PERTURBATIONS OF ATTRACTORS
OF DIFFERENTIAL EQUATIONS

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Dedicated to Henry Antosiewicz on the occasion of his 65th birthday.

Abstract. In this paper we study small $C^1$-perturbations of a differential equation that has a hyperbolic attractor $\mathcal{K}$. We show that if $\mathcal{K}$ has a suitable Lipschitz property and if the perturbation is small enough, then there is a homeomorphism $h : \mathcal{K} \to \mathcal{K}^Y$, where $\mathcal{K}^Y$ is a hyperbolic attractor for the perturbed equation. Examples are included.

Key words. approximation dynamics, attractors, hyperbolic sets

AMS(MOS) subject classifications.

1. Introduction. In recent years researchers in differential equations have devoted considerable efforts to the study of methods of reducing high dimensional problems to low dimensional problems. These efforts have been both of the theoretical level, where for example, one tries to use the techniques of ordinary differential equations to study problems arising in partial differential equations, and on the practical level, say in numerical analysis, where one tries to reduce problems involving systems of thousands (or millions) of ordinary equations to an equivalent problem for a handful of such equations. In the area of the long-time dynamics of the differential equations, these recent studies have been an outgrowth of the Reduction Principle paradigm, see Pliss (1964).

The Reduction Principle, as it can be applied to the system of equations

(1.1) \[ p' = F(p, q), \quad q' = G(p, q), \]

where $p \in \mathcal{P}$, $q \in \mathcal{Q}$, $F : \mathcal{P} \times \mathcal{Q} \to \mathcal{P}$ and $G : \mathcal{P} \times \mathcal{Q} \to \mathcal{Q}$ are $C^1$-functions and $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces, takes on the following form. Assume that the surface $M = \text{graph}\ \phi$ is invariant and exponentially stable for (1.1), where $\phi : \mathcal{P} \to \mathcal{Q}$ is a Lipschitz continuous function. In this case the long-time dynamics of the system (1.1) is completely described by the solutions of the reduced system

\[ p' = F(p, \phi(p)), \quad p \in \mathcal{P}. \]

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The center manifold theory, as well as the theory of inertial manifolds, are illustrations of the Reduction Principle, see for example, Carr (1981), Foias, Sell and Temam (1988), Henry (1981), and Mallet-Paret and Sell (1988).

The theory of approximate inertial manifolds is based on a variation of the procedure described above. In order to explain this we recall that $M = \text{graph } \phi$ is an invariant set for (1.1) if and only if $\frac{dq}{dt} = \nabla \phi \frac{dp}{dt}$, where $q = \phi(p)$. In other words one has

$$\nabla \phi(p)F(p, \phi(p)) = G(p, \phi(p)), \quad \text{a.a. } p \in \mathcal{P}.$$ 

As a result, for any given Lipschitz continuous mapping $q = \phi_a(p)$, the manifold $\mathcal{M}_a = \text{graph } \phi_a$ is an invariant set for the perturbed system

$$p' = F(p, q), \quad q' = G(p, q) + E(p)$$

if and only if

$$E(p) = \nabla \phi_a(p)F(p, \phi_a(p)) - G(p, \phi_a(p)), \quad \text{a.a. } p \in \mathcal{P}.$$ 

If in addition, the manifold $M_a$ is exponentially stable for (1.2), then the long-time dynamics of (1.2) is completely described by the solutions of the reduced equation

$$p' = F(p, \phi_a(p)).$$

The term $E$ appearing in (1.2) can be viewed as an error term. The manifold $\mathcal{M}_a$ is an invariant manifold for (1.1) if and only if $E(p) \equiv 0$. By introducing a suitable norm one can use $\|E\|$ to measure how far $\mathcal{M}_a$ is from being an invariant manifold. As $\|E\|$ gets smaller, $\mathcal{M}_a$ gets closer to being invariant.

This paradigm for approximate inertial manifolds is fully justified for such infinite dimensional dynamical systems as the two-dimensional Navier-Stokes equations (2DNS) on a suitable bounded region $\Omega \subset \mathbb{R}^2$, see Foias and Temam (1988), Marion and Temam (1988), Sell (1990), Temam (1989), and Titi (1988). (In this setting $\mathcal{P}$ is a finite dimensional space and $\Omega$ is infinite dimensional.) It is because of the fact that for the 2DNS the error term $E$ in (1.2) can be made small, in an appropriate norm, one can hope to see the long-time dynamics of (1.1) be well-approximated by the solutions of (1.2), or equivalently, by the solutions of the reduced equation (1.3).

We emphasize here that we are interested in the theory of approximating the long-time dynamics of a given differential equation. This theory, which we call Approximation Dynamics, differs significantly from the classical approximation theories wherein one seeks approximate solutions for a given differential equation on a finite time interval $0 \leq t \leq T$, where $T < \infty$. 

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The theory of Approximation Dynamics is of interest in the study of infinite dimensional systems like the Navier-Stokes equations as well as in the study of finite dimensional systems of ordinary differential equations (ODE). We emphasize that the theory of Approximation Dynamics for ODEs, as well as the associated theories of approximate inertial manifolds, plays a central role in the numerical integration of partial differential equations. Typical space discretizations of the 2DNS, for example, oftentimes lead to systems of thousands of ODEs.

In this paper we will study two systems of ODEs:

\begin{equation}
\frac{dx}{dt} = X(x)
\end{equation}

and

\begin{equation}
\frac{dy}{dt} = X(y) + Y(y),
\end{equation}

where $x, y \in \mathbb{R}^n$ and $X$ and $Y$ are $C^1$-functions from $\mathbb{R}^n$ into itself. For any $x_0 \in \mathbb{R}^n$ we shall let $x(t, x_0)$ (or $y(t, x_0)$) denote the maximally defined solution of (1.4) (or (1.5)) that satisfies $x(0, x_0) = y(0, x_0) = x_0$.

In order to simplify the notation we shall assume that the perturbation term $Y$ is bounded in the $C^1$-norm and satisfies

\begin{equation}
\|Y\|_{C^1} \leq \delta
\end{equation}

for a suitable $\delta > 0$.

Let $\mathcal{K}$ denote a given compact invariant set for (1.4). For each $x_0 \in \mathcal{K}$ we let $\Phi(t, x_0)$ denote the fundamental operator solution of the linear system

\begin{equation}
\frac{dx}{dt} = \frac{\partial X(x(t, x_0))}{\partial x} x
\end{equation}

that satisfies $\Phi(0, x_0) = I$, where $I$ is the identity operator on $\mathbb{R}^n$. We shall say that the linear system (1.7) is \textbf{hyperbolic along} $x(t, x_0)$ on an interval $J$ with constants $a$, $\lambda_1$, and $\lambda_2$ provided $\lambda_2 < \lambda_1$, $\lambda_1 > 0$, $a \geq 1$ and there exist complementary linear spaces $\mathcal{U}^n(t, x_0)$ and $\mathcal{U}^s(t, x_0)$, $\dim \mathcal{U}^n = k$, $\dim \mathcal{U}^s = n - k$, and

\[\Phi(t, x_0)\mathcal{U}^s(0, x_0) = \mathcal{U}^s(t, x_0), \quad \Phi(t, x_0)\mathcal{U}^n(0, x_0) = \mathcal{U}^n(t, x_0)\]

for all $t \in J$. Furthermore if $\bar{x} \in \mathcal{U}^s(\tau, x_0)$, then

\begin{equation}
|\Phi(t, x_0)\Phi^{-1}(\tau, x_0)\bar{x}| \leq a|\bar{x}|e^{-\lambda_1(t-\tau)}, \quad \text{for } t \geq \tau, \quad t, \tau \in J,
\end{equation}
and if \( \bar{x} \in \mathcal{U}^n(\tau, x_0) \), then

\[
|\Phi(t, x_0)\Phi^{-1}(\tau, x_0)\bar{x}| \leq a|\bar{x}|e^{-\lambda_2(t-\tau)}, \quad \text{for } t \leq \tau, \quad t, \tau \in J.
\] (1.9)

We shall say that a compact, invariant set \( \mathcal{K} \) for (1.4) is a **hyperbolic attractor** if \( \mathcal{K} \) satisfies the following two properties:

1. The linear system (1.7) is hyperbolic along \( x(t, x_0) \) on \( R \) with constants \( a, \lambda_1, \) and \( \lambda_2 \) for every \( x_0 \in \mathcal{K} \).
2. There exists an \( r > 0 \) such that for each \( x_0 \in \mathcal{K} \) there exists a \( k \)-dimensional locally invariant disk \( \mathcal{D}(x_0) \subset \mathcal{K} \) with the center at the point \( x_0 \) and radius \( r \) such that if \( x \in \mathcal{D}(x_0) \) then at the point \( x \) the disk \( \mathcal{D}(x_0) \) is tangent to the linear space \( \mathcal{U}^n(0, x) \).

A hyperbolic attractor \( \mathcal{K} \) is said to satisfy the **Lipschitz property** if the mapping \( x_0 \to \mathcal{U}^n(0, x_0) \) is a Lipschitz continuous mapping of \( \mathcal{K} \) into the Grassmanian of \( k \)-dimensional subspaces of \( R^n \). The notion of hyperbolicity introduced here has some antecedents in the literature, see for example, Fenichel (1971), Hirsch, Pugh and Shub (1977), Meyer and Sell (1989), Sacker (1969), Sell (1978), and Smale (1967). In the sequel we shall let \( \mathcal{K} \) denote a given hyperbolic attractor for (1.4) that satisfies the Lipschitz property.

The Lipschitz property for the neutral manifolds \( \mathcal{U}^n(0, x_0) \) plays an important role in the theory we present here. As we shall see later the Lipschitz property implies that the disks \( \mathcal{D}(x_0) \) described in item (2) above are uniquely determined. Examples of hyperbolic attractors that satisfy the Lipschitz property will be given in Section 6.

Let for \( x_0 \in \mathcal{K} \) we define the sets \( S_1(x_0), S_2(x_0), S_3(x_0), \ldots, S(x_0) \) by

\[
S_1(x_0) = \cup_{x \in \mathcal{D}(x_0)} \mathcal{D}(x), \quad S_{i+1}(x_0) = \cup_{x \in S_i(x_0)} \mathcal{D}(x), \quad \text{for } i \geq 1,
\]

and

\[
S(x_0) = \bigcup_{i=1}^{\infty} S_i(x_0).
\]

Note that the set \( S(x_0) \) is invariant and \( S(x_0) \subset \mathcal{K} \), for every \( x_0 \in \mathcal{K} \). Furthermore, if \( x_1 \in S(x_0) \), then \( S(x_1) = S(x_0) \), because of the uniqueness. This implies that if \( S(x_0) \cap S(x_1) \) is nonempty, then \( S(x_0) = S(x_1) \). The set \( S(x_0) \) is referred to as the **leaf** of \( \mathcal{K} \) through \( x_0 \).

The principle objective of this paper is to prove the following result:

**Main Theorem.** Let \( \mathcal{K} \) be a given hyperbolic attractor of (1.4) that satisfies the Lipschitz property. For every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that the \( Y \) satisfies (1.6) for this \( \delta \), then there is a one-to-one, continuous mapping

\[
h : \mathcal{K} \to R^n
\]
that satisfies $|h(x) - x| \leq \epsilon$, and the image $\mathcal{K}' = h(\mathcal{K})$ is a compact, invariant set for (1.5). Furthermore for each leaf $\mathcal{S} \subset \mathcal{K}$, the restriction $h|_{\mathcal{S}}$ is a locally Lipschitz continuous mapping of $\mathcal{S}$ onto a leaf $\mathcal{S}' \subset \mathcal{K}'$. In addition, $\mathcal{K}'$ is a hyperbolic attractor for (1.5).

**Remark.** The inequality (1.6) is stated in a simplified form. What is really used in the proof of the Main Theorem is that

$$\max (\|Y(y)\|, \|DY(y)\|) \leq \delta, \quad y \in U(\mathcal{K}, \beta),$$

where $U(\mathcal{K}, \beta)$ is a fixed $\beta$-neighborhood of $\mathcal{K}$.

This paper is organized as follows. In Section 2 we shall show that the disks $\mathcal{D}(x_0)$ are uniquely determined when $\mathcal{K}$ satisfies the Lipschitz property. In Section 3 we present the basic construction of the mapping $h : \mathcal{K} \to \mathbb{R}^n$. We show here that $h$ is a locally Lipschitz continuous mapping of each leaf $\mathcal{S}(x_0)$ of $\mathcal{K}$ onto an invariant set $\mathcal{S}'(h(x_0))$ for (1.5) and that $|h(x) - x| \leq \epsilon$ for all $x \in \mathcal{K}$. We also show that $h$ is continuous as a mapping of $\mathcal{K}$ into $\mathbb{R}^n$. In Section 4 we argue that the image set $\mathcal{K}' = h(\mathcal{K})$ is an hyperbolic attractor for (1.5). The proof of the main theorem is completed in Section 5 wherein we show that $h$ is one-to-one whenever $\epsilon$ is sufficiently small. Examples are included in Section 6.

Let $\mathcal{U}(x_0) = \mathcal{U}(0, s_0), \mathcal{U}(x_0) = \mathcal{U}(0, x_0)$. It is clear that $\mathcal{U}(t, x_0) = \mathcal{U}(x(t, x_0))$ and $\mathcal{U}(t, x_0) = \mathcal{U}(x(t, x_0))$ for all $t$. Let $N(x), x \in \mathcal{K}$ be the $(n - k)$-dimensional hyperplane perpendicular to $\mathcal{U}(x)$ at the point $x$.

**2. Uniqueness of Disks.** Let $\mathcal{K}$ be a given hyperbolic attractor for (1.4) that satisfies the Lipschitz property, and let $L$ be the Lipschitz constant for the mapping $x_0 \to \mathcal{U}(0, x_0)$. We claim that if $\mathcal{D}_1(x_0)$ and $\mathcal{D}_2(x_0)$ are two disks that satisfy the condition (2) in the definition of a hyperbolic attractor, then $\mathcal{D}_1(x_0) = \mathcal{D}_2(x_0)$.

In order to prove this, let us move the origin to the point $x_0$ and rotate the coordinate axis, to obtain a new coordinate system $(u, v)$, where $u$ is a $k$-dimensional vector and $v$ is an $(n - k)$-dimensional vector. We assume that the coordinate axis have been fixed so that the space $v = 0$ coincides with the space $\mathcal{U}(x_0)$. The disks $\mathcal{D}_1(x_0)$ and $\mathcal{D}_2(x_0)$ can then be represented in the form

$$\mathcal{D}_1(x_0) = \{v = f(u) : |u| \leq r\}, \quad \mathcal{D}_2(x_0) = \{v = g(u) : |u| \leq r\}.$$ 

Let $u_1$ be an arbitrary $k$-dimensional vector with $|u_1| < r$ and let $w(s) = f(su_1) - g(su_1), 0 \leq s \leq 1$. Then

$$\frac{dw}{ds} = \left(\frac{\partial f}{\partial u} - \frac{\partial g}{\partial u}\right) u_1,$$

and

$$\frac{d|w|}{ds} \leq \left|\frac{dw}{ds}\right| \leq \left|\frac{\partial f}{\partial u} - \frac{\partial g}{\partial u}\right| |u_1|.$$
From the Lipschitz property one has
\[
\left| \frac{\partial f}{\partial u} \right| \leq L |f(su_1) - g(su_1)| = L |w|,
\]
and consequently
\[
\frac{d|w|}{ds} \leq L |u_1||w|.
\]
After integrating this inequality on the interval \(0 \leq s \leq 1\) we find \(|w(1)| \leq |w(0)| e^{L |u_1|}\). Since \(w(0) = 0\), we have \(w(1) = 0\), or \(f(u_1) = g(u_1)\) and \(D_1(x_0) = D_2(x_0)\).

3. Existence of \(h\). In this section we first prove the existence of a continuous mapping \(h : \mathcal{K} \rightarrow \mathbb{R}^n\) such that the image \(\mathcal{K}^y = h(\mathcal{K})\) is a compact, invariant set for (1.5). In the following sections we shall show that \(\mathcal{K}^y\) is a hyperbolic attractor, and that \(h\) is a homeomorphism.

**Theorem 1.** For each \(\varepsilon > 0\) there exists a \(\delta > 0\) such that if \(Y\) satisfies (1.6) with this \(\delta\), then the system (1.5) has an invariant set \(\mathcal{K}^y\) and there exists a continuous mapping \(h\) of \(\mathcal{K}\) onto \(\mathcal{K}^y\) such that \(h(x) \in N(x) \cap \mathcal{K}^y\) and \(|h(x) - x| \leq \varepsilon\), for all \(x \in \mathcal{K}\). Furthermore, for each leaf \(\mathcal{S} \subset \mathcal{K}\), the restriction \(h|_{\mathcal{S}}\) is a locally Lipschitz continuous mapping of \(\mathcal{S}\) into \(\mathcal{K}^y\). In addition there is an induced continuous flow \(\tilde{S}(t)x\) on \(\mathcal{K}\) such that

\[
(3.1) \quad h(\tilde{S}(t)x) = y(t, h(x)), \quad \text{for all } x \in \mathcal{K} \text{ and } t \in \mathbb{R}.
\]

**Proof.** Define \(\sigma\) by \(11\sigma = \min(\lambda_1, \lambda_1 - \lambda_2)\), where \(a, \lambda_1,\) and \(\lambda_2\) are given by (1.8) and (1.9). Next we note that there exists an \(\alpha, 0 < \alpha \leq \frac{\pi}{2}\), such that the angle \(\angle (\mathcal{U}^s(x_0), \mathcal{U}^n(x_0))\) satisfies

\[
(3.2) \quad \angle (\mathcal{U}^s(x_0), \mathcal{U}^n(x_0)) \geq \alpha, \quad x_0 \in \mathcal{K}.
\]

Let \(T\) be a positive number satisfying both

\[
(3.3) \quad e^{-(\lambda_1 - \sigma)T} \leq \frac{\sin \frac{\alpha}{2}}{6a}, \quad e^{-(\lambda_1 - 3\sigma)T} \leq \frac{3}{16a}.
\]

Next fix \(c, 0 < c \leq \frac{1}{2}\), so that whenever the vector \(\zeta\) is satisfying

\[
(3.4) \quad \angle (\mathcal{U}^n(x_0), \zeta) \leq c\alpha,
\]

then

\[
(3.5) \quad \angle (\mathcal{U}^n(x(t, x_0)), \Phi(t, x_0)\zeta) \leq \frac{\alpha}{4}, \quad 0 \leq t \leq 2T,
\]

and

\[
(3.6) \quad \angle (\mathcal{U}^n(x(t, x_0)), \Phi(t, x_0)\zeta) \leq \frac{1}{4} c\alpha, \quad T \leq t \leq 2T,
\]
where as earlier $\Phi(t, x_0)$ is a fundamental operator solution of the system (1.7).

With $c$ given by (3.4) we fix $r > 0$, where $r$ is the radius of the disk $\mathcal{D}(x)$ in $\mathcal{K}$, so that if $x_0, x_1$ are distinct points in some leaf $\mathcal{S}$ with $|x_0 - x_1| \leq r$, then one has

$$\mathcal{L}(\mathcal{U}^r(x_0), x_0 - x_1) \leq c \alpha.$$

In accordance with Perron's Stable Manifold Theorem for each $x_0 \in \mathcal{K}$ there exists $(n - k)$-dimensional disk $\mathcal{D}^p(x_0)$ (the so called Perron disk) such that, if $x_1 \in \mathcal{D}^p(x_0)$ then

$$|x(t, x_1) - x(t, x_0)| \leq 2a|x_1 - x_0|e^{-(\lambda_1 - \sigma)t}, \quad \text{for } t \geq 0,$$

where $\sigma$ is given above. The radius $b$ of the Perron disk $\mathcal{D}^p(x_0)$ does not depend on $x_0$, and $\mathcal{D}^p(x_0)$ is arbitrarily close to the linear space $\mathcal{U}^s(x_0)$ provided $b$ is sufficiently small. Finally the disk $\mathcal{D}^p(x_0)$ depends continuously on $x_0 \in \mathcal{K}$.

Next we will choose $\beta$, $0 < \beta < \min(r, b)$, so that in $U(x(t, x_0), \beta)$, the $\beta$-neighborhood of the point $x(t, x_0)$, the linear system (1.7) dominates the nonlinearity in (1.4). Recall that if $\beta$ is small enough, then the Perron disks form a foliation of the neighborhood $U(x_0, \beta)$, i.e., one has

$$U(x_0, \beta) \subset \bigcup_{x_1 \in \mathcal{D}(x_0)} \mathcal{D}^p(x_1), \quad \text{for all } x_0 \in \mathcal{K},$$

see Pliss (1964, 1977). For any $x_0 \in \mathcal{K}$ we define

$$\Gamma(x_0, \beta) \overset{\text{def}}{=} \{x + y : x \in \mathcal{D}(x_0), y \in \mathcal{N}(x), |y| \leq \beta\}.$$

Because of the Lipschitz property, it follows from the Tubular Neighborhood Theorem, see REF, that there is a small $\beta > 0$ such that for any $x_0 \in \mathcal{K}$ there is a Lipschitz continuous mapping

$$\phi = \phi_{x_0} : \Gamma(x_0, \beta) \rightarrow \mathcal{D}(x_0), \quad x_0 \in \mathcal{K}$$

that satisfies

$$y \in \mathcal{N}(\phi(y)), \quad y \in \Gamma(x_0, \beta).$$

While the mapping $\phi_{x_0}$ depends on $x_0$, we note that if $x_1 \in \mathcal{D}(x_0)$, then $\phi_{x_0}$ and $\phi_{x_1}$ agree on the set $\Gamma(x_0, \beta) \cap \Gamma(x_1, \beta)$. Also the Lipschitz constant $\tilde{L}$ for $\phi_{x_0}$ is independent of $x_0$.

Let $x_1$ and $y_0$ be such points that $x_1 \in \mathcal{K}$ and $|y_0 - x_1| \leq \beta$. It follows from (3.8) that there is an $x_0 \in \mathcal{D}(x_1)$ such that $y_0 \in \mathcal{D}^p(x_0)$. Since $\mathcal{D}^p(x_0)$ is arbitrarily close to $\mathcal{U}^s(x_0)$, it follows from (3.2) that

$$|x_0 - y_0| \leq \frac{|y_0 - x_1|}{\sin \frac{2\alpha}{3}}.$$
For the remainder of the argument we fix $\beta > 0$ so that (3.8)-(3.11) hold.

We now turn to the nonlinear unperturbed equation (1.4). We will use the continuity of solutions of (1.4) to fix a $\bar{c}, 0 < \bar{c} \leq 1$, so that the following 4 properties hold: First, whenever $x_1 \in \mathcal{K}$ and $y_0$ satisfies $|y_0 - x_1| \leq \bar{c}\beta$, then

$$x(t, y_0) \in \Gamma((x(t, x_0), \beta), \quad 0 \leq t \leq 2T,$$

where $x_0 \in \mathcal{D}(x_1)$ is chosen so that $y_0 \in \mathcal{D}^p(x_0)$. Furthermore, for sufficiently small $\bar{c}$ the linear system (1.7) is hyperbolic along $x(t, y_0)$ on the interval $0 \leq t \leq 2T$ with constants $2a$, $\lambda_1 - \sigma$, and $\lambda_2 + \sigma$. This implies that the mappings $x_0 \rightarrow \mathcal{U}_n(x_0), \mathcal{U}_s(x_0)$ have continuous extensions from $\mathcal{K}$ to $U(\mathcal{K}, \bar{c}\beta)$, the $\bar{c}\beta$-neighborhood of $\mathcal{K}$. As a second condition on $\bar{c}$ we require that

$$\mathcal{L}(\mathcal{U}_n(x_0), \mathcal{U}_n(y_0)) \leq \frac{ca}{8}, \quad x_0 \in \mathcal{K}, y_0 \in \mathcal{N}(x_0), |y_0| \leq \bar{c}\beta.$$

Thirdly, we ask the $\bar{c}$ be chosen so that if $x_0 \in \mathcal{K}$, $|x_0 - x_i| \leq \bar{c}\beta$, $i = 1, 2$, and the angle satisfies $\mathcal{L}(\mathcal{U}_n(x_0), x_1 - x_2) \leq \bar{c}\beta$ then one has

$$\mathcal{L}(\mathcal{U}_n(x(t, x_0)), x(t, x_1) - x(t, x_2)) \leq \frac{\alpha}{3}, \quad 0 \leq t \leq 2T,$$

and

$$\mathcal{L}(\mathcal{U}_n(x(t, x_0)), x(t, x_1) - x(t, x_2)) \leq \frac{1}{2}c\alpha, \quad T \leq t \leq 2T.$$

Let $y_0$ satisfy $|y_0 - x_1| \leq \bar{c}\beta$ for some $x_1 \in \mathcal{K}$, and let $y_0 \in \mathcal{D}^p(x_0)$, where $x_0 \in \mathcal{D}(x_1)$. By extending the mapping $\phi$ given by (3.10), it follows from (3.12) that

$$x(t, y_0) \in \mathcal{N}(\phi(x(t, y_0))), \quad 0 \leq t \leq 2T,$$

and from (3.7) and (3.11) one obtains

$$|x(t, y_0) - \phi(x(t, y_0))| \leq |x(t, x_0) - x(t, y_0)| \leq \frac{2a |y_0 - x_1| e^{-(\lambda_1 - \sigma)t}}{\sin \frac{2\alpha}{3}},$$

for $0 \leq t \leq 2T$. From (3.3) and (3.14) we get

$$|x(t, y_0) - \phi(x(t, y_0))| \leq \frac{|y_0 - x_1|}{3}, \quad T \leq t \leq 2T.$$

Fourthly, we ask that $\bar{c}$ satisfy $\bar{c} \leq 3(\bar{L} + 3)^{-1}$, where $\bar{L}$ is the Lipschitz constant for $\phi$.

With $T, c, r, \beta,$ and $\bar{c}$ so chosen, we let $\epsilon$ be given so that $0 < \epsilon \leq \bar{c}\beta$. We then pick $\delta > 0$ so that the following 4 properties hold for the equations (1.5) and the linear system

$$\frac{dy}{dt} = \frac{\partial (X(y(t, y_0)) + Y(y(t, y_0)))}{\partial y} y,$$
where (1.6) is satisfied with this δ:

I) If \(|y_0 - x| \leq \varepsilon\) for some \(x \in \mathcal{K}\), then

\[
(3.17) \quad |y(t, y_0) - x(t, y_0)| \leq \frac{\varepsilon}{6}, \quad 0 \leq t \leq 2T.
\]

II) If \(x_0 \in \mathcal{K}\) and if \(|y_0 - x_0| \leq \varepsilon\), then

\[
(3.18) \quad |y(t, y_0) - x(t, x_0)| \leq \frac{\beta}{2}, \quad 0 \leq t \leq 2T.
\]

III) Next we consider next the linear system (3.16). Let \(\Psi(t, y_0)\) denote the fundamental operator solution of the system (3.16) that satisfies \(\Psi(0, y_0) = I\). We note that if \(x_0 \in \mathcal{K}\) and \(|x_0 - y_0| \leq \bar{c}\beta\), then for sufficiently small \(\delta\) the linear system (3.16) is hyperbolic along \(y(t, y_0)\) on the interval \(0 \leq t \leq 2T\) with constants \(3a, \lambda_1 - 2\sigma\) and \(\lambda_2 + 2\sigma\). These constants are independent of \(x_0\) and \(y_0\). Let \(\mathcal{U}^a_Y\(y(t, y_0)\)\) and \(\mathcal{U}^s_Y\(y(t, y_0)\)\) denote the associate neutral and stable spaces for \(0 \leq t \leq 2T\). The linear spaces \(\mathcal{U}^a_Y\(y_0)\) and \(\mathcal{U}^s_Y\(y_0)\) can be made to be close to the corresponding linear spaces \(\mathcal{U}^n\(y_0)\) and \(\mathcal{U}^s\(y_0)\) by choosing \(\delta\) sufficiently small. As a further condition on \(\delta\) we require that

\[
(3.19) \quad \angle (\mathcal{U}^n_Y(y_0), \mathcal{U}^n(y_0)) \leq \frac{ca}{8}, \quad y_0 \in U(\mathcal{K}, \bar{c}\beta).
\]

In addition for small \(\delta\), it follows from (3.5) and (3.6) that if the vector \(\eta\) is such that

\[
\angle (\mathcal{U}^n(x_0), \eta) \leq c\alpha,
\]

then

\[
(3.20) \quad \angle (\mathcal{U}^n(x(t, x_0)), \Psi(t, y_0)\eta) \leq \frac{2}{5}c\alpha, \quad 0 \leq t \leq 2T,
\]

and

\[
(3.21) \quad \angle (\mathcal{U}^n(x(t, x_0)), \Psi(t, y_0)\eta) \leq \frac{3}{5}c\alpha, \quad T \leq t \leq 2T.
\]

IV) Let \(x_i, y_i, i = 0, 1\), be points that satisfy \(x_0 \in \mathcal{K}\), \(x_1 \in \mathcal{D}(x_0)\), \(y_i \in \mathcal{N}(x_i), i = 0, 1\), \(|x_1 - x_0| \leq \varepsilon\), \(|y_1 - y_0| \leq \varepsilon\), and \(|y_1 - x_1| \leq \varepsilon\). From (3.20) and (3.21) it follows that if \(\delta\) is sufficiently small and

\[
\angle (\mathcal{U}^u_Y\(y_0), y_1 - y_0) \leq c\alpha,
\]

then

\[
(3.22) \quad \angle (\mathcal{U}^u_Y\(y(t, y_0)), y(t, y_1) - y(t, y_0)) \leq \frac{\alpha}{2}, \quad 0 \leq t \leq 2T,
\]

and

\[
(3.22) \quad \angle (\mathcal{U}^u_Y\(y(t, y_0)), y(t, y_1) - y(t, y_0)) \leq \frac{3}{4}c\alpha, \quad T \leq t \leq 2T.
\]
Now let \( x_1 \in \mathcal{K} \) and \( y_0 \) be chosen so that \( |x_0 - y_1| \leq \epsilon \). From (3.18) we see that
\[
y(t, y_0) \in \Gamma(x_0, \frac{\beta}{2}), \quad 0 \leq t \leq 2T,
\]
and that
\[
y(t, y_0) \in \mathcal{N}(\phi(y(t, y_0))), \quad 0 \leq t \leq 2T,
\]
as well as
\[
|y(t, y_0) - \phi(y(t, y_0))| \leq \frac{\beta}{2}, \quad 0 \leq t \leq 2T.
\]
Using the Lipschitz constant \( \bar{L} \) for \( \phi \), together with (3.15) and (3.17) we obtain
\[
(3.23) \quad |y(t, y_0) - \phi(y(t, y_0))| \leq K\epsilon, \quad T \leq t \leq 2T.
\]
where \( K = \frac{1}{\epsilon}(\bar{L} + 3) \). (Note that \( K \) depends only on \( \mathcal{K} \).)

Let \( \mathcal{E} \) denote the collection of \((w, x)\) in \( \mathcal{K} \times \mathcal{K} \) such that \( w \in \mathcal{K} \) and \( x \in \mathcal{S}(w) \). Let \( Q \) denote the collection of all functions \( f : \mathcal{E} \to \mathbb{R}^n \) that satisfy:

1. \( f \) is continuous.
2. \( f(w, x) \in \mathcal{N}(x) \) for all \((w, x) \in \mathcal{E} \).
3. \( |f(w, x)| \leq K\epsilon \) for all \((w, x) \in \mathcal{E} \).
4. For every \((w_1, x_1) \in \mathcal{E} \) there is an \( \epsilon_1 = \epsilon_1(w, x_1, f), 0 < \epsilon_1 \leq \epsilon \) such that
\[
|f(w, x_1) - f(w, x_2)| \leq \ell|x_1 - x_2|
\]
for all \((w, x_2) \in \mathcal{E} \) with \( |x_1 - x_2| \leq \epsilon_1 \), where \( \ell \leq \tan(\frac{3}{4}\alpha) < 1 \).
5. \( f(w_1, x_1) = f(w_2, x_2) \) whenever \((w_1, x_1), (w_2, x_2) \in \mathcal{E} \) and \( x_1 = x_2 \).

Introduce in \( Q \) the usual metric:
\[
(3.24) \quad d(f_1, f_2) = \sup_{(w, x) \in \mathcal{E}} |f_1 - f_2|.
\]

Since \( Q \) is a closed subset of \( C^0(\mathcal{E}, \mathbb{R}^n) \cap L^\infty(\mathcal{E}, \mathbb{R}^n) \), \( Q \) is complete with this metric.

Property (5) is equivalent to saying that \( f(w, x) \) is independent of \( w \), provided \( w \in \mathcal{S}(x) \). As a result, if \( f \in Q \), then one has
\[
f(w, x) = f(x, x), \quad \text{for all } x \in \mathcal{K}, \ w \in \mathcal{S}(x).
\]

In addition the mapping \( x \to f(x, x) \) is a continuous mapping of \( \mathcal{K} \) into \( \mathbb{R}^n \).

Let \( f \in Q \) and \( w \in \mathcal{K} \) be fixed. For \( x_1 \in \mathcal{S}(w) \), let \( y_1 = x_1 + f(w, x_1) \). For \( 0 \leq \tau \leq 2T \) we define \( \tilde{f}_\tau = A_\tau f \) by \( \tilde{f}_\tau(w, \phi(y(\tau, y_1))) = y(\tau, y_1) - \phi(y(\tau, y_1)) \). Since the mapping \( x_1 \to \phi(y(\tau, y_1)) \) is a Lipschitz continuous mapping which is the identity mapping for
\( \tau = 0 \), it follows from the Inverse Function Theorem that this mapping is open, and therefore it is a mapping of \( S(w) \) onto itself for small \( \tau > 0 \). By iteration, this is a mapping of \( S(w) \) onto itself for each \( \tau > 0 \).

We want to show that \( \tilde{f}_\tau \in Q \) for \( T \leq \tau \leq 2T \). From (3.23) one obtains

\[
|\tilde{f}_\tau(w, x)| \leq K\epsilon, \quad (w, x) \in \mathcal{E}, \quad T \leq \tau \leq 2T.
\]

From the definition of \( \phi \) one has \( f(w, x) \in \mathcal{N}(x) \) for all \( (w, x) \in \mathcal{E} \). Hence \( \tilde{f}_\tau \) satisfies properties (2) and (3).

Since the mappings \( (w, x_1) \rightarrow y_1 \), \( (w, x_1) \rightarrow y(\tau, y_1) \), and \( (w, x_1) \rightarrow \phi(y(\tau, y_1)) \) are continuous mappings of \( \mathcal{E} \) into \( R^n \), it follows that \( \tilde{f}_\tau \) is continuous on \( \mathcal{E} \). (In fact, \( \tilde{f}_\tau(w, \phi(y(\tau, y_1))) \) is jointly continuous in \( (\tau, w, x_1) \).) Since \( f(w, x_1) \) is independent of \( w \) for \( w \in S(x_1) \), it follows that \( y_1, y(\tau, y_1), \phi(y(\tau, y_1)) \), and \( \tilde{f}_\tau(w, \phi(y(\tau, y_1))) \) are also independent of \( w \) for \( w \in S(x_1) \). Hence \( \tilde{f}_\tau \) satisfies properties (1) and (5).

To complete the proof that \( \tilde{f}_\tau \in Q \), we need to verify that \( \tilde{f}_\tau \) satisfies the Lipschitz condition (4). Fix \( w \in \mathcal{K} \) and let \( x_0 \in S(w), \ x_1 \in \mathcal{D}(x_0) \), and \( y_i = x_i + f(w, x_i), \ i = 0, 1 \). Let \( \bar{y}_i = y(\tau, y_i) \) and \( \bar{x}_i = \phi(y(\tau, y_i)), \ i = 0, 1 \). From the definition of \( \tilde{f}_\tau \) it follows that

\[
\bar{y}_i = \bar{x}_i + \tilde{f}_\tau(w, \bar{x}_i), \quad i = 0, 1.
\]

Since the mapping \( x_1 \rightarrow \bar{x}_1 \) is open, there is an \( \epsilon_1(w, \bar{x}_1, \tilde{f}_\tau), 0 < \epsilon_1(w, \bar{x}_1, \tilde{f}_\tau) \leq \epsilon \), such that for each \( \bar{x}_0 \) satisfying

\[
|\bar{x}_1 - \bar{x}_0| \leq \epsilon_1(w, \bar{x}_1, \tilde{f}_\tau),
\]

there is an \( x_0 \in \mathcal{D}(x_1) \) satisfying \( |x_1 - x_0| \leq \epsilon_1(w, x_1, f) \leq \epsilon \) and such that \( \bar{x}_0 = \phi(y(\tau, y_0)) \).

In order to estimate the difference \( |\tilde{f}_\tau(w, x_1) - \tilde{f}_\tau(w, x_0)| \), we note that

\[
\bar{y}_1 - \bar{y}_0 = (\bar{x}_0 - \bar{y}_0) + (\bar{x}_1 - \bar{x}_0) + (\bar{y}_1 - \bar{x}_1).
\]

Let \( P_n \) be the orthogonal projection onto the space \( \mathcal{U}^n(\bar{x}_0) \). Since \( \bar{y}_0 \in \mathcal{N}(\bar{x}_0) \) one has \( P_n(\bar{x}_0 - \bar{y}_0) = 0 \). Consequently, it follows from (3.27) that

\[
P_n(\bar{y}_1 - \bar{y}_0) = P_n(\bar{x}_1 - \bar{x}_0) + P_n(\bar{y}_1 - \bar{x}_1).
\]

Since \( \mathcal{K} \) is a hyperbolic attractor, \( \mathcal{U}^n(\bar{x}_0) \) is the tangent space of \( \mathcal{D}(\bar{x}_0) \) at the point \( \bar{x}_0 \), and since the angle \( \angle(\mathcal{U}^n(\bar{x}_0), \bar{x}_1 - \bar{x}_0) \) is small, we have

\[
|P_n(\bar{x}_1 - \bar{x}_0)| = |\bar{x}_1 - \bar{x}_0| + O(|\bar{x}_1 - \bar{x}_0|^2).
\]

Likewise we find

\[
|P_n(\bar{y}_1 - \bar{x}_1)| = O(\epsilon|\bar{x}_1 - \bar{x}_0|).\]
From (3.26), (3.28), and (3.29) we obtain
\[ |P_n(\bar{y}_1 - \bar{y}_0)| = |\bar{x}_1 - \bar{x}_0| + O(\epsilon|\bar{x}_1 - \bar{x}_0|). \]

Next let \( P_N \) be the orthogonal projection onto the space \( N(\bar{x}_0) \). From (3.27) we find
\[ P_N(\bar{y}_1 - \bar{y}_0) = P_N(\bar{x}_0 - \bar{y}_0) + P_N(\bar{x}_1 - \bar{x}_0) + P_N(\bar{y}_1 - \bar{x}_1). \]

It is clear that
\[ |P_N(\bar{x}_1 - \bar{x}_0)| = O(|\bar{x}_1 - \bar{x}_0|^2). \]

By definition \( \bar{f}_r(w, \bar{x}_0) = \bar{y}_0 - \bar{x}_0 \) and \( \bar{f}_r(w, \bar{x}_0) \in N(\bar{x}_0) \). Therefore
\[
\begin{align*}
\left\{ \begin{array}{l}
P_N(\bar{x}_0 - \bar{y}_0) = -\bar{f}_r(w, \bar{x}_0) \\
P_N(\bar{y}_1 - \bar{x}_1) = P_N\bar{f}_r(w, \bar{x}_1) = \bar{f}_r(w, \bar{x}_1) + O(\epsilon|\bar{x}_1 - \bar{x}_0|) \\
P_N(\bar{y}_1 - \bar{y}_0) = |\bar{f}_r(w, \bar{x}_1) - \bar{f}_r(w, \bar{x}_0)| + O(\epsilon|\bar{x}_1 - \bar{x}_0|).
\end{array} \right.
\]

From (3.26), (3.30), and (3.31) one obtains
\[ |P_N(\bar{y}_1 - \bar{y}_0)| = |\bar{f}_r(w, \bar{x}_1) - \bar{f}_r(w, \bar{x}_0)| + O(\epsilon|\bar{x}_1 - \bar{x}_0|). \]

However
\[ \frac{|P_N(\bar{y}_1 - \bar{y}_0)|}{|P_N(\bar{y}_1 - \bar{y}_0)|} = \tan[\zeta(u^n(\bar{x}_0), \bar{y}_1 - \bar{y}_0)]. \]

Since \( f \in Q \), it follows that
\[ \zeta(u^n(x_1), y_1 - y_0) \leq \frac{3}{4}c\alpha. \]

From (3.13), (3.19), and (3.32) we obtain
\[ \zeta(u^n(y_1), y_1 - y_0) \leq c\alpha. \]

From (3.13), (3.19), and (3.32), we see that
\[ \zeta(u^n(x_0), y_1 - y_0) \leq \frac{3}{4}c\alpha, \quad T \leq \tau \leq 2T. \]

Consequently, we have
\[ |\bar{f}_r(w, \bar{x}_1) - \bar{f}_r(w, \bar{x}_0)| \leq \tan\left(\frac{3}{4}c\alpha\right)|\bar{x}_1 - \bar{x}_0|, \quad T \leq \tau \leq 2T, \]

which completes the proof that \( A_T Q \subset Q \), for \( T \leq \tau \leq 2T \).
Since for every \( y_0 \in U(\mathcal{K}, \tilde{c}\beta) \), the linear system (3.16) is hyperbolic along \( y(t, y_0) \) on the interval \( 0 \leq t \leq 2T \) with constants \( 3\alpha, \lambda_1 - 2\sigma, \) and \( \lambda_2 + 2\sigma, \) a straight-forward adaptation of the arguments presented in Perron (1928, 1930) can be used to prove the following results: For every \( f \in Q \) and for each \( y_0 = x_0 + f(w, x_0), \) where \( x_0 \in S(w) \), there exists an \((n - k)\)-dimensional Perron disk \( D_{Y,f}(y_0) \) such that if \( y_1 \in D_{Y,f}(y_0) \) then one has

\[
|y(t, y_1) - y(t, y_0)| \leq 4\alpha|y_1 - y_0|e^{-(\lambda_1 - 3\sigma)t}, \quad 0 \leq t \leq 2T.
\]

We now repeat the argumentation leading to (3.14), but now we use it for the surface \( y = x + f_1(w, x) \) over a given leaf \( S(w) \) in \( \mathcal{K} \). In particular, for each \( y_1 = x_1 + f_1(w, x_1) \), \( x_1 \in S(w) \subset \mathcal{K} \), on the surface \( y = x + f_1(w, x) \), we use the fact that the Perron disks \( D_{Y,f_1}(y) \) form a foliation of a neighborhood \( U(y_1, \frac{\beta}{2}) \) of \( y_1 \), and we construct a continuous mapping \( \phi_{f_1} = \phi_{f_1, y_1} \) of \( \Gamma_{f_1}(y_1, \frac{\beta}{2}) \) onto \( f_1(D(x_1)) \). As argued for (3.14), one then obtains

(3.33) \[
|y(t, y_2) - \phi_{f_1}(y(t, y_2))| \leq |y(t, y_1) - y(t, y_2)| \leq 4\alpha|y_1 - y_2|e^{-(\lambda_1 - 3\sigma)t},
\]

for \( 0 \leq t \leq 2T \), provided \( y_2 \in D_{Y,f_1}(y_1) \). In particular for \( f_1, f_2 \in Q \) and \( y_2 = f_2(w, x_1) \) by using (3.3) and (3.33), we obtain

\[
|y(t, y_2) - \phi_{f_1}(y(t, y_2))| \leq \frac{3}{4}|y_1 - y_2| = \frac{3}{4}|f_1(w, x_1) - f_2(w, x_1)| \leq \frac{3}{4}d(f_1, f_2),
\]

provided that \( T \leq t \leq 2T \). It follows then that

\[
d(A_{\tau}f_1, A_{\tau}f_2) \leq \frac{3}{4}d(f_1, f_2), \quad f_1, f_2 \in Q \quad T \leq \tau \leq 2T.
\]

Thus the operator \( A_\tau \) is contractive for \( \tau \in [T, 2T] \), and therefore it has a fixed point \( g_\tau \in Q \), i.e.,

\[
A_\tau g_\tau = g_\tau, \quad T \leq \tau \leq 2T,
\]

and set \( g \overset{\text{def}}{=} g_T \).

Since \( A_\tau \) is continuous in \( \tau \), it follows that \( g_\tau \) is continuous in \( \tau \) for \( T \leq \tau \leq 2T \). Now for any rational number \( \rho = \frac{p}{q} \in [0, 1] \) one has \( A_{\tau+\rho T} = A_{\tau T} \). Hence \( g = gT+\rho T \) for all rational \( \rho \in [0, 1] \). By continuity, \( g = g_\tau \), for \( T \leq \tau \leq 2T \). For \( 0 < \tau < T \) one has \( A_\tau g = A_\tau A_T g = A_{\tau T} g = g \). Hence \( A_\tau g = g \) for all \( \tau > 0 \).

Set \( h(x) \overset{\text{def}}{=} x + g(x, x), S^Y = h(S), \) and \( \mathcal{K}^Y = h(\mathcal{K}) \). Since \( A_\tau g = g \) for all \( \tau > 0 \), it follows that the sets \( S^Y \) and \( \mathcal{K}^Y \) are invariant sets of the system (1.5). Furthermore (3.25) implies that \( |h(x) - x| \leq K \varepsilon \) for all \( x \in \mathcal{K} \). By rescaling \( \varepsilon \) we have \( |h(x) - x| \leq \varepsilon \) for all \( x \in \mathcal{K} \). Since the mapping \( x \rightarrow g(x, x) \) is continuous on \( \mathcal{K} \), it follows that \( h \) is continuous on \( \mathcal{K} \). Moreover, the restriction \( h|_S \) is Lipschitz continuous on the leaf \( S \). Also \( h(x) \in \mathcal{N}(x) \cap \mathcal{K}^Y \) for all \( x \in \mathcal{K} \). Note that for any \( x \in \mathcal{K} \) one has

(3.34) \[
h(\phi(y(t, h(x)))) = y(t, h(x)), \quad t \in R.
\]

It is easily seen that the mapping \( (x, t) \rightarrow \hat{S}(t)x \overset{\text{def}}{=} \phi(y(t, h(x))) \) is a continuous flow on \( \mathcal{K} \), i.e., \( \hat{S}(t)x \) is continuous in each variable, \( \hat{S}(0)x = x \) for all \( x \in \mathcal{K} \), and the group property \( \hat{S}T\hat{S}(t)x = \hat{S}(\tau + t)x \) is valid. Finally (3.1) is simply a reformulation of (3.34). \[\square\]
REMARK. It is easily seen that the function \( w(t) = \phi(y(t, h(x))) \) is a solution of the differential equation
\[
\frac{dw}{dt} = \nabla \phi(h(w)) [X(h(w)) + Y(h(w))]
\]
satisfying \( w(0) = x \), where \( x \in \mathcal{K} \).

4. Hyperbolicity of \( \mathcal{K}^Y \). Let \( S, S^Y, \mathcal{K}^Y \), and \( h \) have the same sense as in Theorem 1. Let \( y_0 \in S^Y, x_0 \in S \) and \( h(x_0) = y_0 \). Let \( w = \phi(y(t, h(x_0))) \) be given by (3.34), and let \( \Psi(t, y_0) \) be the fundamental operator solution of the linear system (3.16).

THEOREM 2. For each \( \sigma > 0 \) there exists a \( \delta > 0 \) such that if (1.6) is satisfied with this \( \delta \), then the system (3.16) is hyperbolic along \( y(t, y_0) \) on \( R \) with constants \( (a + \sigma), (\lambda_1 - \sigma), \) and \( (\lambda_2 + \sigma) \) for every \( y_0 \in \mathcal{K}^Y \). Let \( \mathcal{U}^n_Y(y(t, y_0)) \) and \( \mathcal{U}^s_Y(y(t, y_0)) \) denote the associated neutral and stable linear spaces for (3.16). Then \( \mathcal{U}^n_Y(y(t, y_0)) \) and \( \mathcal{U}^s_Y(y(t, y_0)) \) are arbitrarily close to the spaces \( \mathcal{U}^n(\phi(y(t, y_0))) \) and \( \mathcal{U}^s(\phi(y(t, y_0))) \), respectively, for all \( t \in R \) provided \( \delta \) is sufficiently small.

The proof of this theorem is the same as the proof in Pliss (1977; Theorem 1.3, Chap. 4, p. 257) so we will omit the details.

THEOREM 3. Let \( y_0 = h(x_0) \) for \( x_0 \in \mathcal{K} \). The space \( \mathcal{U}^n_Y(y_0) \) is tangent to the set \( h(\mathcal{D}(x_0)) \) at the point \( y_0 \), provided \( \epsilon \) and \( \delta \) are sufficiently small.

Proof. Assume on the contrary that there exists a sequence of points \( x_i \in \mathcal{D}(x_0), i = 1, 2, \ldots \) such that \( x_i \to x_0 \), as \( i \to \infty \), and for each \( i \)

(4.1) \[
\angle (\mathcal{U}^n_Y(y_0), h(x_i) - y_0) \geq \gamma_1
\]

for some \( \gamma_1 > 0 \). Let \( y_i = h(x_i) \). From (1.8) and (1.9) (as applied to \( \Psi(t, y_0) \)) and (4.1) it follows that there exists a \( T < 0 \) such that

(4.2) \[
\angle (\mathcal{U}^s_Y(y(T, y_0)), \Psi(T, y_0)y_i) \leq 0.1\alpha,
\]

where \( \alpha \) is given by (3.2). From (4.2) and Theorem 2 it follows that for small \( \delta \) one has

(4.3) \[
\angle (\mathcal{U}^s(\phi(y(T, y_0))), \Psi(T, y_0)y_i) \leq 0.2\alpha.
\]

Since \( h \) is continuous on \( \mathcal{D}(x_0) \) one has \( y_i \to y_0 \), as \( i \to \infty \). Therefore there exists an \( i \), say \( i = j \), such that

\[
|y(t, y_0) - y(t, y_j)| \leq \epsilon
\]

and

\[
|\phi(y(t, y_0)) - \phi(y(t, y_j))| \leq \epsilon
\]

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for all $t \in [T, 0]$.

From (4.3) it follows that for small $\delta$ one has

$$(4.4) \quad \mathcal{L}(\mathcal{U}(\phi(y(T, y_0))), y(T, y_j) - y(T, y_0)) \leq 0.3\alpha.$$ 

From inequalities (3.2) and (4.4) we have

$$(4.5) \quad \mathcal{L}(\mathcal{U}(\phi(y(T, y_0))), y(T, y_j) - y(T, y_0)) \geq 0.7\alpha.$$ 

Let $\bar{y}_i = y(T, y_t)$ and $\bar{x}_i = \phi(y(T, y_t))$, $i = 0, j$. We have

$$(4.6) \quad (\bar{y}_j - \bar{y}_0) = (\bar{x}_0 - \bar{y}_0) + (\bar{x}_j - \bar{x}_0) + (\bar{y}_j - \bar{x}_j).$$

As before let $P_n$ and $P_M$ be orthogonal projections onto the spaces $\mathcal{U}(\bar{x}_0)$ and $\mathcal{N}(\bar{x}_0)$, respectively. Since $P_n(\bar{x}_0 - \bar{y}_0) = 0$, it follows from (4.6) that

$$(4.7) \quad P_n(\bar{y}_j - \bar{y}_0) = P_n(\bar{x}_j - \bar{x}_0) + P_n(\bar{y}_j - \bar{x}_j).$$

Since $\mathcal{U}(x_0)$ is the tangent space of $\mathcal{D}(x_0)$ at the point $x_0$, one has

$$(4.8) \quad P_n(\bar{x}_j - \bar{x}_0) = |\bar{x}_j - \bar{x}_0| + O(|\bar{x}_j - \bar{x}_0|^2).$$

Likewise one has

$$|P_n(\bar{y}_j - \bar{x}_j)| = |g(\bar{x}_j, \bar{x}_j)| \cdot O(|\bar{x}_j - \bar{x}_0|),$$

where as before $g(x, x) = h(x) - x$. However $|g(x, x)| \leq \epsilon$, and therefore

$$(4.9) \quad |P_n(\bar{y}_j - \bar{x}_j)| = O(\epsilon|\bar{x}_j - \bar{x}_0|).$$

From (4.7), (4.8), and (4.9) we have the estimate

$$(4.10) \quad |P_n(\bar{y}_j - \bar{y}_0)| \leq |\bar{x}_j - \bar{x}_0| + O(\epsilon|\bar{x}_j - \bar{x}_0|).$$

Similarly from (4.6) we have

$$(4.11) \quad P_M(\bar{y}_j - \bar{y}_0) = P_M(\bar{x}_0 - \bar{y}_0) + P_M(\bar{x}_j - \bar{x}_0) + P_M(\bar{y}_j - \bar{x}_j).$$

It is clear that

$$(4.12) \quad |P_M(\bar{x}_j - \bar{x}_0)| = O(|\bar{x}_j - \bar{x}_0|^2).$$

Since $g(\bar{x}_i, \bar{x}_i) = \bar{y}_i - \bar{x}_i \in \mathcal{N}(\bar{x}_i)$, $i = 0, j$, we have

$$(4.13) \quad P_M(\bar{x}_0 - \bar{y}_0) = -g(\bar{x}_0, \bar{x}_0),$$

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and

\begin{equation}
(4.14) \quad P_N(\bar{y}_j - \bar{x}_j) = g(\bar{x}_j, \bar{x}_j) + O(\epsilon|\bar{x}_j - \bar{x}_0|)
\end{equation}

From (4.11)-(4.14) we obtain

\begin{equation}
(4.15) \quad |P_N(\bar{y}_j - \bar{y}_0)| = |g(\bar{x}_j, \bar{x}_j) - g(\bar{x}_0, \bar{x}_0)| + O(\epsilon|\bar{x}_j - \bar{x}_0|).
\end{equation}

From (4.10) and (4.15) we obtain

\[
\tan \left[ \varangle \left( U^n(\bar{x}_0), \bar{y}_j - \bar{y}_0 \right) \right] = \frac{|P_N(\bar{y}_j - \bar{y}_0)|}{|P_n(\bar{y}_j - \bar{y}_0)|} = \frac{|g(\bar{x}_j, \bar{x}_j) - g(\bar{x}_0, \bar{x}_0)|}{|\bar{x}_j - \bar{x}_0|} + O(\epsilon)
\]

From Section 3 we obtain

\[
\tan \left[ \varangle \left( U^n(\bar{x}_0), \bar{y}_j - \bar{y}_0 \right) \right] \leq \tan \left( \frac{3}{4} c \alpha \right) + O(\epsilon),
\]

which contradicts (4.5) for small \( \epsilon \), since \( c \leq \frac{1}{2} \). \( \square \)

5. One-to-One Property. We conclude the proof by showing that for \( \epsilon \) sufficiently small, \( h \) is a homeomorphism.

Theorem 4. Let the Hypotheses of Theorem 1 be satisfied. Then for sufficiently small \( \epsilon \) the mapping \( h : \mathcal{K} \to \mathcal{K}^Y \) is one to one.

Proof. As in Section 3, we fix \( \beta > 0 \) such that in the \( \beta \)-neighborhood \( U(x(t, x_0), \beta) \) of the point \( x(t, x_0) \) the linear system (1.7) dominates the nonlinear system (1.4). In particular, these exists a \( T \) (which does not depend on \( x_0 \)) such that \( T < 0 \) and if points \( x_0 \) and \( x_1 \) satisfy \( \varangle (N(x_0), x_1 - x_0) \leq \frac{\pi}{4} \) and \( |x(t, x_1) - x(t, x_0)| \leq \beta \), for \( T \leq t \leq 0 \), then

\begin{equation}
(5.1) \quad |x(T, x_1) - x(T, x_0)| \geq \frac{12}{\sin \alpha} |x_1 - x_0|
\end{equation}

and

\begin{equation}
(5.2) \quad \varangle (U^n(x(T, x_0)), x(T, x_1) - x(T, x_0)) \geq \frac{3\alpha}{4},
\end{equation}

where \( \alpha \) is given by inequality (3.2).

Choose the radius \( r \) of the disks \( D(x_0) \) given in the definition of a hyperbolic attractor small enough so that the following two inequalities hold:

\begin{equation}
(5.3) \quad r \leq \min \left\{ \frac{\ln 2}{L}, \frac{1}{2L} \sin \frac{\alpha}{2}, \frac{\beta}{2} \right\},
\end{equation}

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where \( L \) is the Lipschitz constant of the mapping \( x_0 \to U^n(x_0) \), and

\[
L(U^n(x_0), U^n(x_1)) \leq \alpha \frac{x_0 \in \mathcal{K}, \ x_1 \in \mathcal{D}(x_0)}{4},
\]

It follows from the continuity of solutions that there exists a \( \mu \in (0, \frac{1}{3}) \) such that

\[
U(x(t,x_0), \mu r) \subset x(t, U(x_0, r)), \quad \text{for } T \leq t \leq 0, \text{ and all } x_0 \in \mathcal{K}.
\]

Take the numbers \( \epsilon \) and \( \delta \) so small, that the following condition is satisfied: If (1.6) is satisfied with this \( \delta \), \( |y_0 - x_0| \leq 2\epsilon \), and \( x_0 \in \mathcal{K} \), then

\[
|x(t, x_0) - y(t, y_0)| \leq \frac{\mu r}{3}, \quad \text{for } T \leq t \leq 0.
\]

Assume, in contradiction of the statement of the theorem, that there exist two points \( x_0 \in \mathcal{K} \) and \( x_1 \in \mathcal{K} \) such that \( x_0 \neq x_1 \) and \( h(x_0) = h(x_1) = y_0 \). Since \( |y_0 - x_0| \leq \epsilon \) and \( |y_0 - x_1| \leq \epsilon \), one has \( |x_0 - x_1| \leq 2\epsilon \). We will use this and the fact that \( \epsilon \) is small.

Our first objective is to show that the angles

\[
\rho \overset{\text{def}}{=} \angle(N(x_0), x_1 - x_0), \quad \gamma \overset{\text{def}}{=} \angle(y_0 - x_0, y_0 - x_1)
\]

are small. From the definition of \( h \) it follows that \( (y_i - x_i) \in N(x_i), \ i = 0, 1 \). Consequently, it follows from the Lipschitz property one has \( \tan \gamma \leq L|x_1 - x_0|. \) There exist now two cases:

1. \( |y_0 - x_i| \geq \max(|y_0 - x_j|, |x_1 - x_0|) \), where \( i \neq j \) and \( i, j \in \{0, 1\} \).

2. \( |x_1 - x_0| \geq \max(|y_0 - x_1|, |y_0 - x_0|). \)

In case (1) there is no loss in generality in assuming \( i = 0 \) and \( j = 1 \). One then has \( \gamma \geq \sin \gamma \geq \frac{|x_1 - x_0| \sin \rho}{2|x_1 - y_0|} \). Therefore \( \frac{|x_1 - x_0| \sin \rho}{2|x_1 - y_0|} \leq \arctan(L|x_1 - x_0|) \). For small \( \epsilon \), the last inequality becomes \( \sin \rho \leq 2L\epsilon + O(\epsilon^2) \). In case (2) one has \( \rho \leq \gamma \leq \arctan 2L\epsilon \). In either case, one can insure that both \( \rho \) and \( \gamma \) are small by choosing \( \epsilon \) small. In what follows we will assume that \( \epsilon \) is so small that

\[
L(N(x_0), x_1 - x_0) < \frac{\pi}{4}.
\]

For \( x \in \mathcal{K} \) let \( d(x) \) be the \( k \)-dimensional disk with the center at \( x \), radius \( \mu r \) and such that \( d(x) \subset \mathcal{D}(x) \). Using the facts that the mapping \( h \) is continuous, and \( \phi(y(t, h(x_i))) \in x(t, \mathcal{D}(x_i)), \) for \( i = 0, 1 \) and \( T \leq t \leq 0 \), together with (5.5), for small \( \epsilon \) one obtains

\[
\phi(y(t, h(x_i))) \in d(x(t, x_i)) \quad i = 0, 1, \quad T \leq t \leq 0.
\]
Let \( \tilde{x}_i = x(T, x_i) \) and \( x^{(1)}_i = \phi(y(T, h(x_i))) \), \( i = 0, 1 \). From (5.6) it follows that 
\( x^{(1)}_i \in d(\tilde{x}_i) \), therefore \( \tilde{x}_i \in d(x^{(1)}_i) \subset D(x^{(1)}_i) \), \( i = 0, 1 \). Let us transform the origin into the point \( x^{(1)}_0 \) and rotate the coordinate axis to obtain the new coordinate system \((u, v)\), where \( u \) is \( k \)-dimensional vector and \( v \) is an \((n - k)\)-dimensional vector. We will assume that in the new coordinate system the space \( v = 0 \) coincides with the linear space \( \mathcal{U}^n(x^{(1)}_0) \).

In these coordinates the disks \( D(x^{(1)}_0) \) and \( D(x^{(1)}_i) \) can be represented in the form

\[
D(x^{(1)}_i) = \{v = f_i(u) : |u| \leq r\}, \quad i = 0, 1,
\]

where \( f_i \) are continuously differentiable \((n - k)\)-dimensional vector-valued functions, \( i = 0, 1 \), and \( f_0(0) = 0 \). In this coordinate system the points \( \tilde{x}_i \) have representations \((u_i, f_i(u_i))\), \( i = 0, 1 \). Let \( \tilde{x}_2 = (u_1, f_0(u_1)) \). Consider vectors \( z_0 = \tilde{x}_0 - \tilde{x}_2, \ z_1 = \tilde{x}_1 - \tilde{x}_0, \ z_2 = \tilde{x}_1 - \tilde{x}_2 \). It is clear that \( z_2 = z_0 + z_1 \). Let \( P \) be the projector onto the space \( ((0, v)) \). We then have \( Pz_2 = Pz_0 + Pz_1 \) and

\[
(Pz_2) \geq |Pz_1| - |Pz_0|.
\]

From (5.2) it follows that \( L(\mathcal{U}^n(\tilde{x}_0), z_1) \geq \frac{3\alpha}{4} \). By using the additivity of the angles and (5.4) we find \( L(\mathcal{U}^n(x^{(1)}_0), z_1) \geq \frac{\alpha}{2} \). From this inequality it follows that

\[
|Pz_1| \geq |z_1| \sin \frac{\alpha}{2}.
\]

Let us estimate \( |Pz_0| \). Since \( Pz_0 = (0, f_0(u_0) - f_0(u_1)) \) we get

\[
|Pz_0| = |f_0(u_0) - f_0(u_1)|.
\]

Now

\[
|Pz_0| = |f_0(u_0) - f_0(u_1)| \leq \max_{|u| \leq r} \left| \frac{\partial f_0}{\partial u} \right| |u_1 - u_0| \leq Lr |z_1|,
\]

since the Lipschitz property implies that \( \left| \frac{\partial f_0}{\partial u} \right| \leq Lr \) for \( |u| \leq r \). From this inequality, (5.7) and (5.8) it follows that \( |Pz_2| \geq (\sin \frac{\alpha}{2} - Lr) |z_1| \), and (5.3) implies that

\[
|Pz_2| \geq \frac{1}{2} \sin \frac{\alpha}{2} |z_1|,
\]

which can be rewritten in the form

\[
|f_1(u_1) - f_0(u_1)| \geq \frac{1}{2} \sin \frac{\alpha}{2} |x(T, x_1) - x(T, x_0)|.
\]

Using (5.1) we then find

\[
|f_1(u_1) - f_0(u_1)| \geq 6|x_1 - x_0|.
\]
Let \( w(s) = f_1((1 - s)u_1) - f_0((1 - s)u_1), \) \( 0 \leq s \leq 1. \) Then

\[
\frac{dw}{ds} = \left( \frac{\partial f_0}{\partial u} - \frac{\partial f_1}{\partial u} \right) u_1,
\]

which, together with the Lipschitz property, implies that

\[
\left| \frac{dw}{ds} \right| \leq \left| \frac{\partial f_0}{\partial u} - \frac{\partial f_1}{\partial u} \right| \left| u_1 \right| \leq L \left| w(s) \right| \left| u_1 \right| \leq L r \left| w(s) \right|,
\]

since \( \left| u_1 \right| \leq r. \) By using (5.3) we have

\[
\frac{d|w|}{ds} \geq - \ln 2 |w|.
\]

From the Gronwall inequality we obtain \( 2 |w(1)| \geq |w(0)|, \) or equivalently,

\[
|f_1(0) - f_0(0)| \geq \frac{1}{2} |f_1(u_1) - f_0(u_1)|.
\]

Combining this with (5.9) and using the fact that \( f_0(0) = 0, \) we obtain

(5.10)

\[
|f_1(0)| \geq 3 |x_1 - x_0|.
\]

In the \((u, v)\) coordinates \( x^{(1)}_0 = (0, 0), \) and for some \( u_2 \) with \( |u_2| \leq \frac{r}{3}, \) one has \( x^{(1)}_1 = (u_2, f_1(u_2)). \) Therefore

(5.11)

\[
|x^{(1)}_1 - x^{(1)}_0|^2 = |u_2|^2 + |f_1(u_2)|^2.
\]

Since \( |u_2| \leq \frac{r}{3}, \) we have the inequality

\[
|f_1(u_2)| \geq - \max_{|u| \leq |u_2|} \left| \frac{\partial f_1(u)}{\partial u} \right| |u_2| + |f_1(0)| \geq |f_1(0)| - \frac{L r}{3} |u_2|.
\]

As a result, (5.11) implies that

\[
|x^{(1)}_1 - x^{(1)}_0|^2 \geq |u_2|^2 + \left( |f_1(0)| - \frac{L r}{3} |u_2| \right)^2.
\]

By expanding and using the Young inequality we get

\[
|x^{(1)}_1 - x^{(1)}_0|^2 \geq \left( 1 - \frac{4}{45} L^2 r^2 \right) |u_2|^2 + \frac{4}{9} |f_1(0)|^2,
\]
which together with (5.3) and (5.10) implies that

\begin{equation}
(5.12) \quad |x_1^{(1)} - x_0^{(1)}| \geq \frac{2}{3} |f_1(0)| \geq 2|x_1 - x_0|.
\end{equation}

Let \( y_0^{(1)} = y(T, y_0) \). Since \( h(x_0) = h(x_1) = y_0 \), it follows from (3.1) and (3.34) that \( h(x_0^{(1)}) = h(x_1^{(1)}) = y_0^{(1)} \). Since \( |h(x) - x| \leq \epsilon \), we have, \( |x_0^{(1)} - y_0^{(1)}| \leq \epsilon \) and \( |x_1^{(1)} - y_0^{(1)}| \leq \epsilon \).

Using the same considerations for the points \( x_0^{(1)}, x_1^{(1)} \) and \( y_0^{(1)} \) we obtain points \( x_0^{(2)}, x_1^{(2)} \) and \( y_0^{(2)} \) with the same properties, but (5.12) yields

\[ |x_1^{(2)} - x_0^{(2)}| \geq 2|x_1^{(1)} - x_0^{(1)}| \geq 2^2|x_1 - x_0|. \]

Continuing this process we will obtain a sequence of points \( x_0^{(m)}, x_1^{(m)}, y_0^{(m)} \) lying in a bounded set with the properties \( |x_0^{(m)} - y_0^{(m)}| \leq \epsilon, |x_1^{(m)} - y_0^{(m)}| \leq \epsilon \), and

\[ |x_1^{(m)} - x_0^{(m)}| \geq 2^m|x_1 - x_0|, \]

which is impossible for \( x_1 \neq x_0 \). The contradiction proves the theorem. \( \Box \)

**Remark.** Since \( h(x) \in \mathcal{N}(x) \cap S^Y \) for \( x \in S \) and \( |h(x) - x| \leq \beta \), the inverse \( h^{-1} \) is given by \( h^{-1}(y) = \phi(y) \) for any \( y \in S^Y \subset K^Y \).

6. **Examples.** A stable fixed point and a stable periodic orbit are hyperbolic attractors that satisfy the Lipschitz property. More generally, a smooth, stable, normally hyperbolic, invariant manifold is a hyperbolic attractor that satisfies the Lipschitz property. The fact that a normally hyperbolic, invariant manifold persists under a small \( C^1 \)-perturbation is, of course, well-known. See for example, Fenichel (1971), Hirsch, Pugh, and Shub (1977), and Sacker (1969). In this section we shall give several examples of differential equations with hyperbolic attractors that satisfy the Lipschitz property. These examples include:

1. a set \( K \) which is not a manifold, and
2. a set \( K \) which is an invariant manifold, but which is not normally hyperbolic.

Before doing this though, we note that the theory described above extends readily to the study of diffeomorphism, or Poincaré mappings. Since a small perturbation of a diffeomorphism preserves a time synchronization, the theory for diffeomorphisms is somewhat simpler than that of differential equations. For example, in the case of perturbations of diffeomorphisms the induced flow \( \hat{S}(t)x_0 \) on \( K \), as described at the end of Section 3, agrees with the original flow \( x(t, x_0) \).

6.1 **Attractors for Diffeomorphisms.** Let \( A \) be a \( 2 \times 2 \) matrix with integer entries, \( \det A = 1 \), and distinct real eigenvalues. Denote the eigenvalues by \( \mu_1, \mu_2 \). One then has \( \mu_1 \mu_2 = 1 \). We will assume that \( 0 < \mu_2 < 1 < \mu_1 \). For example, let

\[ A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}. \]
The linear mapping \( u \rightarrow Au \) preserves the integer lattice \( \mathbb{Z}^2 \) in \( \mathbb{R}^2 \), and therefore this mapping induces a diffeomorphism \( u \rightarrow F(u) \) on the 2-dimensional torus \( T^2 \). This diffeomorphism is referred to as an Anosov diffeomorphism, see Arnold (1983), and Smale (1967).

Fix \( \mu, 0 < \mu < 1 \). On the product space \( T^2 \times (-1, 1) \), we introduce the diffeomorphism

\[
(\mathbf{u}, v) \rightarrow (F(\mathbf{u}), \mu v).
\]

Then \( M = T^2 \times \{0\} \) is a hyperbolic attractor for (6.1) that satisfies the Lipschitz property. For each \((u, 0) \in M\), the leaf \( \mathcal{S}(u,0) \) is the unstable manifold through \((u,0)\), and \( \dim \mathcal{S}(u,0) = 1 \). The fact that \( M \) persists under small \( C^1 \)-perturbations is a consequence of the classical theory of structural stability, see Smale (1967).

6.2 Suspensions of Diffeomorphisms. By using the suspension method any diffeomorphism can be imbedded into a \( C^1 \)-flow, see Smale (1967). For example, the suspension of the diffeomorphism \( u \rightarrow F(u) \) on the torus \( T^2 \) described above gives rise to a \( C^1 \)-flow \( \pi(t,x) \) on a 3-dimensional manifold \( M^3 \). This is an example of an Anosov flow, see Arnold (1983). From the Whitney Imbedding Theorem, \( M^3 \) can be imbedded smoothly into the Euclidean space \( \mathbb{R}^7 \), and the image of \( M^3 \) has a collared neighborhood \( U \), which can be described in local coordinates as \((x,z)\), where \( x \in M^3 \) and \( z \) belongs to a 4-dimensional disk \( \mathcal{D}(x) \). Let \( \nu \) be a given positive number. By introducing the flow

\[
(x,z) \rightarrow (\pi(t,x), ze^{-\nu t}), \quad t \in \mathbb{R},
\]

in this neighborhood, we see that \( M^3 \) becomes an attractor for an ordinary differential equation \( X' = G(X) \) on \( U \). Furthermore, for all \( \nu > 0 \), \( M^3 \) is a hyperbolic attractor that satisfies the Lipschitz property, and the associated disks \( \mathcal{D}(x,0) \) are two dimensional. However, \( M^3 \) is a normally hyperbolic manifold for \( X' = G(X) \) if and only if \( \nu > \ln \mu_1 > 0 \), see Hirsch, Pugh and Shub (1977), for example.

Figure 1. The Looking-Glass Attractor

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6.3 The Looking-Glass Attractor. The next example is a hyperbolic attractor \( \mathcal{K} \) for a \( C^\infty \)-flow in the plane \( \mathbb{R}^2 \), see Figure 1. This attractor is not a manifold. We let \( \Gamma_1 \) and \( \Gamma_2 \) denote two stable periodic orbits in \( \mathbb{R}^2 \), and we let \( P \) denote a hyperbolic fixed point which lies outside both \( \Gamma_1 \) and \( \Gamma_2 \). Assume that the two branches \( \mathcal{W}^u_i(P) \), \( i = 1, 2 \) of the unstable manifold of \( P \) have \( \Gamma_1 \) and \( \Gamma_2 \) as their respective limits. Then

\[
\mathcal{K} = \{P\} \cup \mathcal{W}^u_1(P) \cup \mathcal{W}^u_2(P) \cup \Gamma_1 \cup \Gamma_2
\]

is a hyperbolic attractor that satisfies the Lipschitz property.

6.4 Products of Attractors. The \( k \)-dimensional torus \( T^k \) can be imbedded into an open set \( \Omega \) in \( \mathbb{R}^{k+\ell} \) by using the coordinate system

\[
(\theta; \sigma) = (\theta_1, \ldots, \theta_k; \sigma_1, \ldots, \sigma_\ell), \quad 0 \leq \theta_i < 2\pi, \quad \sigma = (\sigma_1, \ldots, \sigma_\ell) \in \mathcal{D}^\ell,
\]

where \( \mathcal{D}^\ell \) denotes an \( \ell \)-dimensional disk.

Let \( \mathcal{K}_0 \) denote a given hyperbolic attractor with constants \( a \), \( \lambda_1 \), and \( \lambda_2 \) for an ordinary differential equation \( x' = X(x) \) on \( \mathbb{R}^n \). Then \( \mathcal{K}_1 = \mathcal{K}_0 \times T^k \) is a hyperbolic attractor for the product flow

\[
\begin{align*}
x' &= F(x, \theta, \sigma) \\
\theta' &= \Theta(x, \theta, \sigma) \\
\sigma' &= -A\sigma + G(x, \theta, \sigma)
\end{align*}
\]

on \( \mathbb{R}^n \times \Omega \), provided \( A \) is an \((\ell \times \ell)\) matrix whose eigenvalues \( \mu_i \) satisfy \( \text{Re} \mu_i > \max(0, \lambda_2) \) and \( F \), \( \Theta \), and \( G \) are \( C^1 \)-functions that satisfy \( F(x, \theta, 0) = X(x) \), \( \Theta(x, \theta, 0) = \omega \), \( G(x, \theta, 0) = 0 \), and \( D_3 G(x, \theta, 0) = 0 \), where \( D_3 = \frac{\partial}{\partial z} \) and \( \omega = (\omega_1, \ldots, \omega_k) \) is a constant vector. If \( \mathcal{K}_0 \) satisfies the Lipschitz property, then so does \( \mathcal{K}_1 \).

Let \( \mathcal{D}_i(x_0) \), \( x_0 \in \mathcal{K}_0 \), denote the disks in \( \mathcal{K}_i \), \( i = 0, 1 \), as prescribed in the definition of a hyperbolic attractor. Let \( \dim \mathcal{D}_0 = k_0 \), then one has \( \dim \mathcal{D}_1 = k_0 + k \). If there is an eigenvalue \( \mu_i \) that satisfies \( \max(0, \lambda_2) < \text{Re} \mu_i \leq \lambda_1 \), then the attractor \( \mathcal{K}_1 \) is not normally hyperbolic, see Hirsch, Pugh and Shub (1977). Even though \( \mathcal{K}_1 \) need not be normally hyperbolic, as the in the last two examples, our Main Theorem is applicable to perturbations of the underlying equation.

Remark. It is instructive to compare our theory as it applies to (6.2) with the examples constructed in Fenichel (1971) of perturbations of invariant manifolds which are not normally hyperbolic. To be specific assume that \( \mathcal{K}_0 \) is the hyperbolic attractor in \( \mathbb{R}^7 \) constructed in Section 6.2 with \( 0 < \nu \leq \ln \mu_1 \). Our Main Theorem asserts that if one adds any small \( C^1 \)-perturbation term \( Y \) to (6.2), then there is a homeomorphism \( h \) of \( \mathcal{K}_1 \) onto a hyperbolic attractor \( \mathcal{K}' \) for the perturbed equation. The Fenichel construction suggests that since \( \mathcal{K}_1 \) is not normally hyperbolic, there exists a small perturbation such that the
perturbed equation does not have a smooth invariant manifold near $\mathcal{K}_1$. In such a case, we conclude that, while $h : \mathcal{K}_1 \to \mathcal{K}^Y$ and $h^{-1} : \mathcal{K}^Y \to \mathcal{K}_1$ are both continuous, at least one of them is not differentiable. As we have shown in our theory, for each leaf $\mathcal{S} \subset \mathcal{K}_1$, the restriction $h|_\mathcal{S}$ and its inverse are locally Lipschitz continuous. It follows that a breakdown in smoothness can only occur in the direction normal to $\mathcal{S}$. Wrinkles can only arise in the stable direction.

REFERENCES


G. R. SELL (1990), Optimal approximate inertial manifolds for the 2D Navier-Stokes equations, In preparation.


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