5) results from the volume balance. It says that, at time $t$, the total volume expanded due to swelling is equal to the volume occupied by penetrate in the swollen polymer region.

In a special case $\lambda = 0$, which corresponds to the controlled release without volume change, (1.1)–(1.6) with (1.5) replaced by $l(t) \equiv 0$ has been studied by several authors; see, [1] [2] [4] [6] and references therein.

The fact that in problem 1, the two free boundaries start at the same point $(0, 0)$ causes some difficulties. For this reason we shall first consider an auxiliary problem whereby instead of (1.6), we require that $l(0) < 0$ and $r(0) > 0$.

In section 2, we establish a local existence and uniqueness for the auxiliary problem; in section 3 we extend the solution to all times, and also prove the regularity of the solution $u$ and of the free boundaries. In section 4, we prove the existence and uniqueness for Problem 1. Finally, the asymptotic behavior for $t \to \infty$ and $\lambda \to 0$ is studied in section 5.

We assume throughout this paper that

$$1 - \lambda(\mu + \varepsilon) > 0.$$  

This is equivalent to

$$1 - C_0 \cdot \overline{v} > 0,$$  

a condition which is satisfied in most cases [8].

2. An Auxiliary Problem. Differentiating (1.5) in $t$ and using (1.1), (1.2), (1.3), we obtain

$$\dot{l}(t) = \beta [u_x(l(t), t) + (1 - \mu) \dot{r}(t)] \quad \text{for } t > 0,$$  

where $\beta = \lambda/[1 - \lambda(\mu + \varepsilon)]$; notice that $\beta > 0$ by the condition (1.7). Conversely, (1.5) follows from equations (1.1)-(1.4), (1.6) and (2.1). It will be more convenient to work with (2.1) rather than (1.5).

As mentioned in the introduction we shall first study an auxiliary problem where the polymer matrix is initially penetrated, that is, at time $t = 0$, there is a swollen polymer region $(-b, b)$ for some constant $b > 0$:

**Problem 2.** Find a triple $\{u(x, t), l(t), r(t)\}$ such that for any $0 < T < \infty, l(t), r(t) \in C^1[0, T], u(x, t) \in C^2, 1(D_T) \cap C(\bar{D}_T), D_T = \{(x, t) : l(t) < x < r(t), 0 < t < T\}$, $u_x$ is continuous up to $x = l(t)$ and $x = r(t)$, and such that (1.1) – (1.4)(2.1) hold and

\begin{align*}
(2.2) & \quad l(0) = -b, \quad r(0) = b, \\
(2.3) & \quad u(x, 0) = h(x) \quad \text{for} \quad x \in (-b, b).
\end{align*}

By a solution $\{u, l, r\}$ of Problem 2 we mean that the triple $\{u, l, r\}$ satisfies (1.1)–(1.4) (2.1)–(2.3) in the classical sense and has the following properties: $0 \leq u \leq 1$, $0 \leq -l, r \leq C$.

**Theorem 2.1.** Assume that $h(x) \in C^2[-b, b]$ with $h(-b) = 1, h'(-b) = -h_1 < 0$, $h(b) = c_0 > 0$, $h'(b) = -h_2 < 0$, and $h''(x) \leq 0$ for $-b \leq x \leq b$ such that $l_0 = \beta [-h_1 + (1 - \mu)c_0^\alpha] < 0$, and $h_2 = (1 + \varepsilon c_0)c_0^\alpha$. Then there exists a unique solution to Problem 2 for some small $T > 0$.

We shall need several lemmas.

Let $B$ be the Banach space $C[0, T] \times C^1[0, T]$ with the norm

$$
\|\langle l(t), r(t)\rangle\|_B = \|l(t)\|_{C[0, T]} + \|r(t)\|_{C^1[0, T]}.
$$

Set

\begin{align*}
\mathcal{L}(T) &= \left\{ l(t) \in C[0, T] : l(0) = -b, 2l_0 \leq \frac{l(t_1) - l(t_2)}{t_1 - t_2} \leq l_0/2, \quad \forall \quad t_1, t_2 \in [0, T] \right\}, \\
\mathcal{R}(T) &= \left\{ r(t) \in C^1[0, T] : r(0) = b, c_0^\alpha/2 \leq \frac{r(t_1) - r(t_2)}{t_1 - t_2} \leq 1, \quad \forall \quad t_1, t_2 \in [0, T] \right\}.
\end{align*}

It is clear that $\mathcal{L}(T) \times \mathcal{R}(T)$ is a closed convex subset of $B$.

For each pair $\langle l(t), r(t)\rangle \in \mathcal{L}(T) \times \mathcal{R}(T)$, consider the following problem:

\begin{align*}
(2.4) & \quad \varepsilon w_t = w_{xx}, \quad \text{for} \quad l(t) < x < r(t), \quad 0 < t < T, \\
(2.5) & \quad w = 1 \quad \text{for} \quad x = l(t), \quad 0 < t < T,
\end{align*}
(2.6) \[ -w_x = (1 + \varepsilon w)w^n \quad \text{for} \quad x = r(t), \quad 0 < t < T, \]
(2.7) \[ w(x, 0) = h(x) \quad \text{for} \quad -b < x < b. \]

**Lemma 2.2.** There exists a unique solution $w(x, t)$ to problem (2.4) – (2.7). Moreover, there are positive constants $\alpha, C$ and $T$ ($0 < \alpha < 1$), which depend on $h(x), c_0, l_0, b$, but not on $l(\cdot), r(\cdot)$, such that

(2.8) \[ \|w\|_{C^{\alpha, \alpha/2}(\overline{Q}_T)} \leq C, \]

and

(2.9) \[ \|w_x\|_{C^{\alpha, \alpha/2}(\overline{Q}_T \cap \{x \leq 0\})} \leq C, \]

where, $Q_T = \{(x, t) : l(t) < x < r(t), 0 \leq t < T\}$.

**Proof.** Introducing a new variable $y = (x - l(t))/(r(t) - l(t))$, and setting $v(y, t) = w(x, t) \equiv w(l(t) + y(r(t) - l(t)), t)$, we transform (2.4)–(2.7) into the following problem:

(2.10) \[ \varepsilon v_t = \frac{v_{yy}}{(r - l)^2} + \frac{\dot{i} + y(\dot{r} - \dot{i})}{r - l} v_y \quad \text{for} \quad 0 < y < 1, 0 < t < T, \]
(2.11) \[ v(0, t) = 1 \quad \text{for} \quad 0 < t < T, \]
(2.12) \[ -v_y(1, t) = (r(t) - l(t))(1 + \varepsilon v(1, t))v^n(1, t) \quad \text{for} \quad 0 < t < T, \]
(2.13) \[ v(y, 0) = h([2y - 1)b] \quad \text{for} \quad 0 < y < 1. \]

From a contraction fixed point argument, it follows that (2.10)–(2.13) admits a unique solution $v$; hence, there exists a unique solution $w$ to (2.4)–(2.7) (see Theorem 2.5 in [6] for details). By the maximum principle,

(2.14) \[ 0 \leq w(x, t) \leq 1, \quad \text{for} \quad (x, t) \in \overline{Q}_T, \]
(2.15) \[ -C \leq w_x(x, t) < 0, \quad \text{for} \quad (x, t) \in \overline{Q}_T; \]

here and in the sequel in this section $C$ will denote a constant depending on $h(\cdot), b, l_0, c_0$, but not on $l(\cdot)$ and $r(\cdot)$.

Applying Hölder estimates and $L^p$ estimates for parabolic PDEs to the Problem (2.10)–(2.13) [7], we have that

(2.16) \[ \|v\|_{C^{\alpha, \alpha/2}(0, 1] \times [0, T]} \leq C, \]
(2.17) \[ \|v\|_{W_2^{1,1}(0, 1/2] \times (0, T]} \leq C, \]

for some $0 < \alpha < 1$ and $p > 1$ independent of $l(\cdot)$ and $r(\cdot)$. Due to Sobolev’s embedding theorem, (2.17) implies that

\[ \|v_y\|_{C^{\alpha, \alpha/2}(0, 1/2]) \times [0, T]} \leq C. \]
This inequality and (2.16) yield the assertions (2.8) and (2.9). □

We now define a map $\mathcal{M}$ on $\mathcal{L}(T) \times \mathcal{R}(T)$ such that

$$
\mathcal{M}(l(t), r(t)) = (L(t), R(t)), \quad \text{for } (l(t), r(t)) \in \mathcal{L}(T) \times \mathcal{R}(T).
$$

where

$$
R(t) = b + \int_0^t w^n(r(\tau), \tau) d\tau,
$$

$$
L(t) = -b + \beta \int_0^t \left[ w_x(l(\tau), \tau) + (1 - \mu) \hat{R}(\tau) \right] d\tau.
$$

**Lemma 2.3.** If $T$ is small enough, then $\mathcal{M}$ maps $\mathcal{L}(T) \times \mathcal{R}(T)$ into itself and the image is precompact.

**Proof.** We first compute

$$
w(r(t), t) = \left[ w(r(0), 0) - w(r(0), 0) + w(r(t), t) \right]
\geq c_0 - C(|r(t) - b|^\alpha + t^{\alpha/2}) \quad \text{(by Lemma 2.2)}
\geq c_0/2^{1/n},
$$

for $0 \leq t \leq T$, provided $T$ is small.

Therefore, by (2.14)

$$
c_0^{\alpha/2} \leq w(r(t), t) \leq 1 \quad \text{for } t \in [0, T],
$$

and then by (2.18),

$$
c_0^{\alpha/2} \leq \hat{R}(t) \leq 1 \quad \text{for } t \in [0, T] \quad \text{and} \quad R(0) = b,
$$

which implies $R(t) \in \mathcal{R}(T)$. Moreover,

$$
|\hat{R}(t_1) - \hat{R}(t_2)| = \left| w^n(r(t_1), t_1) - w^n(r(t_2), t_2) \right|
\leq N \left| w(r(t_1), t_1) - w(r(t_2), t_2) \right|
\leq NC(|r(t_1) - r(t_2)|^{\alpha} + |t_1 - t_2|^{\alpha/2}) \quad \text{(by Lemma 2.2)}
\leq 2NC|t_1 - t_2|^{\alpha/2}
$$

for any $t_1, t_2 \in [0, T]$ and $N = n$ (if $n \geq 1$) or $N = n(c_0^{\alpha}/2)^{n-1}$ (if $0 < n < 1$).

On the other hand, by (2.19), $L(0) = -b$, \( \hat{L}(0) = l_0 \), and

$$
\hat{L}(t) = \beta \left[ w_x(l(t), t) + (1 - \mu) \hat{R}(t) \right]
= l_0 + \beta \left[ w_x(l(t), t) - w_x(l(0), 0) + (1 - \mu)(\hat{R}(t) - \hat{R}(0)) \right]
\leq l_0 + \beta \left[ C(|l(t) - l(0)|^{\alpha} + t^{\alpha/2}) + 2(1 - \mu)NCt^{\alpha/2} \right] \quad \text{(by Lemma 2.2)}
\leq l_0/2,
$$
provided \( T \) is small and \( t \in [0, T] \).

Similarly, \( \dot{L}(t) \geq 2l_0 \quad t \in [0, T] \) for small \( T \). Hence, \( L(t) \in L(T) \). Furthermore,

\[
|\dot{L}(t_1) - \dot{L}(t_2)| = \beta|w_x(l(t_1), t_1) - w_x(l(t_2), t_2) + (1 - \mu)(\dot{R}(t_1) - \dot{R}(t_2))|
\leq \beta \left[ C(|l(t_1) - l(t_2)|^\alpha + |t_1 - t_2|^{\alpha/2}) + 2(1 - \mu)NC|t_1 - t_2|^{\alpha/2} \right]
\leq \beta C|t_1 - t_2|^{\alpha/2},
\]

where we again used Lemma 2.2. Since the embedding \( C^{1+\alpha/2}[0, T] \) into \( C^1[0, T] \) is compact, the proof is complete. \( \square \)

The following lemma is useful in proving the continuity of the map \( M \).

**Lemma 2.4.** If \( w(x, t) \) is the solution to (2.4) – (2.7), then there exists a constant \( C \) independent of \( r(\cdot) \) and \( l(\cdot) \) such that

\[
(2.20) \quad |w_{xx}(x, t)| \leq C \quad \text{for} \quad (x, t) \in (Q_T \cap \{x \geq 0\}),
\]

where \( T \) is small enough so that \( w(x, t) \geq c_0/2^{1/n} \).

**Proof.** To estimate the second derivative of \( w_{xx} \) near \( x = r(t) \), we first assume that \( r(t) \in C^2 \). Then \( w_{xxx} \) is continuous up to the free boundary \( r(t) \). Applying the Schauder estimates [5] we get

\[
|w_{xx}(0, t)| \leq C, \quad \text{for} \quad 0 \leq t \leq T.
\]

Differentiating (2.6) along \( x = r(t) \) and setting \( \varphi = w_{xx} \), we deduce that

\[
(2.21) \quad \varphi_x + (\dot{r} + \varepsilon(n + 1)w^n + nw^{n-1})\varphi = ((n + 1)\varepsilon w^n + nw^{n-1})w_x \dot{r} \quad \text{for} \quad x = r(t).
\]

Set \( \xi(x, t) = \varphi(x, t) - C \), where

\[
C = \max \{ ||w_{xx}(0, \cdot)||_{L^\infty[0, T]}, ||h_{xx}||_{L^\infty[-b, b]}, \varepsilon(2\varepsilon + n)||w_x||_{L^\infty(Q_T)}/\delta \},
\]

and \( \delta \) is the positive lower bound on \( [\dot{r} + \varepsilon(n + 1)w^n + nw^{n-1}] \) for \( x = r(t), t \in [0, T] \). It is clear that \( \xi(x, t) \) satisfies:

\[
\begin{align*}
\varepsilon \xi_t & = \xi_{xx} \quad \text{for} \quad (x, t) \in (Q_T \cap \{x > 0\}), \\
\xi(0, t) & = w_{xx}(0, t) - C \leq 0 \quad \text{for} \quad t \in [0, T], \\
\xi(x, 0) & = h_{xx} - C \leq 0 \quad \text{for} \quad x \in [0, b], \\
\xi_x + (\dot{r} + \varepsilon(n + 1)w^n + nw^{n-1})\xi & = \\
\varepsilon(2\varepsilon w^n + nw^{n-1})w_x \dot{r} - C(\dot{r} + \varepsilon(n + 1)w^n + nw^{n-1}) & \leq 0 \quad \text{for} \quad x = r(t), t \in [0, T].
\end{align*}
\]
By maximum principle, \( \xi \leq 0 \) for \((x, t) \in (Q_T \cap \{x \geq 0\})\); that is,
\[
\varphi(x, t) = w_{xx}(x, t) \leq C \quad \text{for} \quad (x, t) \in (Q_T \cap \{x \geq 0\}).
\]
Similarly,
\[
-\varphi(x, t) = -w_{xx}(x, t) \leq C \quad \text{for} \quad (x, t) \in (Q_T \cap \{x \geq 0\}).
\]
Therefore, the conclusion (2.20) follows under the assumption that \( r(t) \in C^2 \). For general \( r(t) \) in \( C^1 \), we approximate it by functions \( r_k(t) \in C^2[0, T] \cap \mathcal{R}(T) \) in \( C^1 \) topology, and apply the previous method to the convergent sequence \( r_k's \).

We now investigate the continuity of \( M \) as a map from \( B \) into \( B \). Let \((l_i(t), r_i(t)) \in \mathcal{L}(T) \times \mathcal{R}(T)(i = 1, 2)\), and \( w_i(x, t)(i = 1, 2)\), be corresponding solutions to (2.4)–(2.7), and let \( M(l_i(t), r_i(t)) = (L_i(t), R_i(t))(i = 1, 2)\).

**Lemma 2.5.** If \( T \) is sufficiently small, then the map \( M \) is Lipschitz continuous on \( \mathcal{L}(T) \times \mathcal{R}(T) \) under the norm of \( B \).

**Proof.** Let
\[
p(t) = \max(l_1(t), l_2(t)),
\]
\[
q(t) = \min(r_1(t), r_2(t))
\]
and
\[
D^* \equiv \{(x, t) : p(t) < x < q(t), 0 < t < T\}.
\]
Set \( v(x, t) = w_1(x, t) - w_2(x, t) \). It is clear that
\[
v(p(t), t) = w_1(p(t), t) - w_2(p(t), t) \\
\leq C|l_1(t) - l_2(t)| \quad \text{(by (2.15))}.
\]
Without loss of generality, we may assume that \( q(t) = r_1(t) \). Let \( f(w) = (1 + \varepsilon w)w^n \).

Then
\[
v_x(q(t), t) = w_{1x}(q(t), t) - w_{2x}(q(t), t) \\
= w_{1x}(r_1(t), t) - w_{2x}(r_2(t), t) + w_{2x}(r_2(t), t) - w_{2x}(r_1(t), t) \\
= f(w_2(r_2(t), t)) - f(w_1(r_1(t), t)) + w_{2x}(r_2(t), t) - w_{2x}(r_1(t), t)(\text{by (2.6)}) \\
\leq \int_0^1 f'[w_1(t) + \theta(w_2(t) - w_1(t))]d\theta \cdot (w_2(r_2(t), t) - w_1(r_1(t), t)) \\
+ C|r_1(t) - r_2(t)| \quad \text{(by (2.20))} \\
\leq -\int_0^1 f'[w_1(t) + \theta(w_2(t) - w_1(t))]d\theta \cdot v(q(t), t) + C|r_1(t) - r_2(t)|,
\]
here we used (2.14) and (2.15) to get the last inequality. From the last inequality we obtain
\[ v_x(q(t), t) + \int_0^1 f'[w_1(t) + \theta(w_2(t) - w_1(t))] d\theta \cdot v(q(t), t) \leq C|R_1(t) - R_2(t)|. \]

It now follows that \( v(x, t) \) is dominated in \( D^* \) by the solution \( V(x, t) \) of
\[ \epsilon V_t = V_{xx} \quad \text{for} \quad (x, t) \in D^*, \]
\[ V(p(t), t) = C\|l_1 - l_2\|_{C[0, T)}, \quad \text{for} \quad 0 < t < T, \]
\[ V_x(q(t), t) + \int_0^1 f'[w_1(t) + \theta(w_2(t) - w_1(t))] d\theta \cdot V(q(t), t) \]
\[ = C\|r_1 - r_2\|_{C[0, T]}, \quad \text{for} \quad 0 < t < T, \]
\[ V(x, 0) = 0, \quad \text{for} \quad -b < x < b. \]

By maximum principle (the argument is similar to that used in Lemma 2.4),
\[ 0 \leq V(x, t) \leq C(\|l_1 - l_2\|_{C[0, T]} + \|r_1 - r_2\|_{C[0, T]}) \quad \text{for} \quad (x, t) \in D^*. \]

Since also \( w_1(x, t) - w_2(x, t) = v(x, t) \leq V(x, t) \) for \( (x, t) \in \overline{D}^* \),
\[ w_1(x, t) - w_2(x, t) \leq C(\|l_1 - l_2\|_{C[0, T]} + \|r_1 - r_2\|_{C[0, T]}) \quad \text{for} \quad (x, t) \in \overline{D}^*. \]

Switching \( w_1 \) and \( w_2 \), we actually have
\[ w_1(x, t) - w_2(x, t) \leq C(\|l_1 - l_2\|_{C[0, T]} + \|r_1 - r_2\|_{C[0, T]}) \quad \text{for} \quad (x, t) \in \overline{D}^*. \]

By (2.15), (2.18) and (2.22),
\[ |\dot{R}_1(t) - \dot{R}_2(t)| = |w_1(r_1(t), t) - w_2(r_2(t), t)| \]
\[ \leq |w_1(r_1(t), t) - w_1(q(t), t)| + |w_1(q(t), t) - w_2(q(t), t)| \]
\[ + |w_2(q(t), t) - w_2(r_2(t), t)| \]
\[ \leq C(\|l_1 - l_2\|_{C[0, T]} + \|r_1 - r_2\|_{C[0, T]}). \]

To derive an estimate on \( \|L_1 - L_2\|_{C[0, T]} \), we shall use a representation of \( L(t) \). Let \( w(x, t) \) be the solution of (2.4)–(2.6). Then
\[ 0 = \iiint_{Q_T} (w_{xx} - \epsilon w_t) dxdt \]
\[ = \int_{\partial Q_T} w_{xx} dt + \epsilon wdx \]
\[ = \int_b^b \epsilon hdx + \int_0^t [w_x + \epsilon w\tau]_{x=r(t)} d\tau - \int_0^t \epsilon wdx - \int_0^t [w_x + \epsilon w\tau]_{x=l(t)} d\tau \]
\[ = \int_b^b \epsilon hdx + \int_0^t [-\epsilon w(x) w^n + \epsilon w\tau]_{x=r(t)} d\tau \]
\[ - \int_0^t \epsilon wdx - \int_0^t w_x((l(t), \tau)d\tau - \epsilon(l(t) + b). \]
Therefore,

\begin{equation}
\int_0^t w_x(l(\tau), \tau) d\tau = \int_0^t [\epsilon w(1 + \epsilon w)w^n]_{x = r(\tau)} d\tau - \int_{l(t)}^{r(t)} \epsilon w dx + \int_b^b \epsilon h dx - \epsilon (l(t) + b).
\end{equation}

Combining (2.19) and (2.24), we deduce that

\begin{align*}
L(t) = b & + \beta (1 - \mu) (R(t) - b) \\
& + \beta \int_0^t [\epsilon w r(\tau) - (1 + \epsilon w)w^n]_{x = r(\tau)} d\tau \\
& - \beta \int_{l(t)}^{r(t)} \epsilon w dx + \beta \int_b^b \epsilon h dx - \beta \epsilon (l(t) + b).
\end{align*}

Hence, by (2.22), (2.23),

\begin{align*}
|L_1(t) - L_2(t)| &= \beta \left| \int_0^t [\epsilon w_1(r_1(\tau), \tau) r_1(\tau) - (1 + \epsilon w_1(r_1(\tau), \tau)) w_1^n(r_1(\tau), \tau)] d\tau \\
& - \int_0^t [\epsilon w_2(r_2(\tau), \tau) r_2(\tau) - (1 + \epsilon w_2(r_2(\tau), \tau)) w_2^n(r_2(\tau), \tau)] d\tau \\
& + \int_{l_1(t)}^{r_1(t)} \epsilon w_1(x, t) dx - \int_{l_2(t)}^{r_2(t)} \epsilon w_2(x, t) dx \\
& + \epsilon (l_1(t) - l_2(t)) + (1 - \mu) (R_1(t) - R_2(t)) \right| \\
& \leq C(||l_1 - l_2||_{C[0, T]} + ||r_1 - r_2||_{C^1[0, T]}).
\end{align*}

(2.25)

It now follows from (2.23) and (2.25) that

\begin{equation}
||L_1 - L_2||_{C[0, T]} + ||R_1 - R_2||_{C^1[0, T]} \leq C(||l_1 - l_2||_{C[0, T]} + ||r_1 - r_2||_{C^1[0, T]}),
\end{equation}

which proves Lemma 2.5. \qed

Proof of Theorem 2.2. In view of Lemma 2.3 and Lemma 2.5, we can apply the Schauder fixed point Theorem to deduce that the map $\mathcal{M}$ has a fixed point on $\mathcal{L}(T) \times \mathcal{R}(T)$. Clearly every fixed point $(l(t), r(t))$ of $\mathcal{M}$ determines the solution $(u, l, r)$ of the Problem 2, where $u$ is obtained by solving (2.4)-(2.7) with the $(l(t), r(t))$.

To prove uniqueness, suppose that \{u_i, l_i, r_i\} \((i = 1, 2)\), are two solutions to Problem 2. Using (1.5), we deduce that

\begin{align*}
l_1(t) - l_2(t) &= \lambda \left[ \int_{l_2(t)}^{r_2(t)} (\mu + \epsilon u_2) dx - \int_{l_1(t)}^{r_1(t)} (\mu + \epsilon u_1) dx \right] \\
& = \lambda \mu (l_1(t) - l_2(t)) + \lambda \mu (r_2(t) - r_1(t)) \\
& + \lambda \epsilon \left[ \int_{l_2(t)}^{r_2(t)} u_2 dx - \int_{l_1(t)}^{r_1(t)} u_1 dx \right],
\end{align*}
By (2.22),
\begin{align*}
|l_1(t) - l_2(t)| & \leq \lambda(\mu + \varepsilon)\|l_1 - l_2\|_{C[0,T]} \\
& + CT(\|l_1 - l_2\|_{C[0,T]} + \|r_1 - r_2\|_{C[0,T]}) \\
& + \lambda(\mu + \varepsilon)\|r_1 - r_2\|_{C[0,T]},
\end{align*}
(2.26)

Recalling (1.7) and (2.23), we find from (2.26) that for any \( T > 0 \)
\[ \|l_1 - l_2\|_{C[0,T]} + \|r_1 - r_2\|_{C[0,T]} \leq CT(\|l_1 - l_2\|_{C[0,T]} + \|r_1 - r_2\|_{C[0,T]}), \]
for some constant \( C \) independent of \( T \). This implies uniqueness for small \( T \). Uniqueness for general \( T \) follows by a standard step-by-step argument. \( \square \)

**Remark 2.1.** Let \( u(x,t), l(t), r(t) \) be the solution constructed in Theorem 2.1. Then we have \( l(t), r(t) \in C^{1+\alpha/2}[0,T] \). In addition, by parabolic Schauder estimates [5], we have that \( u(x,t) \in C^{2+\alpha,1+\alpha/2} \) near the conner of \((b,0)\), which implies \( r(t) \in C^{2+\alpha/2}[0,T] \).

3. Global existence for the auxiliary problem. Before we establish the global existence, we shall obtain an a priori estimate on the solutions of Problem 2.

**Theorem 3.1.** Assume that \( \{u, l, r\} \) solve the Problem 2 for a given \( T < \infty \). Then, there exist constants \( \bar{c}_0 = \bar{c}_0(T) \) and \( C \), both positive, such that
\begin{align*}
(3.1) \quad \bar{c}_0 < u(x,t) < 1, \quad -C < u_x(x,t) < 0 \quad \text{for } (x,t) \in Q_T.
\end{align*}

**Proof.** If \( u(x,t) \) attains the value 0 (necessarily at \( x = r(t) \)) for the first time at some point \((r(t_0), t_0)\), then we have \( u_x(r(t_0), t_0) = 0 \), contradicting the maximum principle. Hence, there exists a \( \bar{c}_0 \) such that \( u(x,t) \geq \bar{c}_0 \). By maximum principle again, it follows that \( u < 1 \) and \( -C < u_x < 0 \) for \( (x,t) \in Q_T \). \( \square \)

The next Theorem establishes the regularity of the solution.

**Theorem 3.2.** Let \( \{u, l, r\} \) be the solution of Problem 2 for \( 0 < t < T \leq \infty \). Then \( u, l, r \in C^\infty \) for \( 0 < t < T \).

**Proof.** Recalling Remark 2.1, we see that \( l, r \in C^{1+\alpha/2}[0,T] \). By the Schauder estimates [5], we immediately have
\[ u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{D^T_\delta}), \]
for any \( \delta > 0 \), where \( D^T_\delta = \{(x,t) : l(t) < x < r(t), \delta < t < T\} \). This further implies that
\begin{align*}
\dot{r}(t) &= u^n(r(t), t) \in C^{1+\alpha/2}[\delta, T], \\
\dot{l}(t) &= \beta[u_x(l(t), t) + (1 - \mu)\dot{r}(t)] \in C^{(1+\alpha)/2}[\delta, T],
\end{align*}

for any \( \delta > 0 \). The next Theorem shows the existence of solutions up to \( T = \infty \).
that is, \( r(t) \in C^{2+\alpha/2}([\delta, T]) \) and \( l(t) \in C^{(3+\alpha)/2}([\delta, T]) \).

By the Schauder estimates again, \( u(x, t) \in C^{3+\alpha, (3+\alpha)/2}(\overline{D}_T^\delta) \), and by a "bootstrap" technique, \( u, l, r \in C^\infty \) for \( t > 0 \). □

We now proceed to prove the strict monotonicity of the free boundaries.

**Theorem 3.3.** Assume the \( \{u, l, r\} \) is the solution of Problem 2 with \( h_{xx} \leq 0 \) and \( \dot{i} \leq 0 \). Then

\[
\dot{r}(t) < 0 \quad \text{for} \quad 0 \leq t \leq T,
\]

and

\[
\dot{i}(t) < 0 \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** Introduce a function \( v(x, t) = [\ln(1 + \varepsilon u)]_{xx} \), as in [4]. A direct calculation shows that

\[
\varepsilon v_t - v_{xx} - 2[\ln(1 + \varepsilon u)]_{xx}v_x - 2v^2 = 0 \quad \text{for} \quad (x, t) \in Q_T,
\]

\[
v(l(t), t) = \frac{\varepsilon^2}{1 + \varepsilon}[-u_x l - u_x^2/(1 + \varepsilon)] < 0 \quad \text{for} \quad 0 \leq t \leq T \quad (\dot{l} \leq 0),
\]

\[
v(r(t), t) = \frac{\varepsilon^2}{1 + \varepsilon} \cdot \frac{\ddot{r}}{nu^{n-1}} \quad \text{for} \quad 0 \leq t \leq T,
\]

\[
v(x, 0) = \frac{\varepsilon h_{xx}}{1 + \varepsilon h} - \frac{\varepsilon^2 h_x^2}{(1 + \varepsilon h)^2} < 0 \quad \text{for} \quad -b < x < b.
\]

On the other hand,

\[
v_x(r(t), t) = -\ddot{r} \left[ \frac{\varepsilon^2}{1 + \varepsilon} (1 + (1 + 1/n)\varepsilon u) - \frac{2\varepsilon^3 u}{n(1 + \varepsilon u)} \right].
\]

Differentiating (1.4) along \( x = r(t) \), and recalling Remark 2.1, we find that

\[
\ddot{r}(0) = nc_0^{n-1}[\varepsilon h_{xx}(b) - c_0^n h_2] < 0.
\]

If there exists a \( t_0 > 0 \) such that \( \ddot{r}(t) < 0 \), for \( 0 \leq t < t_0 \), but \( \ddot{r}(t_0) = 0 \), then \( v(x, t) \) assumes maximum value 0 at \( (r(t_0), t_0) \), and then by maximum principle, \( v_x(r(t_0), t_0) > 0 \). Since the last inequality contradicts (3.8), the assertion (3.2) follows.

To show (3.3), we consider the function

\[
w(x, t) = u_x(x, t) + (1 - \mu)\ddot{r}.
\]

It is clear that \( w(x, t) \) satisfies following equations:

\[
\varepsilon w_t - w_{xx} = \varepsilon(1 - \mu)\ddot{r} < 0 \quad \text{for} \quad (x, t) \in Q_T,
\]

\[
w(l(t), t) = \dot{l}(t)/\beta \quad \text{for} \quad t \in [0, T],
\]

\[
w(r(t), t) = -(\mu + \varepsilon u)\ddot{r} < 0 \quad \text{for} \quad t \in [0, T],
\]

\[
w(x, 0) = h_x + (1 - \mu)c_0^n < -h_1 + (1 - \mu)c_0^n < 0 \quad \text{for} \quad x \in [-b, b].
\]
Notice that \( \dot{l}(0) = l_0 < 0 \), if there exists a \( t_0 > 0 \) such that \( \dot{l}(t) < 0 \), for \( 0 \leq t < t_0 \) but \( \dot{l}(t_0) = 0 \), then \( w(x, t) \) assumes maximum value 0 at \( (l(t_0), t_0) \), hence, maximum principle implies \( w_x(l(t_0), t_0) < 0 \). But in fact, \( w_x(l(t_0), t_0) = u_{xx}(l(t_0), t_0) = \varepsilon u_t(l(t_0), t_0) = -\varepsilon u_x(l(t_0), t_0) \dot{l}(t_0) = 0 \), which is a contradiction. \( \square \)

**Theorem 3.4.** Problem 2 admits a unique solution for any \( T > 0 \).

**Proof.** Suppose that we can continue the local solution \( \{u, l, r\} \) up to \( T^* < \infty \). Since \( 0 < \dot{r}(t) < 1, \dot{r}(t) < 0 \) for \( 0 < t < T^* \), both \( \lim_{t \to T^*} r(t) \) and \( \lim_{t \to T^*} \dot{r}(t) \) exist. Recalling Lemma 2.2, we see that \( \|u_x\|_{C^{0,\alpha/2}(Q_T \cap \{x \geq 0\})} \leq C \) uniformly in \( T < T^* \). Therefore, \( \lim_{t \to T^*} u_x(l(t), t) \) exists, and then so does \( \lim_{t \to T^*} \dot{l}(t) \) by (2.1). It is clear that \( \lim_{t \to T^*} l(t) \) also exists by (3.3). At this point, we can extend the solution \( \{u, l, r\} \) up to \( t = T^* \). Furthermore, by (3.1) and Theorem 3.3, it is easy to see that \( \{u, l, r\} \) satisfy the following inequalities: \( \dot{l}(T^*) < 0 \) and \( \dot{r}(T^*) = w^n(r(T^*), T^*) > 0 \), which enable us to continue the solution \( \{u, l, r\} \) beyond \( t = T^* \) by Theorem 2.1. Finally, uniqueness follows from Theorem 2.1. \( \square \)

### 4. Global existence and uniqueness for Problem 1.

In this section, we prove our main result. By a solution \( \{u, l, r\} \) of Problem 1 we mean that the triple \( \{u, l, r\} \) satisfies (1.1)-(1.6) in the classical sense and has the following properties: \( 0 \leq u \leq 1 \), \( 0 \leq -\dot{l}, \dot{r} \leq C \).

**Theorem 4.1.** There exists one unique solution to Problem 1 for any \( T > 0 \).

We begin with an approximate problem of Problem 1. For each \( b > 0 \), consider the following system:

\[(4.1) \quad \varepsilon u_t = u_{xx} \quad \text{for } Q_T,\]
\[(4.2) \quad u = 1 \quad \text{for } x = l(t), 0 < t < T,\]
\[(4.3) \quad -u_x = (1 + g(b) + \varepsilon u) \dot{r} \quad \text{for } x = r(t), 0 < t < T,\]
\[(4.4) \quad \dot{r} = u^n \quad \text{for } x = r(t), 0 < t < T,\]
\[(4.5) \quad \dot{l} = \beta[u_x + (1 - \mu) \dot{r}] \quad \text{for } x = l(t), 0 < t < T,\]
\[(4.6) \quad u(x, 0) = h_b(x) \quad \text{for } -b < x < b, \quad l(0) = -b, \quad r(0) = b,\]

here \( h_b(x) = A + Bx + Cx^2 \) and

\[
A = 1 - (1 + \varepsilon)b - \frac{3\varepsilon(1 + \varepsilon)sb^2}{2(1 - 2\varepsilon sb)},
\]
\[
B = -(1 + \varepsilon) \left[ 1 + \frac{\varepsilon sb}{1 - 2\varepsilon sb} \right],
\]
\[ C = \frac{\varepsilon(1 + \varepsilon)s}{2(1 - 2\varepsilon sb)} < 0, \]

and \( s \) is a parameter in the interval \([-2\beta(\mu + \varepsilon), \beta(\mu + \varepsilon)/2]\), \( g(b) = (1 + \varepsilon)/h_b^n(b) - (1 + \varepsilon h_b(b)) \), \( Q_T = \{(x, t) : l(t) < x < r(t), t \geq 0\} \).

For \( b \) small enough, a direct calculation shows that

\[
\begin{align*}
    h_b(-b) &= 1, & h_{bx}(b) &= -(1 + \varepsilon), \\
    h_{bx}(-b) &= -(1 + \varepsilon) - 2\varepsilon(1 + \varepsilon) sb/(1 - 2\varepsilon sb), \\
    h_b(b) &= 1 - 2(1 + \varepsilon)b - 2\varepsilon(1 + \varepsilon) s b^2/(1 - 2\varepsilon sb), \\
    l_0 &= \beta[h_{bx}(-b) + (1 - \mu)h_b^n(b)] < 0, \\
    -h_{bx}(b) &= (1 + g(b) + \varepsilon h_b(b))h_b^n(b), \\
    0 < g(b) &\to 0, \quad \text{as} \quad b \to 0, \\
    h_{bx} &= 2C < 0 \quad \text{for} \quad -b \leq x \leq b.
\end{align*}
\]

Hence, by essentially following the proof of Theorem 3.4 we have:

**Lemma 4.1.** For any \( T > 0 \), there exists a unique solution \( \{u_b(x, t), l_b(t), r_b(t)\} \) to the system (4.1) – (4.6).

Next, we shall obtain uniform estimates on \( \{u_b(x, t), l_b(t), r_b(t)\} \) for all small positive \( b \). By Theorem 3.1,

\[ 0 < u_b(x, t) < 1 \quad \text{for} \quad (x, t) \in \overline{Q}_T, \tag{4.7} \]

and

\[ -C \leq u_{bx}(x, t) < 0 \quad \text{for} \quad (x, t) \in \overline{Q}_T, \tag{4.8} \]

where \( C \) is a positive constant independent of \( b \).

By (1.4) (4.7),

\[ 0 < \dot{r}_b(t) \leq 1 \quad \text{for} \quad 0 \leq t \leq T, \tag{4.9} \]

and by (2.1) (4.8),

\[ -C \leq \dot{\dot{r}}_b(t) < \beta(1 - \mu) \quad \text{for} \quad 0 \leq t \leq T. \tag{4.10} \]

Moreover, since \( \dot{\dot{r}}_b(0) = \beta[h_{bx}(-b) + (1 - \mu)h_b^n(b)] = l_0 \), which uniformly goes to \(-\beta(\mu + \varepsilon)\) as \( b \to 0 \) for all \( s \in [-2\beta(\mu + \varepsilon), -\beta(\mu + \varepsilon)/2] \), a simple calculation shows that, for each \( b \) \( (0 < b \leq b_0 \text{ provided } b_0 \text{ is small}) \), we can always take one \( s \) from the
interval \([-2\beta(\mu + \epsilon), -\beta(\mu + \epsilon)/2]\) such that \(s = \hat{l}_b(0)\). It is then easy to check that first order compatibility condition is satisfied at \((-b, 0)\) for the system (4.1)-(4.6), and this fact implies that \(u_b(x, t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)\) for some \(\alpha \in (0,1)\) by the standard theory of parabolic PDEs [5]. The regularity of \(u_b\) yields \(r_b \in C^2\), which in turn implies that \(u_{bxx}\) is continuous up to the boundary \(x = r_b(t)\). Now we set \(\varphi = u_{bxx}\). Since \(\varphi = -\epsilon u_{bxx}\hat{l}_b(t)\) on \(x = l_b(t)\) and

\[
\varphi_x + (\hat{r} + \epsilon(n + 1)w^n + n(1 + g(b))w^{n-1})\varphi
= ((n + 1)\epsilon w^n + n(1 + g(b))w^{n-1})w_x \hat{r}
\]
on \(x = r_b(t)\), it follows then by the maximum principle that \(|\varphi(x, t)| \leq C\) with \(C\) independent of \(b\) (cf. Lemma 2.4.), that is,

(4.11) \[|u_{bxx}(x, t)| \leq C \quad \text{for} \quad (x, t) \in \overline{Q}_T.\]

This yields, by (1.1),

(4.12) \[|u_{bt}(x, t)| \leq C \quad \text{for} \quad (x, t) \in \overline{Q}_T.\]

Proof of Theorem 4.1. By (4.7), (4.8), (4.9), (4.10), (4.11) and (4.12), we can pass to the limit as \(b \to 0\) (if necessary, take a subsequence from \(\{u_b, l_b, r_b\}\)) to get a solution \(\{u, l, r\}\) to Problem 1. Then global existence follows.

Uniqueness is proved by the argument used in the proof of Theorem 2.1. \(\square\)

**Theorem 4.3.** Let \(\{u(x, t), l(t), r(t)\}\) be the unique solution of Problem 1. Then \(u(x, t) \in C^{2,1}(\overline{D}_T) \cap C^\infty(D_T)\) and \(l(t) \in C[0, T] \cap C^\infty(0, T), r(t) \in C^2[0, T] \cap C^\infty(0, T).\) Moreover, \(\hat{l}(t) < 0\) and \(\hat{r}(t) < 0\) for \(t \in [0, T].\)

**Proof.** It follows from (4.11) and (4.12) that \(u(x, t) \in C^{2,1}(\overline{D}_T).\) Notice that

(4.13) \[\hat{r}(t) = nu^{n-1}(u_x(r(t), t) + u_t(r(t), t)).\]

Combining (4.8) and (4.12) yields the assertion \(\hat{r}(t) \in C^2[0, T].\) The remaining conclusions of the theorem can be derived in the same way as done for Theorem 3.2 and Theorem 3.3. \(\square\)

**Remark 4.2.** It is clear that our methods apply to the case \(\lambda = 0.\) Hence we have given a different proof of the global existence and uniqueness theorems for one free boundary problem considered in [4]; their methods seem not work for Problem 1.
5. Asymptotic behavior of the solution. Let \( \{u(x, t), l(t), r(t)\} \) be the solution to Problem 1 for any \( T > 0 \). Then by Green's identity [4],

\[
0 = \int_{D_1} \int_{D_t} x(u_{xx} - \varepsilon u_t)dx\,d\tau = \int_{\partial D_t} [(xu_x - \varepsilon u)\,d\tau + \varepsilon xudx] \quad \text{for} \ t > 0.
\]

By (1.2), (1.3) and (2.1), equation (5.1) yields

\[
(5.2) \quad \frac{1}{2} \left[ r^2(t) + (\varepsilon + \frac{1}{\beta}) l^2(t) \right] = \varepsilon t - \int_0^t \varepsilon u(r(\tau), r(\tau))d\tau \\
+ \int_0^t l(\tau)(1 - \mu) \dot{r}(\tau) - \varepsilon \int_{l(t)}^{r(t)} xudx, \quad \text{for} \ t > 0.
\]

**Theorem 5.1.** There hold:

\[
(5.3) \quad \lim_{t \to \infty} l(t) = -\infty \quad \text{and} \quad \lim_{t \to \infty} r(t) = +\infty.
\]

**Proof.** The existence of both limits is a consequence of monotonicity of \( l(t) \) and \( r(t) \). To find the limit values, we first recall (1.5) and the fact that \( 0 \leq u \leq 1 \) to get

\[
(5.4) \quad \frac{\lambda \mu}{1 - \lambda \mu} r(t) \leq -l(t) \leq \frac{\lambda(\mu + \varepsilon)}{1 - \lambda(\mu + \varepsilon)} r(t) \quad \text{for} \ t > 0,
\]

and then deduce from (5.2) and \( \dot{l}(t) < 0, u(x, t) \leq 1 \) that

\[
(5.5) \quad \frac{1}{2} \left[ r^2(t) + (\varepsilon + \frac{1}{\beta}) l^2(t) \right] \\
\geq \varepsilon t(1 - \frac{1}{t} \int_0^t u(r(\tau), r(\tau))d\tau) \\
+ l(t)(1 - \mu)r(t) - \frac{\varepsilon}{2}(r^2(t) - l^2(t)).
\]

Notice that \( \lim_{t \to \infty} \frac{1}{t} \int_0^t u(r(\tau), r(\tau))d\tau \) is zero, if the integral is bounded in \( t \); and, otherwise, is equal to \( \lim_{t \to \infty} u(r(t), t) = \lim_{t \to \infty} (\dot{r}(t))^{1/n} = c_1 \) such that \( 0 \leq c_1 < 1 \) due to \( \dot{r}(t) < 0 \). Now if \( \lim_{t \to \infty} r(t) \) is finite, then both \( r(t) \) and \( l(t) \) are bounded because of (5.4), and we get a contradiction by letting \( t \) go to infinity on both sides of (5.5). Therefore, \( \lim_{t \to \infty} r(t) = \infty \), and \( \lim_{t \to \infty} l(t) = -\infty. \)

**Theorem 5.2.** There hold:

\[
(5.6) \quad \lim_{t \to \infty} \dot{r}(t) = 0,
\]

and

\[
(5.7) \quad c_2(t - \sigma(t))^{1/2} \leq -l(t), r(t) \leq (2\varepsilon)^{1/2} t^{1/2} \quad \text{for} \ t > 0,
\]
for some positive constant $c_2$ and some function $\sigma(t)$ which goes to 0 as $t \to \infty$.

**Proof.** Since $\bar{r}(t) < 0$, the limit $\lim_{t \to \infty} \bar{r}(t)$ exists. Dividing (5.2) by $t^2$ both sides and using the fact that the right hand side of (5.2) is less than $\varepsilon t$, we obtain that

$$\lim_{t \to \infty} \frac{1}{t^2} \left[ \frac{r^2(t)}{t^2} + \left( \varepsilon + \frac{1}{\beta} \right) \frac{r^2(t)}{t^2} \right] \leq 0.$$

Hence, $\lim_{t \to \infty} r^2(t)/t^2 = 0$, $\lim_{t \to \infty} r^2(t)/t^2 = 0$. On the other hand, by (5.3),

$$\lim_{t \to \infty} \frac{r^2(t)}{t^2} = \lim_{t \to \infty} \frac{r(t)}{t} \bar{r}(t) = \left( \lim_{t \to \infty} \bar{r}(t) \right)^2.$$

The conclusion (5.6) then follows. In view of the fact that the right hand side of (5.2) is less than $\varepsilon t$ again, the upper bound of $-l(t), r(t)$ in (5.7) follows from (5.2) and (5.4). Let $\sigma(t) = \frac{1}{t} \int_0^t u(r(\tau), \tau) d\tau$. Then we can deduce the lower bound of $-l(t), r(t)$ in (5.7) by (5.4) (5.5). Moreover, (5.6) implies that $\lim_{t \to \infty} \sigma(t) = 0$. The proof is complete. $\blacksquare$

Finally, we investigate the asymptotic behavior of the solution $\{u(x, t), l(t), r(t)\}$ as $\lambda \to 0$. Let $\{U(x, t), R(t)\}$ be the unique solution of following free boundary problem for arbitrarily given $T < \infty$.

$$\begin{align*}
(5.8) & \quad \varepsilon u_t = u_{xx} \quad \text{for } 0 < x < r(t), \quad 0 < t < T, \\
(5.9) & \quad u = 1 \quad \text{for } 0 < t < T, \\
(5.10) & \quad -u_x = (1 + \varepsilon u)u^n \quad \text{for } x = r(t), \quad 0 < t < T, \\
(5.11) & \quad \bar{r} = u^n \quad \text{for } x = r(t), \quad 0 < t < T, \\
(5.12) & \quad r(0) = 0.
\end{align*}$$

Then we have

**Theorem 5.3.** Assume that $\{u_\lambda, l_\lambda, r_\lambda\}$ solves the Problem 1 for each $\lambda > 0$. Then $\{u_\lambda(x, t), l_\lambda(t), r_\lambda(t)\} \to \{U(x, t, 0, R(t)\}$ as $\lambda \to 0$ uniformly in $t \in [0, T]$.

**Proof.** Since by the maximum principle $0 < u_\lambda(x, t) \leq 1$ and $-(1 + \varepsilon) \leq u_{\lambda xx}(x, t) < 0$, we immediately have that $l_\lambda(t) \to 0$ as $\lambda \to 0$ (due to (2.1) and (1.4)). Moreover, it is easy to check that $|u_{\lambda x}|$ and $|u_{\lambda xx}|$ are uniformly bounded in $\lambda, l_\lambda(t) < x < r_\lambda(t), t \in [0, T]$ (cf. section 4), which further implies that the $\bar{r}_\lambda(t)$ are uniformly bounded by (4.13). Therefore, we can take a subsequence of $\{u_\lambda(x, t), r_\lambda(t)\}$, say $\{u_\lambda(x, t), r_\lambda(t)\}$, such that $\{u_\lambda(x, t), r_\lambda(t)\}$ converges to a solution of (5.8)-(5.12) as $\lambda' \to 0$. By the uniqueness, the entire family $\{u_\lambda(x, t), r_\lambda(t)\}$ is convergent to $\{U(x, t), R(t)\}$ as $\lambda \to 0$. $\blacksquare$

**Remark 5.1.** The method of this paper can be extended to the case of more
general driving force on the free boundary $x = r(t)$: $\dot{r}(t) = \varphi(u(r(t), t))$ with $\varphi \in C^1(0, 1], \varphi'(u) > 0$ for $u \in (0, 1]$ and $\varphi(0) = 0$.

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