ON DEGENERATE DIFFUSION WITH VERY STRONG ABSORPTION

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IMA Preprint Series # 835

June 1991

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Consider the Cauchy problem

$$u_t = (u^m)_{xx} - u^{-p} \cdot \chi_{\{u>0\}} \qquad \text{in } \mathbb{R}^2_+, \tag{1}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R}, \tag{2}$$

where $\mathbb{R}^2_+ = \{\ (x,t) \mid x \in \mathbb{R},\ t > 0\ \}$, where $m \geq 1$ and $p < m$ are positive real numbers, and where $0 \leq u_0(x) \leq M_0 < +\infty$ on $\mathbb{R}$. Moreover $\chi_{\{u>0\}}(x,t)$ is equal to one if $u(x,t) > 0$ and it vanishes elsewhere.

Definition. A function u is said to be a weak solution of (1) in $\mathbb{R}^2_+$ if it is defined, real, nonnegative bounded and continuous in all nonempty bounded rectangles of the form $P = [x_1, x_2] \times [t_1, t_2] \subset \mathbb{R}^2_+$ and if it satisfies the identity

$$\begin{aligned} I(u,\phi,P) := & \int_P [u^m \phi_{xx} - u^{-p}\chi_{\{u>0\}}\phi + u\chi_{\{u>0\}}\phi_t]\, dx\, dt \\ & - \int_{x_1}^{x_2} [u(x,t_2)\phi(x,t_2) - u(x,t_1)\phi(x,t_1)]dx \\ & - \int_{t_1}^{t_2} [u^m(x_2,t)\phi_x(x_2,) - u^m(x_1,t)\phi_x(x_1,t)]\, dt = 0 \end{aligned} \tag{3}$$

for all $\phi(x,t) \in C^{2,1}(P)$ s.t. $\phi(x_1,t) = \phi(x_2,t) = 0$ in $[t_1,t_2]$.

A function u is said to be a weak solution of (1) (2), if it is a weak solution of (1), if it is continuous as $t \to 0$ and if $u(x,0) = u_0(x)$.

Remark 1. This definition requires that the integral $\int\limits_P u^{-p}\phi\chi_{\{u>0\}}\, dxdt$ exists, because all the other terms in (3) are finite. We do not (and cannot) exclude the existence of some physically interesting solutions to (1) for which the said integral fails to exist.

Existence of solutions

Theorem 1. *Suppose that $0 < p < m$ and that $(u_0^{1/\beta})_x$ is uniformly bounded, where*

$$\beta := \begin{cases} 2/(m+p) & \text{if } m < 2+p, \\ 1/(m-1) & \text{if } m \geq 2+p. \end{cases}$$

Then there exists a weak solution u to problem (1) (2). Moreover, it is a classical solution in those points (x,t) in which it is positive.

Remark 2. The question of uniqueness appears to be open for $p > 0$. For $p \leq 0$ and for many results on the case $p \leq 0$ we refer to [CMM] and the references therein. The distinction between $m < 2+p$ and $m \geq 2+p$ plays also a major role in the related problem $u_t = (u^m)_{xx} + u^{-p} \cdot \chi_{\{u>0\}}$, see[PV].

To prove Theorem 1 we use some ideas from [A2] [BN] [P] [OKC]. Without loss of generality we may assume that $u_0(x)$ is sufficiently smooth. Otherwise we can use standard approximation arguments. Let

$$M_1 = \sup u_0(x) + 1 = M_0 + 1 \qquad \text{and } \varepsilon \in (0,1). \tag{4}$$

Our aim is to regularize the singularity. Therefore we set $v_0(x,\varepsilon) = u_0(x) + \varepsilon$ (note that $\varepsilon \leq v_0 \leq M_1$) and $Q_\varepsilon := (-1/\varepsilon, 1/\varepsilon) \times (0, 1/\varepsilon)$. Since the solution v of (5) (6) (7) below depends on ε, we shall denote it by $v(x,t,\varepsilon)$.

Lemma 1. *The first boundary value problem*

$$\mathcal{L}(v,\varepsilon) := -v_t + (v^m)_{xx} - \frac{v}{\varepsilon + v^{1+p}} = 0 \qquad \text{in } Q_\varepsilon, \tag{5}$$

$$v(x,0,\varepsilon) = v_0(x) \qquad \text{for } x \in (-1/\varepsilon, 1/\varepsilon), \tag{6}$$

$$v(\pm 1/\varepsilon, t, \varepsilon) = v_0(\pm 1/\varepsilon, \varepsilon) \qquad \text{for } t \in (0, 1/\varepsilon) \tag{7}$$

has a unique classical solution in Q_ε. *Moreover, for every fixed positive* ε

$$0 < \varepsilon \le v(x,t,\varepsilon) \le M_1 \qquad \text{in } Q_\varepsilon. \tag{8}$$

Proof of Lemma 1: Define $\phi(r) \in C^\infty(\mathbb{R})$ by

$$\phi(r) = \begin{cases} r & \text{for } r \ge \varepsilon, \\ \frac{\varepsilon}{2} & \text{for } r \le 0, \\ \text{increasing on } [0,\varepsilon]. \end{cases}$$

Instead of (5) consider the first boundary value problem for

$$v_t = m[\phi(v)]^{m-1} v_{xx} + m(m-1)[\phi(v)]^{m-2} v_x^2 - \frac{\phi(v)}{\varepsilon + [\phi(v)]^{1+p}} \tag{9}$$

with conditions (6) and (7). Due to known results (see e.g. [LSU]) this problem has a unique classical solution v with $|v| \le M_1$. To see that $v \ge \varepsilon$ in Q_ε set

$$z = z(t,x,\varepsilon) = e^{-at}(v - \varepsilon)$$

with $a > 0$ to be chosen later. One has

$$z(x,0,\varepsilon) = v_0(x,\varepsilon) - \varepsilon \ge 0$$
$$z(\pm 1/\varepsilon, t, \varepsilon) = e^{-at}[u_0(\pm 1/\varepsilon) + \varepsilon - \varepsilon] \ge 0.$$

Thus $Q_\varepsilon \ge 0$ on the parabolic boundary of Q_ε. Suppose that $z(x,t,\varepsilon)$ attains negative values in Q_ε. Then it has to have a negative minimum at some (x_0, t_0) and there $z_t \le 0$ and $z_{xx} \ge 0$, which implies $z_t - m[\phi(v)]^{m-1} z_{xx} \le 0$.

But from (9) we have at this point, using that $z_x(x_0, t_0) = 0$:

$$z_t - m[\phi(v)]^{m-1} z_{xx} = -az(x_0,t_0) - e^{-at} \frac{\phi(e^{at}z + \varepsilon)}{\varepsilon + [\phi(e^{at}z + \varepsilon)]^{p+1}}$$

By definition of ϕ and z we obtain

$$e^{-at} \frac{\phi(e^{at}z + \varepsilon)}{\varepsilon + [\phi(e^{at}z+\varepsilon)]^{p+1}} \le \frac{e^{-at} M_1}{\varepsilon + (\varepsilon/2)^{p+1}} < 1$$

and $z_t - m[\phi(v)]^{m-1} z_{xx} > 0$ for $a > 0$ large enough, a contradiction. Thus v has to be greater or equal to ε in Q_ε, and since (9) and (5) coincide for functions $v \geq \varepsilon$, the proof of Lemma 1 is complete.

Lemma 2. *(Monotonicity lemma)*

If $\varepsilon_1 > \varepsilon_2$ *then* $v(x,t,\varepsilon_1) > v(x,t,\varepsilon_2)$ *in* Q_{ε_1}.

Proof of Lemma 2: We use the comparisons

$$v(x,0,\varepsilon_1) = u_0(x) + \varepsilon_1 > u_0(x) + \varepsilon_2 = v(x,0,\varepsilon_2)$$

initially and

$$v(\pm 1/\varepsilon_1, t, \varepsilon_1) = u_0(\pm 1/\varepsilon_1) + \varepsilon_1 > u_0(\pm 1/\varepsilon_1) + \varepsilon_2 v(\pm 1/\varepsilon_1, t, \varepsilon_2)$$

on the boundary as well as the differential equations

$$\mathcal{L}(v(x,t,\varepsilon_1), \varepsilon_1) = 0, \qquad \mathcal{L}(v(x,t,\varepsilon_2), \varepsilon_2) = 0 \quad \text{in } Q_{\varepsilon_1}.$$

If we show that $\mathcal{L}(v(x,t,\varepsilon_2), \varepsilon_1) \geq 0$ in Q_{ε_1} then our lemma follows from a standard comparison principle, see e.g. [F] or [W]. Introducing the notation $v(\varepsilon_i) = v(x,t,\varepsilon_i)$ for $i = 1,2$ we have

$$\begin{aligned}
\mathcal{L}(v_{\varepsilon_2}, \varepsilon_1) &= -v_t(\varepsilon_2) + (v^m(\varepsilon_2))_{xx} - \frac{v(\varepsilon_2)}{\varepsilon_1 + [v(\varepsilon_2]^{p+1}} \\
&= \mathcal{L}(v_{\varepsilon_2}, \varepsilon_2) + \frac{v(\varepsilon_2)}{\varepsilon_2 + [v(\varepsilon_2]^{p+1}} - \frac{v(\varepsilon_2)}{\varepsilon_1 + [v(\varepsilon_2]^{p+1}} \\
&= v(\varepsilon_2)\, \frac{\varepsilon_1 - \varepsilon_2}{[\varepsilon_2 + [v(\varepsilon_2]^{p+1}][\varepsilon_1 + [v(\varepsilon_2]^{p+1}]} > 0,
\end{aligned}$$

and the proof of Lemma 2 is complete.

Now we have a monotone bounded sequence $\{v(x,t,\varepsilon)\}_{\varepsilon>0}$, whose limit $\lim\limits_{\varepsilon \to 0} v(x,t,\varepsilon)$ exists at any point in $\mathbb{R}^2_+$. Let us denote this limit by $u(x,t)$.

To conclude the proof of Theorem 1 we have to show that u is a weak solution of (1) (2). This will be done in several steps, partly following ideas of Phillips [P].

Step 1: $u^{-p}\chi_{\{u>0\}} \in L^1_{loc}(\mathbb{R}^2_+)$

The functions $v(x,t,\varepsilon)$ satisfy equation (5) in the classical sense, thus

$$\int\limits_P [v^m\phi_{xx} + v\phi_t]\, dxdt - \int\limits_{t_1}^{t_2} (v^m\phi_x)|_{x_1}^{x_2}\, dt - \int\limits_{x_1}^{x_2} (v\phi)|_{t_1}^{t_2}\, dx = \int\limits_P \frac{v\phi\chi_{\{v>0\}}}{\varepsilon + v^{1+p}}\, dxdt \tag{10}$$

holds. We can pass to the limit in the left hand side of this equation. So it follows from Fatou's Lemma that $u^{-p}\chi_{\{u>0\}} \in L^1_{loc}(\mathbb{R}^2_+)$, since

$$\int\limits_P u^{-p}\chi_{\{u>0\}}\ \phi\ dxdt \le \underline{\lim}_{\varepsilon\to 0} \int\limits_P \frac{v\chi_{\{v>0\}}}{\varepsilon + v^{1+p}}\phi\ dxdt \le M < \infty. \tag{11}$$

Step 2: u solves (1) in the sense of distributions, i.e. (3) holds for any $\phi \in C_0^\infty(\mathbb{R}^2_+)$.

To see this, for every $\delta > 0$ let $\eta_\delta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a smooth, monotone nondecreasing function which satisfies $\eta_\delta(s) = 0$ for $0 \le s \le \delta$, $\eta_\delta(s) = s - \delta$ for $s \ge 2\delta$, $\eta_\delta' \le 2$ and $\eta_\delta'' \le 2/\delta$.

Then

$$\begin{aligned}
\int\limits_{\mathbb{R}^2_+} [u^m\phi_{xx} + u\phi_t]\, dxdt &= \int\limits_{\mathbb{R}^2_+} u^m\phi_{xx} + \eta_\delta(u)\phi_t]\, dxdt + g_1(\delta) \\
&= \int\limits_{\mathbb{R}^2_+} [v^m\phi_{xx} + \eta_\delta(v)\phi_t]\, dxdt + g_2(\delta,\varepsilon) \\
&= \int\limits_{\mathbb{R}^2_+} [v^m\phi_{xx} - \{\eta_\delta(v)\}_t\phi]\, dxdt + g_2(\delta,\varepsilon) \\
&= \int\limits_{\mathbb{R}^2_+} [\eta_\delta'(v)\frac{v\phi}{\varepsilon + v^{1+p}} + mv^{m-1}v_x\eta_\delta''(v)v_x\phi]\, dxdt + g_3(\delta,\varepsilon),
\end{aligned}$$

where g_i, ($i = 1, 2, etc.$) are continuous functions which vanish at the origin. If we send ε to zero in the first term of the last line, this term converges uniformly on suppϕ, so that

$$\int_{\mathbb{R}^2_+} [u^m \phi_{xx} + u\phi_t] \, dxdt = \int_{\{ u>\delta \}} \eta'_\delta(u) \frac{u\phi}{\varepsilon + u^{1+p}} \, dxdt + \int_{\{ \delta<v<2\delta \}} m v^{m-1} v_x \eta''_\delta(v) v_x \phi \, dxdt \; + \; g_4(\delta, \varepsilon)$$

.

If we send δ to zero, the first term on the right hand side converges to the desired quantity, because $\eta'_\delta(s) \to \chi_{\{s>0\}}$ and because $u^{-p}\chi_{\{u>0\}}$ is locally integrable. Moreover, $g_3(\delta, \varepsilon)$ certainly approaches zero if both ε and δ go to zero.

To verify that u is a distributional solution of (1), it remains to show that the term involving η''_δ vanishes in the limit. Observing that

$$\int_{\{ \delta<v<2\delta \}} m v^{m-1} v_x \eta''_\delta(v) v_x \phi \, dxdt \; \leq \frac{2}{\delta} \int_{\{ \delta<v<2\delta \}} m v^{m-1} v_x^2 \phi \, dxdt \tag{12}$$

we distinguish now two cases which are dictated by Theorem 2.

If $2 + p > m$, Theorem 2 implies,

$$m v^{m-1} v_x^2 \leq M_3 \; v^{m+1-2/\beta} \leq M_4 v^{1-p}, \tag{13}$$

and this and Step 1 imply that the right hand side of (12) tends to zero as $\delta \to 0$.

For $2 + p \leq m$ we observe that $1/\beta = m - 1$ and obtain

$$m v^{m-1} v_x^2 \leq M_3 \; v v_x.$$

Thus the right hand side of (12) is estimated by

$$M_4 \int_{\{\delta<v<2\delta\} \cap \mathrm{supp}\phi} |v_x| \; dx,$$

a quantity which tends to zero as $\delta \to 0$, because v has locally finite lap-number, see [CMM].

Step 3: u is a weak solution of (1) (2). This follows from standard arguments and is left as an exercise.

Regularity

Theorem 2. *Let $v(x,t,\varepsilon)$ and $u(x,t)=v(x,t,0)$ be as constructed above and suppose that $0<p<m$ and*

$$\beta := \begin{cases} 2/(m+p) & \text{if } m<2+p, \\ 1/(m-1) & \text{if } m \geq 2+p. \end{cases}$$

Then for every $\tau>0$ there exists a constant M_2 such that

$$|(u^{1/\beta})_x| \leq M_2 < \infty \qquad \text{in } (\tau,T)\times I\!R. \tag{14}$$

Moreover, if $(u_0^{1/\beta})_x \leq M_2$ then (14) is true in $[0,T)\times I\!R$.

Remark 3. The derivatives in Theorem 2 have to be understood in the sense of distributions. Inequality (14) implies that $u(\cdot,t)\in C^\beta(I\!R)$. The condition $\beta m>1$ makes sure that the flux $(u^m)_x$ is continuous. Notice that the case $m=p=1$ is not covered by Theorem 2, whereas the case $p<m<1$ is covered.

Remark 4. In the proof we use the so-called Bernstein method, which was perfected in [A1], [K]. It is interesting to compare our results with the ones in [K].

Proof of Theorem 2. It is sufficient to derive (14) for the approximating sequence $v=v(x,t,\varepsilon)$ as long as the upper bound M_2 in (14) does not depend on ε. In fact since the derivatives are taken in the sense of distributions we can pass to the limit $\varepsilon\to 0$ in the inequality $\int_{I\!R} v^{1/\beta}\phi_x\,dx \leq M_2\int\phi\,dx$ for $0\leq\phi\in C_0^\infty(I\!R)$. We set $h=h(x,t,\varepsilon)=v^{1/\beta}$, i.e. $v=z^\beta$ with $\beta>0$ and calculate $v_t=\beta h^{\beta-1}h_t$, $(v^m)_x=\beta m h^{\beta m-1}h_x$ and $(v^m)_{xx}=\beta m(\beta m-1)h^{\beta m-2}h_x^2+\beta m h^{\beta m-1}h_{xx}$. Using (5) we obtain the following equation for h:

$$\begin{aligned} -h_t &+ m h^{\beta(m-1)}h_{xx} \\ &+ m(\beta m-1)h^{\beta(m-1)-1}h_x^2 - \frac{h}{\beta(\varepsilon+h^{\beta(1+p)})} = 0. \end{aligned} \tag{15}$$

Following Aronson we define

$$f(y)=\frac{N_0}{3}y(4-y), \qquad \text{where } N_0=M_1^{1/\beta}.$$

For $y \in [0,1]$ one has

$$\frac{2N_0}{3} \le f' \le \frac{4N_0}{3}, \qquad f'' = -\frac{2N_0}{3} \qquad \text{and} \quad \left(\frac{f''}{f'}\right)' \le -\frac{1}{4}. \tag{16}$$

The function f is invertible and has range $[0, N_0]$ on the unit interval $y \in [0,1]$. Therefore we may define the function $w = w(x,t,\varepsilon)$ by

$$h = f(w)$$

and derive the following equation for w from (15):

$$\begin{aligned} &-w_t + mf^{\beta(m-1)}w_{xx} \\ &\quad + mf^{\beta(m-1)}\frac{f''}{f'}w_x^2 + m(\beta m - 1)f^{\beta(m-1)-1}f'w_x^2 \\ &\quad - \frac{1}{\beta f'}\,\frac{f}{\varepsilon + f^{\beta(1+p)}} = 0. \end{aligned} \tag{17}$$

Differentiate (17) with respect to x to obtain

$$\begin{aligned} &-w_{tx} + mf^{\beta(m-1)}w_{xxx} + m\beta(m-1)f^{\beta(m-1)-1}f'w_xw_{xx} \\ &\quad + mw_x^3\frac{\partial}{\partial w}\left[f^{\beta(m-1)}\frac{f''}{f'} + (\beta m - 1)f^{\beta(m-1)-1}f'\right] \\ &\quad - \frac{1}{\beta}w_x\frac{\partial}{\partial w}\left[\frac{f}{f'(\varepsilon + f^{\beta(1+p)})}\right] \\ &\quad + mf^{\beta(m-1)}\frac{f''}{f'}2w_xw_{xx} + m(\beta m - 1)f^{\beta(m-1)-1}f'2w_xw_{xx} = 0. \end{aligned} \tag{18}$$

Recall that (18) holds in Q_ε and that we want to estimate h_x or equivalently w_x. So suppose that w_x is "large" in a point x_0 at some time $t_0 > \tau$. Then let $P = [x_0 - 2, x_0 + 2] \times (0,T)$ and $P_1 = [x_0 - 1, x_0 + 1] \times (\tau, T)$ be two rectangles containing (x_0, t_0), and let $\zeta(x,t)$ be a smooth cutoff-function with the properties $0 \le \zeta \le 1$, $\zeta \equiv 1$ in P_1, $\zeta \equiv 0$ in the neighbourhood of the lines $t = 0$ and $x = x_0 \pm 2$, $|\zeta_t| + |\zeta_x| + |\zeta_{xx}| \le M_3$ with M_3 independent of ε. Now consider the function $g(x,t) = \zeta^2 w_x^2$ in $\overline{P}$. In a maximal point of g one has $g_x = 0$ and $g_t - mf^{\beta(m-1)}g_{xx} \ge 0$, and this leads to

$$\zeta w_x w_{xx} = -\zeta_x w_x^2, \tag{19}$$

and, using (19), also to

$$\zeta^2 w_x(w_{tx} - mf^{\beta(m-1)}w_{xxx}) \geq [-\zeta\zeta_t + mf^{\beta(m-1)}(-\zeta\zeta_{xx} - 3\zeta_x^2)]w_x^2. \quad (20)$$

Multiplying (18) by $\zeta^2 w_x$, applying (19) and (20), one has at the maximum point of g:

$$\begin{aligned}
& m\zeta^2 w_x^4 \Big\{ (\beta m - 1)[1 - \beta(m-1)] f^{\beta(m-1)-2} f'^2 \\
& + (1 + \beta - 2m\beta) f^{\beta(m-1)-1} f'' - f^{\beta(m-1)} \left(\frac{f''}{f'}\right)' \Big\} \\
\leq & \zeta^2 w_x^3 \left[\zeta\zeta_t - mf^{\beta(m-1)}(-\zeta\zeta_{xx} - 3\zeta_x^2)\right] \\
& - \zeta\zeta_x w_x^3 \left\{ (m\beta(m-1) + 2m(\beta m - 1)) f^{\beta(m-1)-1} f' + 2mf^{\beta(m-1)} \frac{f''}{f'} \right\} \\
& - \frac{1}{\beta}\zeta^2 w_x^2 \frac{\partial}{\partial w}\left[\frac{f}{f'(\varepsilon + f^{\beta(1+p)})}\right]
\end{aligned} \quad (21)$$

Notice that after division by w_x^2 we have the factor $g = \zeta^2 w_x^2$ on the left hand side of (21), which we want to be bounded.

Now we distinguish two cases. In the first case suppose that $2 + p > m$ and set $\beta = 2/(m+p)$. Then $1 - \beta(m-1) = (2 + p - m)/(m + p) > 0$ and $\beta m > 1$ by assumption. Multiplying (21) by $f^{2-\beta(m-1)}$ and using (16) we can see

a) that the left hand side of (21) is bounded below by a positive constant times $\zeta^2 w_x^4$ (here we have used $m \geq 1$ to show that $1 + \beta - 2m\beta \leq 0$) and

b) that each coefficient of w_x^2 and w_x^3 on the right hand side of (21) is bounded as long as $m \geq 1$ and as long as

$$f^{2-\beta(m-1)} \frac{\partial}{\partial w}\left[\frac{f}{f'(\varepsilon + f^{\beta(1+p)})}\right]$$

is bounded uniformly with respect to ε or has the right (positive) sign. But this term is equal to

$$f^{2-\beta(m-1)} \left\{ -\frac{f''}{(f')^2}\{\frac{f}{\varepsilon + f^{\beta(1+p)}} + \frac{1}{\varepsilon + f^{\beta(1+p)}} - \frac{\beta(1+p)f^{\beta(1+p)}}{(\varepsilon + f^{\beta(1+p)})^2} \right\}. \quad (22)$$

The first two terms in $\{\quad\}$ have the right sign, and so we can drop them. The remaining term is bounded with respect to ε as $\varepsilon \to 0$ if and only if $2 - \beta(m-1) - \beta(1+p) \geq 0$, i.e. $2 - \beta(m+p) \geq 0$. This settles the first case.

Remark 5. If $m = 1 > p$ then $\beta = 2/(1+p)$ and we recover the results of [Ph,p.257] or [FK,Thm.2.4]. Phillips result for this special m is actually derived in more than one space dimension, and our method of proof works in the general n-dimensional case, too.

To complete the proof of Theorem 2 we still have to treat the second case, in which $2 + p \leq m$ and $\beta = 1/(m-1)$. In this case again all coefficients in (21) are bounded except possibly the last one on the right hand side. We show that this term has the proper (negative) sign and can thus be dropped. Equivalently, the quantity in $\{\quad\}$ of (22) is positive. The first term is positive by definition of f''. The sum of the second and third is

$$\frac{\varepsilon + f^{\beta(1+p)} - \beta(1+p)f^{\beta(1+p)}}{(\varepsilon + f^{\beta(1+p)})^2} \geq 0$$

if and only if $1 - \beta(1+p) \geq 0$, i.e. $m \geq 2 + p$. This completes the proof of the first statement in Theorem 2.

To show the last claim of Theorem 2 it suffices to choose $\zeta(x,t) = \zeta(x)$ independent of t.

Regularity is Optimal

The following examples show that our results are optimal. Note that in both examples $u^{-p}\chi_{\{u>0\}} \in L^1_{loc}(I\!R)$.

Example 1. Let $m \geq 2$ and $p = m - 2$. Then

$$u(x,t) = (\sigma t - x)_+^{1/(m-1)}$$

is a solution of (1) with initial values $u_0(x) = (-x)_+^{1/(m-1)}$. Here $\sigma = -(m-1) + m/(m-1)$. In particular for $p = 0$ and $m = 2$ one obtains a solution of (23)

$$u_t = (u^2)_{xx} - \chi_{\{u>0\}} \tag{23}$$

with initial data $u_0(x) = (-x)_+$. Notice that u_0 is unbounded and not in $L^1(I\!R)$.

Example 2. This is a variant of the example in [Ke]. Let $m \geq 2$, $p = m-2$ and $k_0 = 2m(m+1)/(m-1)$. Then for any positive constants k and l the following function $u(x,t)$ is a solution of (1):

$$u(x,t) = (k_0 t + k)^{-1/(m-1)}$$
$$\cdot \left[c(k,l,m)\,(k_0 t + k)^{2/(m+1)} - \frac{(m-1)^2}{4m^2}(k_0 t + k)^2 - x^2 \right]_+^{1/(m-1)}$$

where

$$c(k,l,m) = \frac{(m-1)^2 k^2 + 4m^2 l^2}{4m^2 k^{2/(m+1)}}$$

with initial values

$$u_0(x) = k^{-1/(m-1)} (l^2 - x^2)_+^{1/(m-1)}.$$

In particular for $p = 0$ and $m = 2$ one obtains a solution of (20) with initial data $u_0(x) = k^{-1}(l^2 - x^2)_+$ of parabolic shape and with finite support.

Acknowledgement: This research was financially supported in part by the IMA in Minneapolis and by the DFG via SFB 123, SFB 256 and a Heisenberg award.

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