ON THE EXISTENCE OF SMOOTH BREATHERS FOR NONLINEAR WAVE EQUATIONS

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Abstract. Previous work has shown that there is a finite-dimensional manifold of generalized solutions of the nonlinear wave equation $u_{tt} - \Delta u + \alpha g(u) = f$, subject to $T$-periodicity in $t$ plus radial symmetry and decay to zero at infinity in $x$. The generalized solutions may lack regularity at the origin. If $f \equiv 0$ and $u$ depends on $t$, these solutions are called breathers. Numerical evidence for the existence of smoother solutions is presented. The data was obtained by using a previously documented algorithm to investigate the manifold of generalized solutions.

Key words. bifurcation, breather solution, distributional solution, exponential decay, nonlinear wave equation, radial symmetry, sine-Gordon equation, time-periodicity.

AMS(MOS) subject classifications. 35B10, 35L05, 35Q20, 65N99

§1. Smooth and Generalized Breathers. The question of existence of breathers for nonlinear wave equations is one of interest both mathematically and physically. Currently this question remains only partially resolved. Here we use the term breather to denote a time-dependent solution of the problem

\begin{align*}
(1.1) & \quad u_{tt} - \Delta u + \alpha g(u) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
(1.2) & \quad \lim_{|x| \to +\infty} u(t, x) = 0, \quad t \in \mathbb{R}, \\
(1.3) & \quad u(t + T, x) = u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.
\end{align*}

In these equations $u \in \mathbb{R}, \Delta u = \text{div} (\text{grad} u)$, and $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(0) = 0$. The number $T > 0$ is considered fixed, while $\alpha \in \mathbb{R}$ plays the role of a (bifurcation) parameter. Alternatively one can fixed $\alpha$ (i.e. $\check{g}(u) = \alpha g(u)$) and consider $T$ as the parameter. Clearly the zero function $u = 0$ satisfies (1.1) - (1.3). Thus breathers exist only when the problem admits multiple solutions. In this paper we present numerical evidence showing this is possible in a variety of instances. Some recent theoretical work [5] [6] forms the basis for these investigations.

One fact that is clear is that the value of $\alpha g'(0)$ plays a crucial role. We have included the parameter $\alpha$ for convenience in modifying this value. Otherwise one may think of modifying the graph of $g(u)$ only near zero, while keeping the graph of $g$ the same outside of a neighborhood about the origin. The effect on the theoretical results would remain

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unaltered. It is only the value of the derivative of the nonlinearity at the origin, in relation to the "spectral values" \( \theta_n^2 = (2\pi n/T)^2 \), that is important.

In [1] it was shown that breathers could not exist when \( \alpha g'(0) < \left( \frac{2\pi}{T} \right)^2 \), assuming other reasonable conditions hold. However when \( \alpha g'(0) > \left( \frac{2\pi}{T} \right)^2 \), and \( g(u) = \sin u \), an explicit solution can be found (cf. [3]). In fact there is an infinite sequence of solutions that arises for the problem. For convenience set \( T = 2\pi \). Then \( \theta_n^2 = n^2 \), and \( \alpha g'(0) > \left( \frac{2\pi n}{T} \right)^2 = \theta_n^2 \) is equivalent to \( \alpha > n^2 \). Set \( \beta_n = (\alpha - n^2)^{1/2} \), assuming \( \alpha > n^2 \). Then the function \( u = \varphi_n(t, x) \),

\[
(1.4) \quad \varphi_n(t, x) = 4 \arctan \left( \frac{\beta_n \sin nt}{n \cosh(\beta_n x)} \right),
\]

is a (smooth) solution of (1.1) - (1.3). Clearly these solutions bifurcate from the trivial solution as \( \alpha \) crosses the value \( n^2 \). Plotting \( ||u||_\infty \) as a function of \( \alpha \), where \( u = \varphi_n \) with \( \alpha > n^2 \), produces the plot displayed in Figure 1.

Subsequent work [4] has shown that non-existence persists in the radially symmetric higher dimensional case, under the same conditions \( \alpha g'(0) < \left( \frac{2\pi}{T} \right)^2 \) as in [1]. See also [2], [10] for the 1-dimensional case, and [9] for the higher dimensional case. Existence results, requiring less regularity of solutions than is normally required, have also appeared [11], [12], [6], [8]. Below we shall describe what is meant by a solution in this setting. Throughout we will refer to these solutions as generalized solutions. We also describe what is meant by smooth solution in the context of this paper. It will be clear that all smooth solutions are also generalized solutions.

Throughout our work we will consider only radially symmetric solutions. Nothing seems to be known, from an existence point view, without this hypothesis. Our purpose is to discuss the results of a numerical investigation into the existence of smooth breathers. Our conjecture is that smooth breathers exist as submanifolds in the manifold of generalized breathers described below. Moreover we believe that these submanifolds appear through bifurcation from the trivial solution, as in the known example described above. We also give evidence showing that this behavior persists when the equation is perturbed by a small forcing term.

The essential difference between smooth and generalized solutions, as we refer to them here, is their behavior at the origin. Consider the differential equation

\[
(1.5) \quad w_{tt} - w_{rr} + q_N(r)w + \alpha r^{\frac{N-1}{2}} g \left( \frac{w}{r^{\frac{N-1}{2}}} \right) = 0, \quad (t, r) \in \mathbb{R} \times \mathbb{R}^+,
\]

which is obtained from (1.1) by the standard change of variables \( w(t, r) = r^{\frac{N-1}{2}} u(t, x) \). In (1.5), \( q_N(r) = (N-1)(N-3)/4r^2 \), and we use the notation \( \mathbb{R}^+ = (0, +\infty) \). We say \( u(t, x) \) is a generalized radially symmetric solution of (1.1) - (1.3) if \( u(t, x) = w(t, r)/r^{\frac{N-1}{2}} \), where
The function $w : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfies

\begin{align}
(1.6) \quad & w \in L^2((0,T) \times \mathbb{R}^+), \\
(1.7) \quad & w(t+T,r) = w(t,r), \quad \text{(a.e.)} \quad (t,r) \in \mathbb{R} \times \mathbb{R}^+, \\
(1.8) \quad & \int_0^T \int_0^{+\infty} \left\{ w(\psi_{tt} - \psi_{rr} + q_N(r)\psi) + \alpha r^{N-1} g \left( \frac{w}{r^{N-1}} \right) \psi \right\} dr dt = 0,
\end{align}

for all $\psi \in \mathcal{D}^+ = \{ \psi \in C^\infty(\mathbb{R} \times \mathbb{R}^+) : \psi(t+T,r) = \psi(t,r) \text{ for all } (t,r) \in \mathbb{R} \times \mathbb{R}^+, \text{ and } \psi(t,\cdot) \in C_0^\infty(\mathbb{R}^+) \}$. We call $u(t,x)$ a smooth radially symmetric solution of (1.1) - (1.3) if, in the above definition, we can replace $\mathcal{D}^+$ by $\mathcal{D} = \{ \psi \in C^\infty(\mathbb{R} \times \mathbb{R}^+) : \psi(t,r) = r^{N-1} \varphi(t,x), \text{ for all } (t,r) \in \mathbb{R} \times \mathbb{R}^+, \text{ and } \varphi(t,\cdot) \in C_0^\infty(\mathbb{R}^N) \}$. The later definition is the standard distributional formulation. Observe that the former definition only requires square-integrability at the origin, and in particular does not require the solution to satisfy the equation at the origin. Clearly $\mathcal{D}^+ \subset \mathcal{D}$, so that smooth solutions are generalized solutions.

Although our original work [6] has been extended to all dimensions $N \geq 1$ (cf. [8]) we focus here on the cases $N = 1, 3$. Observe that if $\psi \in \mathcal{D}$ then $\psi = O(r^{N-1})$, as $r \to 0^+$. Obviously $\psi_r = r^{N-2} \varphi + r^{N-1} \varphi_r$. Since $\varphi$ is radially symmetric $\varphi_r = O(r)$, as $r \to 0^+$. Thus $\psi_r = O(r)$ when $N = 1$, and $\psi_r = O(1)$ when $N = 3$. From these considerations it is evident that a (classical) smooth radially symmetric solution of (1.1) - (1.3) should satisfy $u(t,x) = w(t,r)/r^{N-1}$, where

\begin{align}
(1.9) \quad & w(t,r) = O(1), \quad w_r(t,r) = O(r), \quad \text{as } r \to 0^+ \text{ when } N = 1, \\
(1.10) \quad & w(t,r) = O(r), \quad w_r(t,r) = O(1), \quad \text{as } r \to 0^+ \text{ when } N = 3.
\end{align}

These conditions can be used numerically to validate smoothness (at the origin) of a generalized solution. Thus our strategy will be to examine the manifold of generalized solutions described below and determine whether any of these solutions enjoy the additional smoothness indicated by (1.9) or (1.10) respectively. As we point out below, any (small norm) smooth solutions must lie on the manifold of generalized solutions.

The following notation is needed. Let $\alpha_1 \in \mathbb{R}$ be fixed and set $\lambda = \alpha_1 g'(0)$. For integers $n \geq 0$ set $\theta_n = 2\pi n/T$. Assuming $\lambda > 0$, let $m$ be the integer such that $\theta_m^2 < \lambda \leq \theta_{m+1}^2$. Set $\beta_n = (\lambda - \theta_n)^{1/2}$, $0 \leq n \leq m$. If $\lambda \leq 0$ we use the conventions $m = -1, \beta_{-1} = +\infty$. Given $\delta > 0$ and $h \in L^2((0,T) \times \mathbb{R}^+)$ let

\begin{equation}
(1.11) \quad ||h||_\delta = \left\{ \frac{2}{T} \int_0^T \int_0^{+\infty} |h(t,r)|^2 e^{2\delta r} dr dt \right\}^{1/2}.
\end{equation}

Define $H_\delta = \{ h \in L^2((0,T) \times \mathbb{R}^+) : ||h||_\delta < +\infty \}$ and $H^1_\delta = \{ h \in H_\delta : h_t, h_r \in H_\delta \}$. The derivatives are distributional derivatives relative to $\mathcal{D}^+$. 

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THEOREM 1. (cf. [6]) Assume $N = 1$ or 3, and let $\delta \in (0, \beta_m)$. Let $g \in C^2(\mathbb{R})$, with $g(0) = 0$, and both $g'(u)$ and $g''(u)$ bounded on $\mathbb{R}$. If $m \geq 0$, there is a $(2m+1)$-dimensional manifold $M_{\lambda, \delta} \subset H^1_\delta$ of generalized solutions of (1.5)-(1.7), for all $\alpha$ sufficiently close to $\alpha_1$. In addition there is a ball $\mathcal{B} \subset H^1_\delta$, centered at the origin, such that $w \in \mathcal{B}$ satisfies (1.5)-(1.7) if and only if $w \in M_{\lambda, \delta}$.

Remarks.

1) If $m = -1$, there is a $\mathcal{B} \subset H^1_\delta$ centered at the origin in which the zero function is the only solution.

2) If any smooth solutions exist, and belong to $\mathcal{B} \subset H^1_\delta$, then they must lie on the manifold $M_{\lambda, \delta}$.

3) $M_{\lambda, \delta}$ has a 2$m$-dimensional submanifold of time-dependent solutions, and a 1-dimensional submanifold of time-independent solutions.

4) We employ the notation $M_{\lambda, \delta}$ to indicate the dependence of this manifold on both $\lambda = \alpha_1 g'(0)$ and $\delta \in (0, \beta_m)$. A brief description of this dependence is given in the next section.

5) All solutions $w \in M_{\lambda, \delta}$ are a priori bounded; in fact $\sup \{ e^{\delta r} |w(t, r)| : (t, r) \in \mathbb{R} \times \mathbb{R}^+ \} < +\infty$. Thus the first condition in (1.9) is automatic.

6) The zero function can be replaced on the right in (1.5) by $h \in H^1_\delta$, with the norm of $h$ sufficiently small.

§2. Algorithmic Considerations and Strategies. The methods used to establish Theorem 1 are constructive and form the basis for the algorithm used in our computations to determine generalized solutions. This algorithm has been documented in [7]. We briefly describe it here. Throughout we assume $N = 1$ or 3. First consider (cf. [5]) the associated linear operator $L_\lambda w = w_{tt} - w_{rr} + \lambda w$, together with the boundary conditions of $T$-periodicity in time and decay to zero as $r \to +\infty$. Considered as an unbounded operator in $H^1_\delta, L_\lambda$ is a Fredholm operator of non-negative index onto $H^1_\delta$. The index is $2m + 1$, when $0 \leq \theta^2_m < \lambda \leq \theta^2_{m+1}$, and zero otherwise. Thus $L_\lambda$ is invertible when $\lambda \leq 0$, and has a $(2m + 1)$-dimensional null space when $\lambda > 0$. In the later case $L_\lambda$ has a continuous partial inverse $K_\lambda : H^1_\delta \to N_{\lambda, \delta}^+$, where $N_{\lambda, \delta}^+$ is the orthogonal complement in $H^1_\delta$ of the null space $N_{\lambda, \delta} = \{ w \in H^1_\delta : L_\lambda w = 0 \}$. We define an orthogonal projection $P : H^1_\delta \to N_{\lambda, \delta}$ in this case. Setting $\psi(\alpha, r, w) = \alpha_1 g'(0) w - \alpha r^{\frac{N-1}{2}} g(w/r^{\frac{N-1}{2}})$, equation (1.5) may be re-written as $L_\lambda w = \psi(\alpha, r, w)$. If $\lambda > 0$ we set $w = w_0 + w_1$ where $w_0 = P w \in N_{\lambda, \delta}$ and $w_1 = (I - P) w$. Then using the partial inverse $K_\lambda$ we see that $w \in H^1_\delta$ is a solution if and only if $w = w_0 + w_1$, where

\[
(2.1) \quad w_1 = F(w_1) = K_\lambda \psi(\alpha, r, w_0 + w_1).
\]

Clearly the fixed points of $F$ are precisely the solutions of (1.5)-(1.7).
The analysis leading to Theorem 1 shows that there exists an interval \( J \subseteq \mathbb{R} \) centered at \( \alpha_1 \) and balls \( \mathcal{B}_0 \subseteq N_{\lambda,\delta}, \mathcal{B}_1 \subseteq N_{\lambda,\delta}^1 \) centered at the origin, such that \( F \) is a contraction on \( \mathcal{B}_1 \) for all \( (\alpha, w_0) \in J \times \mathcal{B}_0 \). Thus (2.1) implicitly defines a map \( S : J \times \mathcal{B}_0 \rightarrow \mathcal{B}_1 \), where \( w_1 = S(\alpha, w_0) \) is the unique fixed point of \( F \) in \( \mathcal{B}_1 \). In particular \( w \) is a solution of (1.5)-(1.7) in \( \mathcal{B}_0 + \mathcal{B}_1 \) if and only if

\[
(2.2) 
\\
\quad w = w_0 + S(\alpha, w_0)
\]

The balls \( \mathcal{B}_0, \mathcal{B}_1 \) must be sufficiently small to guarantee the contraction property. However in practice (i.e. numerically) they need not be very small. Solutions with \( H^1_\delta \) norms between 5 and 10 were computable, at least in some instances. None the less there does appear to be a limit to the parameter range (or size of solutions) on which the algorithm is effective.

With \( \alpha \in J \) fixed the map \( S(\alpha, \cdot) : \mathcal{B}_0 \rightarrow \mathcal{B}_1 \) serves as a local chart for the manifold \( M_{\lambda,\delta} \subset H^1_\delta \) of generalized solutions, after \( \mathcal{B}_0 \) is identified with a neighborhood of 0 \( \in \mathbb{R}^{2m+1} \). In particular this done through the correspondence

\[
(2.3) 
\\
(p, q) \leftrightarrow w_0 = \frac{1}{2} p_0 u_0(r) + \sum_{n=1}^{m} [p_n u_n(t, r) + q_n v_n(t, r)], \quad \text{where}
\]

\[
(2.4) 
\\
\begin{align*}
\quad u_0(r) &= c_0 \exp(-\beta_0 r), \\
\quad u_n(t, r) &= c_n \exp(-\beta_n r) \cos \theta_n t, & 1 \leq n \leq m, \\
\quad v_n(t, r) &= d_n \exp(-\beta_n r) \sin \theta_n t, & 1 \leq n \leq m.
\end{align*}
\]

The numbers \( c_0, \ldots, c_m \) and \( d_1, \ldots, d_m \) are normalizing constants (cf. [7]) chosen so that \( \{u_0, \ldots, u_m, v_1, \ldots, v_m\} \) is an orthonormal basis for \( N_{\lambda,\delta} \subset H^1_\delta \).

The parameters \( \lambda, \delta \) influence the manifold in the following way. Since \( \beta_n = \sqrt{\lambda - \theta_n^2} \), the spanning functions of \( N_{\lambda,\delta} \) vary with \( \lambda \). On the other hand the metric varies with \( \delta \), since the norm being used is the \( H^1_\delta \) norm. Although we have suggested fixing \( \alpha_1 \) and choosing \( \alpha \) near \( \alpha_1 \), in practice one is interested in a range of \( \alpha \)'s and chooses the value of \( \alpha_1 \) accordingly. Hence varying \( \alpha \) over a large interval requires the value of \( \alpha_1 \) to vary and hence also \( \lambda \). Moreover the requirement \( \delta \in (0, \beta_m) \) is essential for the theory to be applicable. Since \( \beta_m = \beta_m(\lambda) \) the interval \( (0, \beta_m) \) will also be varying. If \( \lambda = a g'(0) \) approaches \( \theta_m^2 \) from above we have \( \beta_m \to 0^+ \). Thus to investigate the set of solutions near the bifurcation points \( \lambda = \theta_m^2 \), the parameter \( \delta \) must be varied. Throughout most of our computations we used the rule \( \delta = (0.45)\beta_m \) to determine \( \delta \in (0, \beta_m) \). Only in Figure 3 did we use \( \delta = \beta_m/2 \). In any case the choice of \( \delta \) is arbitrary and does not effect the qualitative nature of the results.

In our computations we were interested in investigating the nature of solutions as they varied with respect to both \( \alpha \) and \( w_0 \). For fixed \( \alpha \) we believe there is an isolated
set of points, after allowing for phase shift symmetry, on $M_{\lambda, \delta}$ corresponding to smooth breathers. As $\alpha$ varies we believe these isolated points vary continuously and are such that the number of points is equal to $m$ if $\alpha_m < \alpha < \alpha_{m+1}$, where $\alpha_m = \theta_m^2 / g'(0)$ are the bifurcation values. At these points the solution set of smooth breathers undergoes a bifurcation analogous to that observed in semilinear elliptics problems (see Figure 1).

We investigated the dependence of $w = w_0 + w_1$ on $\alpha, w_0$ by moving along paths in $M_{\lambda, \delta}$. This is done by varying $\alpha$ near $\alpha_1$ or $w_0 \leftarrow (p, q) \in \mathbb{R}^{1+2m}$ near zero, and numerically determining $w_1$ according to (2.1). Of course the determination of $w_1$ is an iterative process in which a generalized solution of the equation is determined. To locate smooth solutions on $M_{\lambda, \delta}$ we used conditions (1.9) or (1.10) respectively. While following a path in $M_{\lambda, \delta}$ the validity of these conditions can be monitored by computing a functional. Let $F_N : \mathbb{R} \times \mathbb{R}^{1+2m} \to \mathbb{R}$ be defined by

\begin{equation}
F_N(\alpha, p, q) = \begin{cases} 
\sup_t |w_r(t, 0; \alpha, p, q)|, & N = 1 \\
\sup_t |w(t, 0; \alpha, p, q)|, & N = 3
\end{cases}
\end{equation}

Here we have used $w(t, r; \alpha, p, q)$ to denote the generalized solution $w = w_0 + w_1$ with $w_1 = S(\alpha, w_0)$. Recall that $S$ is implicitly defined by (2.1) so that (2.2) is valid. A generalized solution is a smooth solution precisely when $F_N(\alpha, p, q) = 0$. If $N = 3$ then one should also check that $\sup_{t > 0} \{|w_r(t, 0; \alpha, p, q)|\} < +\infty$.

Set in this framework we see that the problem of determining smooth breathers, and their variation with respect to $(\alpha, p, q) \in \mathbb{R} \times \mathbb{R}^{1+2m}$, has a familiar form. We are simply interested in determining the zeros of a function $F_N : \mathbb{R} \times \mathbb{R}^{1+2m} \to \mathbb{R}$. The validity of this strategy when applied to the case of the known breathers described in section one, will be described in the next section.

Let us mention three typical paths used in the computations. First to locate a smooth solution we often followed a path of the form $s \to (\alpha, 0, q(s))$, where $q(s) = (q_1(s), \cdots, q_m(s))$ with $q_i(s) = \delta_{ij}s$ for some fixed $j$. Here $\delta_{ij} = 0$ or 1 depending on whether $i \neq j$ or $i = j$ respectively. This path corresponds to moving along one of the coordinate axes. Once the coordinates of a smooth breather have been found one can attempt to move along a path $\alpha \to (\alpha, p(\alpha), q(\alpha))$ for which $F_N(\alpha, p(\alpha), q(\alpha)) = 0$. Thus subsequent points $(\alpha, p(\alpha), q(\alpha))$ are found by solving $F_N = 0$ for $(p, q)$. To see that such points $(\alpha, p(\alpha), q(\alpha))$ are isolated within $M_{\lambda, \delta}$ we attempted to determine paths along level curves $F_N = \text{constant}$. Again this type of path requires a zero finding algorithm, where the function is $\tilde{F}_N = F_N$ - constant. The algorithm we used was a descent algorithm which accounted for the fact the $F_N$ is not differentiable at its zeros. Notice that finding zeros of $F_N$ is analogous to finding zeros of $f(s) = |s|$, with the additional constraint that derivatives would be extremely expensive to evaluate.

The number of independent variables used in the calculations can be reduced by symmetry. In general $\tilde{w}(t, r) = w(t + \theta, r)$ is a solution whenever $w(t, r)$ is. Since $\sup \{|\tilde{w}(t, 0)| : t \in \mathbb{R}\} = \sup \{|w(t, 0)| : t \in \mathbb{R}\}$, $\sup \{|\tilde{w}_r(t, 0)| : t \in \mathbb{R}\} = \sup \{|w_r(t, 0)| : t \in \mathbb{R}\}$ and the
coordinates \((p, q)\) of a solution are determined by Fourier coefficient integrals, it follows that \(F_N\) inherits the following symmetry. Let \((p, q)\) and \((\tilde{p}, \tilde{q})\) denote the coordinates (cf. (2.3)) of \(w\) and \(\tilde{w}\), respectively. Then

\[
(2.6) \quad F_N(\alpha, p, q) = F_N(\alpha, \tilde{p}, \tilde{q}), \quad \text{where}
\]

\[
(2.7) \begin{cases}
\tilde{p}_0 &= p_0, \\
\tilde{p}_n &= p_n \cos n\theta + q_n \sin n\theta, & 1 \leq n \leq m, \\
\tilde{q}_n &= q_n \cos n\theta - p_n \sin n\theta, & 1 \leq n \leq m.
\end{cases}
\]

If in addition \(g(w)\) is odd then \(\tilde{w} = -w\) is a solution whether \(w\) is, from which it follows that \(F_N(\alpha, -p, -q) = F_n(\alpha, p, q)\).

\section{Algorithm Validation.} A PASCAL program has been developed which implements the routines described briefly above. In order to test the algorithm (and the program) we considered the problem \(w_{tt} - w_{xx} + \alpha \sin w = 0\), which has the known breathers given by (1.4), assuming \(T = 2\pi\). Results of test runs of the program are presented below. All computations reported on in this and subsequent sections were performed on the HDS 9180 at Iowa State University.

Regardless of the problem, approximate solutions computed by the program were checked for accuracy in two ways. As a first check we computed the residuals in a finite difference approximation of (1.5). Let \(w^i_j = w(t_i, r_j)\). Using 2nd order centered differences to approximate the derivatives in (1.5) we obtain the residual at \((t_i, r_j)\) as

\[
(3.1) \quad R^i_j = h^{-2} [w^{i+1}_j - 2w^i_j + w^{i-1}_j] - k^{-2} [w^{i+1}_j - 2w^i_j + w^{i-1}_j] + \alpha r_j^{N-1} g \left( \frac{w^i_j}{r_j^{N-1}} \right),
\]

where \(h = \Delta t\) and \(k = \Delta r\). If \(\max |R^i_j|\) is small then \(\{w^i_j\}\) may be judged to be an accurate approximation of a solution \(w(t, r)\) of (1.5).

For our second check we employed the energy functional

\[
(3.2) \quad E_N(t) = \int_0^{+\infty} \left\{ w_t^2 + w_r^2 + 2\alpha G_N(r, w) \right\} dr, \quad \text{where}
\]

\[
(3.3) \quad G_N(r, w) = \begin{cases}
G(w) = \int_0^w g(s)ds, & N = 1, \\
r^2 G(w/r), & N = 3.
\end{cases}
\]

A straight forward (formal) computation shows that along generalized solutions \(E'_N(t) = -2w_t(t, 0)w_r(t, 0)\). We observe that along a smooth breather we should have \(E'_N(t) = 0\) for all \(t\). To monitor accuracy using (3.2), we considered the residuals in the integrated energy equation

\[
(3.4) \quad E_i = E_N(t_i) - E_N(0) - 2 \int_0^t w_t(s, 0)w_r(s, 0)ds.
\]
Of course the integrals appearing in (3.2), (3.4) are computed numerically. When $E_N(t)$ had mean value different from zero (if fact bounded away from zero) we made use of the relative residual error $\max\{E_i\}/MVE$, with $MVE$ denoting the mean value of $E_N(t)$. If an approximate solution $\{w_j\}$ has $\max\{E_i\}$ small, at least in a relative sense, then one can judge it to be an accurate approximation of a solution $w(t,r)$ of (1.5)-(1.7).

Obviously additional measurements of accuracy are available when the solution is known. We considered various norms of the error $e = w - \varphi_n$, where $w$ denotes an approximation of the known breather $\varphi_n$. For convenience in labeling graphs we set

\begin{align}
H_0(w) &= \|w\|_{L^2((0,T) \times \mathbb{R}^+)} , \\
H_1(w) &= \|w\|_{H^1((0,T) \times \mathbb{R}^+)} , \\
M(w) &= \|w\|_{L^\infty((0,T) \times \mathbb{R}^+)} , \\
MVE(w) &= \frac{1}{T} \int_0^T \int_0^{+\infty} \{w_i^2 + w_r^2 + 2\alpha \mathcal{G}_N(r,w)\} \, dr \, dt ,
\end{align}

where $\mathcal{G}_N$ is defined in (3.3). We point out that these are the standard norms and are independent of $\delta$ and $\alpha$, except in the case of (3.8) in which $\alpha$ explicitly appears. For test purposes we initially fixed $\alpha \in (1,4)$. Recall that $T = 2\pi$ is assumed, so that $\alpha_n = n^2$ are the bifurcation values. Thus $M_{\lambda,\delta}$ is 3-dimensional. According to (2.6)-(2.7) phase-shift symmetry can be used to reduce the dimension to 2. Let $(p_0, p_1, q_1)$ denote the point in $\mathbb{R}^3$ corresponding to the point $w = w_0 + S(\alpha,w_0)$ on $M_{\lambda,\delta}$ as described by (2.2)-(2.3). From the form of $\varphi_1$, it is natural to look for $\varphi_1 \in M_{\lambda,\delta}$ as corresponding to a point $(0,0,q_1^*) \in \mathbb{R}^3$, for some $q_1^*$. This suggests we follow a path $q_1 \rightarrow (0,0,q_1)$, that is $q_1 \rightarrow w(q_1) = q_1 v_1 + S(\alpha,q_1 v_1)$, and seek a zero of the functional $F_1$ defined in (2.5) to locate $\varphi_1$. Following this strategy one finds a value $q_1^*$ such that $F_1(\alpha,0,0,q_1^*) = 0$. This turns out to be, according to error computations (see below), the point being sought. Thus we find $\varphi_1 = q_1^* v_1 + S(\alpha,q_1^* v_1)$.

The values of $F_1$ along a path of this type are recorded in the graph of Figure 2. In particular $\alpha$ was assigned the value 1.1 and $q_1$ was incrementally varied through the interval $(0,6.4)$. As $q_1 \rightarrow q_1^* \equiv 4.56$, we see $F_1(\alpha,0,0,q_1) \rightarrow 0$, and that $F_1(\alpha,0,0,q_1) \neq 0$ if $q_1 \neq q_1^*$. Thus, at least in this direction, the smooth (known) breather $\varphi_1$ is isolated. Of course phase shifts (of $\theta$) produce a curve of zeros, $\theta \rightarrow (0,q_1^* \sin \theta,q_1^* \cos \theta)$, according to (2.6)-(2.7).

In an effort to determine whether the breather $\varphi_1$ is isolated (up to phase shift) on $M_{\lambda,\delta}$, we computed level curves of the functional $F_1$ in the $p_0,q_1$-plane. The restriction $p_1 = 0$ fixes the phase shift at $\theta = 0$. The level curves, $F(\alpha,p_0,0,q_1) = c$, with $c \in \{0.15,0.35,0.65\}$ and $\alpha = 1.2$ are depicted in Figure 3. The phase shift symmetry appears in this graph as symmetry with respect to the $p_0$-axis, while the fact that $g(u) = \sin u$ is odd produces the symmetry with respect to the origin. The graph is clearly indicative of a function having isolated zeros.
A continuation process was used both in determining level curves of \( F_1 \) and in determining curves \( \alpha \to w(\alpha) \) of smooth breathers. This process was tested by varying \( \alpha > 1 \) and determining \( q_1 = q_1^\ast(\alpha) \) according to \( F_1(\alpha, 0, 0, q_1) = 0 \), so that \( w(\alpha) = q_1^\ast(\alpha)v_1 + S(\alpha, q_1^\ast(\alpha)v_1) \) is an approximation of \( \varphi_1(\alpha) \). As each point \( w(\alpha) \in M_{\lambda, \delta} \) was determined, the error \( e(\alpha) = w(\alpha) - \varphi_1(\alpha) \) in the approximation was checked by evaluating the functionals (3.5)-(3.8) on \( e(\alpha) \). This was also done for some larger values of \( \alpha \). When \( \alpha > 4 \) an approximation of \( \varphi_2 \) can be found by solving \( F_1(\alpha, 0, 0, 0, q_2) = 0 \) for \( q_2 = q_2^\ast(\alpha) \). When \( \alpha > 9 \) an approximation of \( \varphi_3 \) can be found by solving \( F_1(\alpha, 0, 0, 0, 0, 0, q_3) = 0 \) for \( q_3 = q_3^\ast(\alpha) \). Table 1 below contains various measures of the errors encountered for some (random) choices of \( \alpha \). The corresponding discretization parameters are also included.

The program uses a time-space grid consisting of three subgrids (cf. [7]). An inner subgrid is used on \( (0, T) \times (0, R_1) \) with \( R_1 \) small, so that behavior near the origin was adequately captured. A middle subgrid is used on \( (0, T) \times (R_1, R_2) \) with \( R_2 \) of moderate size, to capture the solution behavior at intermediate distances from the origin. And an outer subgrid is used on \( (0, T) \times (R_2, R_3) \) where \( R_3 \) was large and meant to approximate infinity. Since all solutions have exponential decay, with a known minimum rate, the choice of \( R_3 \) does not present any problems. The mesh points of the subgrids were obtained by evenly dividing these sub-regions. In the inner subgrid we have \( \Delta t = T/N_1 \), and \( \Delta r = R_1/N_1 \), in the middle \( \Delta t = T/N_2 \) and \( \Delta r = (R_2 - R_1)/N_2 \), and in the outer \( \Delta t = T/N_3 \), and \( \Delta r = (R_3 - R_2)/N_3 \). Table 1 contains the values of \( N_1, N_2, N_3 \) and the values of \( R_1, R_2, R_3 \).

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Table 1

Error measurements of \( e_n(\alpha) = w_n(\alpha) - \varphi_n(\alpha) \), where \( w_n(\alpha) \) is the approximate solution and \( \varphi_n(\alpha) \) the exact solution (1.4). The correspondence between \( n \) and \( \alpha \) is: \( n = 1 \) for \( 1 < \alpha < 4 \), \( n = 2 \) for \( 4 < \alpha < 9 \), and \( n = 3 \) for \( \alpha > 9 \).
For later comparison, we present in Figure 4 the values of the functions (3.5)-(3.8) as functions of \( \alpha \), by evaluating them along the curve \( \alpha \to w(\alpha) \cong \varphi_1(\alpha) \) of known breathers. Clearly the curves are reminiscent of a pitchfork bifurcation.

Actually there are two curves plotted for each functional. For \( \alpha \in (1.1, 2.0) \) functional values of the approximate solution, obtained by continuation in \( \alpha \), were computed and plotted. And through the entire range displayed, functional values of the exact solution were computed and plotted. Due to the accuracy of the approximations (cf. Table 1), the graphs for each functional are superimposed and there only appears to be a single curve.

To further illustrate the nature of these solutions we present time-slices, \( t = \text{constant} \), of the graph of \( \varphi_1 \) in Figure 5, for some different values of \( \alpha \). As \( \alpha \) increases, this function develops steeper gradients and becomes more localized about the origin. Not surprisingly mesh widths must be refined (cf. Table 1) in order to maintain accuracy as \( \alpha \) increases. This must also be done as the wave number increases due to steep gradients in the \( t \) direction. This was generally observed throughout our computations.

§4. 1-Dimensional Problems. There are two questions addressed in this section. We assume throughout that \( N = 1 \). The first is whether there are other functions \( g(u) \) for which there are smooth breathers of (1.1)-(1.3). The second is whether the presence of smooth solutions persists under perturbations by a (small) nonhomogeneous term. The existence theory presented in [6] applies in both of these situations, and so our algorithm and strategies for using the algorithm continue to be applicable in these cases.

To investigate the first question we considered an assortment of nonlinearities with varying types of asymptotic behavior. In particular we considered

\[
\begin{align*}
(4.1) & \quad g(u) = -u(u - 1)(u + 1) \\
(4.2) & \quad g(u) = \arctan u \\
(4.3) & \quad g(u) = u/(1 + u^2) \\
(4.4) & \quad g(u) = u \exp(-u^2) \\
(4.5) & \quad g(u) = \tanh u
\end{align*}
\]

Clearly (4.1) is similar to \( g(u) = \sin u \) near \( u = 0 \), (4.2) is similar (4.5), and (4.3) is similar (4.4). However there are also subtle differences in each of these pairings. All of these functions share the properties: \( g(0) = 0, g'(0) = 1 \).

Anticipating that any (smooth) breather should have mean value zero, and reducing the number of independent variables by phase-shift symmetry, it is reasonable to consider the functional \( F_1(\alpha, 0, 0, q_1) \) of the single variable \( q_1 > 0 \). As before \( q_1 = q^*_1(\alpha) \) is a zero of \( F_1 \) if and only if \( w(\alpha) = q^*_1(\alpha)v_1 + S(\alpha, q^*_1(\alpha)v_1) \) is a smooth breather of (1.1)-(1.3) (cf. (2.2)-(2.3)).

For each of the choices (4.1)-(4.5) we found that the graph of \( F_1(\alpha, 0, 0, q_1) \), as a function of \( q_1 \), was analogous to Figure 2. Portions of all of these graphs, with \( \alpha = 1.1, \)

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are given in Figure 6. These graphs indicate that smooth breathers exist in each case, and apparently the behavior of \( g(u) \) away from the origin does not change the situation qualitatively. Portions of the corresponding bifurcation diagrams for (4.2) and (4.3) are given in Figure 7, where both the sup norm and the \( H^1 \) norm are used as the vertical measure. We do not present curves for the other functions since they are similar.

In addressing the second question we have thus far only considered a few forcing terms. Generally one could consider a \( T \)-periodic function \( f \in L^2((0, T) \times \mathbb{R}^N) \) which is radially symmetric on \( \mathbb{R}^N \), and replace 0 on the right in (1.1) by \( f \). In (1.5) 0 is replaced by \( h = r^{\frac{N-1}{2}} f \). The particular choices of \( f \) we considered were (again with set \( T = 2\pi \))

\[
(4.6) \quad f(t, x) = A e^{-|x|} \sin t,
\]

\[
(4.7) \quad f(t, x) = A e^{-x^2} \sin t,
\]

\[
(4.8) \quad f(t, x) = A \chi_{(-1,1)}(x) \sin t,
\]

where \( \chi_{(-1,1)} \) is the characteristic function for the interval \((-1,1)\). Again we attempted to determine smooth breathers by considering the function \( F_1(\alpha, 0, 0, q_1) \) of the single variable \( q_1 \). Notice that here the group of symmetries due to phase shifts is lost. However since each function (4.6)-(4.8) oscillates with the frequency and phase shift associated to \( v_1(t, r) \) (cf. (2.4)) the choice of curve \( q_1 \rightarrow (0, 0, q_1) \) seemed reasonable. The graphs of \( F_1 \) corresponding to each of these choices, with \( A = 0.10, \alpha = 1.1 \) and \( g(u) = \sin u \), are plotted in Figure 8. They show that \( F_1 \) now has a second zero at a closer proximity to the origin. Thus there are two smooth solutions of the forced problem, with small forcing amplitude.

The appearance of the second zero can also be thought of as appearing through a bifurcation, where now the bifurcation parameter is the amplitude \( A \). We considered the case in which \( f(t, x) \) is given by (4.8), and determined the zeros \( q^*_i(A)(i = 1, 2) \) of \( F_1(\alpha, 0, 0, q_1) \) for \( A \geq 0 \). During the computations \( \alpha \) was held fixed at \( \alpha = 1.1 \). This produced two functions of \( A \), one corresponding to the smaller zero, \( A \rightarrow q^*_{1,1}(A) \), and one corresponding to the larger zero, \( A \rightarrow q^*_{1,2}(A) \). The graphs of these functions are presented in Figure 9. Near the origin \( A = 0 \), these graphs are as one might expect with \( q^*_{1,1}(A) \rightarrow 0 \) and \( q^*_{1,2}(A) \rightarrow q^*_{1} \) as \( A \rightarrow 0^+ \), where \( q^*_{1} \) is the nontrivial zero of \( F_1(\alpha, 0, 0, q_1) \) corresponding to the smooth solution \( \varphi_1 \) (cf. Figure 2). Near \( A = 0.20 \) something unusual appears. The two zeros appear to coalesce and then disappear. This is further illustrated in Figure 10 where graphs of \( F_1(\alpha, 0, 0, q_1) \), as functions of \( q_1 \), are presented, corresponding to different choices of the amplitude \( A \) near \( A = 0.20 \). Clearly these graphs show that the graph of \( F_1 \) lifts off the plane \( F_1 = 0 \) as \( A \) increases from 0.18 to 0.24. Thus both smooth solutions disappear as \( A \) increases beyond \( A \approx 0.20 \).
§5. The Problem in Dimension Three. The 3-dimensional problem is more delicate numerically, since the nonlinear term is poorly behaved at the origin in this case. Moreover one generally expects less regular behavior of $H^1$ functions in higher dimensions, even in the presence of radial symmetry. With these cautionary remarks made we now report on the results of our computations.

We primarily considered the case in which $g(u) = \sin u$, so that (1.5)-(1.7) becomes the 3-dimensional radially symmetric sine-Gordon equation. To locate breathers on the manifold $M_{\Lambda, \delta}$ we now use the functional $F_3(\alpha, p, q)$ defined in (2.5). Clearly if a generalized solution $w$ satisfies the asymptotic conditions (1.10) then $F_3(\alpha, p^*, q^*) = 0$, where $(p^*, q^*)$ are the coordinates of $w$. Since the asymptotic conditions (1.10) require more than $F_3 = 0$ we checked these conditions when a zero $(p^*, q^*)$ of $F_3$ was found. Numerically we used polynomial interpolation for this purpose. More precisely we used the values of the approximate solution $u = w/r$ and its derivative, at nodal points adjacent to $r = 0$, to construct a quadratic interpolating polynomial. The trace values of the polynomial were then used as limits.

Initially we set $\alpha = 1.1$, a value slightly larger than the expected bifurcation value $\alpha_1 = 1$. The manifold $M_{\Lambda, \delta}$ of generalized solutions is 3-dimensional in this case, and again we can eliminate one of the dimensions by phase-shift symmetry. Assuming solutions have mean value zero we are let to consideration of the function $q_1 \to F_3(\alpha, 0, 0, q_1)$. In Figure 11 we present a graph of this function. Also included in this plot are the graphs of $q_1 \to F_3(\alpha, 0, 0, q_1)$ when $g(u)$ is given by (4.2) and (4.3). Obviously these graphs have the same qualitative shape as the graph in Figure 2, which corresponds to the case of the known breather. A check of the asymptotic conditions (1.10) showed that they are also satisfied. This apparently shows the existence of a smooth breather $w = q_1^* v_1 + S(\alpha, q_1^* v_1)$ (cf. (2.2)) for the sine-Gordon equation corresponding to the value $q_1^* \cong 14.7$, and with $\alpha = 1.1$, as well as for the problems in which $g(u)$ is given by (4.2) and (4.3).

Having determined one solution satisfying (1.10), we attempted to vary $\alpha$ and obtain a (bifurcation) curve $\alpha \to q^*(\alpha) \to w(\alpha)$ of smooth breathers. The results of this are presented in Figure 12. Although one may expect this diagram to be in qualitative agreement with the corresponding diagram, Figure 4, for the 1-dimensional, this is clearly not the case.

In Figure 12 we have actually given graphs corresponding to both $w(t, r)$, which satisfies (1.5)-(1.7), and $u(t, x) = w(t, r)/r$, which satisfies (1.1)-(1.3). In both equations (1.1), (1.5) we used $g(u) = \sin u$. The $H^1$ norm used for $w$ is that defined in (3.6), while for $u$ the standard $H^1((0, T) \times \mathbb{R}^3)$ norm is used. Due to the radial symmetry the later norm turns out to be a simple multiple of the former norm, by a factor $2\sqrt{\pi}$. The supremum norms are not related in such a simple way. Figure 12 indicates that the bifurcation is from infinity in the $H^1$ norm, and from zero in the supremum norm. We also observe in this plot that the curve corresponding to $\sup |u|$ is not monotonic. At present we can offer no explanation for this, other than to say it does not seem to be due to a lack of
grid refinement. However it may be due to numerical inaccuracies at the origin. Almost uniformly in $\alpha$, the supremum was attained at the origin. This is not true of the supremum of $w$.

To illustrate the different nature of these solutions we have plotted some time-slices, $t = \text{constant}$, in Figure 13. This has been done for two different choices of $\alpha$. The nonlinearity used was $g(u) = \sin u$. The solution $u = w/r$ of (1.1)–(1.3) again is seen to have steeper gradients at larger values of $\alpha$.

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REFERENCES

Figure 1

Graph of $\sup |u|$ as a function of $\alpha$, where $u = \varphi_n$ (cf. (1.4)) for $n = 1, 2, 3$. 
Figure 2

Graph of $I^{(a,0,q_1)}_1$ as defined in (2.9), with $q = 1.1$ and $a = a_0$. Above $q^*_1 = 4.65$. 

$F^*_1$
Level curves of the function $f(x, y)$ defined in $\mathbb{R}^2$ in the $P^0$ plane, with $a = 1.5$ and $b = 0.3$.
Graphs of the functionals defined in (6.6)-(6.9) as functions of $\alpha$. Figure 4.
Graphs of $u = \varphi_1(x,t)$ as defined in (1.4), with $t = \pi/2$ and 3 different choices of $\alpha$. 

Figure 5
Figure 6

Graph of the functional $F(\alpha, \beta, \gamma)$ defined in (5.2) for the choice of $\gamma(n)$ given in (4.1)-(4.7), with $\alpha = 1.1$.

- $\gamma(n)$
- $\gamma^{0.8} n = (n)^{0.8}$
- $\gamma^{0.8} n = (n)^{0.8} 2^n$
- $(\gamma + 1)/n = (n)^{0.8}$
- $\gamma(n)$ approximation
- $(1 + n)(1 - n)n = (n)^{0.8}$
Figure 7

(\(n\))H = |n| \text{ and } \left(\frac{\text{arcsec}(n)}{g} + 1\right)/1 = (\(n\))g

(\(n\))H = |n| \text{ and } \left(\frac{\text{arcsec}(n)}{g} + 1\right)/1 = (\(n\))g

(\(n\))H = |n| \text{ and } \left(\frac{\text{arcsec}(n)}{g} + 1\right)/1 = (\(n\))g
Figure 8

Graphs of $P^I(a,0,q)$, $I_b$, and $P^I(a,q)$, with $a = 0.1$, $q = 0.1$, $a = 1.1$, and $a = 6$. Corresponding to the formulae given in (4.9)-(4.10).
Figure 9

Zeroes of $P_{a}(a^2-a^1)$ plotted as functions of the amplitude $A$ for the problem with $a = 1.1, 0$. Given by (4.8).
Figure 10

Graph of the functional $F'(u, x, a)$ with $a = 0$, $a = 0.02$, $a = 0.05$, $a = 0.2$, $a = 0.19$, and $a = 0.18$. The curves show the amplitude $a$.
Figure II

Graph of $F^{(a, 0.94)}$ as defined in (3.6) with $a = 1.1$ and 3 different slopes at $x(0)$.

$\frac{\sqrt{n + 1}}{n} = (n)^{\beta}$

$\sin(n) = (n)^{\beta}$

$\arctan(n) = (n)^{\beta}$
Figure 12

\[ \text{Explanation or description of the graph content here.} \]
Figure 13

Graphs of smooth breather solutions $u = \psi(r)$ of (1.1)-(1.3) and the associated functions $w$, with $N = 3$ and $d(0) = \sin(0)$. 

$\alpha = 1.01$ 

$\alpha = 1.23$
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