ROBUST STABILIZATION OF SYSTEMS
GOVERNED BY SINGULAR
INTEGRO-DIFFERENTIAL EQUATIONS

By

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Robust Stabilization of Systems Governed by Singular Integro-Differential Equations

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Abstract

In this paper we consider the problem of stabilization of systems described by a class of singular integro-differential equations. Systems of this type appear, for example, in the modelling of certain aeroelastic control problems. We show the existence of finite dimensional stabilizing controllers for the above systems and provide a procedure for their construction. The main idea is to represent the original distributed plant as an $H^\infty$ coprime factor perturbation of a finite dimensional system and to use the theory of robustness optimization in the gap metric.

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1 Introduction

In this paper we are interested in the control of systems described by the following integro-differential equations:

\[
\frac{d}{dt} x_1(t) = Ax_1(t) + b_1 x_2(t) + b_2 \int_{-\infty}^{0} \kappa_2(\tau)x_2(\tau + t) d\tau + bu(t) \tag{1.1}
\]

\[
\frac{d}{dt} \int_{-\infty}^{0} \kappa_1(\tau)x_2(\tau + t) d\tau = c_0 x_1(t) \tag{1.2}
\]

where \( x_1(t) \in \mathbb{R}^n \) and \( x_2(t) \in \mathbb{R}, t \geq 0 \), and \( A, b, b_1, b_2, c_0 \) are appropriate size matrices. We want to find a command signal \( u(t) \in \mathbb{R}, t \geq 0 \), such that the system is stable (in the sense defined below). Given all the matrices \( A, b, b_1, b_2, c_0 \), and the kernels \( \kappa_1(t), \kappa_2(t), t < 0 \) we can find the solution to (1.1)-(1.2) from the initial conditions \( x_1(0), x_2(t) \ t < 0 \), and the input \( u(t), t \geq 0 \). The above system is a linear time invariant system whose “state space” is infinite dimensional. In this paper we will use frequency domain methods to analyze the system and synthesize a controller generating \( u \). We will not discuss the state space realizations of the plant.

Note that if we define \( k_i(t) = \kappa_i(-t), t > 0, i = 1, 2, \) then we have

\[
\int_{-\infty}^{0} \kappa_i(\tau)x_2(\tau + t) d\tau = \int_{0}^{\infty} k_i(t)x_2(t - \tau) dt = (k_i \ast x_2)(t)
\]

where \( \ast \) denotes the usual convolution operator. Assuming zero initial conditions \( (x_2(t) = 0, t \leq 0) \) and taking the Laplace transform of both sides of (1.2) we obtain

\[
s\hat{k}_1(s)\hat{x}_2(s) = c_0\hat{x}_1(s) \tag{1.3}
\]

where \( s \) is the Laplace transform variable and \( \hat{\cdot} \) denotes the Laplace transform of a time signal, e.g.

\[
\hat{k}_i(s) = \int_{0}^{\infty} e^{-st}k_i(t) dt, \quad i = 1, 2,
\]

we assume that the right hand side converges in a half plane \( \text{Re } s \geq \sigma \) for some \( \sigma \in \mathbb{R} \).

Consequently, we can express \( \hat{x}_2(s) \) in terms of \( \hat{x}_1(s) \):

\[
\hat{x}_2(s) = \frac{1}{s\hat{k}_1(s)} c_0 \hat{x}_1(s). \tag{1.4}
\]
Similarly, taking the Laplace transform of (1.1) we obtain (assuming $x_1(0) = 0$)

$$s \hat{x}_1(s) = A\hat{x}_1(s) + b_1\hat{x}_2(s) + b_2k_2(s)\hat{x}_2(s) + b\hat{u}(s).$$

(1.5)

Substituting (1.4) into (1.5) we obtain an expression relating the input, $\hat{u}(s)$, to the “states” $\hat{x}_1(s)$, and $\hat{x}_2(s)$:

$$\hat{x}_1(s) = \left( sI - A - \frac{b_1c_0 + b_2c_0k_2(s)}{sk_1(s)} \right)^{-1} b\hat{u}(s)$$

(1.6a)

$$\hat{x}_2(s) = \frac{1}{sk_1(s)}c_0\hat{x}_1(s).$$

(1.6b)

Equations (1.6a-b) will be used to find a “stabilizing” controller which generates an appropriate command signal $\hat{u}(s)$.

In our synthesis we will consider a feedback control scheme which is shown in Figure 1.

![Figure 1](image)

In this configuration $y$ represents the output of the plant $P$, whose dynamics are described by (1.6a-b); $y$ is a measured physical quantity, usually a combination of the states, $x_1$ and $x_2$. The signal $u$ is the command input to the plant, $d$ is the disturbance, $r$ is the reference input, and $e$ is the measured error between the reference and the output. The controller (to be designed) is represented by the block $C$.

According to our zero initial conditions assumption the plant $P$ responds to $u$ only, and the closed loop system has two exogenous inputs, namely $r$ and $d$. We will consider finite energy reference and and disturbance signals, i.e. $r, d \in L^2[0, \infty)$. 

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Definition 1.1 The closed loop system shown in Figure 1 is stable if all external inputs \( r, d \in L^2[0, \infty) \) give rise to signals \( e, u, y \in L^2[0, \infty) \), (i.e. finite energy inputs generate finite energy outputs), and the maximum energy amplification in the system is finite, i.e.

\[
\max \left\{ \sup_{0 \neq r \in L^2} \frac{\|e\|_2 + \|u\|_2}{\|r\|_2} \mid d = 0; \sup_{0 \neq d \in L^2} \frac{\|e\|_2 + \|u\|_2}{\|d\|_2} \right\} < \infty.
\]

It is well known that this definition of the closed loop stability is equivalent to having the entries of the transfer function from \( \begin{bmatrix} r \\ d \end{bmatrix} \) to \( \begin{bmatrix} e \\ u \end{bmatrix} \):

\[
T_{(P,C)} := \begin{bmatrix} (I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix}
\]

in \( H^\infty \), i.e. analytic in the right half plane \( \text{Re } s > 0 \) and bounded on the extended imaginary axis: \( \{j\omega : \omega \in \mathbb{R}\} \cup \{\infty\} \). All these transfer functions relate Laplace transforms of the inputs to the Laplace transforms of the outputs, so they are functions of the complex variable \( s \).

An important point to remark here is that the plant transfer function \( P(s) \) (which gives the output \( \hat{y}(s) \) as \( P(s) \hat{u}(s) \)) need not be a rational function. It was shown that (cf. [22]) the plant is stabilizable (i.e. there exists a controller \( C \), whose entries are in the quotient field of \( H^\infty \), satisfying the above definition of the stability) if and only if there exists strongly left and right coprime factorizations for \( P \) in \( H^\infty \), i.e. \( P = NM^{-1} = \tilde{M}^{-1}\tilde{N} \), for some appropriate size matrices \( N, M, \tilde{N}, \tilde{M} \) with entries in \( H^\infty \) and \( \begin{bmatrix} M \\ N \end{bmatrix} \) (resp. \( [M \quad \tilde{N}] \)) has a left (resp. right) inverse whose entries are in \( H^\infty \). From now on we write \( N \in H^\infty \), if a matrix \( N \) has entries in \( H^\infty \). If the plant is stabilizable then there exist (cf. [22]) appropriate size \( U, V, \tilde{U}, \tilde{V} \in H^\infty \) such that

\[
\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} V \\ -N \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.7}
\]

and the set of all stabilizing controllers is given by

\[
\{ C = (U - MQ)(V + NQ)^{-1} : Q \in H^\infty \} = \{ C = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} - Q\tilde{M}) : Q \in H^\infty \}.
\]
For the system described by (1.6a-b) we will assume that the output of the plant $P$ consists of a combination of the "states" $x_1$ and $x_2$: $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $y_1 = c_1 x_1$ and $y_2 = c_2 x_2$ with $c_1 : 1 \times n$ non-zero constant vector and $c_2$ is a non-zero scalar. For simplicity we assume $c_1 = c_0$, i.e. the right hand side of (1.2) is the measured output $y_1$. Then, the plant transfer function is of the form $P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix}$ where

$$P_1(s) := c_1 \left( sI - A - \frac{b_1 c_1 + b_2 c_1 \hat{k}_2(s)}{s \hat{k}_1(s)} \right)^{-1} b$$

$$P_2(s) := \frac{c_2}{s \hat{k}_1(s)} P_1(s).$$

To obtain a simplified expression for $P_1(s)$ we define

$$F(s) = \left( sI - A - \frac{b_2 c_1 \hat{k}_2(s)}{s \hat{k}_1(s)} \right)^{-1}.$$  

Then simple calculations lead to $P_1(s) = c_1 \left( I - \frac{F(s) b_1 c_1}{s \hat{k}_1(s)} \right)^{-1} F(s) b$. Applying the matrix inversion lemma (see e.g. [19], p.19) to $F(s)$ and to $\left( I - \frac{F(s) b_1 c_1}{s \hat{k}_1(s)} \right)^{-1}$ we see that

$$F(s) = \left( sI - A \right)^{-1} + \frac{\left( sI - A \right)^{-1} b_2 c_1 \left( sI - A \right)^{-1}}{s \hat{k}_1(s) / \hat{k}_2(s) - c_1 \left( sI - A \right)^{-1} b_2}$$

and, after simplifications,

$$P_1(s) = \frac{c_1 \left( sI - A \right)^{-1} b}{1 - \left( 1/s \hat{k}_1(s) \right) \left( \hat{k}_2(s) c_1 \left( sI - A \right)^{-1} b_2 - c_1 \left( sI - A \right)^{-1} b_1 \right)}, \quad (1.8a)$$

and hence

$$P_2(s) = \frac{c_2}{s \hat{k}_1(s)} \frac{c_1 \left( sI - A \right)^{-1} b}{1 - \left( 1/s \hat{k}_1(s) \right) \left( \hat{k}_2(s) c_1 \left( sI - A \right)^{-1} b_2 - c_1 \left( sI - A \right)^{-1} b_1 \right)}. \quad (1.8b)$$

Note that the transfer function from $u$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ contains strictly proper rational terms $c_1 \left( sI - A \right)^{-1} b$, $c_1 \left( sI - A \right)^{-1} b_1$ and $c_1 \left( sI - A \right)^{-1} b_2$, along with the terms $\frac{1}{s \hat{k}_1(s)}$ and $\hat{k}_2(s)$ which may be irrational. Therefore, depending on the kernels $k_i(t)$, $t > 0$, the plant transfer function $P(s)$ will, possibly, be a non-rational function of $s$. This illustrates the infinite dimensionality
of the plant. Our objective is to find a finite dimensional controller, (i.e. \( C(s) \) rational), stabilizing this infinite dimensional plant.

Before going into details we want to summarize the idea briefly. First of all in order to find such a controller the plant \( P(s) \) must be stabilizable. This means that we should be able to find strongly coprime factorizations \( P(s) = N(s)M(s)^{-1} = \tilde{M}(s)^{-1}\tilde{N}(s) \) for some \( N, M, \tilde{N}, \tilde{M} \in H^\infty \) and \( (N, M) \) and \( (\tilde{M}, \tilde{N}) \) strongly coprime. This is equivalent to having (see [20] pp. 292–295)

\[
\lambda := \inf\limits_{\text{Re } s > 0} \sigma_{\text{min}} \left[ \begin{array}{c} N(s) \\ M(s) \end{array} \right] > 0,
\]

and

\[
\tilde{\lambda} := \inf\limits_{\text{Re } s > 0} \sigma_{\text{min}}[\tilde{N}(s) \quad \tilde{M}(s)] > 0,
\]

where \( \sigma_{\text{min}}(\cdot) \) denotes the minimum singular value. Then, we will approximate the plant by a finite dimensional system (a rational transfer function); i.e. given arbitrarily small number \( \epsilon > 0 \) we will find two rational matrices \( N_f(s) \) and \( M_f(s) \) with entries in \( H^\infty \) and coprime satisfying \( N = N_f + \Delta_N, M = M_f + \Delta_M \) and \( \| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \|_\infty < \epsilon. \) So the underlying assumptions on \( k_i \)'s are such that this kind of approximation of the plant is possible. Then, we know that by choosing \( \epsilon > 0 \) sufficiently small we can guarantee the existence of a rational controller stabilizing \( P_f = N_fM_f^{-1} \) as well as \( P = NM^{-1} \). This controller can be computed from the theory of robustness optimization in the gap metric, and using standard techniques of \( H^\infty \) optimal control, the details of this argument are given in Section 3.

We conclude this section with a few remarks to put our work here in perspective for the reader. The system described by (1.1) and (1.2) has been proposed by several authors (see e.g. [3], [2], [4], [5]) to model the elastic motions of three degrees of freedom thin airfoils in two dimensional unsteady flows. Our zero initial conditions assumption corresponds to the so-called indicial problem (cf. [3] p. 291). For the sake of completeness we want to mention some relevant work on the finite delay version of the system (1.1)-(1.2) (the integrals are taken from a finite time, say \(-r\) to 0, instead of \(-\infty\) to 0). For example well-posedness of these type of systems in different state spaces are studied in [5], [6], [14]. Approximation and
control issues are discussed in detail in [16], [17] using semigroup framework and non-zero initial conditions. Also, recently a well-posedness result for scalar equations of the type (1.2), in weighted $L^1$ spaces, is obtained in [15].

The remaining part of this paper is organized as follows. In Section 2 we provide a brief discussion of the aeroelastic control problem that motivates the study of the singular integro-differential system (1.1)-(1.2). Section 3 contains the existence result (Theorem 3.3) for the finite dimensional stabilizing controllers; in Section 4 we summarize an algorithm for construction of these controllers. In Section 5 we make some remarks on the stability definition used in this paper and relations to the "boundedness" of the components of $x_1$ and $x_2$; also in the same section we discuss the assumptions used in Section 3 and their relations to the aeroelastic model described in Section 2. Finally in the last section we draw our conclusions.

2 An Aeroelastic System

Consider the thin airfoil (typical cross-section) shown in Figure 2.

![Figure 2](image)

We assume that the airfoil is placed in an incompressible, inviscid two-dimensional flow. The
basic equations for the pitching, plunging and flap motions (see e.g. [4]) can be written as

$$M \frac{d^2 z(t)}{dt^2} + B \frac{dz(t)}{dt} + K z(t) = \frac{1}{m} F(t) + G u(t)$$

(2.1)

where $z(t) = [h(t), \alpha(t), \beta(t)]^T$ represents the plunging, pitching and flap motions of the airfoil, $u(t)$ is a control torque on the flap hinge line, $F(t) = [L(t), M_\alpha(t), M_\beta(t)]^T$ is the vector of aerodynamic loads corresponding to lift, pitching moment and flap moment, respectively, $M$, $B$, and $K$ are constant matrices and $m$ is a constant. Let $U$ denote the undisturbed stream velocity, $\phi(t, x, y)$ the disturbance velocity potential and define the downwash function $w_a(t, x)$ by

$$w_a(t, x) = \left\{ \begin{array}{ll} -\frac{d}{dt} h(t) - (x - a) \frac{d}{dx} \alpha(t) - U \alpha(t); & -1 < x < c \\
-\frac{d}{dt} h(t) - (x - a) \frac{d}{dx} \alpha(t) - U \alpha(t) - (x - c) \frac{d}{dx} \beta(t) - U \beta(t) & c < x < 1. \end{array} \right.$$  

(2.2)

(The downwash function represents the vertical velocity component of the airfoil.)

The disturbance velocity is given by the gradient of the potential function $\phi$ which in turn satisfies Laplace’s equation

$$\frac{\partial^2}{\partial x^2} \phi(t, x, y) + \frac{\partial^2}{\partial y^2} \phi(t, x, y) = 0; \quad t > 0$$

(2.3)

with boundary conditions

$$\frac{\partial}{\partial y} \phi(t, x, 0^+) = w_a(t, x), \quad -1 < x < 1, \quad t > 0$$

(2.4)

$$\frac{\partial}{\partial y} \phi(t, x, 0^+) + U \frac{\partial}{\partial t} \phi(t, x, 0^+) = 0, \quad |x| \geq 1, \quad t > 0;$$

(2.5)

$$\frac{\partial}{\partial t} \phi(t, 1^-, 0^+) + U \frac{\partial}{\partial x} \phi(t, 1^-, 0^+) = 0, \quad t > 0.$$  

(2.6)

To derive a model (appropriate for control design) one seeks a solution of (2.3)-(2.6) of the form

$$\phi(t, x, y) = -\frac{1}{2\pi} \int_{-1}^{\infty} \gamma(t, \zeta) \tan^{-1}(\frac{y}{x - \zeta}) d\zeta,$$

(2.7)

where the integral is taken in the Cauchy sense and the function $\gamma(t, x), t > 0, \quad -1 < x < \infty$, represents the circulation per unit distance, and it is decomposed into

$$\gamma(t, x) = \left\{ \begin{array}{ll} \gamma_a(t, x) & -1 \leq x \leq 1, \quad t > 0 \\
\gamma_w(t, x) & 1 < x < +\infty, \quad t > 0; \end{array} \right.$$  

(2.8)
moreover it is assumed that the function \( t \rightarrow \int_{-1}^{1} \gamma_a(t, x) dx \) is continuously differentiable for \( t > 0 \). Given an “initial wake history” \( q = (\eta, \phi(\cdot)) \in \mathbb{R} \times L^1(\mathbb{R}) \) (\( q \) is assumed to be zero in this paper, this problem is called the indicial problem), define the extended total airfoil circulation, \( \Gamma_q : (-\infty, \infty) \rightarrow \mathbb{R} \), by

\[
\Gamma_q(t) = \begin{cases} 
\int_{-\infty}^{t} \phi(\tau) d\tau, & t < 0 \\
\eta + \int_{0}^{t} \frac{\partial}{\partial \tau} \Gamma_q(\tau, \zeta) d\zeta d\tau, & t \geq 0.
\end{cases}
\] (2.9)

It can be shown that the aerodynamic loads, \( F(t) \), can be represented as “functionals” of \( \Gamma_q \) and its history. In particular (see [4]) one has

\[
L(t) = \frac{d}{dt} \int_{-\infty}^{0} c_1(\tau) \Gamma_q^\prime(\tau + t) d\tau + c_1 \left[ \frac{d}{dt} z(t) \right],
\] (2.10)

where \( c_1(\tau) = -\rho \sqrt{U} \tau^2 - 2U \tau \) (\( \rho \) is the fluid density), \( c_1 \) is a \( 1 \times 6 \) vector of constants, and \( \Gamma_q^\prime(t + \tau) := \frac{d}{dt} \Gamma_q(t + \tau) \).

Likewise, it can be shown that \( M_\alpha(t) \) and \( M_\beta(t) \) have the form

\[
M_\alpha(t) = D_0 \left[ \frac{\Gamma_q^\prime(t)}{\Gamma_q(t)} \right] + \int_{-\infty}^{0} D_0(\tau) \Gamma_q^\prime(t + \tau) d\tau + D_1 \left[ \frac{z(t)}{\frac{d}{dt} z(t)} \right] + \frac{d}{dt} \int_{-\infty}^{0} D_1(\tau) \Gamma_q^\prime(t + \tau) d\tau \] (2.11)

and

\[
M_\beta(t) = E_0 \left[ \frac{\Gamma_q^\prime(t)}{\Gamma_q(t)} \right] + \int_{-\infty}^{0} E_0(\tau) \Gamma_q^\prime(t + \tau) d\tau + E_1 \left[ \frac{z(t)}{\frac{d}{dt} z(t)} \right] + \frac{d}{dt} \int_{-\infty}^{0} E_1(\tau) \Gamma_q^\prime(t + \tau) d\tau. \] (2.12)

Equations (2.10)-(2.12) provide “constitutive” equations relating the aerodynamic loads to the circulation function \( \Gamma_q(t) \). All that remains is to derive an equation of evolution for \( \Gamma_q(t) \). Following the procedure found in [4] one obtains the equation

\[
-\frac{1}{2\pi} \int_{-1}^{1} \frac{\gamma_a(t, \zeta)}{(x - \zeta)} d\zeta = w_a(t, x) - \frac{1}{2\pi} \int_{-\infty}^{0} \Gamma_q^\prime(t + \tau) \left( \frac{1}{x - 1 + U\tau} \right) d\tau
\] (2.13)

that is “inverted” by using the Söhnge inversion formula (see [14] for the details). In particular (2.13) can be solved for \( \gamma_a(t, x) \) giving

\[
\gamma_a(t, x) = \frac{2}{\pi} \sqrt{\frac{1 - x}{1 + x}} \left( \int_{-1}^{1} \frac{1 + y}{1 - y} \left( \frac{w_a(t, y)}{x - y} + \int_{-\infty}^{0} \Gamma_q^\prime(t + \tau) \left( \frac{1}{y - 1 + U\tau} \right) d\tau \right) dy \right).
\] (2.14)
Integrating (2.14) one obtains
\[
\Gamma_q(t) = \int_{-1}^{1} \gamma_a(t, x) = 2 \int_{-1}^{1} \sqrt{\frac{1 + y}{1 - y}} w_a(t, y) dy - \frac{1}{2\pi} \int_{-\infty}^{0} H(\tau) \Gamma'_q(t + \tau) d\tau,
\]  
(2.15)
where \( H(\tau) = 2\pi \left( \sqrt{\frac{U_T^2 - 2}{U_T^2}} - 1 \right) \). Equation (2.15) can be reduced to the form
\[
\int_{-\infty}^{0} K(\tau) \Gamma'_q(t + \tau) d\tau = \left( \int_{-\infty}^{0} \phi(\tau)d\tau - \eta \right) + B \left[ \frac{z(t)}{dt} \right],
\]  
(2.16)
where \( K(\tau) = \sqrt{\frac{U_T^2 - 2}{U_T^2}} \). Note that (2.16) provides the basic evolution equation for \( \Gamma_q(t) \). The model is completed by differentiating (2.16) to obtain
\[
\frac{d}{dt} \int_{-\infty}^{0} K(\tau) \Gamma'_q(t + \tau) d\tau = B \left[ \frac{z(t)}{dt} \right]
\]  
(2.17)
and augmenting (2.17) to (2.1). The constitutive relations (2.10)-(2.12) relate \( F(t) \) to \( \Gamma_q(t) \). Thus, if one defines
\[
x_1(t) = [h(t), \alpha(t), \beta(t), \frac{dh(t)}{dt}, \frac{d\alpha(t)}{dt}, \frac{d\beta(t)}{dt}, \Gamma_q(t)]^T
\]  
(2.18)
and
\[
x_2(t) = \Gamma'_q(t) = \frac{d}{dt} \Gamma_q(t),
\]  
(2.19)
the resulting equations can be written as
\[
\frac{d}{dt} \left( A_1 \dot{x}_1(t) + \int_{-\infty}^{0} A_1(\tau) x_2(t + \tau)d\tau \right) = B_1 \dot{x}_1(t) + B_2 x_2(t) + \int_{-\infty}^{0} B_1(\tau) x_2(t + \tau)d\tau + Gu(t)
\]  
(2.20)
and
\[
\frac{d}{dt} \int_{-\infty}^{0} K(\tau) x_2(t + \tau)d\tau = C_2 \dot{x}_1(t)
\]  
(2.21)
where \( A_1, B_1, B_2, C_2 \) and \( A_1(\tau), B_1(\tau) \) are appropriate size constant matrices and matrix valued functions respectively, and \( K(\tau) = \sqrt{\frac{U_T^2 - 2}{U_T^2}} \). Furthermore, since \( A_1 \) is nonsingular it can be taken to the right hand side of (2.20), also since \( A_1(\tau) \) is smooth, the term \( \frac{d}{dt} \int_{-\infty}^{0} A_1(\tau) x_2(t + \tau)d\tau \) can be taken to the left hand side of (2.20) and it can be combined with the term involving \( B_1(\tau) \). Thus (2.20)-(2.21) can be represented in the form (1.1)-(1.2).
with \( b_2 \kappa_2(\tau) \) of (1.1) being a \( 1 \times n \) valued function obtained from (2.20) using above arguments. In order to have a relatively simpler model we assume that \( b_2 \) is a \( 1 \times n \) constant vector and \( \kappa_2(\tau) \) is a scalar function.

It seems that (1.1) and (1.2) is a “feasible” mathematical model to study various control problems associated with the aeroelastic systems, where the motions of the structure are coupled to the motion of the fluid. Here we consider the indicial problem (zero wake history, in other words the unsteady motion begins at \( t = 0 \)) and our main interest is to find finite dimensional controllers (say e.g. for flutter suppression, [1]) that robustly stabilize the systems in the sense of the set-up shown in Figure 1.

3 Stabilization of the Plant

In this section we consider the plant described by equations (1.1)-(1.2) with zero initial conditions. In the frequency domain the plant transfer function, \( P(s) \), is given by (1.8a-b). We will now discuss in detail the problem of stabilizing \( P \) by a finite dimensional controller \( C \).

Let us begin by studying the stabilizability conditions for the plant \( P \). The issue is to find strongly coprime factorizations for

\[
P(s) = \begin{bmatrix} P_1(s) \\ c_2(s \hat{k}_1(s))^{-1} P_1(s) \end{bmatrix}.
\]

Therefore it is necessary that \( P_1 \) is stabilizable, and the transfer function \( (s \hat{k}_1(s))^{-1} \) admits a coprime factorization in \( H^\infty \). So, we will assume that there exists \( N_{k_1}, M_{k_1} \in H^\infty \) such that \( N_{k_1}(s)/M_{k_1}(s) = c_2(s \hat{k}_1(s))^{-1} \) and the pair \( (N_{k_1}, M_{k_1}) \) is coprime, i.e.

\[
\inf_{\Re s > 0} (|N_{k_1}(s)|^2 + |M_{k_1}(s)|^2) > 0.
\]

In order to obtain simple conditions on stabilizability of \( P_1 \) we make certain mild assumptions. Suppose that the kernel \( k_2(t), t \geq 0 \), is in \( L^1[0, \infty) \); this guarantees that \( \hat{k}_2 \in H^\infty \). Then, assume that the pairs \( (A, b), (A, b_1) \) and \( (A, b_2) \) are “stabilizable” (in the classical
finite dimensional linear system theoretic sense), and the pair \((c_1, A)\) is detectable. With this assumption we can find rational functions \(N_b, N_{b_1}, N_{b_2}, M_a \in H^\infty\) such that
\[
\frac{N_b(s)}{M_a(s)} = c_1(sI - A)^{-1}b \quad \text{and} \quad \frac{N_{b_i}(s)}{M_a(s)} = c_1(sI - A)^{-1}b_i, \quad i = 1, 2,
\]
and the pairs \((N_b, M_a), (N_{b_1}, M_a), (N_{b_2}, M_a)\) are coprime. Thus, \(P_1(s)\) can be rewritten as
\[
P_1(s) = \frac{N_b(s)M_{k_1}(s)}{M_{k_1}(s)M_a(s) - N_{k_1}(s)(N_{b_2}(s) + \hat{k}_2(s)N_{b_1}(s))}.
\]
(3.1)

Defining
\[
N_1(s) = N_b(s)M_{k_1}(s)
\]
(3.2a)
and
\[
M_1(s) = M_{k_1}(s)M_a(s) - N_{k_1}(s)(N_{b_2}(s) + \hat{k}_2(s)N_{b_1}(s))
\]
(3.2b)
we see that stabilizability of \(P_1\) is equivalent to having \((N_1, M_1)\) strongly coprime.

Recall that \(N_b(s)\) is rational, so it has zeros in the extended closed right half plane at only finitely many points, say \(s_1, \ldots, s_{\ell}\), with multiplicities \(m_1, \ldots, m_{\ell}\) respectively. Then, for \((N_1, M_1)\) to be strongly coprime we have the following two necessary conditions
\[
(i) \quad \left. \frac{\partial^i}{\partial s^i} \left[ M_{k_1}(s)M_a(s) - N_{k_1}(s)(N_{b_2}(s) + \hat{k}_2(s)N_{b_1}(s)) \right] \right|_{s = s_j} \neq 0
\]
for all \(i = 1, \ldots, m_j\), and \(j = 1, \ldots, \ell\);
\[
(ii) \quad \text{the pair } (M_{k_1}, (N_{b_2} + \hat{k}_2N_{b_1})) \text{ is strongly coprime.}
\]
It is also not difficult to see from (3.1) that if both (i) and (ii) are satisfied then the pair \((N_1, M_1)\) is strongly coprime hence \(P_1\) is stabilizable.

Now returning back to the original plant \(P\) we see that, with the above notation and assumptions, we have
\[
P(s) = \begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} M_1(s)^{-1},
\]
where \(N_1(s) = N_b(s)M_{k_1}(s), N_2(s) = N_b(s)N_{k_1}(s)\), and \(M_1(s)\) is given by (3.2b). Therefore, for stabilizability of \(P_2\), in addition to condition (i) we need
\[
(iii) \quad \text{the pair } (N_{k_1}, M_a) \text{ is strongly coprime.}
\]
Moreover (i) and (iii) guarantees that \((N_2, M_1)\) is coprime, and hence \(P_2\) is stabilizable. Thus, we have

**Lemma 3.1** Consider the plant \(P(s)\) described by (1.8a,b), and suppose that \(k_2 \in L^1[0, \infty)\) and \((A, b), (A, b_1), (A, b_2)\) are stabilizable and \((c_1, A)\) is detectable. Then, the plant \(P\) is stabilizable if and only if (i), (ii) and (iii) hold. Moreover, if these conditions are satisfied a right (resp. left) strongly coprime factorization is given by

\[
P = NM^{-1}, \quad \text{where } N := \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, M := M_1,
\]

(resp.)

\[
P = \tilde{M}^{-1} \tilde{N}, \quad \text{where } \tilde{N} := N, \tilde{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.
\]

Furthermore, we have

\[
\lambda^2 := \inf_{\Re s > 0} \left( |N_1(s)|^2 + |N_2(s)|^2 + |M_1(s)|^2 \right) > 0,
\]

and

\[
\bar{\lambda}^2 := \inf_{\Re s > 0} \min\{(|N_1(s)|^2 + |M_1(s)|^2), (|N_2(s)|^2 + |M_1(s)|^2)\} > 0.
\]

**Proof:** Immediate from the above arguments and the results of [22]. \(\Box\)

These conditions guarantee stabilizability of the plant \(P\), i.e. there exists a controller, possibly infinite dimensional, stabilizing the closed loop system. However, our purpose is to find, if possible, a finite dimensional controller, rational \(C(s)\), stabilizing the closed loop system. In order to guarantee the existence of such a controller we need some more conditions on the "infinite dimensional" parts \((sk_1(s))^{-1}\) and \(k_2(s)\). We now want to discuss this issue. As mentioned in Section 1 we will "approximate" these terms by rational functions to obtain a new finite dimensional plant which is "close" to the original plant in the gap metric, (see [9] for a precise definition and details). Then, we will find a rational controller (from the new plant) which stabilizes the approximate plant as well the original plant.

A simple sufficient condition for the existence of rational approximations is that the functions \(M_{k_1}(j\omega), N_{k_1}(j\omega)\) and \(\hat{k}_2(j\omega)\) are continuous for all \(\omega \in \mathbb{R} \cup \{\infty\}\). So we will assume
that this condition holds. Then, we know that these functions are uniformly approximable by rational functions in $H^\infty$ (see e.g. [13]), i.e. given arbitrary small $\epsilon > 0$, there exists rational functions $M^f_{k_1}$, $N^f_{k_1}$, $\hat{k}^f_2 \in H^\infty$ such that

$$\|M_{k_1} - M^f_{k_1}\|_{\infty} + \|N_{k_1} - N^f_{k_1}\|_{\infty} + \|\hat{k}_2 - \hat{k}^f_2\|_{\infty} < \epsilon. \quad (3.3)$$

Therefore, the numerator, $N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, and the denominator, $M_1$, are uniformly approximable in $H^\infty$ by rational functions. Thus, for any $\epsilon > 0$ we can find a rational transfer function

$$P_f(s) = \begin{bmatrix} N^f_{1}(s) \\ N^f_{2}(s) \end{bmatrix} M^f_{1}(s)^{-1} \quad (3.4a)$$

such that

$$P(s) = \begin{bmatrix} N^f_{1}(s) + \Delta_{N_1}(s) \\ N^f_{2}(s) + \Delta_{N_2}(s) \end{bmatrix} (M^f_{1}(s) + \Delta_{M_1}(s))^{-1} \quad (3.4b)$$

with

$$\| \begin{bmatrix} \Delta_{N_1} \\ \Delta_{N_2} \\ \Delta_{M_1} \end{bmatrix} \|_{\infty} < \epsilon, \quad (3.4c)$$

where $N^f_{1}, N^f_{2}, M^f_{1} \in H^\infty$ are rational functions approximating $N_1, N_2, M_1$. We shall make use of the following.

**Lemma 3.2** Consider the equations (3.4a,b,c). Then, for $\epsilon > 0$ sufficiently small, $P$ is stabilizable implies that $P_f$ is also stabilizable.

**Proof:** Define

$$\lambda_f^2 := \inf_{\text{Re } s > 0} (|N^f_{1}(s)|^2 + |N^f_{2}(s)|^2 + |M^f_{1}(s)|^2) > 0,$$

and

$$\bar{\lambda_f}^2 := \inf_{\text{Re } s > 0} \min\{|N^f_{1}(s)|^2 + |M^f_{1}(s)|^2, (|N^f_{2}(s)|^2 + |M^f_{1}(s)|^2)\} > 0.$$

Clearly by choosing $\epsilon > 0$ sufficiently small we can make $\lambda_f \geq \frac{1}{2}$ and $\bar{\lambda_f} \geq \frac{1}{2}$. On the other hand, by the fact that $P$ is stabilizable we have $\lambda > 0$ and $\bar{\lambda} > 0$. Therefore, $\lambda_f > 0$ and $\bar{\lambda_f} > 0$, which means that $P_f$ is stabilizable (see [22] and [20]). $\Box$
Although we have coprime factorizations for $P_f$ from the above Lemma 3.2, we need to have normalized coprime factorizations in order to apply the theory of robustness optimization in the gap metric, [9]. So, let us now find normalized coprime factorizations for $P_f$ by constructing rational functions $G^f, G_1^f, G_2^f \in H^\infty$, with $(G^f)^{-1}, (G_1^f)^{-1}, (G_2^f)^{-1} \in H^\infty$, satisfying

\begin{align}
(G^f)^*G^f &= |G^f_(j\omega)|^2 = (|N_1^f(j\omega)|^2 + |N_2^f(j\omega)|^2 + |M_1^f(j\omega)|^2)^{-1} \\
(G_1^f)^*G_1^f &= |G_1^f(j\omega)|^2 = (|N_1^f(j\omega)|^2 + |M_1^f(j\omega)|^2)^{-1} \\
(G_2^f)^*G_2^f &= |G_2^f(j\omega)|^2 = (|N_2^f(j\omega)|^2 + |M_1^f(j\omega)|^2)^{-1}. 
\end{align}

Note that since $\lambda_f > 0$ and $\bar{\lambda}_f > 0$ these functions exist, and they can be computed using spectral factorization techniques. Then, $P_f$ can be rewritten as

\begin{equation}
P_f = N_f M_f^{-1} = \tilde{M}_f^{-1} \tilde{N}_f \tag{3.6a}
\end{equation}

where

\begin{equation}
N_f = \begin{bmatrix} N_1^f G^f \\ N_2^f G^f \end{bmatrix}, \quad M_f = M_1^f G^f \tag{3.6b}
\end{equation}

and

\begin{equation}
\tilde{N}_f = \begin{bmatrix} G_1^f N_1^f \\ G_2^f N_2^f \end{bmatrix}, \quad \tilde{M}_f = \begin{bmatrix} G_1^f M_1^f & 0 \\ 0 & G_2^f M_1^f \end{bmatrix}. \tag{3.6c}
\end{equation}

Since we have

\[N_f^* N_f + M_f^* M_f = 1,
\]

and

\[
\tilde{N}_f \tilde{N}_f^* + \tilde{M}_f \tilde{M}_f^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[\begin{bmatrix} N_f \\ M_f \end{bmatrix}\] is inner and \[\begin{bmatrix} \tilde{M}_f & \tilde{N}_f \end{bmatrix}\] is co-inner. Moreover, we can find $U_f, \tilde{U}_f, V_f, \tilde{V}_f \in H^\infty$ appropriate size matrices satisfying

\[
\begin{bmatrix} \tilde{V}_f & -\tilde{U}_f \\ -\tilde{N}_f & \tilde{M}_f \end{bmatrix} \begin{bmatrix} M_f & U_f \\ N_f & V_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
(See for example [8], pp. 22–24, and [23], pp. 82–84, on how to compute these matrices.)

The set of all stabilizing controllers for $P_f$ is given by

$$\{C_f = (U_f - M_f Q)(V_f + N_f Q)^{-1} : Q = [Q_1, Q_2], Q_1, Q_2 \in H^\infty\}.$$

It is also known (cf. [9], [24]) that the controller $C_f = (U_f - M_f Q)(V_f + N_f Q)^{-1}$ stabilizes all plants of the form

$$P_t = (\tilde{M}_f + \Delta M_f)^{-1}(\tilde{N}_f + \Delta N_f)$$

where $\Delta P_t := [\Delta M_f, \Delta N_f]$ has arbitrary entries in $H^\infty$, subject to $\|\Delta P_t\|_\infty < \epsilon$, if and only if

$$\| [U_f \atop V_f] - [M_f \atop N_f] Q \|_\infty \leq \frac{1}{\epsilon}.$$

In view of the above fact one can define a quantity $\gamma_f$, for the plant $P_f$, as

$$\gamma_f^{-1} := \inf_{Q \in H^\infty} \| [U_f \atop V_f] - [M_f \atop N_f] Q \|_\infty.$$

The quantity $\gamma_f$ characterizes the largest amount of uncertainty level $\epsilon$ tolerated by $P_f$, in the sense that there exists a controller stabilizing $P_f$ and all plants of the form $P_t$ if and only if $\epsilon \leq \gamma_f$. In fact the same statement is true, [9], for plants of the form

$$P_t = (N_f + \Delta N_f)(M_f + \Delta M_f)^{-1}$$

with $\Delta P_t := [\Delta M_f \atop \Delta N_f]$ having arbitrary entries in $H^\infty$, and $\|\Delta P_t\|_\infty < \epsilon$. Since $P_f$ is stabilizable we have $\lambda_f \geq \gamma_f > 0$. (We refer to [9] for all the details of these facts and other related references.) In a similar fashion one can define the quantity $\gamma$ for the plant $P$ as the largest uncertainty level tolerated by $P$. Again, since $P$ is stabilizable we have $\lambda \geq \gamma > 0$.

Note that the approximate plant can be written as

$$P_f = N_f M_f^{-1} = \begin{bmatrix} N_1^f G^f \\ N_2^f G^f \end{bmatrix} (M_1^f G^f)^{-1}$$

similarly the original plant is of the form

$$P = \begin{bmatrix} N_1^f + \Delta N_1 \\ N_2^f + \Delta N_2 \end{bmatrix} (M_1^f + \Delta M_1)^{-1} = \begin{bmatrix} N_1^f G^f + G^f \Delta N_1 \\ N_2^f G^f + G^f \Delta N_2 \end{bmatrix} (M_1^f G^f + G^f \Delta M_1)^{-1}.$$
Accordingly we set

\[ \Delta_P := G^f \begin{bmatrix} \Delta_{N_1} \\ \Delta_{N_2} \\ \Delta_{M_1} \end{bmatrix}. \]

We are now ready to state our main result. But first we summarize all the assumptions used in this section.

**Assumption 1:** (On finite dimensional parts of the plant)

\((A, b), \ (A, b_1), \ (A, b_2)\) are stabilizable and \((c_1, A)\) is detectable.

**Assumption 2:** (On the kernel \(k_2(t), t \geq 0\))

\(k_2 \in L^1[0, \infty), \ \text{so} \ \hat{k}_2(s) := \int_0^\infty e^{-st}k_2(t)dt \in H^\infty \ \text{and} \ \hat{k}_2(j\omega) \ \text{is continuous on} \ \omega \in \mathbb{R} \cup \{\infty\}.\)

**Assumption 3:** (On the kernel \(k_1(t), t \geq 0\))

\(\hat{k}_1(s) := \int_0^\infty e^{-st}k_1(t)dt \ \text{is such that} \ c_2(s\hat{k}_2(s))^{-1} = N_{k_1}(s)/M_{k_1}(s) \ \text{with} \ N_{k_1}, M_{k_1} \in H^\infty \ \text{and} \ \ (N_{k_1}, M_{k_1}) \ \text{is strongly coprime in} \ H^\infty, \ \text{moreover} \ N_{k_1}(j\omega) \ \text{and} \ M_{k_1}(j\omega) \ \text{are continuous in} \ \omega \in \mathbb{R} \cup \{\infty\}.\)

**Assumption 4:** (Coupling between finite and infinite dimensional parts)

(4.i): The pair \( (N_b, \ (M_{k_1}M_a - N_{k_1}(N_{b_2} + \hat{k}_2N_{b_1})) ) \) is coprime.

(4.ii): The pair \( (M_{k_1}, \ (N_{b_2} + \hat{k}_2N_{b_1}) ) \) is coprime.

(4.iii): The pair \( (N_{k_1}, \ M_a) \) is coprime.

**Theorem 3.3** Consider the plant described by (1.8a,b) and suppose that the Assumptions 1-4 are satisfied. Then, there exists a controller \(C_f, \ \text{finite dimensional}, \ (\text{i.e.} \ C_f(s) \ \text{is rational}) \ \text{stabilizing the closed loop system with plant} \ P.\)

**Proof:** From the discussion following Lemma 3.2 we know that if \(\|\Delta_P\|_\infty < \gamma_f\) then there exists a controller stabilizing both \(P_f\) and \(P\). Such a controller can be obtained from the finite dimensional plant \(P_f\) in an optimal way. Moreover, this controller, \(C_{f,\text{opt}}, \ \text{is rational, i.e. finite dimensional. This is essentially a consequence of the well known result that} \ H^\infty \ \text{optimal controllers for finite dimensional plants are finite dimensional. Now, we know that, by uniform approximability assumptions on} \ M_{k_1}, N_{k_1} \ \text{and} \ \hat{k}_2 \ \text{we can make} \ \epsilon > 0, \ \text{in} (3.4c), \ \text{as small as we wish. However, we want to prove that we can make} \ \|\Delta_P\|_\infty \ \text{arbitrarily small.} \)
To show this it is sufficient to prove that there exists $\eta' > 0$ and $\epsilon' > 0$ such that for all $\epsilon' > \epsilon > 0$ we have $\|G^f\|_\infty < \eta'$. This is indeed true because from the proof of the Lemma 3.2 we know that $\|G^f\| \leq \frac{1}{2}$ for some sufficiently small $\epsilon$, say for $\epsilon \leq \epsilon'$. Thus, for any $0 < \epsilon \leq \epsilon'$ fixed we can make $\|\Delta_P\|_\infty < \epsilon$, by a suitable choice of the rational functions $M_{k_1}^f, N_{k_1}^f, \hat{k}_2^f \in H^\infty$. Finally in order to prove the existence of a finite dimensional controller stabilizing $P$, we need to ensure that as $\epsilon \to 0$, $\gamma_f$ is bounded below by a strictly positive number. Again, we know that as $\epsilon \to 0$ the distance, in the gap metric, between $P$ and $P_f$ approaches to zero and hence the quantity $\gamma_f$ approaches to $\gamma > 0$, see [9]. This concludes the proof of the existence of a finite dimensional controller which stabilizes the closed loop system with the infinite dimensional plant $P$. □

4 An algorithm to construct $C_{f,opt}^f$

In this section we summarize the procedure described in Section 3 as an algorithm for the construction of a stabilizing finite dimensional controller.

Algorithm: Given the data $A, b, b_1, b_2, c_1, \hat{k}_2(s), c_1(s\hat{k}_1(s))^{-1} = N_{k_1}(s)/M_{k_1}(s)$.

Step 1: Find rational functions $N_b, N_{b_1}, N_{b_2}, M_a \in H^\infty$ such that

$$\frac{N_b(s)}{M_a(s)} = c_1(sI-A)^{-1}b, \quad \frac{N_{b_i}(s)}{M_a(s)} = c_1(sI-A)^{-1}b_i, \quad i = 1, 2;$$

and the pairs $(N_b, M_a), (N_{b_1}, M_a)$ and $(N_{b_2}, M_a)$ are coprime.

Step 2: Pick a small number $\epsilon > 0$, and find $M_{k_1}^f, N_{k_1}^f, \hat{k}_2^f \in H^\infty$, rational functions such that

$$\|M_{k_1}^f - M_{k_1}\|_\infty < \epsilon$$
$$\|N_{k_1}^f - N_{k_1}\|_\infty < \epsilon$$
$$\|\hat{k}_2^f - \hat{k}_2\|_\infty < \epsilon$$
$$\|N_{k_1}^f \hat{k}_2^f - N_{k_1} \hat{k}_2\|_\infty < \epsilon.$$
**Step 3:** Define

\[ N_1^f := N_b M_1^k, \quad N_2^f := N_b N_1^k, \]

\[ M_1^f := M_a M_1^k - N_1^k(N_{b_2} + \hat{k}_2 N_{b_1}) \]

and check that \((N_1^f, M_1^f)\) and \((N_2^f, M_1^f)\) are coprime, otherwise go to step 1 and decrease \(\epsilon\) until this is satisfied.

**Step 4:** Find the rational function \(G^f \in H^\infty\) such that \((G^f)^{-1} \in H^\infty\) and

\[ |G^f(j\omega)|^2 = (|N_1^f(j\omega)|^2 + |N_2^f(j\omega)|^2 + |M_1^f(j\omega)|^2)^{-1}. \]

**Step 5:** Compute \(\|M_b\|_\infty, \|M_a\|_\infty, \|N_{b_2}\|_\infty, \|N_{b_1}\|_\infty, \|G^f\|_\infty\) (or find upper bounds for each of these norms) and check that

\[ \|N_1^f - N_1\|_\infty < \|N_b\|_\infty \ \epsilon \]

\[ \|N_2^f - N_2\|_\infty < \|N_b\|_\infty \ \epsilon \]

\[ \|M_1^f - M_1\|_\infty < (\|M_a\|_\infty + \|N_{b_2}\|_\infty + \|N_{b_1}\|_\infty) \ \epsilon. \]

(Note that from Step 2 these are automatically satisfied.) Using the above bounds find a real number \(\eta\) such that

\[ \eta \geq \|G^f\|_\infty \left(2\|N_b\|_\infty^2 + (\|M_a\|_\infty + \|N_{b_2}\|_\infty + \|N_{b_1}\|_\infty)^2\right)^{1/2}. \]

Then we have \(\|\Delta P\|_\infty < \eta \ \epsilon.\)

**Step 6:** Compute \(\gamma_f\) for the plant \(P_f\) from the formula

\[ \gamma_f = \sqrt{1 - \|\Gamma_f\|^2}, \]

where \(\Gamma_f\) is the Hankel operator with symbol \([M_1^{\ast f}, N_1^{\ast f}, N_2^{\ast f}]\), (cf. [11]).

**Step 7:** Check if \(\gamma_f \geq \eta \epsilon > \|\Delta P\|_\infty\)

* True: go to next step
* False: go to Step 1, decrease $\epsilon$ and repeat the procedure.

**Step 8:** From $P_f := \begin{bmatrix} N_1^f \\ N_2^f \end{bmatrix} (M_1^f)^{-1}$ compute the optimal controller $C_{f,\text{opt}}$ which robustly stabilize the gap ball around $P_f$ of radius $\gamma_f$, see [9], [11], etc. Note that $C_{f,\text{opt}}$ is rational.

**End of the algorithm,** $C_{f,\text{opt}}$ found in Step 8 stabilizes the original plant $P = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} M_1^{-1}$.

We now want to make some remarks about the above algorithm. There are several ways to perform the computations required in Step 2; see for example [13] and [12] for different approximation schemes and further references on this subject.

From the Lemma 3.2 we have stabilizability of $P_f$ for $\epsilon > 0$ sufficiently small, so it is guaranteed that the algorithm will pass the test in Step 3.

There are also several methods to perform a spectral factorization which gives $G^f$ in Step 4, see for example [8] pp. 90–93. Computing the $H^\infty$ norm of the scalar rational transfer functions $N_b$, $N_{b_1}$, $N_{b_2}$, $M_a$ and $G^f$ is not difficult. For example a simple plot of their magnitudes on the imaginary axis (Bode plots) would give these norms. Therefore, the constant in Step 5 can be found easily. Similarly computing $\gamma_f$ for a given rational plant $P_f$ is rather easy, see for example [7], [8] and [21] for different methods and further references. We know that for $\epsilon > 0$ sufficiently small we have $\gamma_f \geq \gamma/2 > 0$, therefore the algorithm will eventually pass the test in Step 7.

In Step 8 computing $C_{f,\text{opt}}$ requires finding the singular values and vectors of the Hankel operator of Step 6. These also can be obtained from the standard methods of $H^\infty$ optimal control theory, see e.g. [7], [10], [21] for further details and references.

Finally we want to make a remark on the order of the controller. As $\epsilon$ decreases the order of $P_f$, hence the dimension of the controller $C_{f,\text{opt}}$ increases. Therefore, to have a reasonably low order controller one may want to find the smallest $\eta$ in Step 5, and the largest $\epsilon > 0$ satisfying $\gamma_f \geq \eta \epsilon$ (note that $\gamma_f$ also depends on $\epsilon$).
5 Remarks on Stability and the Aeroelastic Model

In this section we discuss the relations between the closed loop stability (which is an external stability concept) and the “boundedness” issue for the components of the “states” $x_1$ and $x_2$, (which can be regarded as internal stability). We also discuss the relationship between the assumptions used in Section 3 and the aeroelastic model described in Section 2.

Recall, from the definition of the closed loop stability, that the controller $C_{f,\text{opt}}$, obtained in Step 8 of the algorithm, guarantees the following: when $r, d \in L^2[0, \infty)$ we have $u, y_1, y_2 \in L^2[0, \infty)$. Note that $y_2 \in L^2[0, \infty)$ implies $x_2 \in L^2[0, \infty)$, because $y_2(t) = c_2 x_2(t)$ and $c_2 \neq 0$; so, $x_2$ has “finite energy.”

We are also concerned with the other “states,” i.e., the entries of $x_1$. Let us define $x_2(t) := u_1(t)$ and $\int_0^t k_2(\tau)x_2(t-\tau)d\tau := u_2(t)$. Then, since $x_2 \in L^2[0, \infty)$ and $k_2 \in L^1[0, \infty)$, (Assumption 2), we have $u_1, u_2 \in L^2[0, \infty)$. Hence, we can re-write (1.1)-(1.2) as

$$\frac{d}{dt} x_1(t) = Ax_1(t) + b_1 u_1(t) + b_2 u_2(t) + bu(t)$$

and

$$y_1(t) = c_1 x_1(t).$$

So,

$$y_1(t) = \int_0^t c_1 e^{A(t-\tau)}(b_1 u_1(\tau) + b_2 u_2(\tau) + bu(\tau))d\tau$$

with $y_1 \in L^2[0, \infty)$ and $u, u_1, u_2 \in L^2[0, \infty)$. Thus, by Assumption 1, detectability of $(c_1, A)$, we conclude that all the entries of $x_1$ are in $L^2[0, \infty)$.

It is also desirable to have “exponential decay” in the output signals when the input signals are of the same type. This also can be put in the framework of Definition 1.1 of stability in the following way. Let us define the set of all finite energy signals that are exponentially decaying, with a rate faster than $\sigma \geq 0$, as

$$L^2_{\sigma} := L^2_{\sigma}[0, \infty) := \{g : g(t) = e^{-\sigma t} f_g(t), t \geq 0, f_g \in L^2[0, \infty)\}. $$
Then, we say that a linear time invariant system is $\sigma$-stable if for all input $v \in L^2_\sigma$ we have the output $z \in L^2_\sigma$ and the maximum energy amplification, $\sup_{0 \neq v \in L^2_\sigma} (\|z\|_2/\|v\|_2)$ is finite. In this context $\sigma$-stability of the closed loop system of Figure 1 is equivalent to having all the entries of $T(p,c)$ in

$$H^\infty_\sigma := \{ \phi : \phi_\sigma \in H^\infty where \phi_\sigma(s) := \phi(s - \sigma) \}.$$ 

With the above notation we can define $\sigma$-coprimeness as follows: two functions $\varphi, \psi \in H^\infty_\sigma$ are $\sigma$-coprime (or the pair $(\varphi, \psi)$ is $\sigma$-coprime) if the pair $(\varphi_\sigma, \psi_\sigma)$ is coprime. Then, replacing $H^\infty$ by $H^\infty_\sigma$, $j\omega$ by $-\sigma + j\omega$, and the word coprime by $\sigma$-coprime in Assumptions 1-4 we can guarantee the existence of a finite dimensional controller which $\sigma$-stabilizes the closed loop system. The proof and the constructive algorithm are exactly the same as before except that we replace $s$ by $s - \sigma$ and $j\omega$ by $-\sigma + j\omega$ wherever these appear.

We now want to spend a few words about the assumptions on the kernels $k_1$ and $k_2$ in the context of the aeroelastic model of Section 2. It seems (cf. [4]) that the choice of $k_2(t)$, $t \geq 0$, is not as important as the choice of $k_1$. In fact comparing (2.20)-(2.21) with (1.1)-(1.2) one can see that we have already made a simplifying assumption on the structure, in that the term

$$A_1^{-1} \left( B_2 x_2(t) - \frac{d}{dt} \int_{-\infty}^{0} A_1(\tau)x_2(t + \tau)d\tau + \int_{-\infty}^{0} B_1(\tau)x_2(t + \tau)d\tau \right)$$

is taken to be

$$b_1 x_2(t) + b_2 \int_{-\infty}^{0} \kappa_2(\tau)x_2(t + \tau)d\tau,$$

with $\kappa_2(\tau)$ a scalar function. In the more general case of (5.1), where $b_2 \kappa_2(\tau)$ of (5.2) is replaced by a $1 \times n$ vector of functions, our approach can still be used, but the procedure and the approximations might be more complicated, depending on the structure of the $1 \times n$ function.

The choice of $k_1(t)$, $t \geq 0$, is rather important. The key in this choice is that the asymptotic behavior of $k_1(t)$ (as $t \to 0$ and $t \to \infty$) should be similar to the asymptotic
behavior of $\sqrt{\frac{2+Ut}{Ut}}$, [4], [2], etc. For example a “reasonable” choice is

$$k_1(t) = 1 + \sqrt{\frac{2}{Ut}}.$$ 

Moreover this particular $k_1$ satisfies the Assumption 3. To see this note that

$$\dot{k}_1(s) = \int_0^\infty e^{-st}(1 + \sqrt{2/Ut})dt = \frac{1}{s} + a_1 \frac{1}{\sqrt{s}},$$

where $a_1 = \sqrt{2/\pi U} > 0$ (see e.g. [18]), so

$$c_2(s\dot{k}_1(s))^{-1} = \frac{c_2}{1 + a_1\sqrt{s}} \in H^\infty,$$  \hspace{1cm} (5.3)

and it is continuous. Furthermore, for this specific $k_1$, (5.3) implies that whenever $P_1 \in H^\infty$ we will have $P_2 \in H^\infty$ because $P_2(s) = c_2(s\dot{k}_1(s))^{-1}P_1(s)$. Therefore, if we can find a controller $C_1$ which stabilizes $P_1$ then we guarantee the following: $r, d \in L^2$ implies $y_1 = c_1x_1 \in L^2$ (this implies all entries of $x_1 \in L^2$, by detectability of $(c_1, A)$, and hence $x_2 \in L^2$, because $\dot{x}_2(s) = (s\dot{k}_1(s))^{-1}\dot{y}_1(s)$ and $(s\dot{k}_1(s))^{-1} \in H^\infty$. Thus, in this case we do not need the measurement of $x_2$ in the output for the “internal” stabilization of the system by $C_1$.

6 Conclusions

In this paper we have considered a system described by the singular integro-differential equations (1.1)-(1.2). The aeroelastic model derived in Section 2 provides a motivation to study of this type of systems. The main restriction of the particular stabilization problem considered in this paper is that the initial conditions in (1.1)-(1.2) are assumed to be zero. This problem corresponds to the indcidual problem appearing in aeroelastic systems, see [3], p. 291.

We have used the frequency domain description of the plant to be controlled. Our main focus was on the stabilization of the plant by a finite dimensional controller. The original plant is infinite dimensional, but we have not used any particular state space realization. By approximating the original plant transfer function by rational functions the problem is
put in the framework of the theory of robust stabilization in the gap metric. An algorithm is given to construct a stabilizing finite dimensional controller. Important steps of the algorithm are finding rational approximates of certain $H^\infty$ functions that are continuous on the boundary, computing the norm of a certain Hankel operator whose symbol is rational, and constructing the controller from the singular values and vectors of this Hankel operator. These are straightforward computations; there are several software packages available for these operations.

As discussed in the previous section the measurement of $x_2$ is not necessary for the aeroelastic model of Section 2. We have also assumed that the measurement of $y_1 = c_0x_1$, ($c_0 = c_1$), is perfect. In the case of a noisy measurement (and/or the presence of a disturbance) one may want to find a controller which not only stabilize the system but also minimizes the effect of the noise/disturbance on certain signals of interest. This can be put in the framework of an $H^\infty$ control problem, where the plant and the signal uncertainties can be incorporated into a single $H^\infty$ optimality criteria, see e.g. [8].

Another possible future research subject along the lines of the present paper is to consider non-zero initial conditions and minimize the effect of the initial conditions while keeping the system stable. This problem has been studied, in the $H^2$ control setting, in [1] using state space realizations. However, robustness to possible approximations of the optimal infinite dimensional controller was not discussed.

Finally we would like to mention that the finite delay version of the system (1.1)-(1.2) can also be studied in the frequency domain using the techniques of this paper. In fact there are several interesting questions associated with the finite delay problem, e.g. the problem of finding the conditions under which we can "approximate" the integrals of (1.1)-(1.2) by certain integrals over a finite time. We will report on this problem elsewhere.

References


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