REMARK ON THE FAST DIFFUSION EQUATION
IN A BALL

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IMA Preprint Series # 830
June 1991
REMARK ON THE FAST DIFFUSION EQUATION IN A BALL

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Abstract. We consider the fast diffusion equation \( u_t = \Delta u^m \) with \( m \in \left( \frac{(N-2)_+}{N+2}, 1 \right) \) in a ball \( x \in B_1 = \{ |x| < 1 \}, t > 0, u = 0 \) on \( \partial B_1, t > 0 \), with a regular enough initial function \( u(x, 0) = u_0(|x|) > 0 \) in \( B_1, u_0(r) \) is nonincreasing. By using intersection comparison techniques we give new simple proof of asymptotic behavior of \( u(|x|, t) \) as \( t \to T, T \) is a finite extinction time.

Key words. Fast diffusion, finite extinction time, asymptotic behavior, intersection comparison

AMS(MOS) subject classifications. 35K55, 35K65

1. Main Result. In this paper we give an application of intersection comparison techniques to study the asymptotic behavior to the problem

\[
(1.1) \quad u_t = \Delta u^m \text{ for } x \in B_1 = \{ |x| < 1 \} \subset \mathbb{R}^N, t > 0,
\]

where

\[
(1.2) \quad \frac{(N-2)_+}{N+2} < m < 1 (\cdot)_+ = \max\{0, (\cdot)\},
\]

with a radially symmetric initial function

\[
(1.3) \quad u(x, 0) = u_0(|x|) > 0 \text{ in } B_1, u_0(1) = 0, u_0 \in C([0, 1]),
\]

and the boundary condition

\[
(1.4) \quad u(x, t) = 0 \text{ for } x \in S_1 = \{ |x| = 1 \}, t > 0.
\]

It is well-known [S1], [S2], [BC] that there exists a finite extinction time \( T \) such that \( u(x, t) > 0 \) is the classical solution in \( B_1 \times (0, T) \) and

\[
(1.5) \quad u(x, T) \equiv 0 \text{ in } B_1.
\]

The asymptotic behavior as \( t \to T \) of the solution to equation (1.1) for \( x \in \Omega \), a smooth bounded domain in \( \mathbb{R}^N \), \( t \in (0, T) \), has been proved in [BH]. We give a very simple proof of such a result for \( \Omega = B_1 \) in a "subcritical" case (1.2).

Denote by \( V_1(x) \) the unique positive in \( B_1 \) classical solution (see [JL]) of the problem

\[
(1.6) \quad \Delta V_1^m + \frac{1}{1-m} V_1 = 0 \text{ in } B_1, \quad V_1 = 0 \text{ on } S_1.
\]

We now state the main result (cf. [BH]).

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**Theorem.** Assume that \( u_0(r) \) is nonincreasing for \( r \in [0,1] \) and for some small \( \varepsilon > 0 \)

\[
(1.7) \quad u_0(r) \in C^1([1-\varepsilon,1]), u_0'(1) < 0,
\]

\[
(1.8) \quad u_0(r) \text{ is Lipschitz continuous in } [0,\varepsilon].
\]

Then

\[
(1.9) \quad v(x,t) \equiv (T-t)^{-\frac{1}{1-m}} u(x,t) \to V_1(x) \text{ as } t \to T
\]

uniformly in \( B_1 \).

The main problems in studying the asymptotic behavior as \( t \to T \) consist of proving the lower and upper bounds of the rescaled function \( v(x,t) \) given by (1.9). Indeed, \( v = v(x,\tau), \quad \tau = -\log(T-t) \to \infty \text{ as } t \to T \), solves the following initial-boundary value problem

\[
(1.10) \quad v_\tau = \Delta v^m + \frac{1}{1-m} v \quad \text{for } x \in B_1, \tau > \tau_0 = -\log T,
\]
 \[
(1.11) \quad v(x,\tau_0) = u_0(x) \equiv T^{-\frac{1}{1-m}} u_0(x) \text{ in } B_1,
\]
 \[
(1.12) \quad v(x,\tau) = 0 \text{ for } x \in S_1, \tau > \tau_0.
\]

Assume for a moment that we have suitable lower and upper bounds for \( v(x,t) \) so that the \( \omega \)-limit set \( \omega(v_0) \) satisfies \( \{0\} \not\in \omega(v_0) \) and \( \omega(v_0) \) is bounded respectively. Then, since the stationary solution \( V_1 \not\equiv 0 \) to (1.6) is unique, we immediately arrive at convergence (1.9) in \( \dot{W}_{1,2}(B_1) \) (by using standard embedding theorems, see [BH]) or uniformly in \( B_1 \) (by using interior regularity results [DB1], [DB2]). This follows from the fact that equation (1.10) admits the Liapunov function (see [BH]) \( F(P)(\tau), P \equiv v^m \),

\[
(1.13) \quad F(P)(\tau) = \int_{B_1} \left( \frac{1}{2} |\nabla P|^2 - \frac{m}{(1-m^2)} \frac{m+1}{m} \right) dx,
\]

which is nonincreasing on any evolutionary trajectory \( \{P(\cdot,\tau), \tau > \tau_0\} \): for \( \tau_2 > \tau_1 > \tau_0 \)

\[
(1.14) \quad F(P)(\tau_2) - F(P)(\tau_1) = -\frac{4m}{(m+1)^2} \int_{B_1} \frac{1}{\tau_2} \int_{B_1} \left[ \left( \frac{m+1}{2m} P^\tau \right) \right] dx d\tau \leq 0.
\]

The proof of lower and upper bounds of \( v(x,t) \) is based on the method of intersection comparison (cf. [SGKM, Chapter IV], [GP1], [GP2]) of the solution \( u(x,t) \) with the family of explicit solutions \( \{u_*(x,t;R), R > 0\} \) to equation (1.1) having the same finite extinction time \( T \). This intersection comparison technique yields directly lower and upper bounds under hypothesis (1.7) and (1.8) respectively. (Notice that (1.7), (1.8) can be weakened, but we keep these in order to show how the method of intersection comparison works for this problem.) In the last Section 3 we discuss the relationships between the intersection comparison for fast diffusion case and a usual comparison technique for the porous medium equation, \( m > 1 \).

By the Maximum Principle we have that \( u = u(r,t) \) does not increase in \( r \geq 0 \) for any \( t \in (0,T) \).
2. Lower and Upper Bounds. Fix arbitrary $R > 0$ and denote by $V_R(x)$ the unique positive classical solution to the problem

\begin{equation}
\Delta V_R^m + \frac{1}{1-m} V_R = 0 \text{ in } B_R, \quad V_R = 0 \text{ on } S_R.
\end{equation}

By invariance of the problem (2.1) we deduce that

\begin{equation}
V_R(x) \equiv R^{-\frac{1}{1-m}} V_1 \left( \frac{x}{R} \right) \text{ in } B_R.
\end{equation}

Thus, we have defined a family of explicit solutions $u_*$ to equation (1.1) of the form

\begin{align*}
(2.3) & \quad u_*(x, t; R) = (T - t)^{-\frac{1}{1-m}} V_R(x) \text{ in } B_R \times (0, T), \\
(2.4) & \quad u_*(x, t; R) = 0 \text{ for } x \in S_R, t \in [0, T),
\end{align*}

having the same extinction time $T$ as $u(x, t)$.

**Lemma 1 (lower bound).** Under hypothesis (1.7) there exists $R_+ > 1$ large enough such that

\begin{equation}
(2.5) \quad u(0, t) > (T - t)^{-\frac{1}{1-m}} V_{R_+}(0) \text{ for } t \in [0, T).
\end{equation}

**Proof.** Given $R > 1$ denote by $N(t; R)$ for a fixed $t \in [0, T)$ the number of intersections for $r \in [0, 1]$ of the functions $u(r, t)$ and $u_*(r, t; R)$ or, which is the same, the number of sign changes of the difference $u(r, t) - u_*(r, t; R)$. Since by (2.2)

\begin{equation}
V_R(r) \to 0, \left| \frac{d}{dr} V_R(r) \right| \to 0 \text{ as } R \to \infty
\end{equation}

uniformly for $r \in [0, 1],$

one can see that if (1.7) is valid then for any $R = R_+$ large enough

\begin{equation}
(2.7) \quad u_0(0) > u_*(0, 0; R_+) = T^{\frac{1}{1-m}} R_+^{-\frac{2}{1-m}} V_1(0)
\end{equation}

and

\begin{equation}
(2.8) \quad N(0; R_+) = 1.
\end{equation}

Using the fact that $u(1, t) - u_*(1, t; R_+) < 0$ for $t \in (0, T)$ we conclude that by the Strong Maximum Principle the number of intersections $N(t; R_+)$ does not increase in time (see references in [GP1], [GP2]) and hence

\begin{equation}
(2.9) \quad N(t; R_+) \leq 1 \text{ for } t \in (0, T).
\end{equation}
We now prove that since the solutions \( u(x, t) \) and \( u_*(x, t; R_+) \) have the same finite extinction time \( T \), there holds (cf. (2.9))

\[
N(t; R_+) \equiv 1 \text{ for } t \in (0, T).
\]

One can see that (2.5) is a straightforward consequence of (2.10).

Assume for a contradiction that (2.10) is not valid and hence by (2.9) there exists \( t_0 \in (0, T) \) such that

\[
N(t_0; R_+) = 0.
\]

This means that since \( R_+ > 1 \)

\[
u(x, t_0) \leq u_*(x, t_0; R_+) \text{ in } \overline{B}_1.
\]

Then by the Strong Maximum Principle applied to the linear parabolic equation for the difference \( u(x, t) - u_*(x, t; R_+) \) in the domain of strict positiveness of both functions we conclude that there exists \( \nu_1 > 0 \) small enough such that

\[
u(x, t_0 + \nu_1) < u_*(x, t_0 + \nu_1; R_+) \text{ in } \overline{B}_1.
\]

Therefore by continuity of \( u_*(x, t; R_+) \) we deduce that there exists small \( \nu_2 > 0 \) such that

\[
u(x, t_0 + \nu_1) \leq u_*(x, t_0 + \nu_1 + \nu_2; R_+) \text{ in } \overline{B}_1.
\]

Thus, by comparison we have that

\[
u(x, t) \leq u_*(x, t + \nu_2; R_+) \text{ in } \overline{B}_1 \times (t_0 + \nu_1, T).
\]

Of course, this contradicts the assumption that both solutions \( u \) and \( u_* \) have the same extinction time. Indeed, letting \( t = T - \nu_2 \) in (2.15) yields

\[
u(x, T - \nu_2) \leq u_*(x, T; R_+) \equiv 0 \text{ in } \overline{B}_1,
\]

whence the contradiction. \( \square \)

**Lemma 2 (upper bound).** Under hypothesis (1.8) there exists \( R_- \in (0, 1) \) small enough such that

\[
u(0, t) < (T - t)^{\frac{1}{1 - m}} V_{R_-}(0) \text{ for } t \in [0, T).
\]
Proof. It is quite similar to the previous one. For a given \( R \in (0, 1) \) we now denote by \( N(t; R) \) the number of intersections for \( r \in [0, R] \) of the functions \( u(r, t) \) and \( u_*(r, t; R) \). Using (2.2) yields

\[
V_R(r) \to \infty \text{ in } [0, R/2] \text{ as } R \to 0,
\]

\[
\left| \frac{d}{dr} V_R(r) \right| \to \infty \text{ in } [R/2, R] \text{ as } R \to 0.
\]

Hence under hypothesis (1.8) there exists \( R = R_- \in (0, 1) \) small enough such that (2.8) is valid with \( R_+ \) replaced by \( R_- \). The rest of the proof is the same. By the Strong Maximum Principle we have \( N(t; R_-) \leq 1 \) for \( t \in (0, T) \). Assume that \( N(t_0; R_-) = 0 \) for some \( t_0 \in (0, T) \). Then (cf. (2.12)) \( u_*(x, t_0; R_-) \leq u(x, t_0) \) in \( B_{R_-} \) and hence for any small \( \nu_1 > 0 \), \( u_*(x, t_0 + \nu_1; R_-) < u(x, t_0 + \nu_1) \) in \( B_{R_-} \). By continuity (cf. (2.14)) \( u_*(x, t_0 + \nu_1; R_-) \leq u(x, t_0 + \nu_1 + \nu_2) \) in \( B_{R_-} \) if \( \nu_2 > 0 \) is small enough. Thus, by comparison \( u_*(x, t; R_-) \leq u(x, t + \nu_2) \) in \( B_{R_-} \times (t_0 + \nu_1, T) \) (cf. (2.15)). This leads to the contradiction since for \( t = T - \nu_2 \) we have \( u_*(x, T - \nu_2; R_-) \leq u(x, T) = 0 \) in \( B_{R_-} \), contradicting the fact that the explicit solution \( u_*(x, t; R_-) \) considered has exactly \( T \) as a finite extinction time and completing the proof of Lemma 2. \( \square \)

By using estimates (2.5) and (2.17) the proof of Theorem can be completed by a standard way. Since by Lemma 2 \( v(x, t) \) is uniformly bounded and hence by general results [DB1, DB2] the trajectory \( \{v(\cdot, \tau), \tau > \tau_0\} \) is compact in \( C(B_1) \), by using Liapunov function (1.13) we conclude that \( \omega(\theta_0) = \{V \in C(B_1), V \geq 0|V \neq 0, V \text{ solves } (1.6)\} \). By uniqueness of such \( V_1 \) we have \( \omega(\theta_0) = \{V_1\} \), completing the proof of the Theorem.

3. Final Remark. It seems to be interesting to compare the results given above with a similar technique for porous medium equation (2.1), (1.4) with

\[
m > 1.
\]

It is well-known [AP] that the asymptotic behavior as \( t \to \infty \) of the solution is described by the explicit self-similar solution of the form (cf. (2.3))

\[
u_*(x, t; R) = (T + t)^{-\frac{1}{m-1}} V_R(x) \text{ in } B_R \times (0, \infty)
\]

with \( R = 1 \), where \( T > 0 \) is arbitrary constant and \( V_R > 0 \) in \( B_R \) is the unique solution to (cf. (2.1))

\[
\Delta V_R^m + \frac{1}{m-1} V_R = 0 \text{ in } B_R, \quad V_R = 0 \text{ on } S_R.
\]

There holds [AP] (cf. (1.9))

\[
v(x, t) \equiv t^{\frac{1}{m-1}} u(x, t) \to V_1(x) \text{ as } t \to \infty
\]
uniformly in $B_R$.

Of course, the proof of (3.5) is also based in main part on the sharp lower and upper bounds of the rescaled function $v(x, t)$ given in (3.5). These bounds can be easily proved by usual comparison (not intersection comparison) with the family of explicit solutions (3.2). Indeed, if $u_0(0) > 0$ (if not we can wait a bit) then one can choose such $0 < R_- < 1 < R_+$ and $0 < T_+ < T_-$ that

$$T_-^{\frac{1}{m-1}}V_{R_-}(x) \leq u_0(x) \text{ in } \overline{B}_{R_-},$$

$$u_0(x) \leq T_+^{\frac{1}{m-1}}V_{R_+}(x) \text{ in } \overline{B}_1.$$

Hence by comparison we have

$$\left(T_- + t\right)^{-\frac{1}{m-1}}V_{R_-}(x) \leq u(x, t) \text{ in } \overline{B}_{R_-} \times (0, \infty),$$

$$u(x, t) \leq \left(T_+ + t\right)^{-\frac{1}{m-1}}V_{R_+}(x) \text{ in } \overline{B}_1 \times (0, \infty).$$

These mean that for large $t > 0$

$$0 < \frac{1}{2}V_{R_-}(0) < v(0, t) < 2V_{R_+}(0) < \infty,$$

whence the desired lower and upper bounds. The rest of analysis is absolutely similar.

Thus, in a ball both equations: the fast diffusion equation with $m$ satisfying (1.2) and the porous medium equation with $m > 1$ can be studied by using some comparison techniques. The only difference is as follows: for fast diffusion, where extinction in a finite time occurs, we need to use the intersection comparison with a family of explicit solutions (2.3) having the same fixed blow-up time $T$ instead of usual comparison working for porous medium equation, where the corresponding family of explicit solutions (3.2) has a free parameter $T > 0$ which does not appear in the asymptotic behavior as $t \to \infty$.

**Acknowledgement.** This paper was written during the visit of the author to the IMA to participate in the Program on Phase Transitions and Degenerate Diffusion. The author thanks Professor L.A. Peletier for drawing his attention to the problem considered and for several helpful discussions. The author also thanks Professor S. Kamin for useful remarks on the asymptotic behavior and Professor J.L. Vazquez for a discussion on regularity results.

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REMARK ON THE FAST DIFFUSION EQUATION IN A BALL

By

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IMA Preprint Series # 830
June 1991
REMARK ON THE FAST DIFFUSION EQUATION IN A BALL

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Abstract. We consider the fast diffusion equation \( u_t = \Delta u^m \) with \( m \in \left( \frac{(N-2)_+}{N+2}, 1 \right) \) in a ball \( x \in B_1 = \{|x| < 1\}, t > 0, u = 0 \) on \( \partial B_1, t > 0 \), with a regular enough initial function \( u(x, 0) = u_0(|x|) > 0 \) in \( B_1, u_0(r) \) is nonincreasing. By using intersection comparison techniques we give new simple proof of asymptotic behavior of \( u(|x|, t) \) as \( t \to T, T \) is a finite extinction time.

Key words. Fast diffusion, finite extinction time, asymptotic behavior, intersection comparison

AMS(MOS) subject classifications. 35K55, 35K65

1. Main Result. In this paper we give an application of intersection comparison techniques to study the asymptotic behavior to the problem

\[
(1.1) \quad u_t = \Delta u^m \text{ for } x \in B_1 = \{|x| < 1\} \subset \mathbb{R}^N, t > 0,
\]

where

\[
(1.2) \quad \frac{(N-2)_+}{N+2} < m < 1 \left( (\cdot)_+ = \max\{0, (\cdot)\} \right),
\]

with a radially symmetric initial function

\[
(1.3) \quad u(x, 0) = u_0(|x|) > 0 \text{ in } B_1, u_0(1) = 0, u_0 \in C([0, 1]),
\]

and the boundary condition

\[
(1.4) \quad u(x, t) = 0 \text{ for } x \in S_1 = \{|x| = 1\}, t > 0.
\]

It is well-known [S1], [S2], [BC] that there exists a finite extinction time \( T \) such that \( u(x, t) > 0 \) is the classical solution in \( B_1 \times (0, T) \) and

\[
(1.5) \quad u(x, T) \equiv 0 \text{ in } B_1.
\]

The asymptotic behavior as \( t \to T \) of the solution to equation (1.1) for \( x \in \Omega \), a smooth bounded domain in \( \mathbb{R}^N, t \in (0, T) \), has been proved in [BH]. We give a very simple proof of such a result for \( \Omega = B_1 \) in a "subcritical" case (1.2).

Denote by \( V_1(x) \) the unique positive in \( B_1 \) classical solution (see [JL]) of the problem

\[
(1.6) \quad \Delta V_1^m + \frac{1}{1-m} V_1 = 0 \text{ in } B_1, \quad V_1 = 0 \text{ on } S_1.
\]

We now state the main result (cf. [BH]).

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Theorem. Assume that $u_0(r)$ is nonincreasing for $r \in [0, 1]$ and for some small $\varepsilon > 0$

\[(1.7) \quad u_0(r) \in C^1([1 - \varepsilon, 1]), u'_0(1) < 0,
\]

\[(1.8) \quad u_0(r) \text{ is Lipschitz continuous in } [0, \varepsilon].\]

Then

\[(1.9) \quad v(x, t) \equiv (T - t)^{-\frac{1}{1 - m}} u(x, t) \to V_1(x) \text{ as } t \to T
\]

uniformly in $B_1$.

The main problems in studying the asymptotic behavior as $t \to T$ consist of proving the lower and upper bounds of the rescaled function $v(x, t)$ given by (1.9). Indeed, $v = v(x, \tau)$, $

\tau = -\log(T - t) \to \infty \text{ as } t \to T,$

solves the following initial-boundary value problem

\[(1.10) \quad v_\tau = \Delta v^m + \frac{1}{1 - m} v \text{ for } x \in B_1, \tau > \tau_0 = -\log T,
\]

\[(1.11) \quad v(x, \tau_0) = v_0(x) \equiv T^{-\frac{1}{1 - m}} u_0(x) \text{ in } B_1,
\]

\[(1.12) \quad v(x, \tau) = 0 \text{ for } x \in S_1, \tau > \tau_0.
\]

Assume for a moment that we have suitable lower and upper bounds for $v(x, t)$ so that the

$\omega$-limit set $\omega(v_0)$ satisfies $\{0\} \not\in \omega(v_0)$ and $\omega(v_0)$ is bounded respectively. Then, since the stationary solution $V_1 \not\equiv 0$ to (1.6) is unique, we immediately arrive at convergence (1.9) in $\dot{W}_{1, 2}(B_1)$ (by using standard embedding theorems, see [BH]) or uniformly in $B_1$ (by using interior regularity results [DB1], [DB2]). This follows from the fact that equation

(1.10) admits the Liapunov function (see [BH]) $F(P)(\tau), P = v^m$,

\[(1.13) \quad F(P)(\tau) = \int_{\bar{B}_1} \left( \frac{1}{2} |\nabla P|^2 - \frac{m}{(1 - m)^2} P^{\frac{m + 1}{m}} \right) dx,
\]

which is nonincreasing on any evolutionary trajectory $\{P(\cdot, \tau), \tau > \tau_0\}$: for $\tau_2 > \tau_1 > \tau_0$

\[(1.14) \quad F(P)(\tau_2) - F(P)(\tau_1) = \frac{4m}{(m + 1)^2} \int_{\tau_1}^{\tau_2} \int_{\bar{B}_1} \left[ P^{\frac{m + 1}{2m}} (\tau) \right]^2 dx d\tau \leq 0.
\]

The proof of lower and upper bounds of $v(x, t)$ is based on the method of intersection comparison (cf. [SGKM, Chapter IV], [GP1], [GP2]) of the solution $u(x, t)$ with the family of explicit solutions $\{u_\ast(x, t; R), R > 0\}$ to equation (1.1) having the same finite extinction time $T$. This intersection comparison technique yields directly lower and upper bounds under hypothesis (1.7) and (1.8) respectively. (Notice that (1.7), (1.8) can be weakened, but we keep these in order to show how the method of intersection comparison works for this problem.) In the last Section 3 we discuss the relationships between the intersection comparison for fast diffusion case and a usual comparison technique for the porous medium equation, $m > 1$.

By the Maximum Principle we have that $u = u(r, t)$ does not increase in $r \geq 0$ for any $t \in (0, T)$. 

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2. Lower and Upper Bounds. Fix arbitrary $R > 0$ and denote by $V_R(x)$ the unique positive classical solution to the problem

$$\Delta V_R^m + \frac{1}{1-m} V_R = 0 \text{ in } B_R, \quad V_R = 0 \text{ on } S_R. \quad (2.1)$$

By invariance of the problem (2.1) we deduce that

$$V_R(x) \equiv R^{-\frac{2}{1-m}} V_1 \left( \frac{x}{R} \right) \text{ in } B_R. \quad (2.2)$$

Thus, we have defined a family of explicit solutions $u_*$ to equation (1.1) of the form

$$u_*(x, t; R) = (T - t)^{\frac{1}{1-m}} V_R(x) \text{ in } B_R \times (0, T), \quad (2.3)$$

$$u_*(x, t; R) = 0 \text{ for } x \in S_R, t \in [0, T), \quad (2.4)$$

having the same extinction time $T$ as $u(x, t)$.

**Lemma 1 (lower bound).** Under hypothesis (1.7) there exists $R_+ > 1$ large enough such that

$$u(0, t) > (T - t)^{\frac{1}{1-m}} V_{R_+}(0) \text{ for } t \in [0, T). \quad (2.5)$$

**Proof.** Given $R > 1$ denote by $N(t; R)$ for a fixed $t \in [0, T)$ the number of intersections for $r \in [0, 1]$ of the functions $u(r, t)$ and $u_*(r, t; R)$ or, which is the same, the number of sign changes of the difference $u(r, t) - u_*(r, t; R)$. Since by (2.2)

$$V_R(r) \to 0, \quad \left| \frac{d}{dr} V_R(r) \right| \to 0 \text{ as } R \to \infty \quad (2.6)$$

uniformly for $r \in [0, 1]$,

one can see that if (1.7) is valid then for any $R = R_+$ large enough

$$u_0(0) > u_*(0, 0; R_+) = T^{\frac{1}{1-m}} R_+^{-\frac{2}{1-m}} V_1(0) \quad (2.7)$$

and

$$N(0; R_+) = 1. \quad (2.8)$$

Using the fact that $u(1, t) - u_*(1, t; R_+) < 0$ for $t \in (0, T)$ we conclude that by the Strong Maximum Principle the number of intersections $N(t; R_+)$ does not increase in time (see references in [GP1], [GP2]) and hence

$$N(t; R_+) \leq 1 \text{ for } t \in (0, T). \quad (2.9)$$
We now prove that since the solutions $u(x,t)$ and $u_*(x,t; R_+)$ have the same finite extinction time $T$, there holds (cf. (2.9))

\[(2.10) \quad N(t; R_+) \equiv 1 \text{ for } t \in (0, T).\]

One can see that (2.5) is a straightforward consequence of (2.10).

Assume for a contradiction that (2.10) is not valid and hence by (2.9) there exists $t_0 \in (0, T)$ such that

\[(2.11) \quad N(t_0; R_+) = 0.\]

This means that since $R_+ > 1$

\[(2.12) \quad u(x, t_0) \leq u_*(x, t_0; R_+) \text{ in } \overline{B}_1.\]

Then by the Strong Maximum Principle applied to the linear parabolic equation for the difference $u(x, t) - u_*(x, t; R_+)$ in the domain of strict positiveness of both functions we conclude that there exists $\nu_1 > 0$ small enough such that

\[(2.13) \quad u(x, t_0 + \nu_1) < u_*(x, t_0 + \nu_1; R_+) \text{ in } \overline{B}_1.\]

Therefore by continuity of $u_*(x, t; R_+)$ we deduce that there exists small $\nu_2 > 0$ such that

\[(2.14) \quad u(x, t_0 + \nu_1) \leq u_*(x, t_0 + \nu_1 + \nu_2; R_+) \text{ in } \overline{B}_1.\]

Thus, by comparison we have that

\[(2.15) \quad u(x, t) \leq u_*(x, t + \nu_2; R_+) \text{ in } \overline{B}_1 \times (t_0 + \nu_1, T).\]

Of course, this contradicts the assumption that both solutions $u$ and $u_*$ have the same extinction time. Indeed, letting $t = T - \nu_2$ in (2.15) yields

\[(2.16) \quad u(x, T - \nu_2) \leq u_*(x, T; R_+) \equiv 0 \text{ in } \overline{B}_1,\]

whence the contradiction. $\square$

**Lemma 2 (upper bound).** Under hypothesis (1.8) there exists $R_- \in (0, 1)$ small enough such that

\[(2.17) \quad u(0, t) < (T - t)^{1-n} V_{R_-}(0) \text{ for } t \in [0, T).\]
Proof. It is quite similar to the previous one. For a given \( R \in (0, 1) \) we now denote by \( N(t; R) \) the number of intersections for \( r \in [0, R] \) of the functions \( u(r, t) \) and \( u_*(r, t; R) \). Using (2.2) yields

\[
V_R(r) \to \infty \text{ in } [0, R/2] \text{ as } R \to 0,
\]

\[
\left| \frac{d}{dr} V_R(r) \right| \to \infty \text{ in } [R/2, R] \text{ as } R \to 0.
\]

Hence under hypothesis (1.8) there exists \( R = R_- \in (0, 1) \) small enough such that (2.8) is valid with \( R_+ \) replaced by \( R_- \). The rest of the proof is the same. By the Strong Maximum Principle we have \( N(t; R_-) \leq 1 \) for \( t \in (0, T) \). Assume that \( N(t_0; R_-) = 0 \) for some \( t_0 \in (0, T) \). Then (cf. (2.12)) \( u_*(x, t_0; R_-) \leq u(x, t_0) \) in \( \overline{B}_{R_-} \) and hence for any small \( \nu_1 > 0 \) \( u_*(x, t_0 + \nu_1; R_-) < u(x, t_0 + \nu_1) \) in \( \overline{B}_{R_-} \). By continuity (cf. (2.14)) \( u_*(x, t_0 + \nu_1; R_-) \leq u(x, t_0 + \nu_1 + \nu_2) \) in \( \overline{B}_{R_-} \) if \( \nu_2 > 0 \) is small enough. Thus, by comparison \( u_*(x, t; R_-) \leq u(x, t + \nu_2) \) in \( \overline{B}_{R_-} \times (t_0 + \nu_1, T) \) (cf. (2.15)). This leads to the contradiction since for \( t = T - \nu_2 \) we have \( u_*(x, T - \nu_2; R_-) \leq u(x, T) \equiv 0 \) in \( \overline{B}_{R_-} \) contradicting the fact that the explicit solution \( u_*(x, t; R_-) \) considered has exactly \( T \) as a finite extinction time and completing the proof of Lemma 2. \( \square \)

By using estimates (2.5) and (2.17) the proof of Theorem can be completed by a standard way. Since by Lemma 2 \( v(x, t) \) is uniformly bounded and hence by general results [DB1], [DB2] the trajectory \( \{v(\cdot, \tau), \tau > \tau_0\} \) is compact in \( C(B_1) \), by using Liapunov function (1.13) we conclude that \( \omega(\theta_0) = \{V \in C(\overline{B}_1), V \geq 0 | V \neq 0, V \text{ solves (1.6)}\} \). By uniqueness of such \( V_1 \) we have \( \omega(\theta_0) = \{V_1\} \), completing the proof of the Theorem.

3. Final Remark. It seems to be interesting to compare the results given above with a similar technique for porous medium equation (2.1), (1.4) with

\[
m > 1.
\]

It is well-known [AP] that the asymptotic behavior as \( t \to \infty \) of the solution is described by the explicit self-similar solution of the form (cf. (2.3))

\[
u_*(x, t; R) = (T + t)^{-\frac{1}{m-1}} V_R(x) \text{ in } B_R \times (0, \infty)
\]

with \( R = 1 \), where \( T > 0 \) is arbitrary constant and \( V_R > 0 \) in \( B_R \) is the unique solution to (cf. (2.1))

\[
\Delta V_R^m + \frac{1}{m - 1} V_R = 0 \text{ in } B_R, \quad V_R = 0 \text{ on } S_R.
\]

There holds [AP] (cf. (1.9))

\[
v(x, t) \equiv t^{-\frac{1}{m-1}} u(x, t) \to V_1(x) \text{ as } t \to \infty
\]
uniformly in $B_R$.

Of course, the proof of (3.5) is also based in main part on the sharp lower and upper bounds of the rescaled function $v(x, t)$ given in (3.5). These bounds can be easily proved by usual comparison (not intersection comparison) with the family of explicit solutions (3.2). Indeed, if $u_0(0) > 0$ (if not we can wait a bit) then one can choose such $0 < R_+ < 1 < R_-$ and $0 < T_+ < T_-$ that

$$
T_+^{-\frac{1}{m-1}} V_{R_+}(x) \leq u_0(x) \text{ in } \overline{B}_{R_-}, \\
u_0(x) \leq T_+^{-\frac{1}{m-1}} V_{R_+}(x) \text{ in } \overline{B}_1.
$$

(3.6)

Hence by comparison we have

$$
(T_+ + t)^{-\frac{1}{m-1}} V_{R_+}(x) \leq u(x, t) \text{ in } \overline{B}_{R_-} \times (0, \infty), \\
u(x, t) \leq (T_+ + t)^{-\frac{1}{m-1}} V_{R_+}(x) \text{ in } \overline{B}_1 \times (0, \infty).
$$

(3.7)

These mean that for large $t > 0$

$$
0 < \frac{1}{2} V_{R_+}(0) < v(0, t) < 2V_{R_+}(0) < \infty,
$$

(3.8)

whence the desired lower and upper bounds. The rest of analysis is absolutely similar.

Thus, in a ball both equations: the fast diffusion equation with $m$ satisfying (1.2) and the porous medium equation with $m > 1$ can be studied by using some comparison techniques. The only difference is as follows: for fast diffusion, where extinction in a finite time occurs, we need to use the intersection comparison with a family of explicit solutions (2.3) having the same fixed blow-up time $T$ instead of usual comparison working for porous medium equation, where the corresponding family of explicit solutions (3.2) has a free parameter $T > 0$ which does not appear in the asymptotic behavior as $t \to \infty$.

Acknowledgement. This paper was written during the visit of the author to the IMA to participate in the Program on Phase Transitions and Degenerate Diffusion. The author thanks Professor L.A. Peletier for drawing his attention to the problem considered and for several helpful discussions. The author also thanks Professor S. Kamin for useful remarks on the asymptotic behavior and Professor J.L. Vazquez for a discussion on regularity results.

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