MONOTONE MAPS OF $T^n \times R^n$
AND THEIR PERIODIC ORBITS

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Abstract. This article presents a theorem of existence of $n + 1$ ($2^n$ if non degenerate) periodic orbits of any given rotation vector for a large class of exact symplectic maps of the $n$-dimensional annulus $T^n \times R^n$. This theorem is global and uses the discrete variational approach of Aubry.

§0. Introduction

In this paper, we present a theorem of existence of periodic orbits with any rational rotation vector for a large class of symplectic diffeomorphisms of the space $A^n = T^n \times R^n$.

A lot of work has been produced recently, after a conjecture by Arnold [Ar 2], to relate the study of fixed points of symplectic maps on a manifold to the Morse theory of that manifold. The latter is seen to live in an underlying variational space to which it transmits some of its topology, often yielding far more fixed points than, say, Lefshetz theory would predict.

In our case, the variational setting is the one of Aubry [A-LeD] for twist maps of the annulus as generalised by Bernstein and Katok [B-K] to higher dimensions. This discrete setting generates symplectic maps that we call monotone here. For these maps, provided some boundary conditions, we find $n + 1$ periodic orbits for each rational rotation vector, and $2^n$ when they are non degenerate. It expands the result of Bernstein and Katok in that it does not require the maps to be close to integrable and that it relaxes their convexity condition.

A source of inspiration in this field is the seminal work of Conley and Zehnder [C-Z, 1 and 2] who found homotopically trivial periodic orbits for Hamiltonian flows on $T^n$ and $A^n$. In this setting, Josellis [J] proved a theorem to which ours is parallel. We also use some refinement of Conley’s Morse theory that Floer [Fl1] introduced in his work on the Arnold conjecture.

For $n = 1$, the maps that we consider here (i.e. compositions of monotone maps) are always time-1 maps of time dependent Hamiltonians, by a theorem of Moser [M]. It is not clear that his theorem can be generalized for $n > 1$, especially in the non-convex case. The aim of this paper is not, however, to fill that hypothetical gap. Rather, our wish is to promote Aubry’s approach as an alternate method for the global study of periodic orbits of exact symplectic maps. This approach, stemming from various physical situations has been used in numerical studies where $2^n$ orbits were indeed observed ([K-M]). Recently, as exposed by Robert MacKay [C, McK, M] in this conference, it has even yielded the first known nontrivial examples of Aubry-Mather sets in higher dimensions.

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In the final section of this paper, we sketch some extensions of the theorem presented here and make a few remarks about the problem of existence of quasiperiodic orbits.

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§1. Monotone maps

Let \( T^n = \mathbb{R}^n / \mathbb{Z}^n \). We consider the space \( A^n = T^n \times \mathbb{R}^n = T^*T^n \) with coordinates \((\theta, r), \theta \in \mathbb{R}^n / \mathbb{Z}^n, r \in \mathbb{R}^n \) endowed with its canonical symplectic form \( \Omega = \sum_{i=1}^{n} dr_i \wedge d\theta_i = d(\alpha) \), where \( \alpha = \sum_{i=1}^{n} r_i d\theta_i \). Corresponding coordinates in the universal covering will be denoted with a tilde.

A diffeomorphism \( F \) of \( A^n \) is symplectic if

\[
F^* \Omega = \Omega
\]

and exact symplectic if

\[
F^* \alpha - \alpha \text{ is exact}
\]

We write: \( F(z) = F(\theta, r) = (\Theta, R) = Z, \)

\[
DF_z = \begin{pmatrix} \Theta_\theta & \Theta_r \\ R_\theta & R_r \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

(1.4) **Definition:** A diffeomorphism \( F \) of \( A^n \) is monotone if it is exact symplectic and if \( \det b(z) \neq 0 \) for all \( z \in A^n \).

**Remark:** This terminology is inspired by, although not quite equivalent to M. Herman’s [H]. He does not assume exact symplectic.

1.5 **Definition:** A point \( z \) of \( A^n \) is called \((w, q)\)-periodic for \( F \) if \((w, q) \in \mathbb{Z}^n \times \mathbb{Z} \) and:

\[
\tilde{F}^q(\tilde{z}) - \tilde{z} = w,
\]
where $\tilde{F}$ and $\tilde{z}$ denote some chosen lift of $F$ and $z$ to the covering space $\tilde{A}^n = \mathbb{R}^{2n}$ of $A^n$. The vector $w/q$ is called the rotation vector of the orbit. When for all $i (w_i, q)$ are relatively prime, we say that $(w, q)$ is relatively prime.

1.6 Remark: The rotation vector is lift-dependant: We fix a lift once and for all (rotation vectors given by different lifts differ by fixed integer vectors).

(1.7) Definition: A $(w, q)$-periodic point (or orbit) $z$ is said to be nondegenerate if 

$$\det(D(F^q)_z - I) \neq 0$$

§2. Generating Functions: The variational setting.

A Monotone diffeomorphism $F$ is often uniquely determined by a single, real valued function on $A^n$. Indeed, since $F$ is exact symplectic, we can write (see 1.2):

(2.1) \[ F^*\alpha - \alpha = dS \]

or:

(2.1') \[ Rd\Theta - rd\theta = dS \]

where $S : A^n \to \mathbb{R}$. If, moreover, $S$ can be expressed as a function of the variables $(\theta, \tilde{\Theta})$, that is, if the map:

$$\psi : (\theta, r) \to (\theta, \tilde{\Theta}) = (\theta, \partial_1 S(\theta, r))$$

is a global diffeomorphism of $A^n$, then $S$ is called a global generating function.

Remark: Here and in the sequel, $\partial_1 S$ (resp. $\partial_2 S$) represents the partial derivative with respect to the first (resp. the second) vector variable.

If $S$ is $C^2$, then $F$ is $C^1$ and we have:

(2.3) \[ r = -\partial_1 S(\theta, \tilde{\Theta}) \]

\[ R = \partial_2 S(\theta, \tilde{\Theta}) \]

If we also denote by $S$ the lift of $S$ to $\mathbb{R}^{2n}$, then (2.3) becomes

(2.4) \[ r = -\partial_1 S(\tilde{\theta}, \tilde{\Theta}) \]

\[ R = \partial_2 S(\tilde{\theta}, \tilde{\Theta}) \]

In that setting, $S$ satisfies:

(2.5) \[ S(\tilde{\theta} + m, \tilde{\Theta} + m) = S(\tilde{\theta}, \tilde{\Theta}), \forall m \in \mathbb{Z}^n \]
Let \( \tilde{z}_i = \tilde{T}^i(\tilde{z}_0) = (\tilde{\theta}_i, r_i) \) then the orbit \( \{\tilde{z}_i\} \) is uniquely determined by the sequence \( \{\tilde{\theta}_i\} \), since

\[
(2.6) \\
r_i = -\partial_1 S(\tilde{\theta}_i, \tilde{\theta}_{i+1}) = \partial_2 S(\tilde{\theta}_{i-1}, \tilde{\theta}_i)
\]

Instead of orbits, we shall therefore look for critical states. These are bi-infinite sequences of \( R^n = \tilde{T}^n : (x) = \ldots x_{-1}, x_0, x_1, x_2, \ldots \) satisfying

\[
(2.7) \\
\partial_1 S(x_i, x_{i+1}) + \partial_2 S(x_{i-1}, x_i) = 0
\]

**Remark:** This construction is readily generalized to exact symplectomorphisms \( F = F_1 \circ F_2 \circ \ldots \circ F_m \) where each \( F_i \) is monotone with global generating function \( S_i \). In that setting, (2.7) becomes:

\[
(2.8) \\
\partial_1 S_i(x_i, x_{i+1}) + \partial_2 S_{i-1}(x_{i-1}, x_i) = 0
\]

(which makes sense for \( i > m \) by setting \( S_{i+km} = S_i \)).

For a monotone \( F \), finding \((w, q)\)-periodic points is equivalent to finding critical states within the space

\[
(2.9) \\
\phi^*_{w, q} = \psi_{w, q}/Z^n
\]

of periodic states, where:

\[
(2.9) \\
\psi_{w, q} = \{ (x) \in (R^n)^Z | x_{i+q} = x_i + w \},
\]

And \( Z^n \) acts on \( \psi_{w, q} \) by \((x + m)_j = x_j + m, \forall m \in Z^n \). Alternatively, we shall think of \( \psi_{w, q} \) as the codimension plane:

\[
(2.10) \\
\psi_{w, q} = \{ (x_0, \ldots, x_q) | x_i \in R^n, x_q = x_0 + w \} \simeq (R^n)^q.
\]

Finding \((w, q)\)-points is therefore done by solving the equation ("discrete variational problem")

\[
(2.11) \\
\nabla L = 0, \text{ for } L(x) = \sum_{i=1}^q S(x_{i-1}, x_i), \text{ and } \nabla L(x) = \partial_1 S(x_i, x_{i+1}) + \partial_2 S(x_{i-1}, x_i)
\]

where \((x) \in \phi^*_{w, q}\). (When dealing with a composition of \( F_i \)'s, one uses \( \sum S_i \)).

Because of the periodicity of \( S \) (2.5), the function \( L \) defined in (2.11) is invariant under the action of \( Z^n \) on \( \psi_{w, q} \), namely:

\[
L(x + m) = L(x), \text{ } m \in Z^n
\]
and hence it is well defined on $\phi^*_{w,q}$.

We will sometimes refer to $L$ as the energy on the set of states. In the case $n = 1$, McKay and Meiss [McK-M] derived a formula analogous to the index formula in classical Morse-theory. It gives a relation between the index of $\nabla^2 L(x)$ (the Hessian of $L$ at a critical point $x$) and the spectrum of $DF^q(z)$ ($z \in A$ corresponds to the sequence $x$). Mather [Ma] later completed the study of that relation by interpreting the contribution of $DF^q(z)$ as the rotation number of its action on the tangent circle-bundle. Recently, Kook and Meiss [K,M] generalized the above mentioned formula for higher dimensional monotone maps. Even though, when $n > 1$ this formula does not completely characterize the stability type of a periodic orbit, using it one can prove [Go]:

**Lemma (2.12):** Let $x$ be a critical state in $\phi^*_{w,q}$ corresponding to the orbit of $z$. Then $z$ is non degenerate if and only if $\det \nabla^2 L(x) \neq 0$ (i.e. the Hessian of $L$ is nondegenerate at $x$).

**2.13 Remark:**

As noted by M.R. Herman [He], the condition:

$$\sup_{x \in A^n} \|b^{-1}(z)\| < +\infty$$

(2.14)

is enough to ensure the existence of a globally defined generating function for a monotone diffeomorphism $F$. See [Go] for more details.

**§3. Examples**

**3.0 Monotone Twist Maps of the Annulus (and their compositions)**

The theory presented here is an attempt to generalise some of the theory of monotone twist maps to higher dimensions. Monotone twist maps are monotone maps for $n = 1$, that is, on the Annulus $A = S^1 \times R$. In that case the non-degeneracy condition $\det b \neq 0$ (see definition 1.4) is, modulo a choice of sign, equivalent to the Twist condition. The main result in our work is an analogue to the Poincaré-Birkhoff theorem in higher dimensions (existence of periodic orbits of all rotation numbers), and one of our aims is to investigate possible generalisations of the Aubry-Mather theorem (existence and topological characterisation of quasi-periodic orbits). The standard example of a twist map is the Standard map of the annulus:

$$f(\theta, r) = (\theta + r + (k/2\pi)\sin(2\pi \theta), r + (k/2\pi)\sin(2\pi \theta))$$

For references on this vast subject, consult, e.g., [Ban],[Ch],[Ha],[Ma1].
3.1 “Completely integrable” diffeomorphisms

We say that $F$ is “completely integrable” if it is of the form:

$$ F(\theta, r) = (\theta + A(r), r), $$

Where $A$ is a diffeomorphism with $A(r) = \nabla V(r)$ for some $V$. When $A$ is linear, it must be symmetric and the generating function for $F$ is:

$$ S(\bar{\theta}, \bar{\Theta}) = \frac{1}{2} < A^{-1}(\bar{\Theta} - \bar{\theta}), \Theta - \theta > $$

indeed $R = \partial_2 S(\bar{\theta}, \bar{\Theta}) = A^{-1}(\bar{\Theta} - \bar{\theta}) = -\partial_1 S(\bar{\theta}, \bar{\Theta}) = r$ and hence $\bar{\Theta} - \bar{\theta} = A(r)$.

Since $A$ is symmetric, it can be diagonalized with real eigenvalues. Thus, in the right frame for $TA^n$, $F$ is a product of shear maps of the annulus $A$:

$$ F = f_1 \times \ldots \times f_n, \ f_i : A \to A $$

where $f_i(\theta_i, R_i) = (\theta_i + \lambda_i r_i, r_i)$, $f_i$ is a positive (resp. negative) twist map if $\lambda_i > 0$ (resp. $\lambda_i < 0$).

For these maps, each torus $\{r = r_0\}$ is invariant and a point $(\theta, r_0)$ has rotation vector $A(r_0)$. When $A(r_0)$ is of the form $\frac{w}{q}$, the whole torus $\{r = r_0\}$ is formed by $(w, q)$-periodic orbits. Irrational $A(r_0)$ correspond to quasi-periodic orbits. The topological structure of the “remnants” of such orbits for general symplectic maps of $A^n$ is still a mystery.

The completely integrable diffeomorphisms occur naturally as time-1 maps of the completely integrable Hamiltonian systems with:

$$ H(\theta, r) = \frac{1}{2} < A(r), r > $$

3.2 The standard map

$$ F(\theta, r) = (\theta + r + \nabla V(\theta), R + \nabla V(\theta)) $$

with $S(\bar{\theta}, \bar{\Theta}) = \frac{1}{2} (\bar{\Theta} - \bar{\theta})^2 + \nabla V(\bar{\theta})$

Example (Froeschle, [Froe]):

$$ V(\theta_1, \theta_2) = \frac{1}{(2\pi)^2} \left\{ K_1 \cos(2\pi \theta_1) + K_2 \cos(2\pi \theta_2) + h \cos(2\pi (\theta_1 + \theta_2)) \right\} $$

this gives a 3-parameter family of monotone maps on $A^2$. This is also studied numerically in Kook-Meiss [K-M].

6
3.3 Hamiltonian systems

If we assume
\[ \text{Det} \frac{\partial^2 H}{\partial r^2} \neq 0, \] and that, for \(|r| > a\), \(H(\theta, r) = \frac{1}{2} < Ar, r > + < Cr, r >\)

for \(A\) symmetric, nondegenerate, and \(C \in \mathbb{R}^n\) (see Conley-Zehnder, Thm 3 [C-Z 2]) then the time-1 map \(F_1\) can always be decomposed into monotone \(F_i\)'s. Moreover, they trivially satisfy the boundary conditions of our theorem. Also note that such Hamiltonian systems fall into the domain of those studied by Josellis [J]

§4. Statement of the theorem

In this section, we assume that our map \(F\) continues, through a family \(\{F_\lambda\}_{\lambda \in \Lambda}\) of monotone maps, to a completely integrable map of the form:

\[ F_0(\theta, r) = (\theta + A^{-1}(r), r) \]

where \(A\) is a symmetric, nondegenerate matrix.

More precisely, we will be working with the following type of family of maps:

(4.1) Consider a family \(\{F_\lambda\}_{\lambda \in \Lambda}\) where each \(F_\lambda\) is a monotone map generated by the function \(S_\lambda\), and \(\Lambda\) is a hausdorff, compact and connected topological space (e.g., a compact region of \(\mathbb{R}^N\)).

Further, assume that:

(4.2) The dependence of \(F_\lambda\) on \(\lambda\) is continuous for the \(C^1\) topology, the dependence of \(S_\lambda\) on \(\lambda\) is continuous for the \(C^2\) topology.

(4.3) there is a point \(0 \in \Lambda\) at which:

\[ S_0(x X) = \frac{1}{2} < A(X - x), (X - x) > \]

where \(A\) is symmetric, non degenerate (that is, \(F_0\) is as in 4.1).

(4.4)

\[ S_\lambda(x, X) = S_0(x, X) + R_\lambda(x, X) \]

and \(R_\lambda\) is either 0 outside a bounded set, i.e.:

(4.4a) \[ R_\lambda(x, X) = 0 \quad \text{when:} \|X - x\| \geq K \]

(That amounts to saying that \(F_\lambda = F_0\) when \(\|A^{-1}(r)\| \geq K\).) or assumes sublinear growth in its derivative:

(4.4b) \[ \lim_{\|X - x\| \to \infty} \frac{\|\partial_\alpha R_\lambda(x, X)\|}{\|X - x\|} = 0, \quad \alpha = 1, 2 \]
4.5 Theorem: Let $(F_\lambda)_\lambda \in \Lambda$ be a family as above. Then, for any $\lambda \in \Lambda$, $F_\lambda$ has at least $n + 1$ $(w, q)$-periodic orbits and $2^n$ when they are non-degenerate. If moreover $F_\lambda$ satisfies 4.4a for some $K$ and $\|w/q\| \leq K$ then all the $(w, q)$-periodic orbits are contained in the compact set $\{ (\theta, r) | \|A^{-1}(r)\| \leq K \}$.

4.6 Example: Let $H_\lambda$ be a compact family of time dependent Hamiltonian functions such that:

$$H_0 = \frac{1}{2} < Ar, r >$$

$$H_\lambda = H_0, \text{ for } \|r\| > C, \text{ and } Det \frac{\partial^2 H}{\partial r^2} \neq 0$$

Then, according to 3.3, the time $\frac{1}{N}$-map at the time $\frac{k}{N}$, say $F_{k, \lambda}$, is monotone, with generating function

$$S_{k, \lambda}(x, X) = \frac{1}{2} < \frac{1}{N} A^{-1}(X - x), (X - x) > + R_{k, \lambda}(X, x)$$

and $R_{k, \lambda} = 0$ when $< \frac{1}{N^2} A^{-2}(X - x), X - x > \geq C^2$. As noted above, the following theory readily generalizes to the composition of such maps, and hence to the time one map of the Hamiltonian $H_\lambda$.

Remark: To simplify the notation, we only consider families of monotone maps. The same statements could be made about families of compositions of monotone maps provided that they satisfy the same boundary conditions (actually, one only needs the “linear parts” $A$ to commute).

We present the proof of theorem 4.5 in the next two sections. The idea is that the gradient flow $\nabla L_0$ corresponding to $F_0$, has a normally hyperbolic invariant set (for that flow), which is isolated and continues, in the sense of Conley, to isolated invariant sets for all $\lambda \in \Lambda$. The invariant set for $\nabla L_0$ is in fact diffeomorphic to the $F_0$-invariant torus of rotation vector $w/q$. We then use a result by Floer which implies that the invariant set itself conserves its cohomology throughout the parameter space. The conclusion follows from Conley-Zehnder’s Morse inequalities. In section , we state a more general theorem where the continuation setting is not needed.

§5. Existence of the isolating block $P$

In this section, it is not necessary to assume that our map $F$ continues to a completely integrable one. All we need is that it be monotone with generating function $S(x, X) = S_0(x, X) + R(x, X)$, and $R$ satisfies (4.4a). We show that the corresponding gradient flow $\nabla L$ has $P(C)$ (see the following definition) as isolating block. In [Go], we also find an
isolating block $B(C)$ when only the more general condition (4.4b) is assumed. The proof will not be reproduced here. At the end of this section we introduce the setting that will enable us to distinguish periodic orbits instead of points. Again, the following result holds for compositions of monotone maps (with good boundary conditions): see [Go].

Remember (2.9, 2.10) that

$$
\psi_{w,q} = \{(x) \in (\mathbb{R}^n)^Z | x_{i+q} = x_i + w\}
\simeq \{(x_0, \ldots, x_q) | x_q = x_0 + w\}
$$

and

$$
\phi^*_{w,q} = \psi_{w,q}/\mathbb{Z}^n
$$

One can also see it as:

$$
\phi^*_{w,q} = \{(x_0, a) | x_0 \in T^n, a_i \in \mathbb{R}^n, \sum_{i=1}^q a_i = w\} \simeq T^n \times (\mathbb{R}^n)^{q-1}.
$$

Where $a_i = x_i - x_{i-1}$.

We think of $q$ as the period dimension, $n$ the number of degrees of freedom. We want to break the degrees of freedom dimension into the eigenspaces of the matrices $A$, so as to compare the map $F$ to a product of shear maps (see example 3.1).

*Without loss of generality, we can choose coordinates on $\mathbb{R}^n$ for which the matrix $A$ (see 4.3) is diagonal. This will be the standing assumption.* In these coordinates we denote by $x_i^k$ the $k^{th}$ component of $x_i$. We also denote by $\lambda_k$ the (real) eigenvalues of $A$. By permuting the coordinate, we can assume that:

$$
\lambda_k > 0 \text{ for } 0 \leq k \leq k_0, \lambda_k < 0 \text{ for } k_0 < k \leq n
$$

note that we do not assume all the $\lambda_k$ to be of the same sign. When this occurs, we say that $F$ satisfies a convexity condition (cf [B-K]), or that it is twist ([K-M],[He]). In that case, finding at least one orbit is trivial (global extremum: see [B-K]).

*5.4 Definition*

$$
P(C) = \{(x) \in \phi^*_{w,q} | \sup_{i \in \{0, \ldots, q\}} \sup_{k \in \{1, \ldots, n\}} |x_i^k - x_{i-1}^k| < C\}
$$

It is not hard to see that $P(C) \simeq T^n \times (D^n)^{q-1}$.

*5.5 Lemma: Let $F$ be a monotone map with generating function*

$$
(5.5a) S(x, X) = S_0(x, X) + R(x, X),
$$
and $R$ satisfies
\begin{equation}
R(x, X) = 0 \text{ whenever } \|X - x\| \geq K
\end{equation}

Then there is a constant $C_0 > 0$, independent of $q$ such that, for any $C \geq C_0$, $P(C)$ is an isolating block for the flow $\nabla L$ on $\phi_{w, q}$, whenever
\begin{equation}
\|w/q\| < K
\end{equation}

5.6 Remark: condition (5.5c) just restricts our attention to interesting orbits; we know completely the other ones, which lay on invariant tori (large $\|r\|$).

Proof:

We will show that the flow $\nabla L$ points out of $P(C)$ for points belonging to the following set of boundary components of $\partial P(C)$; for $C$ large enough:
\begin{equation}
\bigcup_{k=1}^{k_0} \bigcup_{i=0}^{q} \{ (x) \in P(C) \cap \phi_{w, q}^* \mid x_i^k - x_{i-1}^k = \pm C \}
\end{equation}

\[ \overset{\text{def}}{=} \bigcup_{k=1}^{k_0} \bigcup_{i=0}^{q} \partial P_{i,k}^+ \]

The same argument shows that $\nabla L$ points in at the other components of $\partial P(C)$.

Schematically, the picture is the following:

$\partial P_{i,k}^-, \ k > k_0$

\begin{equation}
T^n \times \partial P_{i,k}^+, \ k \leq k_0
\end{equation}

Each $\partial P_{i,k}$ is itself a box, either attracting or repelling.

It is enough to look at one of them, say $\partial P_{i,k}^+$, for $k \leq k_0$ (see def. in 5.7).

Because of the periodicity of $S$, the set $\phi_{w, q}$ is invariant under the flow $\dot{x} = \nabla L(x)$.

Indeed, if $x \in \phi_{w, q}$, then

$$
\dot{x}_{i+q} - \dot{x}_i = \partial_1 S(x_{i+q}, x_{i+q+1}) + \partial_2 S(x_{i+q-1}, x_{i+q})
$$
(5.9) \[ -\partial_1 S(x_i, x_{i+1}) - \partial_2 S(x_{i-1}, x_i) = 0, \text{ by periodicity of } \partial_\alpha S, \alpha = 1, 2 \]

Hence \( \nabla L \) is parallel to \( \phi_{w,q}^* \), and it is enough to show that:

(5.10) \[ N_{i,k}^+ \cdot \nabla L > 0 \quad (k \leq k_0) \]

where \( N_{i,k}^+ \) is the normal vector to \( \partial P_{i,k}^+ \), namely:

(5.11) \[
(N_{i,k}^+ )_{j,l} = \begin{cases} 
1 & \text{if } (j,l) = (i,k) \\
-1 & \text{if } (j,l) = (i-1,k) \\
0 & \text{otherwise} 
\end{cases}
\]

(Both \( \nabla L \) and \( N_{i,k}^+ \) are thought of as vectors in \((\mathbb{R}^n)^{g+1}\). On the other hand:

(5.12) \[
(\nabla L(x))_{i,k} = \lambda_k(2x^k_i - x^k_{i-1} - x^k_{i+1}) + \partial_1 R^k(x_i, x_{i+1}) \\
+ \partial_2 R^k(x_{i-1}, x_i)
\]

Thus:

(5.13) \[
N_{i,k}^+ \cdot \nabla L(x) = \\
\lambda_k(2x^k_i - x^k_{i-1} - x^k_{i+1}) + \partial_1 R^k(x_i, x_{i+1}) + \partial_2 R^k(x_{i-1}, x_i) \\
- \lambda_k(2x^k_{i-1} - x^k_{i-2} - x^k_i) + \partial_1 R^k(x_{i-1}, x_i) + \partial_2 R^k(x_{i-2}, x_{i-1})
\]

Choose \( C \geq K \) (see 4.1a). Then:

(5.14) \[ |x^k_i - x^k_{i-1}| = C \Rightarrow ||x_i - x_{i-1}|| \geq K \]

and therefore \( R^k(x_i, x_{i-1}) = 0 \). For such a \( C \) and for \( x \) in \( \partial P_{i,k}^+(C) \), we have (rearranging terms):

(5.15) \[
N_{i,k}^+ \cdot \nabla L(x) = \\
(2\lambda_k)(x^k_i - x^k_{i-1}) + \lambda_k(x^k_i - x^k_{i+1}) + \partial_1 R^k(x_i, x_{i+1}) \\
- \lambda_k(x^k_{i-1} - x^k_{i-2}) - \partial_2 R^k(x_{i-1}, x_i)
\]

Since in \( \partial P_{i,k}^+(C) \), \( x^k_i - x^k_{i-1} = C \) and that for \( k \leq k_0, \lambda^k_j > 0, \forall j \), this expression will be positive whenever:

(5.16) \[
\sup_{x \in P(C)} |\lambda_k(x^k_i - x^k_{i+1}) + \partial_1 R^k(x_i, x_{i+1})| \leq |\lambda_k|C
\]
\[ \sup_{x \in P(C)} |\lambda_k(x_{i-1}^k - x_{i-2}^k) + \partial_2 R^k(x_{i-2}, x_{i-1})| \leq |\lambda_k| C \]

for all \( i \).

This in turn will be made possible because \( R(x, X) \) and its derivatives are 0 outside of \( \|X - x\| \geq K \) (4.1a): their periodicity implies that they are bounded, say

\[ \|\partial_\alpha R\| < K_1 \quad \alpha = 1, 2 \]

Hence, (5.16) is satisfied whenever

\[ C \geq K + \frac{K_1}{\inf |\lambda_k|} \]

The reader will note that this does not guarantee strict positivity of \( \nabla L.N_{ik}^+ \). In fact, what we have proven so far is:

\( \nabla L.N_{ik}^+ > 0 \) on \( \partial P_{i,k}^+ \), except possibly on

\[ E_1 = \partial P_{i,k}^+ \cap \{x_{i-1}^k - x_{i-2}^k = C\} \cap \{x_{i+1}^k - x_i^k = C\} \]

where it is null. However, we will show that, because \( x \in \phi_{w,q}^* \), \( \nabla L \) has to be positively transverse to at least one of the adjacent faces to \( E_1 \). We start with the face \( \partial P_{i-1,k}^+ \):

\[ x_{i-1}^k - x_{i-2}^k = C \text{ on } E_1 \Rightarrow R^k(x_{i-2}, x_{i-1}) = 0 = R^k(x_i, x_{i+1}) \]

Hence:

\[ \nabla L \bigg|_{E_1} . N_{i-1,k}^+ = (2\lambda_k)(x_{i-1}^k - x_{i-2}^k) \]

\[ + \lambda_k(x_{i-1}^k - x_i^k) - \lambda_k(x_{i-2}^k - x_{i-3}^k) \]

\[ - \partial_2 R^k(x_{i-3}, x_{i-2}) \]

\[ = \lambda_k C - \lambda_k(x_{i-2}^k - x_{i-3}^k) - \partial_2 R^k(x_{i-3}, x_{i-2}) \]

\[ > 0 \]

because of (5.12), except possibly when \( x_{i-2}^k - x_{i-3}^k = C \).

Define:

\[ E_2 = E_1 \cap \{x_{i-2}^k - x_{i-3}^k = C\} \]

and by induction

\[ E_l = E_{l-1} \cap \{x_{i-(l)}^k - x_{i-(l-1)}^k = C\} \]

12
and:

\[ \nabla L \bigg|_{E_{i-1}} . N_{i-1}^{+} > 0, \quad \text{except possibly on } E_i. \]

On \( \phi_{w,q}^* \), this process comes to an end, since \( x_{i-q} = x_i - w \). At the end, we get:

\[ \nabla L \bigg|_{E_{q-1}} . N_{i-(q-1),l}^{+} > 0, \quad \text{except possibly on } E_q \]

But \( E_q = \{ (x) | x_j^k - x_{j-1}^k = C, \forall j \in (0, \ldots, q) \} \) cannot be in \( \phi_{w,q}^* \), because then:

\[ x_{j+q}^k - x_j^k = qC, \]

which contradicts the assumption 5.1c.

We now turn to the problem of distinguishing geometrically distinct periodic orbits: if \( (x_0 \ldots x_{q-1}) \) is critical, so is \( (x_1 \ldots x_q) \). But they represent the same orbit. Hence, to look for distinct periodic orbits, we need to quotient by the action of the following diffeomorphism:

(5.21) \[ T : \phi_{w,q}^* \to \phi_{w,q}^* \text{ defined by } \]

\[ T(x)_i = x_{i+1} \]

then

(5.22) \[ L \circ T = L, \]

Bernstein-Katok ([B-K], proposition 1), show that the quotient map:

(5.23) \[ \phi_{w,q}^* \xrightarrow{\pi} \phi_{w,q}^*/T = \phi_{w,q} \]

is an \( q \)-fold covering map. In fact, \( \phi_{w,q} = \phi_{w,q}^*/T \) is a fiber bundle over \( T^n \) with fiber \( (\mathbb{R}^n)^{q-1} \) whereas \( \phi_{w,q}^* \) is diffeomorphic to \( T^n \times (\mathbb{R}^n)^{q-1} \). For this purpose they define the coordinates \( (v, t_1 \ldots, t_{q-1}) \) of \( \psi_{w,q} \) by:

(5.24) \[ v = \frac{1}{q} (x_0 + \ldots + x_{q-1}) \]

\[ t_i = x_i - x_{i-1} - w/q \]
In these coordinates, the action of $Z^n$ on $\psi_{w,q}$:

\[(5.25) \quad (x_i) \rightarrow (x_i + m) \text{ becomes } (v, t) \rightarrow (v + m, t)\]

and the shift

\[(5.26) \quad T(v, t_1 \ldots t_{q-1}) = (v + w/q, t_2, \ldots, t_{q-1}, -\sum_{i=1}^{q-1} t_i)\]

$L$ induces a function on $\phi_{w,q}$, that we denote by $L$ and the critical points of $L$ on $\phi_{w,q}$ are in 1-1 correspondence with the geometrically different periodic orbits of rotation vector $w/m$ of $F$.

To do some Morse theory on $\phi_{w,q}$, observe that $P = \pi(P)$ is an isolating block for the gradient flow of $L$ on $\phi_{w,q}$ with exit set $P^- = \pi(P^-)$. It is important to note that $L$ (resp. $L$) has no critical points outside of $P$ (resp. $P$).

We end this section with a simple proof of Theorem 4.5 when $F$ satisfies the convexity condition (and either 4.4a or 4.4b). This proof does not assume that $F$ continues to a completely integrable $F_0$.

5.27 Proof of 4.5 in the convex case

Suppose that the matrix $A$ is positive definite. In this case, the classical Morse-Lyusternick-Schnirelman theory applies (see [Mi] and [Kl]). Indeed, $P^- = \partial P$ and $P$ is an attractor block for the flow $\dot{x} = -\nabla L(x)$ on $\phi_{w,q}$. As remarked above, $L$ has no critical points outside of $P$. Let $c = \sup L(x)$, then $P \subset L^{-1}(-\infty, c] = M_c$. This set, by a standard argument in Morse theory ([Mi,2][Th. 3.1]) is a deformation retract of $\phi_{w,q}$, itself homotopically equivalent to $T^n$. The cup long $l(M_c) = n + 1$ and Lyusternick-Schnirelman implies the existence of at least $n + 1$ critical points for $L$.

When the critical points are non degenerate, the classical Morse theory gives at least the sum of Betti numbers $SB(M_c) = 2^n$ critical points inside $M_c$. (All the above holds with an isolating block $B$ instead of $P$ when the boundary condition 4.4b is used instead of 4.4a: see [Go])

§6. Continuation of the invariant set and proof of Theorem 4.5

Going back to the general continuation setting, we denote by $\nabla L_\lambda$ the gradient flow associated to $S_\lambda$, as in (2.11).

6.1 Definition We call a ghost-torus and denote by $G_{w,q}(C)$ (resp. $G_{w,q}^\lambda(C)$) the maximal invariant set for $\nabla L_\lambda$ (resp. $\nabla L_\lambda$) in the isolating blocks $P_{w,q}(C)$ or $B_{w,q}$ (resp. $P_{w,q}(C)$ or $B_{w,q}(C)$) constructed in section 5.
(Note the new rotation vector index). Since we assume $\Lambda$ to be compact, the constant $C$ as above can be chosen uniformly for all $\lambda \in \Lambda$.

Since the $G^\lambda_{w,q}$'s are the maximal isolated invariant sets for the same isolating neighborhood $P_{w,q}$ (or $B_{w,q}$) in the continuous family of flows $\nabla L_\lambda$, they are related by continuation in the sense of Conley. (This notion of continuation is actually much broader and allows for instance, dealing with sequences of such relationships. For the more general definition, we refer to [C] or [Sa]).

The invariant set $G^0_{w,q}$ plays a particular role. In fact:

6.2 Lemma: (a) $G^0_{w,q}$ is diffeomorphic to $T^n$ and all its points are critical,
(b) $G^0_{w,q}$ is a retract of $\phi_{w,q}$,
(c) It is normally hyperbolic for the flow $\nabla L_0$.

Moreover, $G^0_{w,q}$ corresponds naturally to the invariant torus for $F_0$, of rotation vector $w/q$.

Proof: Unless we state otherwise, we use the intrinsic coordinate system $(x_o, \ldots, x_{q-1})$ for $\phi_{w,q}$. The same coordinates will be used for

$$T(\phi_{w,q}^*) \cong T((\mathbb{R}^n)^q/\mathbb{Z}^n) \cong (\mathbb{R}^n)^q$$

In these coordinates, $\nabla L_\lambda(x)$ is:

$$\nabla L_\lambda(x)_i = \partial_1 S_\lambda(x_i, x_{i+1}) + \partial_2 S_\lambda(x_{i-1}, x_i)$$

with $x_q = x_o + w, x_{-1} = x_{q-1} - w$.

(a) To find critical points for $\nabla L_0$, we solve the equation:

$$0 = \partial_1 S_0(x_i, x_{i+1}) + \partial_2 S_0(x_{i-1}, x_i)$$

(6.3)

$$= A(2x_i - x_{i-1} - x_{i+1}), \quad i \in \{0, \ldots, q-1\}$$

with the constraint $x_{i+q} = x_i + w$.

Since 6.3 is equivalent to $x_i - x_{i-1} = x_{i+1} - x_i$, the solutions are given by one of the $x_i$'s, say $x_o$ and are of the form $\{(x_o, x_o + \frac{w}{q}, \ldots, x_o + \frac{q-1}{q}w)\}$ under the identification $(x_i) \sim (x_i + m)$, $m \in \mathbb{Z}^n$ this gives us a torus. One can see that this torus constitutes all of $G^0_{w,q}$: the proof of (c) will show us that, modding out by this eigendirection, $\nabla L_0$ is a linear hyperbolic flow. Hence it can not contain more than a fixed point as isolated invariant set in the space $\phi_{w,q}^*/G^0_{w,q}$. Thus:

$$G^0_{w,q} = \{x \in \phi_{w,q}^* | x_i = x_o + i(w/q)\}$$
It is easy to see that these critical states correspond to the $F_0$-invariant torus of rotation vector $w/q$. This correspondance is in fact a diffeomorphism if one looks at $G_{w,q}^0$ in $\phi_{w,q}$.

(b) In (5.24), we introduced, following Bernstein-Katok, the coordinates $(v,t)$ for $\psi_{w,q}, \phi_{w,q}, \phi_{w,q}$. $v$ represents the torus dimension in both $\phi_{w,q}$ and $\phi_{w,q}$, since the identifications only take place there. The retraction map will be given in these coordinates by:

\[ r : \phi_{w,q}^* \longrightarrow G_{w,q}^0 \]

\[ (v,t) \longrightarrow (v,0) \] (6.5)

(It is easy to see that $t = 0$ is indeed the set $G_{w,q}^0$ in these coordinates). This map is obviously continuous and $r \bigg|_{G_{w,q}^0} = Id_{G_{w,q}^0}$. In fact, it is a strong deformation retract and it commutes with the action of $T$. Hence $r$ gives rise to a retraction:

\[ r : \phi_{w,q} \longrightarrow G_{w,q}^0 \]

\[ r([v],t) \longrightarrow ([v],0) \] (6.6)

(c) We have to study the linearization $T(\nabla L_0)$ of $\nabla L_0$ on the tangent space $T(\phi_{w,q}^*)|_{G_{w,q}^0}$. To do so, it is convenient to split $T(\phi_{w,q}^*)$ according to the eigenspaces of $A$, as we did in lemma 5.5. Say

\[ T(\phi_{w,q}^*) = \oplus_{k=1}^n E_k, \]

where $E_k$ corresponds to the eigenvalue $\lambda_k$ of $A$. This is an orthogonal splitting, since $A$ is symmetric. $T(\nabla L_0)$ obviously preserves this splitting and its expression in each $E_k$ is the $q \times q$ matrix:

\[ T(\nabla L_0)|_{E_k} = \lambda_k \]

\[ \begin{pmatrix} 2 & -1 & & -1 \\ -1 & \ddots & \ddots \\ & \ddots & \ddots & \ddots \\ -1 & \ddots & \ddots & -1 \\ -1 & -1 & 2 \end{pmatrix} \] (6.7)

(bank spaces are 0's) We get this directly from (12.7).
One recognizes the "discrete Laplacian" on the space of q-periodic sequences in $\mathbb{R}$. To find its spectrum is an elementary exercise in the difference equation:

\begin{align}
\begin{cases}
\mu a_n &= 2a_n - a_{n-1} - a_{n+1} \\
\rho_{n+q} &= a_n
\end{cases}
\end{align}

that one solves by setting $a_n = \rho^n$, which imposes the set of eigensequences and eigenvalues:

$$V_j = (\rho_j, \rho_j^2 \ldots \rho_j^q) \text{ with }$$

$$\rho_j = e^{i \frac{2\pi j}{q}} \quad j \in \{a, \ldots, q\}$$

(6.10)

$$\mu_j = 2(1 - \cos \frac{2\pi j}{q})$$

In particular $V_0$ is the vector $(1, \ldots, 1)$ and corresponds to the only 0 eigenvalue. The other eigenvalues are all strictly positive.

Putting it all together, we have found an orthogonal ($T(\nabla L_0)$ is symmetric) basis of eigenvectors for $T(\nabla L_0)$: $(V_0^1, \ldots, V_0^n, V_1^1 \ldots V_1^n \ldots V_{q-1}^1 \ldots V_{q-1}^n)$ where the space spanned by $(V_0^1 \ldots V_0^n)$ is just the tangent space to $G_{w,q}^0$, which corresponds to the 0 eigenvalue (Harmonic sequences!). As for $V_i^k$, when $i \neq 0$, it has eigenvalue $\lambda_k \mu_i$ which is non zero and of the sign of $\lambda_k$. This proves the normal hyperbolicity of $G_{w,q}^0$.

Since the covering map $\pi$ is a local diffeomorphism, the above result also holds for $G_{w,q}^\lambda(C)$. depth3pt

This lemma will enable us to prove the main theorems of this section, of which theorem 4.5 becomes a corollary: the topology of $G_{w,q}^0$ (i.e. $T^n$) continues in $G_{w,q}^\lambda$ for all values of $\lambda$. For this reason, we call $G_{w,q}^\lambda$ a ghost-torus.

6.11 Theorem (Local continuation):

Given $w/q$, there is an open set $U(o) \subset \Lambda$ such that $G_{w,q}^\lambda$ contains a torus which is a graph over $G_{w,q}^0$. If $S_\lambda$ is $C^r$, the torus will be $C^{m-1}$ and $C^{m-1}$ diffeomorphic to $G_{w,q}^\lambda$. This theorem also holds for $G_{w,q}^\lambda$.

Proof: This is an immediate application of classical theorems of perturbation of normally hyperbolic invariant manifolds. The version that we are using here is Fenichel's ([Fe]) theorem (see also [H,P,S]), who worked out carefully the smoothness of the perturbed invariant manifold. In our case, the ghost-torus will be as smooth as $\nabla L_\lambda$, due to the fact that $\nabla L_0 = 0$ on $G_{w,\lambda}^0$. 

17
6.12 Theorem (Global continuation)

For all $\lambda$ in $\Lambda$,

$$
\left( \frac{r}{G_{w,q}^\lambda} \right)^* : H^*(G_{w,q}^0) \to H^*(G_{w,q}^\lambda)
$$

is injective.

(hence $G_{w,q}^\lambda$ "contains" the topology of $T^n$).

This theorem also holds for $G_{w,q}^0$, $G_{w,q}^\lambda$ and $r$.

Proof: This is an immediate consequence of lemma 6.2, the fact that the $G_{w,q}^\lambda$'s are related by continuation and Theorem 2 of Floer [Fl1]. Indeed, we have collected here the three ingredients sufficient for his theorem of global continuation to apply, namely:

1. Normal hyperbolicity of $G_{w,q}^0$ (resp. $G_{w,q}^\lambda(C)$) for $\nabla L_0$ (resp. $\nabla L_0$).
2. $G_{w,q}^0$ (resp. $G_{w,q}^\lambda$) is a retract of the whole space $\phi_{w,q}^*$ (resp. $\phi_{w,q}$).
3. $G_{w,q}^\lambda$ is related to $G_{w,q}^0$ by continuation (resp. $G_{w,q}^\lambda$ with $G_{w,q}^0$).

(Thus this theorem fits more general situations).

Finally, we can conclude the proof of theorem 4.5: By theorem 6.12, $l(G_{w,q}^\lambda) \geq n + 1$ and $SB(G_{w,q}^\lambda) \geq 2^n$. The existence of $n + 1$ or $2^n$ ($w,q$)-periodic orbits follows from Conley-Zehnder’s Morse inequalities (valid for index pairs and thus a fortiori for isolated compact invariant sets) and the fact that we are dealing with a gradient flow (see [C-Z,1 and 2]).

§7. Some extensions of Theorem 4.5 and remarks about irrational rotation vectors

In [Go, Thm. 5.1], we prove a theorem of existence of periodic orbits for maps that are not known to continue to a completely integrable one.*

We now state this theorem and outline its proof.

---

*Let us point to the fact that a theorem of suspension of monotone maps by a (time dependent) Hamiltonian flow satisfying $Det H_{\frac{\partial^2}{\partial^2 t}} \neq 0$ would make theorem 7.1 a corollary of theorem 4.5 in the present paper. Such a theorem of suspension was proven for $n = 1$ (Twist maps) by Moser [Mo]. In higher dimension, one could follow the suggestion of Albert Fathi and find a clever homotopy of a given global generating function to a quadratic one. The hard part is to keep each function in the homotopy a globally generating one.
7.1 Theorem:

Let $F = F_1 \circ \ldots \circ F_s$ where each $F_i$ is a monotone map of $A^n$ generated by a global generating function $S_i$. Suppose that there is a set $\{A_i\}_{i=1}^s$ of symmetric, nondegenerate matrices such that

\begin{equation}
[A_i, A_j] = 0 \quad \forall i, j \in \{1, \ldots, s\},
\end{equation}

and such that all the $A_i$'s are positive definite (resp. negative definite) in the same subspaces of dimension $k_0$ (resp. $n - k_0$). Suppose that we have the following condition:

\begin{equation}
\lim_{||\theta - \tilde{\theta}|| \to \infty} \frac{1}{||\Theta - \tilde{\theta}||} \| \partial_\alpha S_i(\theta, \tilde{\Theta}) - (-1)^\alpha A_i(\Theta - \tilde{\theta}) \| = 0, \quad \alpha = 1 \text{ or } 2
\end{equation}

uniformly in $||\Theta - \tilde{\theta}||$. Then:

(i) $F$ has at least $n + 1$ geometrically distinct $(w, m)$-periodic orbits for each $(w, m) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $(w_i, m)$ are relatively prime, for at least one $i$.

Suppose in the following that all (prime) $(w, m)$-periodic orbits are nondegenerate. Then:

(ii) If either $m$ is odd, $s$ is even, or $k_0$ is even or equal to $n$ or 0, then $F$ has at least $2^n$ $(w, m)$-periodic orbits. Otherwise $F$ has at least $2^{n-1}$ of them.

Sketch of the proof:

As for theorem 4.5, we use the existence of an isolating block $(B(C)$ in this case) which looks like figure 5.8.

Conley and Zehender [C-Z, 2] were able to prove, through a thorough study of the cohomology $H(B, B^-)$ of the index pair, together with the use of their generalised Morse inequalities, the existence of at least $n + 1$ ($2^n$ non-degenerate) critical points for a gradient like flow with such an isolating block. The problem arises when one tries to count orbits and not points in our situation. For this, we have to quotient by the shift map $T$ defined in (5.21), which becomes a deck transformation of a cyclic covering space. Using a crucial part of Conley and Zehender's proof [C-Z, 2], we are able to prove that the cup length $l(G_{w,q} \geq l(G_{w,q})$, thus yielding part (i) of the theorem. To prove statement (ii) of the theorem is harder: one has to prove the theorem 7.5 for equivariant gradient flows. This theorem implies that Morse inequalities of Conley-Zehnder apply equally to the pairs $(B, B^-)$ and $(B, B^+)$, at least in the case where $T^*$ is the identity. This occurs in the cases described in (ii). In the other cases, we are only able to say that $(T^*)^2$ is the identity: we then have to apply theorem 7.5 to $T^2$ which explains the loss of information as one passes to the quotient in these cases.
7.5 Theorem: Let \( \pi: M \to \mathbf{M} \) be an \( q \)-fold \( C^k \) covering map between two \( C^k \) manifolds, with deck transformation group \( \{T_i\}_{i=1}^q \) such that \( T^m = \text{Id} \).

Let \( L: \mathbf{M} \to \mathbf{R} \) be a Morse function and \( L = L \circ \pi \).

Let \((B, B^-)\) be an index pair for \( L \), invariant under \( T \) and such that:

\[
T^*: H^*(B, B^-) \to H^*(B, B^-) \text{ is the identity}
\]

Denote \((\mathcal{B}, \mathcal{B}^-) = (\pi(B), \pi(B^-)). Then \((\mathcal{B}, \mathcal{B}^-)\) is an index pair for \( L \) and

\[
H^*(\mathcal{B}, \mathcal{B}^-) = H^*(B, B^-)
\]

(7.6)

An other extension of the result presented in this paper concerns a more detailed study of the topological structure of the ghost-tori \( G_{w,q} \). One would hope that, at least in the non-degenerate case, they actually contain a torus. If moreover this torus forms a graph over the \( x_0 \) axis, the critical points it contains would be natural candidates for the notion of well ordered, or Birkhoff orbits in higher dimensions. In that spirit, we are able to prove ([Go], section 14):

7.7 Proposition:

Let \( n = 1 \), i.e., let \( F \) be a Twist map of the annulus. The isolating block \( P \) (or \( B \)) contains an invariant subset for the flow \(-\zeta^t\), solution of \( \dot{x} = -\nabla L(x) \), which is homeomorphic to a homotopically non trivial circle. Moreover, this circle is made of unstable manifolds meeting tangentially at their endpoints, and each one projects diffeomorphically onto the \( x_0 \)-axis. of \( \psi_{w,q} \) (and hence of \( \tilde{A} \)).

The last statement of the proposition derives from the work of Angenent [An] on this gradient flow: he proves that it is order preserving. Note that if we could eliminate the possibility of cusps, the circle found in the proposition would be a smooth graph over the \( x_0 \) axis.

We end by making some remarks about possible uses of the Aubry setting in the search for orbits of irrational rotation vector. Let us remind the reader that Chen, Mckay and Meiss [Chen, Mck, M]

have found such orbits for a nontrivial class of monotone maps and that they use the discrete variational approach to do so.

*This theorem appears to be a special case of a recent result of C.MacCord and K.Mikhailov where they develop thoroughly an equivariant version of Conley’s theory for actions of compact groups [McC, M].
One way to attack the problem would be to consider the flow \( \dot{x} = -\nabla L(x) = \partial_1 S(x_i, x_{i+1}) + \partial_2 S(x_{i-1}, x_i) \), on the space of bi-infinite sequences of \( \mathbb{R}^n \). This flow turns out to be defined even though \( L \) is not. Within this big space one can look at the space

\[
Y = \bigcup_{\omega \in \mathbb{R}^n} Y_\omega = \bigcup_{\omega \in \mathbb{R}^n} \{ (x) \in (\mathbb{R}^n)^\mathbb{Z} / |x_j - j\omega| < \infty \}
\]

each slice of which is invariant under \( \nabla L_\lambda \), and diffeomorphic to \( l^\infty \). One could hope that rational \( G^\lambda_{w,q} \)'s would continue onto some invariant set \( G^\lambda_\omega \) in \( Y_\omega \), in the sense of Conley. Critical points in \( G^\lambda_\omega \) would automatically have rotation vector \( \omega \), as elements of \( Y_\omega \). There the main stumbling block is: even though \( Y \subset (\mathbb{R}^n) \) can be endowed with the product topology, for which \( \nabla L_\lambda \) is continuous, the "slices" \( Y_\omega \) are not closed in that topology and hence do not constitute a local product parametrization for the flow \( \nabla L_\lambda \) (with parameter \( \omega \in \mathbb{R}^n \): for a definition of local product parametrization, i.e. the setting in which Conley’s continuation is defined, see [C],[S]). The other problem, supposing one could prove the existence of a ghost-torus in each \( Y_\omega \), is that the flow is not automatically gradient like on this set, making it hard to use the Conley-Zehnder inequalities to find critical points. Note that \( Y_\omega \) contains a diffeomorphic image of the KAM torus of rotation vector \( \omega \) when it exists, as a compact invariant set for the variational flow.

Finally, we state a limiting theorem ([Go], thm. 13.2). \( (\mathbb{R}^n)^\mathbb{Z} \) can be given the coordinates \((\ldots, a_{-1}, x_0, a_1, \ldots) \) where \( a_i = x_i - x_{i-1} \). When we take the quotient by the action of \( \mathbb{Z}^n \), these gives us coordinates for \( T^n \times (\mathbb{R}^n)^\mathbb{Z} \). In these coordinates, we can define a retraction map:

\[
(7.8) \quad R : T^n \times (\mathbb{R}^n)^\mathbb{Z} \to T^n \times \{(0)\}
\]

\[
(x_0, a) \to (x_0, 0)
\]

which is continuous since projection mappings are continuous in the product topology. Also define \( K(C) \) to be a set of the form: \( T^n \times \text{infinite product of Ball of radius } C \), which is compact in the product topology. It is not hard to see that, given any compact range in the parameter set \( \Lambda \) we can find an appropriate \( C \) for which all ghost-tori in a bounded set of rotation vector belong to \( K(C) \). We can now state our result:

**Theorem 7.9:** Assume that \( S_\lambda \) are quadratic outside a bounded set (condition (4.4a)), then:

(a) Any sequence \( G^\lambda_{w_k,q_k} \) has a converging subsequence in \( K \), for the hausdorff metric on compact sets.

(b) If \( \mathcal{G} = \lim_{k \to \infty} G^\lambda_{w_k,q_k} \), then

\[
(\mathcal{R} \big|_\mathcal{G})^* : H^*(T^n \times \{(0)\}) \to H^*(\mathcal{G})
\]
is injective.

The proof uses the property of continuation of the Alexander cohomology. We cannot claim at this point that $\mathcal{G}$ (or any part of it) is included in $Y_\omega$, where $\omega = \lim(w_k/q_k)$.

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<table>
<thead>
<tr>
<th>#</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>506</td>
<td>Scott J. Spector</td>
<td>Linear Deformations as Global Minimizers in Nonlinear Elasticity</td>
</tr>
<tr>
<td>507</td>
<td>Denis Serre</td>
<td>Richness and the classification of quasilinear hyperbolic systems</td>
</tr>
<tr>
<td>508</td>
<td>L. Preziosi and F. Rosso</td>
<td>On the stability of the shearing flow between pipes</td>
</tr>
<tr>
<td>509</td>
<td>Avner Friedman and Wenxiong Liu</td>
<td>A system of partial differential equations arising in electrophotography</td>
</tr>
<tr>
<td>600</td>
<td>Jonathan Bell, Avner Friedman, and Andrew A. Lacey</td>
<td>On solutions to a quasilinear diffusion problem from the study of soft tissue</td>
</tr>
<tr>
<td>601</td>
<td>David G. Schaeffer and Michael Shearer</td>
<td>Loss of hyperbolicity in yield vertex plasticity models under nonproportional loading</td>
</tr>
<tr>
<td>602</td>
<td>Herbert C. Kranzer and Barbara Lee Keyfitz</td>
<td>A strictly hyperbolic system of conservation laws admitting singular shocks</td>
</tr>
<tr>
<td>603</td>
<td>S. Laederich and M. Levi</td>
<td>Qualitative dynamics of planar chains</td>
</tr>
<tr>
<td>604</td>
<td>Milan Miklavčič</td>
<td>A sharp condition for existence of an inertial manifold</td>
</tr>
<tr>
<td>605</td>
<td>Charles Collins, David Kinderlehrer, and Mitchell Luskin</td>
<td>Numerical approximation of the solution of a variational problem with a double well potential</td>
</tr>
<tr>
<td>606</td>
<td>Todd Arbogast</td>
<td>Two-phase incompressible flow in a porous medium with various nonhomogeneous boundary conditions</td>
</tr>
<tr>
<td>607</td>
<td>Peter Poláčik</td>
<td>Complicated dynamics in scalar semilinear parabolic equations in higher space dimension</td>
</tr>
<tr>
<td>608</td>
<td>Bei Hu</td>
<td>Diffusion of penetrant in a polymer: a free boundary problem</td>
</tr>
<tr>
<td>609</td>
<td>Mohamed Sami ElBialy</td>
<td>On the smoothness of the linearization of vector fields near resonant hyperbolic rest points</td>
</tr>
<tr>
<td>610</td>
<td>Max Jodeit, Jr. and Peter J. Olver</td>
<td>On the equation  [ \text{grad} f = M \text{grad} g ]</td>
</tr>
<tr>
<td>611</td>
<td>Shui-Nee Chow, Kening Lu, and Yun-Qiu Shen</td>
<td>Normal form and linearization for quasiperiodic systems</td>
</tr>
<tr>
<td>612</td>
<td>Prabir Daripa</td>
<td>Theory of one dimensional adaptive grid generation</td>
</tr>
<tr>
<td>613</td>
<td>Michael C. Mackey and John G. Milton</td>
<td>Feedback, delays and the origin of blood cell dynamics</td>
</tr>
<tr>
<td>614</td>
<td>D.G. Aronson and S. Kamin</td>
<td>Disappearance of phase in the Stefan problem: one space dimension</td>
</tr>
<tr>
<td>615</td>
<td>Martin Krupa</td>
<td>Bifurcations of relative equilibria</td>
</tr>
<tr>
<td>616</td>
<td>D.D. Joseph, P. Singh, and K. Chen</td>
<td>Couette flows, rollers, emulsions, tall Taylor cells, phase separation and inversion, and a chaotic bubble in Taylor-Couette flow of two immiscible liquids</td>
</tr>
<tr>
<td>617</td>
<td>Artemio González-López, Niky Kamran, and Peter J. Olver</td>
<td>Lie algebras of differential operators in two complex variables</td>
</tr>
<tr>
<td>618</td>
<td>L.E. Fraenkel</td>
<td>On a linear, partly hyperbolic model of viscoelastic flow past a plate</td>
</tr>
<tr>
<td>619</td>
<td>Stephen Schecter and Michael Shearer</td>
<td>Undercompressive shocks for nonstrictly hyperbolic conservation laws</td>
</tr>
<tr>
<td>620</td>
<td>Xinfu Chen</td>
<td>Axially symmetric jets of compressible fluid</td>
</tr>
<tr>
<td>621</td>
<td>J. David Logan</td>
<td>Wave propagation in a qualitative model of combustion under equilibrium conditions</td>
</tr>
<tr>
<td>622</td>
<td>M.L. Zeeman</td>
<td>Hopf bifurcations in competitive three-dimensional Lotka-Volterra Systems</td>
</tr>
<tr>
<td>623</td>
<td>Allan P. Fordy</td>
<td>Isopectral flows: their Hamiltonian structures, Miura maps and master symmetries</td>
</tr>
<tr>
<td>624</td>
<td>Daniel D. Joseph, John Nelson, Michael Renardy, and Yuriko Renardy</td>
<td>Two-Dimensional cusped interfaces</td>
</tr>
<tr>
<td>625</td>
<td>Avner Friedman and Bei Hu</td>
<td>A free boundary problem arising in electrophotography</td>
</tr>
<tr>
<td>626</td>
<td>Hamid Bellout, Avner Friedman and Victor Isakov</td>
<td>Stability for an inverse problem in potential theory</td>
</tr>
<tr>
<td>627</td>
<td>Barbara Lee Keyfitz</td>
<td>Shocks near the sonic line: A comparison between steady and unsteady models for change of type</td>
</tr>
<tr>
<td>628</td>
<td>Barbara Lee Keyfitz and Gerald G. Warnecke</td>
<td>The existence of viscous profiles and admissibility for transonic shocks</td>
</tr>
<tr>
<td>629</td>
<td>P. Szmolyan</td>
<td>Transversal heteroclinic and homoclinic orbits in singular perturbation problems</td>
</tr>
<tr>
<td>630</td>
<td>Philip Boyland</td>
<td>Rotation sets and monotone periodic orbits for annulus homeomorphisms</td>
</tr>
<tr>
<td>631</td>
<td>Kenneth R. Meyer</td>
<td>Apollonius coordinates, the N-body problem and continuation of periodic solutions</td>
</tr>
<tr>
<td>632</td>
<td>Chjian C. Lim</td>
<td>On the Poincare–Whitney circuitspace and other properties of an incidence matrix for binary trees</td>
</tr>
</tbody>
</table>
equations on an infinite interval using piecewise constant arguments

634 Stanley Minkowitz and Matthew Witten, Periodicity in cell proliferation using an asynchronous cell population

635 M. Chipot and G. Dal Maso, Relaxed shape optimization: The case of nonnegative data for the Dirichlet problem

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